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A NEW H(div)-CONFORMING p-INTERPOLATION OPERATOR IN TWO DIMENSIONS *

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Abstract. In this paper we construct a new $\mathbf{H}(\mathrm{div})$-conforming projection-based p-interpolation operator that assumes only $\mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$-regularity ($r > 0$) on the reference element (either triangle or square) K. We show that this operator is stable with respect to polynomial degrees and satisfies the commuting diagram property. We also establish an estimate for the interpolation error in the norm of the space $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$, which is closely related to the energy spaces for boundary integral formulations of time-harmonic problems of electromagnetics in three dimensions.

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1. Introduction and main results

This paper addresses the problem of $\mathbf{H}(\mathrm{div})$-conforming interpolation of low-regular vector fields by high order polynomials. Corresponding p-interpolation operators are relevant for the analysis of high order *boundary element* approximations for time-harmonic problems of electromagnetics.

Aiming at high-order *finite element* (FE) approximations of Maxwell's equations, Demkowicz and Babuška [19] introduced and analysed two projection-based p-interpolation operators satisfying the commuting diagram property (de Rham diagram). These are the H^1-conforming interpolation operator $\Pi_p^1 : H^{1+r}(K) \to \mathcal{P}_p(K)$ and the $\mathbf{H}(\mathrm{curl})$-conforming interpolation operator $\Pi_p^{\mathrm{curl}} : \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{curl}, K) \to \boldsymbol{\mathcal{P}}_p^{\mathrm{Ned}}(K)$; here $r > 0$ in both cases, K is the reference element (either triangle or square), $\mathcal{P}_p(K)$ is the set of polynomials of degree $\leq p$ on K, and $\boldsymbol{\mathcal{P}}_p^{\mathrm{Ned}}(K)$ is the $\mathbf{H}(\mathrm{curl})$-conforming (first) Nédélec space of degree p (precise definitions of all involved Sobolev spaces and polynomial sets are given in Sect. 2.1 below).

In 2D, the operators curl and div are isomorphic. The corresponding polynomial set isomorphic to the Nédélec space $\boldsymbol{\mathcal{P}}_p^{\mathrm{Ned}}(K)$ is the Raviart-Thomas (RT) space denoted by $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$. Therefore, the results of [19] related to the operator Π_p^{curl} can be used also in the $\mathbf{H}(\mathrm{div})$-conforming settings (we will denote the corresponding $\mathbf{H}(\mathrm{div})$-conforming projection-based interpolation operator by Π_p^{div}). In particular, given a vector field $\mathbf{u} \in$

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$\mathbf{H}^r(K)\cap\mathbf{H}(\mathrm{div},K)$ with $r>0$, the interpolant $\tilde{\mathbf{u}}^p=\Pi_p^{\mathrm{div}}\mathbf{u}\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ is defined as the sum of three terms:

$$\tilde{\mathbf{u}}^p=\mathbf{u}_1+\mathbf{u}_2^p+\tilde{\mathbf{u}}_3^p, \tag{1.1}$$

where $\mathbf{u}_1$ is a lowest order interpolant, $\mathbf{u}_2^p$ is the sum of edge interpolants, and $\tilde{\mathbf{u}}_3^p$ is the interior interpolant (a more detailed description of these interpolants is given in Sect. 2.5). As follows from [19], the following diagram commutes:

$$\begin{array}{ccccc} H^{1+r}(K) & \xrightarrow{\mathbf{curl}} & \mathbf{H}^r(K)\cap\mathbf{H}(\mathrm{div},K) & \xrightarrow{\mathrm{div}} & L^2(K) \\ \downarrow\Pi_p^1 & & \downarrow\Pi_p^{\mathrm{div}} & & \downarrow\Pi_{p-1}^0 \\ \mathcal{P}_p(K) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{p-1}(K) \end{array} \tag{1.2}$$

where $\Pi_p^0:\ L^2(K)\to\mathcal{P}_p(K)$ denotes the standard L^2-projection onto the set of polynomials $\mathcal{P}_p(K)$.

The commuting diagram property, and the corresponding p-interpolation error estimates, have immediate applications to the analysis of high-order FE discretisations of time-harmonic Maxwell's equations. In particular, these results are critical to prove the discrete compactness property, which in turn implies the convergence of FE approximations for Maxwell's equations, as well as for the error analysis (see [3,7–9,22]). We note that classical Nédélec or RT interpolation operators (see, *e.g.*, [10]) are not suitable for these purposes, as they are not stable (with respect to the polynomial degree p) for low-regular fields and do not work equally well for triangular and parallelogram elements.

When time-harmonic problems of electromagnetics are posed in infinite domains (*e.g.*, outside a scatterer), it is convenient to reformulate them as a boundary integral equation (on the surface of the scatterer). The energy spaces for such boundary integral equations (BIE) involve Sobolev spaces of negative order for both the vector field and its divergence (a typical example is the space $\mathbf{H}^{-1/2}(\mathrm{div},\Gamma)$ in the case of a smooth (closed) surface Γ). Then, it is common to use the $\mathbf{H}(\mathrm{div})$-conforming boundary elements (*e.g.*, of RT type) to discretise these BIE. The fundamental problem is that the underlying integral operator is not coercive, and the convergence analysis of the boundary element methods (BEM) requires a suitable regular decomposition of the energy space into the space of divergence-free vector fields and the complementary space, *cf.* [11]. In the case of Maxwell's source problem it is possible to use a decomposition, where the complementary space is regular enough even on non-smooth surfaces. Then, the $\mathbf{H}(\mathrm{div})$-conforming p-interpolation operator of Demkowicz and Babuška is applicable for the convergence and error analysis of the p- and the hp-BEM (see [5,6]). However, when considering the boundary integral formulation for the Maxwell eigenvalue problem, the orthogonal Hodge decomposition of the energy space (see [12,14]) must be used to prove the discrete compactness property. In this case, the regularity issues on non-smooth surfaces affect the smoothness of the complementary space and prevent one from using the known $\mathbf{H}(\mathrm{div})$-conforming interpolation operators. Hence, the aim of this paper is to introduce and analyse a new $\mathbf{H}(\mathrm{div})$-conforming p-interpolation operator, which is stable with respect to p and retains the commuting diagram property analogous to (1.2), but assumes less regularity than Π_p^{div} (namely, $\mathbf{H}^r(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)$-regularity with $r>0$). This new interpolation operator will be denoted by $\Pi_p^{\mathrm{div},-\frac{1}{2}}$, and is defined as follows.

Given a vector field $\mathbf{u}\in\mathbf{H}^r(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)$ with $r>0$, we define the interpolant $\mathbf{u}^p=\Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u}\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ in a similar way as the interpolant $\tilde{\mathbf{u}}^p=\Pi_p^{\mathrm{div}}\mathbf{u}\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ (see (1.1)):

$$\mathbf{u}^p=\mathbf{u}_1+\mathbf{u}_2^p+\mathbf{u}_3^p. \tag{1.3}$$

Here, $\mathbf{u}_1$ and $\mathbf{u}_2^p$ are exactly the same as for the interpolant $\Pi_p^{\mathrm{div}}\mathbf{u}$ (see (2.15) and (2.20), respectively), whereas $\mathbf{u}_3^p\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ is determined by solving the following system of equations:

$$\langle\mathrm{div}(\mathbf{u}-(\mathbf{u}_1+\mathbf{u}_2^p+\mathbf{u}_3^p)),\mathrm{div}\,\mathbf{v}\rangle_{\tilde{H}^{-1/2}(K)}=0\qquad\forall\mathbf{v}\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K), \tag{1.4}$$

$$\langle\mathbf{u}-(\mathbf{u}_1+\mathbf{u}_2^p+\mathbf{u}_3^p),\mathbf{curl}\,\phi\rangle_{0,K}=0\qquad\forall\phi\in\mathcal{P}_p^0(K), \tag{1.5}$$

where $\langle\cdot,\cdot\rangle_{\tilde{H}^{-1/2}(K)}$ and $\langle\cdot,\cdot\rangle_{0,K}$ denote the $\tilde{H}^{-1/2}(K)$- and the $\mathbf{L}^2(K)$-inner products respectively.

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Our construction of the interpolation operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ is much in the spirit of [19]. In a more recent publication [18] Demkowicz presents a unified theory of projection-based interpolation operators and their commuting properties. Whereas in [19] the construction of the interpolation operator is based on L^2-inner products (see (2.21) below), in [18] this technique is generalized to inner products of Sobolev spaces of order $\geq -\frac{1}{2}$. We focus precisely on the $\tilde{H}^{-1/2}$-inner product, used in (1.4), which is more natural than the L^2-inner product in the boundary element setting.

For the $\mathbf{H}(\mathrm{curl})$-conforming interpolation operator $\Pi_{-\frac{1}{2}}^{\mathrm{curl}}$, Demkowicz proves an almost optimal error estimate [18], Theorem 4.3, under the same regularity assumption (with curl instead of div) as in our error estimate, Theorem 1.3 below. The operator $\Pi_{-\frac{1}{2}}^{\mathrm{curl}}$ is defined by $H^{-1/2}$-projections on the element level and needs H^{-1}-regularity of the tangential trace of the given function (*cf.* [18], eq. (148) with $s=\frac{1}{2}$). This limit case of the trace theorem (*cf.* [18], No. (154)) causes an additional $(\log p)^{1/2}$-factor in the error bound. Our operator is different in the sense that we are using different inner products in (1.4) and (1.5), namely $\tilde{H}^{-1/2}$ and L^2, respectively, instead of L^2 [19] or $H^{-1/2}$ [18]. In our setting the corresponding regularity makes the normal trace of the given function well defined (see Lem. 2.1 below). Here, not only the different orders of the norms are essential (which has been observed by Demkowicz [18], Nos. (154), (155)) but also the subtle difference between $\tilde{H}^{-1/2}$ and $H^{-1/2}$. By this careful selection of inner products we are able to define the operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ so that it satisfies an optimal error estimate and is uniformly stable in p (for functions in $\mathbf{H}^r$ $(r>0)$ whose divergence is in $\tilde{H}^{-1/2}$).

The $\tilde{H}^{-1/2}(K)$-inner product has to be written in an appropriate explicit form. Of particular importance for our analysis is the following property of the $\tilde{H}^{-1/2}(K)$-inner product $\langle u,v\rangle_{\tilde{H}^{-1/2}(K)}$: for a constant function v, it reduces to the $L^2(K)$-inner product, *i.e.*,

$$\langle u,1\rangle_{\tilde{H}^{-1/2}(K)} = \langle u,1\rangle_{0,K} \qquad \forall u\in \tilde{H}^{-1/2}(K). \tag{1.6}$$

An inner product satisfying this property is presented in the Appendix (see Lems. A.2 and A.3). Other discrete inner products, being equivalent to the continuous inner products for polynomials, are analysed in [17].

In the following three theorems we formulate the main results of the paper – the properties of the operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ (all proofs are given in Sect. 3 below).

The first theorem justifies the definition of the operator and states its continuity.

Theorem 1.1. *For $r>0$ the operator*

$$\Pi_p^{\mathrm{div},-\frac{1}{2}} : \mathbf{H}^r(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)\to \mathbf{L}^2(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)$$

is well defined and bounded, with its operator norm being independent of p, i.e., *there exists a constant $C>0$ independent of p (but depending on r) such that*

$$\left\|\Pi_p^{\mathrm{div},-\frac{1}{2}}\right\|_{\mathcal{L}} \leq C, \tag{1.7}$$

where $\|\cdot\|_{\mathcal{L}}$ is the operator norm in the space $\mathcal{L}\Big(\mathbf{H}^r(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K),\mathbf{L}^2(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)\Big)$. Moreover, the operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ preserves polynomial vector fields, i.e., *$\Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{v}_p=\mathbf{v}_p$ for any $\mathbf{v}_p\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$.*

The next theorem states the commuting diagram property analogous to (1.2).

Theorem 1.2. *For $r>0$ the following diagram commutes:*

$$\begin{array}{ccccc} H^{1+r}(K) & \xrightarrow{\mathbf{curl}} & \mathbf{H}^r(K)\cap\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K) & \xrightarrow{\mathrm{div}} & \tilde{H}^{-1/2}(K) \\ \downarrow \Pi_p^1 & & \downarrow \Pi_p^{\mathrm{div},-\frac{1}{2}} & & \downarrow \Pi_{p-1}^{-1/2} \\ \mathcal{P}_p(K) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{p-1}(K) \end{array} \tag{1.8}$$

where $\Pi_p^{-1/2} : \tilde{H}^{-1/2}(K)\to\mathcal{P}_p(K)$ denotes the $\tilde{H}^{-1/2}$-projector.

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The third theorem provides an error estimate for the interpolation operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ in the norm of the space $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$, which is closely related to the energy space for the electric field integral equation (a boundary integral formulation of Maxwell's equations in 3D).

Theorem 1.3. *If* $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$ *with* $r > 0$, *then there exists a positive constant* C *independent of* $\mathbf{u}$ *and* p *such that*

$$\|\mathbf{u} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\,\mathbf{u}\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)} \le C\, p^{-(r+1/2)}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},K)}. \tag{1.9}$$

Remark 1.1. Using similar constructions it is possible to introduce a stable $\mathbf{H}(\mathrm{div})$-conforming p-interpolation operator for even less regular vector fields $\mathbf{u} \in \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^s(\mathrm{div}, K)$, $r > 0$, $-1 \le s < -\frac{1}{2}$. However, our proof of the commuting diagram property carries over to this operator if the $\tilde{H}^s(K)$-inner product reduces to the L^2-inner product for a constant function (*cf.* property (1.6)), which is an open problem. Although we note that such an operator would be of purely theoretical interest.

Remark 1.2. Theorem 1.3 states the interpolation error estimate for sufficiently regular vector fields, for which one can also apply the operator Π_p^{div}. For BIE of electromagnetics on open surfaces, the solution is less regular and belongs to $\mathbf{H}^r(\mathrm{div}, \Gamma)$, where $r \in (-\frac{1}{2}, 0)$ and Γ is an open surface (see [15], Sect. 4.4, and [4], Appendix A). To obtain the error estimate for the corresponding BEM in this case, one can apply the (global) orthogonal projection P_p with respect to the energy norm $\|\cdot\|_{\mathbf{X}}$. Then the estimate for $\|\mathbf{u} - P_p\mathbf{u}\|_{\mathbf{X}}$ can be reduced to the estimate obtained in Theorem 1.3 in the same way as in [12] or [5]. Note that using the projector P_p locally does not guarantee the conformity of approximations (*i.e.*, the continuity of normal components across inter-element boundaries). Moreover, the projector P_p does not satisfy the commuting diagram property in (1.8), and, thus, it is not suitable for such purposes as the convergence analysis and the proof of the discrete compactness.

The rest of the paper is organised as follows. Section 2 gives necessary preliminaries: we introduce the notation, recall definitions of functional spaces of scalar functions and vector fields, and collect auxiliary results. In particular, we give a more detailed description of the interpolation operators Π_p^1, Π_p^{div} and summarise their properties (see Sect. 2.5). In Section 3 we prove the main theorems formulated above. Finally, in the Appendix we introduce some equivalent norms in the Sobolev spaces $H^r(K)$ and $\tilde{H}^r(K)$ ($r = \pm\frac{1}{2}$), derive expressions for corresponding inner products, and establish the key property (1.6) for the $\tilde{H}^{-1/2}$-inner product.

2. Preliminaries

2.1. Functional spaces and polynomial sets

In what follows, $p \ge 0$ will always specify a polynomial degree and C denotes a generic positive constant which is independent of p and involved functions, unless stated otherwise. Furthermore, throughout the paper, K is either the equilateral reference triangle $T = \{\mathbf{x} = (x_1, x_2);\ x_2 > 0,\ x_2 < x_1\sqrt{3},\ x_2 < (1 - x_1)\sqrt{3}\}$ or the reference square $Q = (0, 1)^2$. A generic edge of K will be denoted by ℓ, and $\mathbf{n}$ denotes the outward normal unit vector to ∂K.

We will use the standard definitions for the Sobolev spaces $H^r(\Omega)$ ($r \ge 0$) of scalar functions on Ω, see, *e.g.*, [23] (hereafter, Ω is either the unit interval $I = (0, 1)$ or the reference element K). The norms in these spaces are denoted by $\|\cdot\|_{H^r(\Omega)}$. For $r \in (0, 1)$ we will also need the Sobolev spaces $\tilde{H}^r(\Omega)$ which are defined by interpolation. We use the real K-method of interpolation (see [23]) to define

$$\tilde{H}^r(\Omega) = \Big(L^2(\Omega), H_0^t(\Omega)\Big)_{\frac{r}{t},2} \quad (1/2 < t \le 1,\ 0 < r < t).$$

Here, $H_0^t(\Omega)$ ($0 < t \le 1$) is the completion of $C_0^\infty(\Omega)$ in $H^t(\Omega)$ and we identify $H_0^1(\Omega)$ with $\tilde{H}^1(\Omega)$. Note that the Sobolev spaces $H^r(\Omega)$ also satisfy the interpolation property

$$H^r(\Omega) = \Big(L^2(\Omega), H^1(\Omega)\Big)_{r,2} \quad (0 < r < 1)$$

with equivalent norms.

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The L^2-inner product and the corresponding L^2-norm on Ω are denoted by $\langle\cdot,\cdot\rangle_{0,\Omega}$ and $\|\cdot\|_{0,\Omega}$, respectively. For $r \in [-1,0)$ the Sobolev spaces and their norms are defined by duality with $L^2(\Omega) = H^0(\Omega) = \tilde{H}^0(\Omega)$ as pivot space:

$$H^r(\Omega) = \big(\tilde{H}^{-r}(\Omega)\big)', \quad \tilde{H}^r(\Omega) = \big(H^{-r}(\Omega)\big)',$$
$$\|u\|_{H^r(\Omega)} = \sup_{0\neq v\in\tilde{H}^{-r}(\Omega)} \frac{|\langle u,v\rangle_{0,\Omega}|}{\|v\|_{\tilde{H}^{-r}(\Omega)}}, \quad \|u\|_{\tilde{H}^r(\Omega)} = \sup_{0\neq v\in H^{-r}(\Omega)} \frac{|\langle u,v\rangle_{0,\Omega}|}{\|v\|_{H^{-r}(\Omega)}}. \tag{2.1}$$

Note that the Sobolev spaces H^r and $\tilde{H}^r$ on any edge $\ell \subset \partial K$ are defined by using the definitions of the corresponding spaces on the interval I.

In the Appendix we consider some other expressions for norms in the Sobolev spaces $H^r(K)$ and $\tilde{H}^r(K)$ with $r = \pm\frac{1}{2}$. We will prove their equivalence to the norms defined above, and we will also derive expressions for corresponding inner products.

Throughout the paper, we use boldface symbols for vector fields. The spaces (or sets) of vector fields are denoted in boldface as well (*e.g.*, $\mathbf{H}^r(K) = (H^r(K))^2$), with their norms and inner products being defined component-wise. Similarly to the scalar case, the norm and inner product in $\mathbf{L}^2(K)$ will be denoted by $\langle\cdot,\cdot\rangle_{0,K}$ and $\|\cdot\|_{0,K}$, respectively, which should not lead to any confusion. The standard notation will be used for differential operators $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $\mathrm{div} = \nabla\cdot$, $\mathrm{curl} = \nabla\times$, and for the Laplace operator $\Delta = \mathrm{div}\,\nabla$.

Furthermore, we will use the following spaces

$$\mathbf{H}^r(\mathrm{div},K) := \{\mathbf{u}\in\mathbf{H}^r(K);\ \mathrm{div}\,\mathbf{u}\in H^r(K)\}, \quad r\geq 0$$

and

$$\tilde{\mathbf{H}}^r(\mathrm{div},K) := \{\mathbf{u}\in\tilde{\mathbf{H}}^r(K);\ \mathrm{div}\,\mathbf{u}\in\tilde{H}^r(K)\}, \quad r\in\Big[-1,-\tfrac{1}{2}\Big].$$

These spaces are equipped with their graph norms $\|\cdot\|_{\mathbf{H}^r(\mathrm{div},K)}$ and $\|\cdot\|_{\tilde{\mathbf{H}}^r(\mathrm{div},K)}$, respectively. For $r=0$ we drop the superscript in the above notation: $\mathbf{H}^0(\mathrm{div},K) = \mathbf{H}(\mathrm{div},K)$.

Finally, we will need two sub-spaces incorporating homogeneous boundary conditions for the trace of the normal component on ∂K. By $\mathbf{H}_0(\mathrm{div},K)$ (resp., $\tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div},K)$) we denote the subspace of elements $\mathbf{u}\in\mathbf{H}(\mathrm{div},K)$ (resp., $\mathbf{u}\in\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)$) such that for all $v\in C^\infty(K)$ there holds

$$\langle\mathbf{u},\nabla v\rangle_{0,K} + \langle\mathrm{div}\,\mathbf{u},v\rangle_{0,K} = 0. \tag{2.2}$$

We note that if $\mathbf{u}\in\tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div},K)$, then identity (2.2) holds for any $v\in H^{3/2}(K)$ by density. In particular, $\tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div},K)$ is a closed subspace of $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)$.

Let us now introduce the polynomial sets we need. By $\mathcal{P}_p(I)$ we denote the set of polynomials of degree $\leq p$ on the interval I, and $\mathcal{P}_p^0(I)$ denotes the subset of $\mathcal{P}_p(I)$ which consists of polynomials vanishing at the end points of I. In particular, these two sets will be used for any edge $\ell\subset\partial K$.

Further, $\mathcal{P}_p^1(T)$ denotes the set of polynomials on T of total degree $\leq p$, and $\mathcal{P}_{p_1,p_2}^2(Q)$ is the set of polynomials on Q of degree $\leq p_1$ in x_1 and of degree $\leq p_2$ in x_2. For $p_1=p_2=p$ we denote $\mathcal{P}_p^2(Q) = \mathcal{P}_{p,p}^2(Q)$, and we will use the unified notation $\mathcal{P}_p(K)$, which refers to $\mathcal{P}_p^1(T)$ if $K=T$ and to $\mathcal{P}_p^2(Q)$ if $K=Q$. The corresponding set of polynomial (scalar) bubble functions on K is denoted by $\mathcal{P}_p^0(K)$.

Let us denote by $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ the RT-space of order $p\geq 1$ on the reference element K (see, *e.g.*, [10,25]), *i.e.*,

$$\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K) = (\mathcal{P}_{p-1}(K))^2 \oplus \mathbf{x}\mathcal{P}_{p-1}(K) = \begin{cases} (\mathcal{P}_{p-1}^1(T))^2 \oplus \mathbf{x}\mathcal{P}_{p-1}^1(T) & \text{if } K=T,\\ \mathcal{P}_{p,p-1}^2(Q)\times\mathcal{P}_{p-1,p}^2(Q) & \text{if } K=Q. \end{cases}$$

The subset of $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ which consists of vector-valued polynomials with vanishing normal trace on the boundary ∂K (vector bubble-functions) will be denoted by $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$.

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2.2. Auxiliary lemmas

First, let us formulate the following result, which will be used frequently in what follows.

Lemma 2.1. *The normal trace mapping* $\mathbf{u} \mapsto \mathbf{u} \cdot \mathbf{n}$ *defines a linear and continuous operator from* $\mathbf{H}^s(K) \cap \tilde{\mathbf{H}}^{-1+s}(\mathrm{div}, K)$ *to* $H^{-1/2+s}(\partial K)$ *for* $s \in [0, \frac{1}{2})$.

Proof. Let us denote by γ_{tr} the standard (scalar) trace operator with $\gamma_{\mathrm{tr}} : H^{1-s}(K) \to H^{1/2-s}(\partial K)$ for $s \in [0, \frac{1}{2})$, and let $\gamma_{\mathrm{tr}}^{-1} : H^{1/2-s}(\partial K) \to H^{1-s}(K)$ be a right inverse of γ_{tr}. Let $\mathbf{u} \in \mathbf{H}^s(K) \cap \tilde{\mathbf{H}}^{-1+s}(\mathrm{div}, K)$. Taking an arbitrary $v \in H^{1/2-s}(\partial K)$ we integrate by parts to obtain

$$\begin{aligned}
\int_{\partial K} (\mathbf{u} \cdot \mathbf{n})\, v \,\mathrm{d}\sigma &= \int_K (\mathrm{div}\, \mathbf{u})\, \gamma_{\mathrm{tr}}^{-1} v \,\mathrm{d}\mathbf{x} + \int_K \mathbf{u} \cdot \nabla(\gamma_{\mathrm{tr}}^{-1} v) \,\mathrm{d}\mathbf{x} \\
&\le \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1+s}(K)} \, \|\gamma_{\mathrm{tr}}^{-1} v\|_{H^{1-s}(K)} + \|\mathbf{u}\|_{\mathbf{H}^s(K)} \, \|\nabla(\gamma_{\mathrm{tr}}^{-1} v)\|_{\mathbf{H}^{-s}(K)} \\
&\le C \left(\|\mathbf{u}\|_{\mathbf{H}^s(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1+s}(K)} \right) \|v\|_{H^{1/2-s}(\partial K)}.
\end{aligned}$$

Hence, $\mathbf{u} \cdot \mathbf{n} \in H^{-1/2+s}(\partial K)$ and we prove the continuity of the normal trace mapping:

$$\begin{aligned}
\|\mathbf{u} \cdot \mathbf{n}\|_{H^{-1/2+s}(\partial K)} &= \sup_{0 \neq v \in H^{1/2-s}(\partial K)} \frac{|\int_{\partial K} (\mathbf{u} \cdot \mathbf{n})\, v \,\mathrm{d}\sigma|}{\|v\|_{H^{1/2-s}(\partial K)}} \\
&\le C \left(\|\mathbf{u}\|_{\mathbf{H}^s(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1+s}(K)} \right).
\end{aligned}$$

□

We will also need the following p-approximation result in 2D (see [1], Lem. 4.1).

Lemma 2.2. *Let* K *be the reference triangle or square. Then there exists a family of operators* $\{\pi_p\}$, $p = 1, 2, \ldots$, $\pi_p : H^r(K) \to \mathcal{P}_p(K)$ *such that for any* $f \in H^r(K)$, $r \ge 0$ *there holds*

$$\|f - \pi_p f\|_{H^t(K)} \le C p^{-(r-t)} \|f\|_{H^r(K)}, \qquad 0 \le t \le r.$$

Moreover, π_p *preserves polynomials of degree* p, i.e., $\pi_p f = f$ *if* $f \in \mathcal{P}_p(K)$.

We use this result, in particular, to prove the next lemma, which provides an optimal error estimate for the $\tilde{H}^{-1/2}$-projector $\Pi_p^{-1/2} : \tilde{H}^{-1/2}(K) \to \mathcal{P}_p(K)$.

Lemma 2.3. *Let* $\phi \in H^r(K)$, $r > -\frac{1}{2}$. *Then for any* $p \ge 0$ *there holds*

$$\|\phi - \Pi_p^{-1/2} \phi\|_{\tilde{H}^{-1/2}(K)} \le C(p+1)^{-(1/2+r)} \|\phi\|_{H^r(K)}. \tag{2.3}$$

Proof. If $p = 0$ then (2.3) is trivial. Let $p \ge 1$. First, we assume that $r > 0$. Using the standard duality argument and the p-approximation result of Lemma 2.2, we estimate the error of the L^2-projection $\Pi_p^0 : L^2(K) \to \mathcal{P}_p(K)$ in the $\tilde{H}^{-1/2}$-norm:

$$\begin{aligned}
\|\phi - \Pi_p^0 \phi\|_{\tilde{H}^{-1/2}(K)} &\le \|\phi - \Pi_p^0 \phi\|_{0,K} \sup_{\varphi \in H^{1/2}(K) \setminus \{0\}} \inf_{\varphi_p \in \mathcal{P}_p(K)} \frac{\|\varphi - \varphi_p\|_{0,K}}{\|\varphi\|_{H^{1/2}(K)}} \\
&\le \|\phi - \Pi_p^0 \phi\|_{0,K} \sup_{\varphi \in H^{1/2}(K) \setminus \{0\}} \frac{\|\varphi - \Pi_p^0 \varphi\|_{0,K}}{\|\varphi\|_{H^{1/2}(K)}} \\
&\le C\,(p+1)^{-(1/2+r)} \, \|\phi\|_{H^r(K)}.
\end{aligned}$$

This estimate yields (2.3) due to the minimization property of the $\tilde{H}^{-1/2}$-projection.

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Now, let $r \in (-\frac{1}{2}, 0]$. Assuming that $\phi \in H^s(K) = \tilde{H}^s(K)$ with some $s \in (0, \frac{1}{2})$ and using the first part of the proof, we have

$$\|\phi - \Pi_p^{-1/2}\phi\|_{\tilde{H}^{-1/2}(K)} \leq C\,(p+1)^{-(1/2+s)}\,\|\phi\|_{H^s(K)}.$$

On the other hand, it is trivial that

$$\|\phi - \Pi_p^{-1/2}\phi\|_{\tilde{H}^{-1/2}(K)} \leq \|\phi\|_{\tilde{H}^{-1/2}(K)}.$$

Therefore, we prove by interpolation that

$$\begin{aligned}\|\phi - \Pi_p^{-1/2}\phi\|_{\tilde{H}^{-1/2}(K)} &\leq C\,(p+1)^{-(1/2+r)}\,\|\phi\|_{\tilde{H}^r(K)} \\ &\leq C(r)\,(p+1)^{-(1/2+r)}\,\|\phi\|_{H^r(K)} \qquad \forall \phi \in H^s(K).\end{aligned}$$

Hence, by density of regular functions in $H^r(K)$, we obtain (2.3), and the proof is finished. □

The following lemma states the inverse inequality for polynomials on the reference element K.

Lemma 2.4. *Let $v_p \in \mathcal{P}_p(K)$. Then for any s, $r \in [-1, 1]$ with $s \leq r$ there holds*

$$\|v\|_{H^r(K)} \leq C\,p^{2(r-s)}\,\|v\|_{H^s(K)},$$

where C is a positive constant independent of p.

For $r \geq 0$, $s = 0$ the proof is based on Schmidt's inequality and given in [20] for both types of reference elements (see Lem. 5.1 and its proof therein). By using interpolation arguments and induction, this result has been extended in [21] to the full range of parameters s, $r \in [-1, 1]$.

2.3. The regularized Poincaré integral operators

In [16], Costabel and McIntosh studied a regularized version of the Poincaré-type integral operator acting on differential forms in $\mathbb{R}^n$. They proved, in particular, that this operator is bounded on a wide range of functional spaces including the whole scale of Sobolev spaces $H^r(\Omega)$ ($r \in \mathbb{R}$) on a bounded Lipschitz domain Ω which is star-like with respect to an open ball. Moreover, the essential polynomial preserving property of the classical Poincaré map is retained by its regularized version. Thus, the results of [16] have immediate applications to the analysis of high-order elements (see, *e.g.*, [3,6,22]).

Let us formulate some results of [16] in two particular cases. Namely, we will define two Poincaré-type integral operators: one operator acts on scalar functions, and the other one acts on divergence-free vector fields. In both cases the functions and vector fields are defined on the reference element K. Denoting by B an open ball in K, let us consider a smoothing function

$$\theta \in C^\infty(\mathbb{R}^2), \quad \operatorname{supp}\theta \subset B, \quad \int_B \theta(\mathbf{a})\,\mathrm{d}\mathbf{a} = 1, \quad \mathbf{a} = (a_1, a_2).$$

Then the first regularized Poincaré-type integral operator $R : C^\infty(\bar{K}) \to (C^\infty(\bar{K}))^2$ (*i.e.*, the operator acting on scalar functions) is defined as $R\psi = (R_1, R_2)$, where

$$R_i(\mathbf{x}) := \int_B \theta(\mathbf{a})\,(x_i - a_i) \int_0^1 t\psi(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))\,\mathrm{d}t\,\mathrm{d}\mathbf{a}, \quad i = 1, 2.$$

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The second operator acting on vector fields is defined as follows:

$$A : (C^\infty(\bar{K}))^2 \to C^\infty(\bar{K}),$$
$$A\mathbf{u}(\mathbf{x}) := \int_B \theta(\mathbf{a}) \Big((x_2 - a_2) \int_0^1 u_1(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))\, \mathrm{d}t - (x_1 - a_1) \int_0^1 u_2(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))\, \mathrm{d}t \Big) \mathrm{d}\mathbf{a},$$

where $\mathbf{u} = (u_1, u_2)$.

The following properties of the operators R and A are easy to check directly (see also [16], Prop. 4.2):

(R1) R is a right inverse of the div operator, *i.e.*,

$$\operatorname{div}(R\psi) = \psi \qquad \forall \psi \in H^r(K), \quad r \geq 0;$$

(A1) if $\mathbf{u}$ is divergence-free, then A is a right inverse of the vector curl, *i.e.*,

$$\mathbf{curl}(A\mathbf{u}) = \mathbf{u} \qquad \forall \mathbf{u} \in \mathbf{H}^r(\operatorname{div}0, K) = \{\mathbf{u} \in \mathbf{H}^r(K);\ \operatorname{div}\mathbf{u} = 0 \text{ in } K\}, \quad r \geq 0.$$

The operators R and A satisfy the following continuity properties (see [16], Cor. 3.4):

(R2) the mapping R defines a bounded operator $H^{r-1}(K) \to \mathbf{H}^r(K)$ for any $r \geq 0$;
(A2) the mapping A defines a bounded operator $\mathbf{H}^r(K) \to H^{r+1}(K)$ for any $r \geq 0$.

Furthermore, the operators R and A preserve polynomials:

(R3) R maps $\mathcal{P}_p(K)$ into $\boldsymbol{\mathcal{P}}^{\mathrm{RT}}_{p+1}(K)$;
(A3) A maps $\boldsymbol{\mathcal{P}}^{\mathrm{RT}}_{p}(K)$ into $\mathcal{P}_p(K)$.

We will use the operators R and A to prove the following auxiliary lemma.

Lemma 2.5. *Let $r > 0$ and $s \geq r - 1$. If $\mathbf{u} \in \mathbf{H}^r(K)$ and $\operatorname{div}\mathbf{u} \in H^s(K)$, then there exist a function $\psi \in H^{r+1}(K)$ and a vector field $\mathbf{v} \in \mathbf{H}^{s+1}(K)$ such that*

$$\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}. \tag{2.4}$$

Moreover,

$$\|\mathbf{v}\|_{\mathbf{H}^{s+1}(K)} \leq C\, \|\operatorname{div}\mathbf{u}\|_{H^s(K)} \qquad \text{and} \qquad \|\psi\|_{H^{r+1}(K)} \leq C\, \|\mathbf{u}\|_{\mathbf{H}^r(K)}. \tag{2.5}$$

Proof. The proof is exactly the same as for Lemma 2.3 in [3]. We use the operators R and A to define ψ and $\mathbf{v}$:

$$\mathbf{v} := R(\operatorname{div}\mathbf{u}) \in \mathbf{H}^{s+1}(K), \quad \psi := A(\mathbf{u} - R(\operatorname{div}\mathbf{u})) \in H^{\min\{r, s+1\}+1}(K) = H^{r+1}(K).$$

Hence, due to properties (R1) and (A1), the vector field $\mathbf{u}$ can be decomposed as in (2.4):

$$\mathbf{u} = (\mathbf{u} - R(\operatorname{div}\mathbf{u})) + R(\operatorname{div}\mathbf{u}) = \mathbf{curl}\,\psi + \mathbf{v}.$$

Inequalities (2.5) are then obtained by using the continuity properties of the Poincaré-type operators and the boundedness of the divergence operator as a mapping $\mathbf{H}^r(K) \to H^{r-1}(K)$ for $r \geq 0$ (*cf.* [3], Lem. 2.3). □

Remark 2.1. Note that $\mathbf{u} \in \mathbf{H}^r(K)$ implies that $\operatorname{div}\mathbf{u} \in H^{r-1}(K)$ for $r > 0$. That is why, it is assumed in Lemma 2.5 that $s \geq r - 1$.

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2.4. Discrete Friedrichs inequalities

In this subsection we improve the discrete Friedrichs inequalities of [19], Theorem 1. This improvement has also become possible due to the properties of the regularized Poincaré integral operators which were established in [16] and summarised in the previous subsection.

Lemma 2.6. *There exist positive constants C_1, C_2 independent of p such that:*

(i) *for any $\mathbf{u} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ satisfying $\langle \mathbf{u}, \mathbf{curl}\, \varphi\rangle_{0,K} = 0$ for all $\varphi \in \mathcal{P}_p^0(K)$, there holds*

$$\|\mathbf{u}\|_{0,K} \le C_1 \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1}(K)}; \tag{2.6}$$

(ii) *for any $\mathbf{u} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ satisfying $\langle \mathbf{u}, \mathbf{curl}\, \varphi\rangle_{0,K} = 0$ for all $\varphi \in \mathcal{P}_p(K)$, there holds*

$$\|\mathbf{u}\|_{0,K} \le C_1 \|\mathrm{div}\, \mathbf{u}\|_{H^{-1}(K)}. \tag{2.7}$$

Proof. Following the idea of [19], Theorem 1, the proof reduces to finding a continuous right inverse of the divergence operator within appropriate polynomial spaces. In particular, in order to prove the first statement of the lemma one needs to construct an operator $\mathcal{T}$ mapping $\mathring{\mathcal{P}}_{p-1}(K) := \{\psi \in \mathcal{P}_{p-1}(K);\ \int_K \psi\, \mathrm{d}\mathbf{x} = 0\}$ into $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ and satisfying the following properties:

$$\mathrm{div}\,(\mathcal{T}\psi) = \psi \qquad \forall \psi \in \mathring{\mathcal{P}}_{p-1}(K), \tag{2.8}$$

$$\|\mathcal{T}\psi\|_{0,K} \le C \|\psi\|_{\tilde{H}^{-1}(K)} \qquad \forall \psi \in \mathring{\mathcal{P}}_{p-1}(K). \tag{2.9}$$

Then, given any $\mathbf{u} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ such that $\langle \mathbf{u}, \mathbf{curl}\, \varphi\rangle_{0,K} = 0$ for all $\varphi \in \mathcal{P}_p^0(K)$, we prove (2.6):

$$\begin{aligned} \|\mathbf{u}\|_{0,K} &= \min_{\varphi \in \mathcal{P}_p^0(K)} \|\mathbf{u} - \mathbf{curl}\, \varphi\|_{0,K} \le \|\mathbf{u} - (\mathbf{u} - \mathcal{T}(\mathrm{div}\, \mathbf{u}))\|_{0,K} \\ &= \|\mathcal{T}(\mathrm{div}\, \mathbf{u})\|_{0,K} \overset{(2.9)}{\le} C \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1}(K)}. \end{aligned}$$

Here, $\mathrm{div}\, \mathbf{u} \in \mathring{\mathcal{P}}_{p-1}(K)$ and the existence of $\varphi \in \mathcal{P}_p^0(K)$ satisfying $\mathbf{curl}\, \varphi = \mathbf{u} - \mathcal{T}(\mathrm{div}\, \mathbf{u})$ follows from two facts:

$$\mathbf{u} - \mathcal{T}(\mathrm{div}\, \mathbf{u}) \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$$

and

$$\mathrm{div}(\mathbf{u} - \mathcal{T}(\mathrm{div}\, \mathbf{u})) \overset{(2.8)}{=} \mathrm{div}\, \mathbf{u} - \mathrm{div}\, \mathbf{u} = 0.$$

Let us construct the operator $\mathcal{T}$ satisfying (2.8), (2.9). Let $\psi \in \mathring{\mathcal{P}}_{p-1}(K)$. Applying the regularized Poincaré operator R we define $\mathbf{v} := R\psi$. Then $\mathbf{v} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$, due to property (R3) of this operator. Moreover, using property (R1) and the fact that $\int_K \psi\, \mathrm{d}\mathbf{x} = 0$ we conclude that $\mathbf{v} \cdot \mathbf{n}$ has zero average along ∂K:

$$\int_{\partial K} \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma = \int_K \mathrm{div}\, \mathbf{v}\, \mathrm{d}\mathbf{x} = \int_K \mathrm{div}(R\psi)\, \mathrm{d}\mathbf{x} = \int_K \psi\, \mathrm{d}\mathbf{x} = 0.$$

Hence, there exists a continuous piecewise polynomial ϕ defined on ∂K such that $\phi|_\ell \in \mathcal{P}_p(\ell)$ for any edge $\ell \subset \partial K$ and $\frac{\partial \phi}{\partial \sigma} = \mathbf{v} \cdot \mathbf{n}$ on ∂K. Therefore, applying the polynomial extension result of Babuška *et al.* [2],

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we find a polynomial $\tilde{\phi} \in \mathcal{P}_p(K)$ such that $\tilde{\phi}|_{\partial K} = \phi$ and there holds

$$\begin{aligned}
\|\mathbf{curl}\,\tilde{\phi}\|_{0,K} &\leq \|\tilde{\phi}\|_{H^1(K)} \leq C\,\|\phi\|_{H^{1/2}(\partial K)/\mathbb{R}} \\
&\leq C\,\Big\|\frac{\partial\phi}{\partial\sigma}\Big\|_{H^{-1/2}(\partial K)} && ([19], \text{Lem. 2}) \\
&= C\,\|\mathbf{v}\cdot\mathbf{n}\|_{H^{-1/2}(\partial K)} && (\tfrac{\partial\phi}{\partial\sigma} = \mathbf{v}\cdot\mathbf{n}) \\
&\leq C\,(\|\mathbf{v}\|_{0,K} + \|\operatorname{div}\mathbf{v}\|_{\tilde{H}^{-1}(K)}) && (\text{Lem. 2.1 with } s=0) \\
&= C\,(\|R\psi\|_{0,K} + \|\operatorname{div}(R\psi)\|_{\tilde{H}^{-1}(K)}) && (\mathbf{v} = R\psi).
\end{aligned}$$

Hence, using properties (R1) and (R2) of the operator R, we obtain

$$\|\mathbf{curl}\,\tilde{\phi}\|_{0,K} \leq C\,\|\psi\|_{\tilde{H}^{-1}(K)}. \tag{2.10}$$

Now we can define the desired operator $\mathcal{T}$ as $\mathcal{T}\psi = R\psi - \mathbf{curl}\,\tilde{\phi}$. It is easy to check that $\mathcal{T} : \mathring{\mathcal{P}}_{p-1}(K) \to \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ and (2.8) holds. Making use of (2.10) and the continuity of the operator $R : H^{-1}(K) \to \mathbf{L}^2(K)$ (see (R2)), we also prove (2.9).

The proof of statement (ii) is analogous. In this case we can use the operator $R : H^{-1}(K) \to \mathbf{L}^2(K)$ for the desired continuous right inverse of div. Then $R \equiv \mathcal{T}$ maps $\mathcal{P}_{p-1}(K)$ into $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ and (2.7) is derived similarly as above. □

2.5. H^1- and H(div)-conforming interpolation operators

Let us briefly sketch the definitions and summarise the properties of the H^1-conforming interpolation operator Π_p^1 and the $\mathbf{H}(\mathrm{div})$-conforming interpolation operator Π_p^{div} from [19].

Let $g \in H^{1+r}(K)$, $r > 0$. To define the interpolant $\Pi_p^1 g$, one starts with the standard linear interpolation of g at the vertices of K:

$$g_1 \in \mathcal{P}_1(K), \quad g_1 = g \quad \text{at each vertex of } K.$$

Then, for each edge $\ell \subset \partial K$, we define a polynomial $g_{2,\ell}$ by using the projection

$$g_{2,\ell} \in \mathcal{P}_p^0(\ell): \quad \|(g - g_1)|_\ell - g_{2,\ell}\|_{\tilde{H}^{1/2}(\ell)} \to \min. \tag{2.11}$$

Extending $g_{2,\ell}$ by zero onto the remaining part of ∂K (and keeping its notation), using some polynomial extension $\mathcal{E}_p$ from the boundary, and summing up over all edges we define

$$g_2^p := \sum_{\ell\subset\partial K} \mathcal{E}_p(g_{2,\ell}) \in \mathcal{P}_p(K). \tag{2.12}$$

Finally, we define the polynomial bubble g_3^p by projection in the H^1-semi-norm

$$g_3^p \in \mathcal{P}_p^0(K): \quad |g - (g_1 + g_2^p + g_3^p)|_{H^1(K)} \to \min. \tag{2.13}$$

Then the interpolant $\Pi_p^1 g$ is defined as the sum

$$\Pi_p^1 g := g_1 + g_2^p + g_3^p \in \mathcal{P}_p(K). \tag{2.14}$$

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Now we proceed to the $\mathbf{H}(\mathrm{div})$-conforming interpolation operator. Given a vector field $\mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$ with $r > 0$, the interpolant $\tilde{\mathbf{u}}^p = \Pi_p^{\mathrm{div}} \mathbf{u} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ is also defined as the sum of three terms:

$$\tilde{\mathbf{u}}^p = \mathbf{u}_1 + \mathbf{u}_2^p + \tilde{\mathbf{u}}_3^p.$$

Here, $\mathbf{u}_1$ is a lowest order interpolant defined as

$$\mathbf{u}_1 = \sum_{\ell \subset \partial K} \left(\int_\ell \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\sigma \right) \boldsymbol{\phi}_\ell, \tag{2.15}$$

where $\boldsymbol{\phi}_\ell$ are the standard basis functions (associated with edges ℓ) for $\boldsymbol{\mathcal{P}}_1^{\mathrm{RT}}(K)$ such that

$$\boldsymbol{\phi}_\ell \cdot \mathbf{n} = \begin{cases} 1 & \text{on } \ell, \\ 0 & \text{on } \partial K \backslash \ell. \end{cases}$$

For any edge $\ell \subset \partial K$ one has

$$\int_\ell (\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n} \, \mathrm{d}\sigma = 0. \tag{2.16}$$

Hence, there exists a function ψ, defined on the boundary ∂K, such that

$$\frac{\partial \psi}{\partial \sigma} = (\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n}, \quad \psi = 0 \ \text{ at all vertices.} \tag{2.17}$$

Then, for each edge ℓ, we define $\psi_2^\ell \in \mathcal{P}_p^0(\ell)$ by projection

$$\langle \psi|_\ell - \psi_2^\ell, \phi \rangle_{\tilde{H}^{1/2}(\ell)} = 0 \quad \forall \phi \in \mathcal{P}_p^0(\ell) \tag{2.18}$$

(see Rem. A.1 for the expression of $\langle \cdot, \cdot \rangle_{\tilde{H}^{1/2}(\ell)}$). Extending ψ_2^ℓ by zero from ℓ onto ∂K (and keeping its notation), we denote by $\psi_{2,p}^\ell \in \mathcal{P}_p(K)$ a polynomial extension of ψ_2^ℓ from ∂K onto K, *i.e.*,

$$\psi_{2,p}^\ell \in \mathcal{P}_p(K), \quad \psi_{2,p}^\ell|_\ell = \psi_2^\ell, \quad \psi_{2,p}^\ell|_{\partial K \backslash \ell} = 0. \tag{2.19}$$

Then we set

$$\mathbf{u}_2^p = \sum_{\ell \subset \partial K} \mathbf{u}_{2,\ell}^p, \quad \text{where} \quad \mathbf{u}_{2,\ell}^p = \mathbf{curl}\, \psi_{2,p}^\ell. \tag{2.20}$$

The interior interpolant $\tilde{\mathbf{u}}_3^p$ is a vector bubble function living in $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ and satisfying the following system of equations:

$$\langle \mathrm{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \tilde{\mathbf{u}}_3^p)), \mathrm{div}\, \mathbf{v} \rangle_{0,K} = 0 \qquad \forall \mathbf{v} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K), \tag{2.21}$$

$$\langle \mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \tilde{\mathbf{u}}_3^p), \mathbf{curl}\, \phi \rangle_{0,K} = 0 \qquad \forall \phi \in \mathcal{P}_p^0(K). \tag{2.22}$$

These interpolation operators satisfy the following properties.

Proposition 2.1. (*cf.* [19], Props. 1–3).

(1) *For $r > 0$ the operators $\Pi_p^1 : \mathbf{H}^{1+r}(K) \to H^1(K)$ and $\Pi_p^{\mathrm{div}} : \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K) \to \mathbf{H}(\mathrm{div}, K)$ are well defined and bounded, with corresponding operator norms independent of the polynomial degree p.*

(2) *The operators Π_p^1 and Π_p^{div} preserve scalar polynomials in $\mathcal{P}_p(K)$ and polynomial vector fields in $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$, respectively.*

(3) *For $r > 0$, the diagram in (1.2) commutes.*

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The next proposition gives optimal interpolation error estimates for the operators Π_p^1 and Π_p^{div}. These estimates are proved in [3] (see Thms. 4.1 and 4.2 therein).

Proposition 2.2.

(i) *Let $g \in H^{1+r}(K)$, $r > 0$. Then there exists a positive constant C independent of p and g such that*

$$|g - \Pi_p^1 g|_{H^1(K)} \leq C\, p^{-r}\, \|g\|_{H^{1+r}(K)}.$$

(ii) *Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, K)$, $r > 0$. Then there exists a positive constant C independent of p and $\mathbf{u}$ such that*

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\, \mathbf{u}\|_{\mathbf{H}(\mathrm{div},K)} \leq C\, p^{-r}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},K)}.$$

3. Proofs of theorems

In this section we prove the main results of the paper.

Proof of Theorem 1.1. Let $\mathbf{u} \in \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$, $r > 0$. We will study each term on the right-hand side of (1.3). Throughout the proof we denote by s a small parameter such that $0 < s < \min\{\frac{1}{2}, r\}$ for given $r > 0$.

Step 1. Fixing an edge $\ell \subset \partial K$ and using a function

$$\phi_\ell \in H^{1-s}(K), \quad \phi_\ell = \begin{cases} 1 & \text{on } \ell, \\ 0 & \text{on } \partial K \backslash \ell \end{cases}$$

as a test function, we integrate by parts to obtain

$$\begin{aligned} \int_\ell \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}\sigma &= \int_{\partial K} (\mathbf{u} \cdot \mathbf{n})\, \phi_\ell\, \mathrm{d}\sigma = \int_K (\mathrm{div}\, \mathbf{u})\, \phi_\ell\, \mathrm{d}\mathbf{x} + \int_K \mathbf{u} \cdot \nabla \phi_\ell\, \mathrm{d}\mathbf{x} \\ &\leq \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1+s}(K)}\, \|\phi_\ell\|_{H^{1-s}(K)} + \|\mathbf{u}\|_{\mathbf{H}^s(K)}\, \|\nabla \phi_\ell\|_{\mathbf{H}^{-s}(K)} \\ &\leq C(\phi_\ell, s) \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1/2}(K)} \right). \end{aligned}$$

Note that if $\mathrm{div}\, \mathbf{u} \in H^{-1+s}(K)$ then an extension to $\mathrm{div}\, \mathbf{u} \in \tilde{H}^{-1+s}(K)$ exists but is not unique. However, by assumption $\mathrm{div}\, \mathbf{u} \in \tilde{H}^{-1/2}(K) \subset \tilde{H}^{-1+s}(K)$, which is a unique extension (see [24] for details). Thus, $\mathbf{u}_1$ in (2.15) is well defined. Moreover, since $\mathbf{u}_1$ is a lowest order interpolant, we find by the equivalence of norms in finite-dimensional spaces that

$$\|\mathbf{u}_1\|_{\mathbf{H}(\mathrm{div},K)} \leq C \sum_{\ell \subset \partial K} \left| \int_\ell \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}\sigma \right| \leq C \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1/2}(K)} \right).$$

Hence, due to the finite dimensionality of $\mathbf{u}_1$, we obtain by using Lemma 2.1

$$\begin{aligned} \|(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n}\|_{H^{-1/2+s}(\partial K)} &\leq C \left(\|\mathbf{u} - \mathbf{u}_1\|_{\mathbf{H}^s(K)} + \|\mathrm{div}(\mathbf{u} - \mathbf{u}_1)\|_{\tilde{H}^{-1+s}(K)} \right) \\ &\leq C \left(\|\mathbf{u}\|_{\mathbf{H}^s(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1+s}(K)} + \|\mathbf{u}_1\|_{\mathbf{H}(\mathrm{div},K)} \right) \\ &\leq C \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1/2}(K)} \right). \end{aligned} \tag{3.23}$$

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Step 2. From the construction of $\mathbf{u}_1$ and from the result of Step 1 we conclude that

$$(\mathbf{u}-\mathbf{u}_1)\cdot\mathbf{n}\in H^{-1/2+s}(\partial K),\quad \int_{\partial K}(\mathbf{u}-\mathbf{u}_1)\cdot\mathbf{n}\,\mathrm{d}\sigma=0.$$

Therefore, due to the isomorphism (see [19], Lem. 2)

$$\frac{\partial}{\partial\sigma}: H^{1/2+s}(\partial K)/\mathbb{R}\to H_*^{-1/2+s}(\partial K)=\{\phi\in H^{-1/2+s}(\partial K);\ \langle u,1\rangle_{0,\partial K}=0\},$$

the function ψ in (2.17) is well defined, $\psi\in H^{1/2+s}(\partial K)$, $\psi|_\ell\in\tilde H^{1/2}(\ell)$ for any edge $\ell\subset\partial K$, and

$$\sum_{\ell\subset\partial K}\|\psi|_\ell\|_{\tilde H^{1/2}(\ell)}\le C\sum_{\ell\subset\partial K}\|\psi|_\ell\|_{H_0^{1/2+s}(\ell)}\le C\,\|\psi\|_{H^{1/2+s}(\partial K)}\le C\,\|(\mathbf{u}-\mathbf{u}_1)\cdot\mathbf{n}\|_{H^{-1/2+s}(\partial K)}. \tag{3.24}$$

Hence, (2.18) is uniquely solvable and

$$\|\psi_2^\ell\|_{\tilde H^{1/2}(\ell)}\le C\,\|\psi|_\ell\|_{\tilde H^{1/2}(\ell)}. \tag{3.25}$$

Furthermore, applying the polynomial extension result of Babuška *et al.* [2], we find the desired polynomial $\psi_{2,p}^\ell\in\mathcal{P}_p(K)$ (see (2.19)) satisfying

$$\|\psi_{2,p}^\ell\|_{H^1(K)}\le C\,\|\psi_2^\ell\|_{\tilde H^{1/2}(\ell)}. \tag{3.26}$$

Thus, $\mathbf{u}_2^p$ in (2.20) is well defined. Putting together (3.24)–(3.26) we find

$$\|\mathbf{u}_2^p\|_{0,K}\le C\sum_{\ell\subset\partial K}\|\mathbf{curl}\,\psi_{2,p}^\ell\|_{0,K}\le C\sum_{\ell\subset\partial K}\|\psi_{2,p}^\ell\|_{H^1(K)}\le C\,\|(\mathbf{u}-\mathbf{u}_1)\cdot\mathbf{n}\|_{H^{-1/2+s}(\partial K)}.$$

Hence, making use of (3.23), we obtain

$$\|\mathbf{u}_2^p\|_{\mathbf{H}(\mathrm{div},K)}=\|\mathbf{u}_2^p\|_{0,K}\le C\left(\|\mathbf{u}\|_{\mathbf{H}^r(K)}+\|\mathrm{div}\,\mathbf{u}\|_{\tilde H^{-1/2}(K)}\right). \tag{3.27}$$

Step 3. The vector bubble function $\mathbf{u}_3^p$ is uniquely defined by (1.4)–(1.5). To estimate the norms of $\mathbf{u}_3^p$ and $\mathrm{div}\,\mathbf{u}_3^p$ we use the discrete Helmholtz decomposition

$$\mathbf{u}_3^p=\mathbf{v}_p+\mathbf{curl}\,\phi_p, \tag{3.28}$$

where $\phi_p\in\mathcal{P}_p^0(K)$ and $\mathbf{v}_p\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ is such that $\langle\mathbf{v}_p,\mathbf{curl}\,\varphi\rangle_{0,K}=0$ for all $\varphi\in\mathcal{P}_p^0(K)$.

From (1.4) one has by using the result of Step 1

$$\begin{aligned}\|\mathrm{div}\,\mathbf{u}_3^p\|_{\tilde H^{-1/2}(K)} &\le C\,\|\mathrm{div}(\mathbf{u}-\mathbf{u}_1)\|_{\tilde H^{-1/2}(K)}\le C\left(\|\mathrm{div}\,\mathbf{u}\|_{\tilde H^{-1/2}(K)}+|\mathrm{div}\,\mathbf{u}_1|\right)\\ &\le C\left(\|\mathbf{u}\|_{\mathbf{H}^r(K)}+\|\mathrm{div}\,\mathbf{u}\|_{\tilde H^{-1/2}(K)}\right).\end{aligned} \tag{3.29}$$

Then, applying Lemma 2.6(i) and recalling that $\mathrm{div}\,\mathbf{v}_p=\mathrm{div}\,\mathbf{u}_3^p$, we find

$$\|\mathbf{v}_p\|_{0,K}\le C\,\|\mathrm{div}\,\mathbf{v}_p\|_{\tilde H^{-1}(K)}\le C\,\|\mathrm{div}\,\mathbf{u}_3^p\|_{\tilde H^{-1/2}(K)}. \tag{3.30}$$

Since $\langle\mathbf{v}_p,\mathbf{curl}\,\phi_p\rangle_{0,K}=0$, we estimate the norm of $\mathbf{curl}\,\phi_p$ by using (1.5) and by employing the results of the first two steps:

$$\|\mathbf{curl}\,\phi_p\|_{0,K}\le\|\mathbf{u}-\mathbf{u}_1-\mathbf{u}_2^p\|_{0,K}\le C\left(\|\mathbf{u}\|_{\mathbf{H}^r(K)}+\|\mathrm{div}\,\mathbf{u}\|_{\tilde H^{-1/2}(K)}\right). \tag{3.31}$$

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Combining (3.29)–(3.31) and applying the triangle inequality we obtain, by making use of decomposition (3.28),

$$\|\mathbf{u}_3^p\|_{0,K} + \|\operatorname{div}\mathbf{u}_3^p\|_{\tilde{H}^{-1/2}(K)} \leq C\left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\operatorname{div}\mathbf{u}\|_{\tilde{H}^{-1/2}(K)}\right).$$

The boundedness of the operator $\Pi_p^{\operatorname{div},-\frac{1}{2}}$ (and inequality (1.7)) now follows by putting together the results of the three individual steps and by applying the triangle inequality.

The polynomial-preserving property of the operator $\Pi_p^{\operatorname{div},-\frac{1}{2}}$ easily follows from its definition. □

It is essential for the proof of Theorem 1.2 given below that the $\tilde{H}^{-1/2}(K)$-inner product satisfies (1.6) (*i.e.*, reduces to the $L^2(K)$-inner product for a constant function). As it follows from Lemma A.3 in the Appendix, the $\tilde{H}^{-1/2}(K)$-inner product given by (A.12) satisfies this property.

Proof of Theorem 1.2. To prove the first part of the diagram, we consider $\mathbf{u} = \mathbf{curl}\, g$, $g \in H^{1+r}(K)$. Let us decompose $\Pi_p^1 g$ and $\Pi_p^{\operatorname{div},-\frac{1}{2}}\mathbf{u}$ as in (2.14) and (1.3), respectively. Then it follows from the definitions of these interpolation operators that $\mathbf{u}_1 = \mathbf{curl}\, g_1$ and $\mathbf{u}_2^p = \mathbf{curl}\, g_2^p$ (*cf.* [19]). Hence, $\operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{u}_1 = \operatorname{div}\mathbf{u}_2^p = 0$ and it follows from (1.4) that $\operatorname{div}\mathbf{u}_3^p = 0$. Therefore, decomposing $\mathbf{u}_3^p$ as in (3.28) and comparing (1.5) with (2.13), we conclude that $\mathbf{u}_3^p = \mathbf{curl}\, g_3^p$. Thus, $\Pi_p^{\operatorname{div},-\frac{1}{2}}(\mathbf{curl}\, g) = \mathbf{curl}(\Pi_p^1 g)$.

Let us prove the second part of the diagram. For any $\varphi \in \mathcal{P}_{p-1}(K)$ there exists $\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ such that $\operatorname{div}\mathbf{v}_p = \varphi$. Therefore, decomposing $\Pi_p^{\operatorname{div},-\frac{1}{2}}\mathbf{u}$ as in (1.3), we need to show that for all $\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ there holds

$$\left\langle \operatorname{div}\left(\mathbf{u} - \Pi_p^{\operatorname{div},-\frac{1}{2}}\mathbf{u}\right), \operatorname{div}\mathbf{v}_p\right\rangle_{\tilde{H}^{-1/2}(K)} = \langle \operatorname{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_3^p)), \operatorname{div}\mathbf{v}_p\rangle_{\tilde{H}^{-1/2}(K)} = 0. \tag{3.32}$$

Let us also decompose $\mathbf{v}_p = \Pi_p^{\operatorname{div},-\frac{1}{2}}\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ as in (1.3):

$$\mathbf{v}_p = \mathbf{v}_1 + \mathbf{v}_2^p + \mathbf{v}_3^p, \quad \operatorname{div}\mathbf{v}_1 = \text{const.}, \quad \operatorname{div}\mathbf{v}_2^p = 0, \quad \mathbf{v}_3^p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K).$$

Then, recalling (1.4), applying Lemma A.3, and integrating by parts, we prove (3.32):

$$\begin{aligned}
&\left\langle \operatorname{div}\left(\mathbf{u} - \Pi_p^{\operatorname{div},-\frac{1}{2}}\mathbf{u}\right), \operatorname{div}\mathbf{v}_p\right\rangle_{\tilde{H}^{-1/2}(K)} \\
&\qquad = \langle \operatorname{div}(\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_3^p), \text{const.}\rangle_{\tilde{H}^{-1/2}(K)} + \left\langle \operatorname{div}\left(\mathbf{u} - \Pi_p^{\operatorname{div},-\frac{1}{2}}\mathbf{u}\right), \operatorname{div}\mathbf{v}_3^p\right\rangle_{\tilde{H}^{-1/2}(K)} \\
&\qquad = \langle \operatorname{div}(\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_3^p), \text{const.}\rangle_{0,K} = \text{const.} \int_{\partial K} (\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_3^p)\cdot\mathbf{n}\,\mathrm{d}\sigma = 0.
\end{aligned}$$

For the last step we used the fact that $\mathbf{u}_3^p \cdot \mathbf{n}|_{\partial K} = 0$ and then applied (2.16). □

For the proof of Theorem 1.3 we will need two auxiliary results regarding the new $\mathbf{H}(\operatorname{div})$-conforming interpolation operator $\Pi_p^{\operatorname{div},-\frac{1}{2}}$. These results are formulated in the next two lemmas: the first one concerns the normal trace of the interpolant $\Pi_p^{\operatorname{div},-\frac{1}{2}}\mathbf{u}$ on the boundary ∂K, and the second one states some auxiliary error estimates for $\Pi_p^{\operatorname{div},-\frac{1}{2}}$.

Lemma 3.1. *Let $\mathbf{u} \in \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\operatorname{div}, K)$ with $r > 0$, and let $\mathbf{u}^p = \Pi_p^{\operatorname{div},-\frac{1}{2}}\mathbf{u} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$. Then for any edge $\ell \subset \partial K$ there holds*

$$\|(\mathbf{u} - \mathbf{u}^p)\cdot\mathbf{n}\|_{\tilde{H}^{-1}(\ell)} \leq C\, p^{-1/2}\, \|(\mathbf{u} - \mathbf{u}^p)\cdot\mathbf{n}\|_{H^{-1/2}(\partial K)}. \tag{3.33}$$

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Proof. If $\mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K)$ with $r > 0$, then it was proved in [5], Lemma 3.3, that

$$\|(\mathbf{u} - \Pi_p^{\mathrm{div}}\mathbf{u}) \cdot \mathbf{n}\|_{\tilde{H}^{-1}(\ell)} \leq C\, p^{-1/2}\, \|(\mathbf{u} - \Pi_p^{\mathrm{div}}\mathbf{u}) \cdot \mathbf{n}\|_{H^{-1/2}(\partial K)}.$$

We note, however, that $\Pi_p^{\mathrm{div}}\mathbf{u} \cdot \mathbf{n} = \Pi_p^{\mathrm{div},-1/2}\mathbf{u} \cdot \mathbf{n} = (\mathbf{u}_1 + \mathbf{u}_2^p) \cdot \mathbf{n}$ on the boundary ∂K, and, as it follows from the proof of Theorem 1.1 above, $\mathbf{u}_1$ and $\mathbf{u}_2^p$ are in fact well defined for $\mathbf{u} \in \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$, $r > 0$. Therefore, the proof of Lemma 3.3 in [5] carries over to the case considered in this paper and inequality (3.33) is valid. □

Lemma 3.2. *Let $r > 0$ and $s > \max\{-\frac{1}{2}, r - 1\}$. If $\mathbf{u} \in \mathbf{H}^r(K)$ and $\mathrm{div}\,\mathbf{u} \in H^s(K)$, then*

$$\|\mathrm{div}(\mathbf{u} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u})\|_{\tilde{H}^{-1/2}(K)} \leq C\, p^{-(1/2+s)}\, \|\mathrm{div}\,\mathbf{u}\|_{H^s(K)} \tag{3.34}$$

and for any $\varepsilon > 0$ there holds

$$\|\mathbf{u} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u}\|_{0,K} \leq C\left(p^{-r}\, \|\mathbf{u}\|_{\mathbf{H}^r(K)} + \varepsilon^{-1}\, p^{-(s+1/2-\varepsilon)} \|\mathrm{div}\,\mathbf{u}\|_{H^s(K)}\right). \tag{3.35}$$

The positive constants C in (3.34) and (3.35) are independent of $\mathbf{u}$ and p.

Proof. Estimate (3.34) is an immediate consequence of the commuting diagram property (1.8) and Lemma 2.3:

$$\|\mathrm{div}\,\mathbf{u} - \mathrm{div}(\Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u})\|_{\tilde{H}^{-1/2}(K)} = \|\mathrm{div}\,\mathbf{u} - \Pi_{p-1}^{-1/2}(\mathrm{div}\,\mathbf{u})\|_{\tilde{H}^{-1/2}(K)} \leq C\, p^{-(1/2+s)}\, \|\mathrm{div}\,\mathbf{u}\|_{H^s(K)}.$$

Let us now prove (3.35). For $p = 1$ this estimate follows trivially from Theorem 1.1. Let $p \geq 2$. Using Lemma 2.5 we decompose $\mathbf{u}$ as follows:

$$\mathbf{u} = \mathbf{curl}\,\psi + \mathbf{v}, \qquad \psi \in H^{r+1}(K), \quad \mathbf{v} \in \mathbf{H}^{s+1}(K). \tag{3.36}$$

Moreover, the norms of $\mathbf{v}$ and ψ are bounded as in (2.5). Then, applying the interpolation operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ and using its commutativity with Π_p^1 (see (1.8)), we write

$$\Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u} = \Pi_p^{\mathrm{div},-\frac{1}{2}}(\mathbf{curl}\,\psi) + \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{v} = \mathbf{curl}(\Pi_p^1\psi) + \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{v}. \tag{3.37}$$

Since $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ is a bounded operator preserving polynomials (see Thm. 1.1), one has for any polynomial $\mathbf{v}_p \in (\mathcal{P}_{p-1}(K))^2 \subset \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$:

$$\begin{aligned}
\|\mathbf{v} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{v}\|_{0,K} &= \|\mathbf{v} - \mathbf{v}_p - \Pi_p^{\mathrm{div},-\frac{1}{2}}(\mathbf{v} - \mathbf{v}_p)\|_{0,K} \\
&\leq C \inf_{\mathbf{v}_p \in (\mathcal{P}_{p-1}(K))^2} \left(\|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}^{\tilde{\varepsilon}}(K)} + \|\mathrm{div}(\mathbf{v} - \mathbf{v}_p)\|_{\tilde{H}^{-1/2}(K)}\right) \\
&\leq C\,\varepsilon^{-1} \inf_{\mathbf{v}_p \in (\mathcal{P}_{p-1}(K))^2} \|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}^{1/2+\varepsilon}(K)},
\end{aligned} \tag{3.38}$$

where $\tilde{\varepsilon} \in (0, \frac{1}{2})$ is fixed, $\varepsilon > 0$ is arbitrarily small, and for the last step we used Lemma 5 of [21] as well as the boundedness of the divergence operator to estimate

$$\begin{aligned}
\|\mathrm{div}(\mathbf{v} - \mathbf{v}_p)\|_{\tilde{H}^{-1/2}(K)} &\leq \|\mathrm{div}(\mathbf{v} - \mathbf{v}_p)\|_{\tilde{H}^{-1/2+\varepsilon}(K)} \\
&\leq C\,\varepsilon^{-1}\, \|\mathrm{div}(\mathbf{v} - \mathbf{v}_p)\|_{H^{-1/2+\varepsilon}(K)} \leq C\,\varepsilon^{-1}\, \|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}^{1/2+\varepsilon}(K)}.
\end{aligned}$$

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Applying now Lemma 2.2 component-wise and using the first inequality in (2.5), we deduce from (3.38) that

$$\|\mathbf{v} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\,\mathbf{v}\|_{0,K} \le C\,\varepsilon^{-1}\,(p-1)^{-(s+1/2-\varepsilon)}\,\|\mathbf{v}\|_{\mathbf{H}^{s+1}(K)} \le C\,\varepsilon^{-1}\,p^{-(s+1/2-\varepsilon)}\,\|\operatorname{div}\mathbf{u}\|_{H^s(K)}. \tag{3.39}$$

On the other hand, applying Proposition 2.2(i) and the second inequality in (2.5) we obtain

$$\|\mathbf{curl}(\psi - \Pi_p^1\psi)\|_{0,K} = |\psi - \Pi_p^1\psi|_{H^1(K)} \le C\,p^{-r}\,\|\psi\|_{H^{1+r}(K)} \le C\,p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(K)}. \tag{3.40}$$

Combining (3.39) and (3.40) we prove (3.35) by making use of decompositions (3.36), (3.37) and the triangle inequality. □

Now we are in a position to prove the main interpolation error estimate.

Proof of Theorem 1.3. For simplicity of notation we denote $\mathbf{u}^p := \Pi_p^{\mathrm{div},-\frac{1}{2}}\,\mathbf{u} \in \mathcal{P}_p^{\mathrm{RT}}(K)$. Let us consider an auxiliary problem: find $\mathbf{u}_0 \in \mathbf{H}(\operatorname{div},K)$ such that

$$\langle \mathbf{u}-\mathbf{u}_0, \mathbf{v}\rangle_{0,K} + \langle \operatorname{div}(\mathbf{u}-\mathbf{u}_0), \operatorname{div}\mathbf{v}\rangle_{0,K} = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0(\operatorname{div},K), \tag{3.41}$$

$$\mathbf{u}_0\cdot\mathbf{n} = \mathbf{u}^p\cdot\mathbf{n} \qquad \text{on } \partial K.$$

Then, using Lemma 4.8 in [12] and applying Lemmas 3.1 and 2.1, we estimate for $t = -1, -\frac{1}{2}$

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_0\|_{\tilde{\mathbf{H}}^{t+1/2}(\operatorname{div},K)} &\le C\,\|(\mathbf{u}-\mathbf{u}^p)\cdot\mathbf{n}\|_{H^t(\partial K)} \le C\,p^{t+1/2}\,\|(\mathbf{u}-\mathbf{u}^p)\cdot\mathbf{n}\|_{H^{-1/2}(\partial K)}\\ &\le C\,p^{t+1/2}\left(\|\mathbf{u}-\mathbf{u}^p\|_{0,K} + \|\operatorname{div}(\mathbf{u}-\mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)}\right).\end{aligned} \tag{3.42}$$

By the triangle inequality one has

$$\|\mathbf{u}-\mathbf{u}^p\|_{\tilde{\mathbf{H}}^{-1/2}(\operatorname{div},K)} \le \|\mathbf{u}-\mathbf{u}_0\|_{\tilde{\mathbf{H}}^{-1/2}(\operatorname{div},K)} + \|\mathbf{u}_0-\mathbf{u}^p\|_{\tilde{\mathbf{H}}^{-1/2}(\operatorname{div},K)}. \tag{3.43}$$

For the first term on the right-hand side of (3.43) we have by using (3.42) with $t=-1$:

$$\|\mathbf{u}-\mathbf{u}_0\|_{\tilde{\mathbf{H}}^{-1/2}(\operatorname{div},K)} \le C\,p^{-1/2}\left(\|\mathbf{u}-\mathbf{u}^p\|_{0,K} + \|\operatorname{div}(\mathbf{u}-\mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)}\right). \tag{3.44}$$

Now, we consider the second term on the right-hand side of (3.43) and prove that

$$\|\mathbf{u}_0-\mathbf{u}^p\|_{\tilde{\mathbf{H}}^{-1/2}(\operatorname{div},K)} \le C\left(p^{-1/2}\,\|\mathbf{u}-\mathbf{u}^p\|_{0,K} + \|\operatorname{div}(\mathbf{u}-\mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)}\right). \tag{3.45}$$

Denote $\mathbf{X} := \tilde{\mathbf{H}}_0^{-1/2}(\operatorname{div},K)$, and let $\mathbf{X}'$ be the dual space of $\mathbf{X}$ (with $\mathbf{L}^2(K)$ as pivot space). From [13], Section 6, we know that any $\mathbf{w} \in \mathbf{X}'$ can be decomposed as follows:

$$\mathbf{w} = \nabla f + \mathbf{curl}\, g, \qquad f \in H^{1/2}(K)/\mathbb{R},\ g \in H_0^1(K)\cap H^{3/2}(K),$$

and

$$\|f\|_{H^{1/2}(K)/\mathbb{R}} + \|g\|_{H^{3/2}(K)} \le C\,\|\mathbf{w}\|_{\mathbf{X}'}. \tag{3.46}$$

Hence, recalling that $\mathbf{u}_0-\mathbf{u}^p \in \mathbf{H}_0(\operatorname{div},K) \subset \mathbf{X}$, we have

$$\begin{aligned}\|\mathbf{u}_0-\mathbf{u}^p\|_{\tilde{\mathbf{H}}^{-1/2}(\operatorname{div},K)} &= \sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{X}'} \frac{\langle \mathbf{u}_0-\mathbf{u}^p, \mathbf{w}\rangle_{0,K}}{\|\mathbf{w}\|_{\mathbf{X}'}} = \sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{X}'} \frac{\langle \mathbf{u}_0-\mathbf{u}^p, \nabla f + \mathbf{curl}\, g\rangle_{0,K}}{\|\mathbf{w}\|_{\mathbf{X}'}}\\ &= \sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{X}'} \frac{-\langle \operatorname{div}(\mathbf{u}_0-\mathbf{u}^p), f\rangle_{0,K} + \langle \mathbf{u}_0-\mathbf{u}^p, \mathbf{curl}\, g\rangle_{0,K}}{\|\mathbf{w}\|_{\mathbf{X}'}}.\end{aligned} \tag{3.47}$$

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Let $g_p \in \mathcal{P}_p^0(K)$. Using (1.5) with $\phi = g_p$ and (3.41) with $\mathbf{v} = \mathbf{curl}\, g_p$, we find that

$$\langle \mathbf{u}_0 - \mathbf{u}^p, \mathbf{curl}\, g_p\rangle_{0,K} = \langle \mathbf{u} - \mathbf{u}^p, \mathbf{curl}\, g_p\rangle_{0,K} - \langle \mathbf{u} - \mathbf{u}_0, \mathbf{curl}\, g_p\rangle_{0,K} = 0 \quad \forall g_p \in \mathcal{P}_p^0(K).$$

Therefore, selecting $g_p := \Pi_p^1 g \in \mathcal{P}_p^0(K)$ and using Proposition 2.2(i), we obtain from (3.47)

$$\begin{aligned}
&\|\mathbf{u}_0 - \mathbf{u}^p\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)} \\
&\quad\le \sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{X}'} \frac{\|\mathrm{div}(\mathbf{u}_0 - \mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)}\, \|f\|_{H^{1/2}(K)/\mathbb{R}} + \|\mathbf{u}_0 - \mathbf{u}^p\|_{0,K}\, |g - \Pi_p^1 g|_{H^1(K)}}{\|\mathbf{w}\|_{\mathbf{X}'}} \\
&\quad\le \sup_{\mathbf{0}\neq\mathbf{w}\in\mathbf{X}'} \frac{\|\mathrm{div}(\mathbf{u}_0 - \mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)}\, \|f\|_{H^{1/2}(K)/\mathbb{R}} + C\, p^{-1/2}\, \|\mathbf{u}_0 - \mathbf{u}^p\|_{0,K}\, \|g\|_{H^{3/2}(K)}}{\|\mathbf{w}\|_{\mathbf{X}'}} \\
&\quad\overset{(3.46)}{\le} C\left(\|\mathrm{div}(\mathbf{u}_0 - \mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)} + p^{-1/2}\, \|\mathbf{u}_0 - \mathbf{u}^p\|_{0,K}\right). \qquad (3.48)
\end{aligned}$$

Both norms on the right-hand side of (3.48) are estimated by applying the triangle inequality and inequalities (3.42) (with $t = -1$ and $t = -\frac{1}{2}$, respectively):

$$\begin{aligned}
\|\mathrm{div}(\mathbf{u}_0 - \mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)} &\le \|\mathbf{u} - \mathbf{u}_0\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)} + \|\mathrm{div}(\mathbf{u} - \mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)} \\
&\le C\left(p^{-1/2}\, \|\mathbf{u} - \mathbf{u}^p\|_{0,K} + \|\mathrm{div}(\mathbf{u} - \mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)}\right) \qquad (3.49)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathbf{u}_0 - \mathbf{u}^p\|_{0,K} &\le \|\mathbf{u} - \mathbf{u}^p\|_{0,K} + \|\mathbf{u} - \mathbf{u}_0\|_{\mathbf{H}(\mathrm{div},K)} \\
&\le C\left(\|\mathbf{u} - \mathbf{u}^p\|_{0,K} + \|\mathrm{div}(\mathbf{u} - \mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)}\right). \qquad (3.50)
\end{aligned}$$

The desired inequality in (3.45) then follows from (3.48)–(3.50).

Now, collecting (3.44) and (3.45) in (3.43), we obtain

$$\|\mathbf{u} - \mathbf{u}^p\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},K)} \le C\left(p^{-1/2}\, \|\mathbf{u} - \mathbf{u}^p\|_{0,K} + \|\mathrm{div}(\mathbf{u} - \mathbf{u}^p)\|_{\tilde{H}^{-1/2}(K)}\right).$$

Hence, recalling that $\mathbf{u}^p = \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u}$ and applying Lemma 3.2 with $s = r$ and $\varepsilon = \frac{1}{2}$, we arrive at estimate (1.9). □

Appendix A: Some equivalent norms and corresponding inner products in the Sobolev spaces H^r and $\tilde{H}^r$ for $r = \pm\frac{1}{2}$

In this appendix we consider the Sobolev spaces H^r and $\tilde{H}^r$ on the reference element K for $r = \pm\frac{1}{2}$. We will derive expressions for norms which are equivalent to those defined in Section 2.1. First, let us introduce some notation.

(1) We denote by D the polyhedron (cube or triangular prism) such that $D = K \times (0,1)$. Thus $\partial D = \cup_{i=1}^{\mathcal{I}} \bar{\Gamma}_i$ ($\mathcal{I} = 5$ if $K = T$ and $\mathcal{I} = 6$ if $K = Q$). Let $K = \Gamma_1 = \{(x_1, x_2, 0);\ (x_1, x_2) \in K\}$, $\Gamma_{\mathcal{I}} = \{(x_1, x_2, 1);\ (x_1, x_2) \in K\}$, and denote $\tilde{K} = \partial D \backslash \bar{\Gamma}_{\mathcal{I}}$. Note that $\tilde{K}$ is an open surface. We will denote by $\boldsymbol{\nu}$ the outward normal unit vector to ∂D, and we will use the standard notation for the gradient ∇ and for the Laplace operator Δ, both acting on scalar functions of three variables.

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(2) Given $u \in H^{-1/2}(K)$, we denote by $\tilde{u}_K$ the solution of the mixed problem: find $\tilde{u}_K \in H^1(D)$ such that

$$\Delta \tilde{u}_K = 0 \text{ in } D, \quad \tfrac{\partial \tilde{u}_K}{\partial \boldsymbol{\nu}} = u \text{ on } K, \quad \tilde{u}_K = 0 \text{ on } \partial D \backslash K.$$

If $u \in H^{-1/2}(\tilde{K})$, then we will use the same notation as above with K replaced by $\tilde{K}$.

(3) Given $u \in H^{1/2}(\partial D)$, we denote by $\tilde{\tilde{u}}$ its harmonic extension, *i.e.*, the solution of the Dirichlet problem: find $\tilde{\tilde{u}} \in H^1(D)$ such that

$$\Delta \tilde{\tilde{u}} = 0 \text{ in } D, \quad \tilde{\tilde{u}} = u \text{ on } \partial D. \tag{A.1}$$

(4) Given $u \in \tilde{H}^{1/2}(K)$, we denote by u° the extension of u by zero onto ∂D. Thus, $u^\circ \in H^{1/2}(\partial D)$.

We make use of standard definitions for the norm and the semi-norm in $H^1(D)$:

$$\|u\|_{H^1(D)} = \Big(\|u\|_{0,D}^2 + |u|_{H^1(D)}^2 \Big)^{1/2}, \quad |u|_{H^1(D)} = \|\nabla u\|_{0,D}.$$

Since $H^{1/2}(\partial D)$ is the trace space of $H^1(D)$, the norm and the semi-norm in $H^{1/2}(\partial D)$ can be equivalently written as follows

$$\begin{aligned} \|u\|_{H^{1/2}(\partial D)} &\simeq \inf_{\substack{U \in H^1(D) \\ U|_{\partial D} = u}} \|U\|_{H^1(D)}, \\ |u|_{H^{1/2}(\partial D)} &\simeq \inf_{\substack{U \in H^1(D) \\ U|_{\partial D} = u}} |U|_{H^1(D)} = \|\nabla \tilde{\tilde{u}}\|_{0,D}. \end{aligned} \tag{A.2}$$

Now we can define equivalent norms in $\tilde{H}^{1/2}(K)$ and $H^{1/2}(K)$:

$$\|u\|_{\tilde{H}^{1/2}(K)} \simeq |u^\circ|_{H^{1/2}(\partial D)} \simeq \Big\| \nabla \widetilde{\widetilde{u^\circ}} \Big\|_{0,D}, \tag{A.3}$$

$$\|u\|_{H^{1/2}(K)} \simeq \inf_{\substack{U \in \tilde{H}^{1/2}(\tilde{K}) \\ U|_K = u}} \|U\|_{\tilde{H}^{1/2}(\tilde{K})}, \tag{A.4}$$

where $\|\cdot\|_{\tilde{H}^{1/2}(\tilde{K})}$ is defined as in (A.3), because $\tilde{K}$ is an open surface.

From (A.3) one can easily derive the expression for the corresponding $\tilde{H}^{1/2}(K)$-inner product. In fact, applying the parallelogram law twice, integrating by parts, and recalling notations (3)−(4), we find (see also [19])

$$\begin{aligned} \langle u, v \rangle_{\tilde{H}^{1/2}(K)} &= \Big\langle \nabla \widetilde{\widetilde{u^\circ}}, \nabla \widetilde{\widetilde{v^\circ}} \Big\rangle_{0,D} = \Big\langle \frac{\partial \widetilde{\widetilde{u^\circ}}}{\partial \boldsymbol{\nu}}, \widetilde{\widetilde{v^\circ}} \Big\rangle_{0,\partial D} \\ &= \Big\langle \frac{\partial \widetilde{\widetilde{u^\circ}}}{\partial \boldsymbol{\nu}}, v \Big\rangle_{0,K} = \Big\langle u, \frac{\partial \widetilde{\widetilde{v^\circ}}}{\partial \boldsymbol{\nu}} \Big\rangle_{0,K} \quad \forall u, v \in \tilde{H}^{1/2}(K). \end{aligned} \tag{A.5}$$

The space $H^{-1/2}(K)$ is the dual space of $\tilde{H}^{1/2}(K)$. We prove the following result regarding an equivalent norm in $H^{-1/2}(K)$.

Lemma A.1. *For any $u \in H^{-1/2}(K)$ there holds*

$$\|u\|_{H^{-1/2}(K)} \simeq \|\nabla \tilde{u}_K\|_{0,D}. \tag{A.6}$$

The $H^{-1/2}$-inner product corresponding to the norm on the right-hand side of (A.6) reads as

$$\langle u, v \rangle_{H^{-1/2}(K)} = \langle u, \tilde{v}_K \rangle_{0,K} = \langle \tilde{u}_K, v \rangle_{0,K} \quad \forall u, v \in H^{-1/2}(K). \tag{A.7}$$

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Proof. Using notations (2)−(4), we integrate by parts to obtain for any $u \in H^{-1/2}(K)$ and any $v \in \tilde{H}^{1/2}(K)$

$$\left\langle \nabla \tilde{u}_K, \nabla \widetilde{\widetilde{v^\circ}} \right\rangle_{0,D} = \left\langle \frac{\partial \tilde{u}_K}{\partial \boldsymbol{\nu}}, \widetilde{\widetilde{v^\circ}} \right\rangle_{0,\partial D} = \left\langle \frac{\partial \tilde{u}_K}{\partial \boldsymbol{\nu}}, \widetilde{\widetilde{v^\circ}} \right\rangle_{0,K} + \left\langle \frac{\partial \tilde{u}_K}{\partial \boldsymbol{\nu}}, \widetilde{\widetilde{v^\circ}} \right\rangle_{0,\partial D\setminus K} = \langle u, v \rangle_{0,K}.$$

Hence, we find from (2.1) and (A.3)

$$\|u\|_{H^{-1/2}(K)} = \sup_{0\neq v\in \tilde{H}^{1/2}(K)} \frac{\left|\left\langle \nabla \tilde{u}_K, \nabla \widetilde{\widetilde{v^\circ}} \right\rangle_{0,D}\right|}{\|v\|_{\tilde{H}^{1/2}(K)}} \simeq \sup_{0\neq v\in \tilde{H}^{1/2}(K)} \frac{\left|\left\langle \nabla \tilde{u}_K, \nabla \widetilde{\widetilde{v^\circ}} \right\rangle_{0,D}\right|}{\left\|\nabla \widetilde{\widetilde{v^\circ}}\right\|_{0,D}}. \tag{A.8}$$

Let $w := \tilde{u}_K|_K$. One has $w \in \tilde{H}^{1/2}(K)$ because $\tilde{u}_K = 0$ on $\partial D\setminus K$. Moreover, $w^\circ = \tilde{u}_K|_{\partial D}$ and, due to the uniqueness of the solution to the Dirichlet problem (A.1), we conclude that $\widetilde{\widetilde{w^\circ}} = \tilde{u}_K$. Therefore,

$$\sup_{0\neq v\in \tilde{H}^{1/2}(K)} \frac{\left|\left\langle \nabla \tilde{u}_K, \nabla \widetilde{\widetilde{v^\circ}} \right\rangle_{0,D}\right|}{\left\|\nabla \widetilde{\widetilde{v^\circ}}\right\|_{0,D}} \geq \frac{\left|\left\langle \nabla \tilde{u}_K, \nabla \widetilde{\widetilde{w^\circ}} \right\rangle_{0,D}\right|}{\left\|\nabla \widetilde{\widetilde{w^\circ}}\right\|_{0,D}} = \|\nabla \tilde{u}_K\|_{0,D}. \tag{A.9}$$

On the other hand, it is easy to see that

$$\sup_{0\neq v\in \tilde{H}^{1/2}(K)} \frac{\left|\left\langle \nabla \tilde{u}_K, \nabla \widetilde{\widetilde{v^\circ}} \right\rangle_{0,D}\right|}{\left\|\nabla \widetilde{\widetilde{v^\circ}}\right\|_{0,D}} \leq \|\nabla \tilde{u}_K\|_{0,D}. \tag{A.10}$$

Now (A.6) immediately follows from (A.8)–(A.10).

Using (A.6) together with the parallelogram law we find

$$\langle u, v \rangle_{H^{-1/2}(K)} = \left\langle \nabla \tilde{u}_K, \nabla \tilde{v}_K \right\rangle_{0,D} \quad \forall u, v \in H^{-1/2}(K).$$

Hence, integrating by parts and using notation (2), we derive (A.7). □

The following lemma states an analogous result for the space $\tilde{H}^{-1/2}(K)$ which is the dual space of $H^{1/2}(K)$.

Lemma A.2. *For any $u \in \tilde{H}^{-1/2}(K)$ there holds*

$$\|u\|_{\tilde{H}^{-1/2}(K)} \simeq \left\|\nabla \widetilde{(u^\circ)}_{\tilde{K}}\right\|_{0,D}. \tag{A.11}$$

The $\tilde{H}^{-1/2}$-inner product corresponding to the norm on the right-hand side of (A.11) reads as

$$\langle u, v \rangle_{\tilde{H}^{-1/2}(K)} = \left\langle u, \widetilde{(v^\circ)}_{\tilde{K}} \right\rangle_{0,K} = \left\langle \widetilde{(u^\circ)}_{\tilde{K}}, v \right\rangle_{0,K} \quad \forall u, v \in \tilde{H}^{-1/2}(K). \tag{A.12}$$

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Proof. Let $u \in \tilde{H}^{-1/2}(K)$. Then $u^\circ \in \tilde{H}^{-1/2}(\tilde{K}) \subset H^{-1/2}(\tilde{K})$. Using (2.1) and (A.4) we have

$$\begin{aligned}
\|u^\circ\|_{H^{-1/2}(\tilde{K})} &= \sup_{0\neq w\in\tilde{H}^{1/2}(\tilde{K})} \frac{|\langle u^\circ, w\rangle_{0,\tilde{K}}|}{\|w\|_{\tilde{H}^{1/2}(\tilde{K})}} = \sup_{0\neq w\in\tilde{H}^{1/2}(\tilde{K})} \frac{|\langle u, w\rangle_{0,K}|}{\|w\|_{\tilde{H}^{1/2}(\tilde{K})}} \\
&= \sup_{0\neq v\in H^{1/2}(K)} \sup_{\substack{V\in\tilde{H}^{1/2}(\tilde{K})\\ V|_K=v}} \frac{|\langle u, V\rangle_{0,K}|}{\|V\|_{\tilde{H}^{1/2}(\tilde{K})}} = \sup_{0\neq v\in H^{1/2}(K)} \frac{|\langle u, v\rangle_{0,K}|}{\inf\limits_{\substack{V\in\tilde{H}^{1/2}(\tilde{K})\\ V|_K=v}} \|V\|_{\tilde{H}^{1/2}(\tilde{K})}} \\
&\simeq \sup_{0\neq v\in H^{1/2}(K)} \frac{|\langle u, v\rangle_{0,K}|}{\|v\|_{H^{1/2}(K)}} = \|u\|_{\tilde{H}^{-1/2}(K)}.
\end{aligned}$$

Hence, using (A.6) with u replaced by u° and with K replaced by $\tilde{K}$, we prove (A.11):

$$\|u\|_{\tilde{H}^{-1/2}(K)} \simeq \|u^\circ\|_{H^{-1/2}(\tilde{K})} \simeq \left\|\nabla\widetilde{(u^\circ)_{\tilde{K}}}\right\|_{0,D} \quad \forall u \in \tilde{H}^{-1/2}(K).$$

Then, applying the parallelogram law, integrating by parts, and making use of notations (2), (4), we derive (A.12). □

Remark A.1. The same arguments as above can be used to find equivalent norms and corresponding inner products in the Sobolev spaces on any edge $\ell \subset \partial K$. In particular, using the notation analogous to (3) and (4), we have (*cf.* (A.3), (A.5))

$$\begin{aligned}
\|u\|_{\tilde{H}^{1/2}(\ell)} &\simeq \left\|\nabla\widetilde{\widetilde{u^\circ}}\right\|_{0,K} \quad \forall u \in \tilde{H}^{1/2}(\ell), \\
\langle u, v\rangle_{\tilde{H}^{1/2}(\ell)} &= \left\langle \frac{\partial\widetilde{\widetilde{u^\circ}}}{\partial\mathbf{n}}, v\right\rangle_{0,\ell} = \left\langle u, \frac{\partial\widetilde{\widetilde{v^\circ}}}{\partial\mathbf{n}}\right\rangle_{0,\ell} \quad \forall u, v \in \tilde{H}^{1/2}(\ell).
\end{aligned}$$

The next lemma states the fact that for a constant function v in (A.12) the $\tilde{H}^{-1/2}(K)$-inner product reduces to the $L^2(K)$-inner product.

Lemma A.3. *For any $u \in \tilde{H}^{-1/2}(K)$ there holds*

$$\langle u, 1\rangle_{\tilde{H}^{-1/2}(K)} = \langle u, 1\rangle_{0,K}.$$

Proof. We have by (A.12)

$$\langle u, 1\rangle_{\tilde{H}^{-1/2}(K)} = \langle u, \varphi|_K\rangle_{0,K}, \tag{A.13}$$

where $\varphi(x)$ ($x = (x_1, x_2, x_3) \in D = K \times (0,1)$) solves the following mixed problem (see (A.12) and notations (1), (2), (4)): find $\varphi \in H^1(D)$ such that

$$\Delta\varphi = 0 \text{ in } D, \quad \tfrac{\partial\varphi}{\partial\boldsymbol{\nu}} = 1 \text{ on } \Gamma_1 = K, \quad \tfrac{\partial\varphi}{\partial\boldsymbol{\nu}} = 0 \text{ on } \Gamma_i \ (i = 2, \dots, \mathcal{I}-1), \quad \varphi = 0 \text{ on } \Gamma_{\mathcal{I}}.$$

It is easy to see that $\varphi = 1 - x_3$. Then $\varphi|_K = \varphi|_{x_3=0} = 1$ and the assertion follows from (A.13). □

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