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Abstract

Let X be a topological space. We consider certain generalized configuration spaces of points on X, obtained from the cartesian product X^n by removing some intersections of diagonals. We give a systematic framework for studying the cohomology of such spaces using what we call 'twisted commutative dg algebra models' for the cochains on X. Suppose that X is a 'nice' topological space, R is any commutative ring, $H_c^\bullet(X, R) \to H^\bullet(X, R)$ is the zero map, and that $H_c^\bullet(X, R)$ is a projective R-module. We prove that the compact support cohomology of any generalized configuration space of points on X depends only on the graded R-module $H_c^\bullet(X, R)$. This generalizes a theorem of Arabia.

1. Introduction

1.1 If X is a Hausdorff space, let $F(X, n)$ denote the *configuration space* of n distinct ordered points on X. A basic question in the study of configuration spaces is how the (co)homology groups of $F(X, n)$ depend on the (co)homology groups of X. This question has a slightly nicer answer if one considers *cohomology with compact support* instead of the usual cohomology. This can be seen already for $n = 2$, in which case there is a Gysin long exact sequence

$$\cdots \to H_c^k(F(X, 2)) \to H_c^k(X^2) \to H_c^k(X) \to H_c^{k+1}(F(X, 2)) \to \cdots$$

associated to the inclusion of $F(X, 2)$ as an open subspace of X^2. If we work with field coefficients, then $H_c^k(X^2) \cong \bigoplus_{p+q=k} H_c^p(X) \otimes H_c^q(X)$ and the map $H_c^k(X^2) \to H_c^k(X)$ is given by multiplication in the cohomology ring $H_c^\bullet(X)$; consequently, the compactly supported cohomology groups of $F(X, 2)$ are completely determined by the compactly supported cohomology of X, *with its ring structure*. No such simple statement is true for the usual cohomology or homology.

1.2 For $n > 2$ it is no longer true that $H_c^\bullet(F(X, n))$ depends only on the ring structure on $H_c^\bullet(X)$. However, as a consequence of the results proved in this paper, we will see that if X is any locally compact Hausdorff space, and R is any ring,[1] then the graded $\mathbb{S}_n$-module $H_c^\bullet(F(X, n), R)$ depends only on the quasi-isomorphism class of the $\mathbb{E}_\infty$-*algebra* given by the *compactly supported cochains* $C_c^\bullet(X, R)$.[2] In fact, the full $\mathbb{E}_\infty$-structure is not needed: there is a forgetful functor from

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[1] From now until forever, all rings are assumed to be commutative.

[2] If $R = \mathbf{Z}$ and the one-point compactification X^* of X is finite type nilpotent, this is an easy consequence of Mandell's theorem [Man06], at least modulo point-set issues. Indeed, we may identify $C_c^\bullet(X, R)$ with the reduced cochains of X^*, so that the $\mathbb{E}_\infty$-structure on the compactly supported cochains determines the homotopy type of X^*. Then $F(X, n)^*$ is given by collapsing the big diagonal inside the n-fold smash product of X^* with itself, which is clearly homotopy invariant.

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$\mathbb{E}_\infty$-algebras to *twisted commutative dg algebras* (tcdgas), and one only needs to know $C_c^\bullet(X, R)$ as a tcdga. If we are not interested in the $\mathbb{S}_n$-module structure on $H_c^\bullet(F(X,n), R)$ we need even less information: we may consider $C_c^\bullet(X, R)$ as what is known as a *commutative shuffle dg algebra*. With field coefficients one can show using homological perturbation theory that $H_c^\bullet(F(X,n))$ depends only on the ring $H_c^\bullet(X)$ *together with its higher Massey products* – in fact, one only needs to know the m-fold Massey products for $m \leqslant n$, which recovers what we said above for $n = 2$. When X is a compact oriented manifold, this is a theorem of Baranovsky and Sazdanovic [BS12].

1.3 The prototype of our first main theorem is a result of Bendersky and Gitler [BG91]. They define for any integer $n \geqslant 1$ an explicit and combinatorially defined functor $\Lambda(n, -)$ from commutative dg algebras (cdgas) to $\mathbb{S}_n$-equivariant cochain complexes. If $A \to A'$ is a cdga quasi-isomorphism, then $\Lambda(n, A) \to \Lambda(n, A')$ is a quasi-isomorphism. If A is a cdga model for the cochains $C^\bullet(X, \mathbf{Q})$ of a space X, then this cochain complex computes the following relative cohomology group:

$$H(\Lambda(n, A)) \cong H^\bullet(X^n, D(X,n); \mathbf{Q}),$$

where $D(X,n) = X^n \setminus F(X,n)$ is the complement of the configuration space (i.e. the 'big diagonal').

Our first main theorem is the construction of two functors $\mathcal{CF}(U, A)$ and $\mathcal{CD}(U, A)$, where U is an upwards closed subset of the nth partition lattice Π_n and A is a *twisted* commutative dg algebra. When $U = \Pi_n \setminus \{\hat{0}\}$ and A is a cdga, we have $\mathcal{CF}(U, A) \simeq \Lambda(n, A)$. The functors $\mathcal{CF}$ and $\mathcal{CD}$ improve upon the construction of Bendersky and Gitler in three directions.

(A) The construction of Bendersky and Gitler works only for cohomology with coefficients in $\mathbf{Q}$. By replacing cdgas with tcdgas, we can treat coefficients in an arbitrary ring (or in fact any sheaf or complex of sheaves) in a uniform manner.

(B) The construction here works for more general 'configuration-like' spaces: to an upwards closed subset $U \subset \Pi_n$ we can associate an open subset of X^n obtained by removing a family of intersections of diagonals, and any such open subset arises from some upwards closed U. Our functors $\mathcal{CF}(U, -)$ and $\mathcal{CD}(U, -)$ compute the cohomology of the resulting generalized configuration space and the cohomology of the 'discriminant', respectively.

(C) We give a construction that works equally well for computing the *compactly supported cohomology* of the configuration space of points on X; that this should be possible is not at all obvious from how the functor $\Lambda(n, -)$ is constructed by Bendersky and Gitler. In particular, if X is an oriented manifold, we obtain a complex which computes the homology of $F(X,n)$ (since Poincaré duality identifies homology and compact support cohomology).

Let us, in turn, comment on each of these points in some more detail.

A. Arbitrary coefficients

1.4 In Bendersky and Gitler's definition of $\Lambda(n, A)$, commutativity of A is used crucially in order to verify the equation $d^2 = 0$. This means that their results can only be used over a field of characteristic 0; otherwise, there will practically never exist a strictly commutative dg algebra model for the cochains on a space. For this reason, Baranovsky and Sazdanovic [BS12] remarked that it would be interesting to understand if the construction of $\Lambda(n, A)$ could be modified somehow to allow A to be an $\mathbb{E}_\infty$-algebra. But the higher coherence conditions in an $\mathbb{E}_\infty$-algebra are unwieldy, no matter what model of $\mathbb{E}_\infty$-operad one chooses, and it is far from immediate how the functor Λ should be modified.

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1.5 It was proven by Sagave and Schlichtkrull [SS12] that $\mathbb{E}_\infty$-algebras, which are only commutative up to coherent homotopy, can be faithfully modeled by *strictly commutative objects in a larger category*, in which the higher coherences are in a sense built into the objects of the category themselves. Specifically, they introduced the notion of an I-algebra; an I-algebra is a commutative monoid in the category of functors from finite sets and injections to cochain complexes. Many readers of this paper are perhaps familiar with the literature on representation stability – in the usual lingo of representation stability, an I-algebra is just a gadget which is simultaneously an FI-module and a twisted commutative algebra. For a suitable model structure on I-algebras, this category is Quillen equivalent to the category of $\mathbb{E}_\infty$-algebras.

1.6 It turns out that Bendersky and Gitler's functor Λ, and our generalized versions $\mathcal{CF}$ and $\mathcal{CD}$, can be readily modified to make sense also for commutative dg I-algebras. In fact, one does not even need the injections, so the full $\mathbb{E}_\infty$-structure is not needed – the functor is well defined already on the category of tcdgas. We shall think of the category of tcdgas as a useful enlargement of the category of cdgas: small enough that its objects are specified by a manageable amount of data, and large enough to contain all examples of interest.

1.7 The passage from cdgas to tcdgas not only allows us to pass from $\mathbf{Q}$-coefficients to $\mathbf{Z}$-coefficients; it is also needed if we wish to deal with cohomology with twisted coefficients. If $\mathcal{F}$ is a complex of sheaves on X, then we define tcdgas $R\Gamma^\otimes(X,\mathcal{F})$ and $R\Gamma_c^\otimes(X,\mathcal{F})$ whose cohomologies are given by

$$\bigoplus_{n\geqslant 1} H^\bullet(X,\mathcal{F}^{\otimes n}) \quad \text{and} \quad \bigoplus_{n\geqslant 1} H_c^\bullet(X,\mathcal{F}^{\otimes n})$$

with multiplication given by $H^k(X,\mathcal{F}^{\otimes n})\otimes H^k(X,\mathcal{F}^{\otimes m})\to H^{k+l}(X,\mathcal{F}^{\otimes(n+m)})$. Evaluating our functor $\mathcal{CF}$ on these tcdgas gives a cochain complex calculating the cohomologies of

$$H^\bullet(X^n, D(X,U);\mathcal{F}^{\boxtimes n}) \quad \text{and} \quad H_c^\bullet(F(X,U),\mathcal{F}^{\boxtimes n}),$$

respectively. The additional generality of being allowed to use arbitrary coefficients is in fact useful. For example, we can recover a construction of Knudsen [Knu17] computing the rational homology of *unordered* configuration spaces of points on a manifold X by taking for $\mathcal{F}$ the orientation local system on X and taking $\mathbb{S}_n$-invariants. More generally, if we let $\mathcal{F}=\mathbb{D}\mathbf{Z}_X$ be the dualizing complex on the space X, then $H_c^\bullet(F(X,U),\mathcal{F}^{\boxtimes n})$ is the integer homology of $F(X,U)$ and we get a spectral sequence calculating the homology of the configuration space of points on an arbitrary space. In general, it is a hard problem to compute $\mathbb{D}\mathbf{Z}_X$ for non-manifold X, but even partial information can be used to obtain qualitative results on the cohomology of configuration spaces. For example, Tosteson [Tos16] shows, using a spectral sequence equivalent to the one here, that if $\mathcal{H}^i(\mathbb{D}\mathbf{Z}_X)=0$ for $i<2$ (X is 'locally $\geqslant$ 2-dimensional') then the configuration spaces of points on X satisfy representation stability.

1.8 Our constructions of the tcdgas $R\Gamma^\otimes(X,\mathcal{F})$ and $R\Gamma_c^\otimes(X,\mathcal{F})$ is quite simple, but we hope that it can be of independent interest. The case when $\mathcal{F}$ is a constant sheaf is particularly interesting – in this case one can give a natural FI-module structure on these tcdgas as well, so that we obtain an I-algebra in the sense of § 1.5. Under the Quillen equivalence between I-algebras and $\mathbb{E}_\infty$-algebras, these I-algebras are equivalent to the cochains (respectively, compactly supported cochains) on X, with their $\mathbb{E}_\infty$-algebra structure (so we recover in a slightly unusual way the $\mathbb{E}_\infty$-structure on cochains). A different explicit I-algebra structure for the cochains on a space was very recently constructed by Richter and Sagave [RS18]; our construction gives a sheaf-theoretic alternative to theirs.

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B. Compact support cohomology

1.9 Let us momentarily consider the simplest case of rational coefficients and the classical configuration spaces $F(X,n)$, where our construction is equivalent to the functor $\Lambda(n,-)$ of Bendersky and Gitler. As we said above, if A is a cdga model for the cochains $C^\bullet(X,\mathbf{Q})$, then $\Lambda(n,A)$ computes the relative cohomology $H^\bullet(X^n, D(X,n);\mathbf{Q})$. A consequence of what we prove here is that if instead A is a cdga model for the *compactly supported cochains* $C_c^\bullet(X,\mathbf{Q})$, then we have an isomorphism

$$H(\Lambda(n,A)) \cong H_c^\bullet(F(X,n),\mathbf{Q}).$$

In other words, the *exact same functor* will, when we plug in the compactly supported cochains of X, calculate the compact support cohomology of the configuration space itself. This is not clear from Bendersky and Gitler's construction of the functor Λ. The same holds integrally if we choose a tcdga model for $C_c^\bullet(X,\mathbf{Z})$, rather than a cdga model (which will practically never exist).

1.10 This is particularly interesting when X is an oriented d-manifold. In this case we have a Poincaré duality isomorphism

$$H_c^k(F(X,n),\mathbf{Z}) \cong H_{nd-k}(F(X,n),\mathbf{Z})$$

between homology and cohomology, and we obtain a canonical cochain complex computing the homology of $F(X,n)$, depending only on a tcdga model for $C_c^\bullet(X,\mathbf{Z})$. After giving this cochain complex a natural filtration, one obtains a spectral sequence whose first page and first differential depend only on the ring $H_c^\bullet(X,\mathbf{Z})$; this spectral sequence is exactly Poincaré dual to the familiar spectral sequence of Cohen and Taylor, Totaro, and Kříž [CT78, Tot96, Kří94]. This recovers some familiar statements: for example, the first differential in the Cohen–Taylor spectral sequence is given by the Gysin map $\Delta_!\colon H^\bullet(X) \to H^{\bullet+d}(X^2)$, which is exactly Poincaré dual to the cup product in compact support cohomology. However, the fact that the higher differentials depend only on the cochain algebra $C_c^\bullet(X)$ appears to be new. For example, we see that if the algebra $C_c^\bullet(X,\mathbf{Q})$ is formal, then the Cohen–Taylor spectral sequence degenerates rationally after the first differential; again, this observation seems to be new.

C. Generalized configuration spaces

1.11 Let Π_n denote the *partition lattice*, that is, the partially ordered set (poset) of all partitions of the set $\{1,\dots,n\}$, ordered by refinement. Each element $T \in \Pi_n$ corresponds to a locally closed subset $X(T) \subset X^n$:

$$X(T) = \{(x_1,\dots,x_n) \in X^n : x_i = x_j \iff i \text{ and } j \text{ are in the same block of } T\}.$$

If $U \subset \Pi_n$ is upwards closed, then we may define

$$D(X,U) = \bigcup_{T\in U} X(T), \quad F(X,U) = X^n \setminus D(X,U).$$

Then $F(X,U)$ is a 'generalized configuration space' of points on X; if U consists of all elements of Π_n except the bottom element $\hat{0}$, then $F(X,U) = F(X,n)$, and in general we get an intermediate open subset between $F(X,n)$ and X^n.

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1.12 We define functors $\mathcal{CF}(U,A)$ and $\mathcal{CD}(U,A)$ such that if $U = \Pi_n \setminus \{\hat{0}\}$ and A is a cdga, then $\mathcal{CF}(U,A) \simeq \Lambda(n,A)$. If A is a tcdga model for the cochains (respectively, the compactly supported cochains) on X then $\mathcal{CF}(U,A)$ computes the cohomology of $H^\bullet(X^n, D(X,U); \mathbf{Z})$ (respectively, $H_c^\bullet(F(X,U),\mathbf{Z})$). Similarly, $\mathcal{CD}(U,A)$ computes the cohomology of $H^\bullet(D(X,U),\mathbf{Z})$ (respectively, $H_c^\bullet(D(X,U),\mathbf{Z})$). Thus the results of this paper provide a uniform way of studying the cohomology and compact support cohomology of spaces of the form $F(X,U)$ and $D(X,U)$, for any upwards closed $U \subseteq \Pi_n$.

1.13 When X is a Euclidean space $\mathbf{R}^d$, then $F(X,U)$ is the complement of a 'hypergraph arrangement' (also called a diagonal arrangement) and our results recover the Goresky–MacPherson formula in this case (as well as its equivariant version). For a general space X we obtain a spectral sequence computing $H_c^\bullet(F(X,U),\mathbf{Z})$ whose first page depends only on the cohomology of X and the cohomology of the lower intervals in the poset U. (A more general form of this spectral sequence was described previously in [Pet17].) The cohomologies of such lower intervals have been studied intensely, precisely because they can be interpreted (via the Goresky–MacPherson formula) as computing the cohomology of hypergraph arrangements in Euclidean space. Our results thus show that whenever we can compute the cohomologies of such lower intervals for some poset U, we can write down an explicit spectral sequence computing the compact support cohomology of $F(X,U)$, whose first page depends only on the ring $H_c^\bullet(X,\mathbf{Z})$. Cases where the cohomologies of such lower intervals are known include k-equals arrangements [BW95] and large classes of orbit arrangements [Koz97].

1.14 Totaro and Kříž both note that the structure of the Cohen–Taylor spectral sequence for an oriented d-manifold X depends on knowing the cohomology of $F(\mathbf{R}^d, n)$, and that this should reflect the locally Euclidean structure of X. It is interesting that we see the exact same phenomenon here – for an arbitrary space X, the combinatorial structure of the spectral sequence computing $H_c^\bullet(F(X,U),\mathbf{Z})$ depends only on knowing the cohomology of the corresponding hypergraph arrangement in a Euclidean space – even though there is no 'locally Euclidean' structure in sight.

1.15 Let us now state the theorem, which is a composite of Theorems 5.12 and 5.18. The comparison with the Cohen–Taylor spectral sequence appears in §§ 5.20–5.23. The two functors $\mathcal{CF}$ and $\mathcal{CD}$ are defined in Definition 5.7. The symbols $[\![-]\!]$ and $[\![-]\!]$ denote two variants of the order complex of a poset, which can be found in § 4.2.

Theorem 1.16 (First main theorem). *There exist explicit functors $\mathcal{CF}(U,A)$ and $\mathcal{CD}(U,A)$, where A is a tcdga over a ring R and $U \subset \Pi_n$ is an upwards closed subset, to cochain complexes over R. If $G \subset \mathbb{S}_n$ preserves U then G acts on $\mathcal{CF}(U,A)$. Let X be a locally contractible paracompact Hausdorff space, $\mathcal{F}$ a bounded-below complex of sheaves of R-modules on X. There are equivariant quasi-isomorphisms of cochain complexes*

$$\begin{aligned}
\mathcal{CF}(U, R\Gamma_c^\otimes(X,\mathcal{F})) &\simeq C_c^\bullet(F(X,U), \mathcal{F}^{\boxtimes n}),\\
\mathcal{CF}(U, R\Gamma^\otimes(X,\mathcal{F})) &\simeq C^\bullet(X^n, D(X,U); \mathcal{F}^{\boxtimes n}),\\
\mathcal{CD}(U, R\Gamma_c^\otimes(X,\mathcal{F})) &\simeq C_c^\bullet(D(X,U), \mathcal{F}^{\boxtimes n}),\\
\mathcal{CD}(U, R\Gamma^\otimes(X,\mathcal{F})) &\simeq C^\bullet(D(X,U), \mathcal{F}^{\boxtimes n}),
\end{aligned}$$

which are natural in X, $\mathcal{F}$ and U. There are natural filtrations on $\mathcal{CF}(U,A)$ and $\mathcal{CD}(U,A)$ such

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that we get spectral sequences

$$E_1^{pq} = \bigoplus_{\substack{T \in J_U \\ |T|=p}} \widetilde{H}^{p+q}(\llbracket J_{\overline{U}}^{\prec T} \rrbracket; A(T)) \implies H^{p+q}(\mathcal{CD}(U, A))$$

and

$$E_1^{pq} = \bigoplus_{\substack{T \in J_{U_0} \\ |T|=p}} \widetilde{H}^{p+q}(\llbracket J_{\overline{U}_0}^{\prec T} \rrbracket; A(T)) \implies H^{p+q}(\mathcal{CF}(U, A)),$$

where $J_U \subseteq U$ is the subposet consisting of all joins of minimal elements of U, $J_{U_0} = J_U \cup \{\hat{0}\}$. The resulting spectral sequence is Poincaré dual to the Cohen–Taylor spectral sequence when $A = R\Gamma_c^{\otimes}(X, \mathbf{Z})$, X is an oriented manifold, and $U = \Pi_n \setminus \{\hat{0}\}$.

1.17 When $U = \Pi_n \setminus \{\hat{0}\}$ consists of the whole partition lattice minus its bottom element – this is the case in which the functor $\mathcal{CF}(U, -)$ computes the cohomology of the usual configuration space of points – then the cochain complex $\mathcal{CF}(U, -)$ can be given a somewhat unexpected interpretation in terms of operadic cohomology. Specifically, we show in the final section of the paper that when $U = \Pi_n \setminus \{\hat{0}\}$, the cochain complex $\mathcal{CF}(U, -)$ may be identified with the Chevalley–Eilenberg chains computing the Lie algebra homology of a certain *twisted* Lie algebra (a left module over the Lie operad). The precise statement is that $\mathcal{CF}(U, A)$ computes the Lie algebra homology of $\mathsf{S}A \otimes_H \mathsf{Lie}$ (the Hadamard tensor product of the suspension of A with the operad Lie, considered as a left module over itself). If we take X to be a manifold, and $A = R\Gamma_c^{\otimes}(X, \mathbb{D}\mathbf{Q})$, then we recover theorems of Knudsen [Knu17] and Hô [Hô17] upon taking $\mathbb{S}_n$-invariants[3] in this result. Here $\mathbb{D}\mathbf{Q}$ denotes the rational dualizing complex of X, that is, the orientation sheaf with $\mathbf{Q}$-coefficients placed in degree $\dim X$. We also indicate how this result can be understood in terms of the Goodwillie calculus. That said, it is not clear to me what the precise relationship is between what is done in this paper, and methods such as higher Hochschild homology or manifold calculus. In particular, recent papers [Hô18, CILW18] contain results that ought to be related to the ones obtained here, although the methods are completely different.

Remark 1.18. Local contractibility is used in Theorem 5.12 only to get an isomorphism between sheaf cohomology and singular cohomology. We could have omitted this hypothesis if the right-hand sides of the quasi-isomorphisms of the theorem had been interpreted as complexes computing derived functor cohomologies.

Configuration spaces of points on *i*-acyclic spaces

1.19 Our second main theorem is a generalization and reinterpretation of a beautiful result due to Arabia [Ara16]. Arabia introduced the notion of an *i-acyclic space*: a topological space X is i-acyclic over a ring R if $H_c^k(X, R) \to H^k(X, R)$ is the zero map for all k. This condition is in fact satisfied in many cases of interest: for example, any open subset of Euclidean space is i-acyclic, and the product of any space with an i-acyclic space is i-acyclic. (For example, if Y is arbitrary then $Y \times \mathbf{R}$ is i-acyclic.) More examples are given in Example 6.3. The remarkable fact about i-acyclicity is that it is exactly the right hypothesis to ensure that the compactly

[3] Knudsen and Hô both work with unordered configuration spaces, which is why we need to take $\mathbb{S}_n$-invariants to recover their results. However, the arguments they use can be modified to obtain an interpretation of the homology of ordered configuration spaces in terms of Lie algebra homology as well, provided that we replace Lie algebras with twisted Lie algebras throughout; in this case their results coincide with the ones obtained here.

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supported cohomology of configuration spaces of points on X depends in the simplest possible way on the compactly supported cohomology of X itself. Although the main focus of Arabia's paper is the rational cohomology, the following theorem is proven with arbitrary field coefficients.

Theorem 1.20 (Arabia). *Let X be an i-acyclic paracompact locally compact Hausdorff space over a field k. Then $H_c^\bullet(F(X,n),k)$ depends only on the graded vector space $H_c^\bullet(X,k)$.*

1.21 On the other hand, a consequence of Theorem 1.16 is that for an arbitrary locally compact Hausdorff space, $H_c^\bullet(F(X,n),\mathbf{Q})$ depends only on the choice of a cdga model for the compactly supported cochains $C_c^\bullet(X,\mathbf{Q})$. It would be appealing to try to 'explain' the rational case of Arabia's result in these terms, instead: that i-acyclicity should force the algebra of compactly supported cochains to be homotopically trivial in some strong sense. This is indeed the case.

Theorem 1.22 (Second main theorem, with $\mathbf{Q}$-coefficients). *Let X be an i-acyclic paracompact locally compact Hausdorff space over $\mathbf{Q}$. Then a cdga model for $C_c^\bullet(X,\mathbf{Q})$ is given by the cohomology $H_c^\bullet(X,\mathbf{Q})$, with identically zero multiplication and differential. Equivalently, $C_c^\bullet(X,\mathbf{Q})$ is formal and the cup product in compact support cohomology vanishes.*

1.23 This result re-proves, reinterprets, and generalizes Arabia's theorem. The proof of Theorem 1.22 is simple enough that it makes sense to outline it here in the introduction. It will be enough to construct an A_∞-quasi-isomorphism between $H_c^\bullet(X,\mathbf{Q})$ (with identically zero differential and multiplication) and the dg algebra $C_c^\bullet(X,\mathbf{Q})$. This means that we have to find maps

$$f_n\colon H_c^\bullet(X,\mathbf{Q})^{\otimes n} \to C_c^\bullet(X,\mathbf{Q}), \quad n \geqslant 1,$$

of degree $1-n$, such that

$$d \circ f_n = \sum_{i+j=n} (-1)^i f_i \cdot f_j.$$

Let $f\colon H_c^\bullet(X,\mathbf{Q}) \to C_c^\bullet(X,\mathbf{Q})$ be a map taking every class to a representing cocycle. Since X is i-acyclic, every cocycle in $C_c^\bullet(X,\mathbf{Q})$ is a coboundary in $C^\bullet(X,\mathbf{Q})$. This means that there exists

$$g\colon H_c^\bullet(X,\mathbf{Q}) \to C^{\bullet-1}(X,\mathbf{Q})$$

such that $d \circ g = -f$. Now define for all $n \geqslant 1$,

$$f_n(x) = f(x) \cdot \underbrace{g(x) \cdots g(x)}_{n-1 \text{ times}}.$$

Since the product of a compactly supported cochain with an arbitrary cochain has compact support, this product is well defined as an element of $C_c^\bullet(X,\mathbf{Q})$, and one can verify that this defines an A_∞-quasi-isomorphism.

1.24 By combining Theorems 1.16 and 1.22 we obtain as a corollary that if X is i-acyclic, then the spectral sequences of Theorem 1.16 converging to $H_c^\bullet(F(X,U),\mathbf{Q})$ and $H_c^\bullet(D(X,U),\mathbf{Q})$ degenerate immediately; there is no differential on any page of the spectral sequence. So, for example, the Totaro spectral sequence degenerates immediately for any i-acyclic oriented manifold. We deduce the following corollary.

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Corollary 1.25. *Let X be an i-acyclic locally contractible paracompact Hausdorff space over $\mathbf{Q}$, and let $U \subset \Pi_n$ be upwards closed. There are isomorphisms*

$$H_c^k(F(X,U),\mathbf{Q}) \cong \bigoplus_{i+j=k} \bigoplus_{T \in J_U} \widetilde{H}^i(\llbracket J_{\overline{U}_0}^{\prec T} \rrbracket, \mathbf{Q}) \otimes H_c^j(X^{|T|},\mathbf{Q})$$

and

$$H_c^k(D(X,U),\mathbf{Q}) \cong \bigoplus_{i+j=k} \bigoplus_{T \in J_U} \widetilde{H}^i(\llbracket J_{\overline{U}}^{\prec T} \rrbracket, \mathbf{Q}) \otimes H_c^j(X^{|T|},\mathbf{Q}).$$

If $G \subset \mathbb{S}_n$ preserves U then the isomorphisms are G-equivariant.

1.26 If we wish to extend our generalization of Arabia's theorem to arbitrary coefficients, we can no longer work with cdgas. An advantage of the framework set up in this paper is that once the formalism is in place, one can give more or less the same proof as in § 1.23 to prove a version of Theorem 1.22 for cohomology with arbitrary coefficients. One might naively hope for the following statement: if X is an i-acyclic space over the ring R – that is, $H_c^\bullet(X,R) \to H^\bullet(X,R)$ is the zero map – then the tcdga $R\Gamma_c^\otimes(X,R)$ is quasi-isomorphic to its cohomology with identically zero differential and multiplication. However, if the tcdga $R\Gamma_c^\otimes(X,R)$ were homotopically trivial in this strong sense, then not only would the spectral sequence of Theorem 1.16 converging to $H_c^\bullet(F(X,U),R)$ degenerate immediately; we would also have an *equivariant* isomorphism between $H_c^\bullet(F(X,U),R)$ and the direct sum of the terms on the E_1-page of the spectral sequence, just as in Corollary 1.25. Thus we see that the above statement is false already for X the real line and $R = \mathbf{Z}$: for the configuration space of two points we have $H_c^2(F(\mathbf{R},2),\mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$, on which $\mathbb{S}_2$ acts by switching the two factors. But the E_1-page of the spectral sequence has $E_1^{1,1} \cong \mathbf{Z}$ (with the trivial representation of $\mathbb{S}_2$) and $E_1^{2,0} \cong \mathbf{Z}$ (with the sign representation). Since $\mathbf{Z} \oplus \mathbf{Z}$ is not the direct sum of the trivial and the sign representation, there is no such equivariant isomorphism in this case.

1.27 What one needs to consider instead is the forgetful functor from tcdgas to what are known as *commutative dg shuffle algebras*; a commutative dg shuffle algebra is essentially a twisted commutative algebra in which one has forgotten the actions of the symmetric group while remembering as much as possible of the remaining structure. Our functors $\mathcal{CF}$ and $\mathcal{CD}$ make sense also on the larger category of cdgas. Moreover, i-acyclicity of a space X *does* imply that $R\Gamma_c^\otimes(X,R)$ is quasi-isomorphic to its cohomology with identically zero multiplication and differential *as a commutative dg shuffle algebra.*

Theorem 1.28 (Second main theorem with arbitrary coefficients). *Let X be a locally compact Hausdorff space, R any ring. Suppose that X is i-acyclic over R, and that $H_c^\bullet(X,R)$ is a projective R-module. Then the tcdga $R\Gamma_c^\otimes(X,R)$ is quasi-isomorphic as a cdga to its cohomology with identically zero multiplication and differential.*

1.29 Again we obtain as a corollary a theorem describing the compact support cohomology of configuration spaces of points on i-acyclic spaces.

Corollary 1.30. *Let X be an i-acyclic locally contractible paracompact Hausdorff space over the ring R, and let $U \subset \Pi_n$ be upwards closed. There are isomorphisms*

$$H_c^k(F(X,U),R) \cong \bigoplus_{i+j=k} \bigoplus_{T \in J_U} \widetilde{H}^i(\llbracket J_{\overline{U}_0}^{\prec T} \rrbracket; H_c^j(X^{|T|},R))$$

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and

$$H_c^k(D(X,U),R) \cong \bigoplus_{i+j=k} \bigoplus_{T\in J_U} \widetilde{H}^i(\llbracket J_{\widehat{U}}^{\prec T} \rrbracket; H_c^j(X^{|T|},R)).$$

If $G \subset \mathbb{S}_n$ preserves U then the left-hand side has a filtration whose associated graded is G-equivariantly isomorphic to the right-hand side.

1.31 In particular, we see under these hypotheses that $H_c^\bullet(F(X,U),R)$ depends only on the graded R-module $H_c^\bullet(X,R)$ and on the cohomology of lower intervals in the poset U. In any situation where one can compute the cohomology of such lower intervals, the theorem can be used for quite explicit computations. In § 7 of this paper we give very explicit generating series for the rational compact support cohomology of k-equals configuration spaces of i-acyclic spaces, considered as a sequence of representations of the symmetric groups $\mathbb{S}_n$. The calculations are expressed in terms of the algebra of symmetric functions; the result is equivalent to computing the 'character polynomials' of the compactly supported cohomology groups of k-equals configuration spaces of i-acyclic spaces. The computations of poset cohomology used here are due to Sundaram and Wachs [SW97a].

2. Twisted commutative algebras and commutative shuffle algebras

Twisted commutative algebras

2.1 The notion of a twisted commutative algebra goes back a long time, at least to Barratt [Bar78] and Joyal [Joy86]. Twisted commutative algebras have received a lot of recent attention because of their role in recent work on stable representation theory by Sam, Snowden and others – see, for example, [SS15] for a possible starting point. In this section we will review the definition and set up our conventions. In particular, we caution the reader about our non-standard choice that *all our twisted commutative algebras will be non-unital by default.*

2.2 Let Fin_+ be the groupoid of *non-empty* finite sets and bijections. We make Fin_+ a non-unital symmetric monoidal category using the disjoint union of finite sets. A *twisted commutative dg algebra* (tcdga) is a lax symmetric monoidal functor $A\colon \mathsf{Fin}_+ \to \mathsf{Ch}_R$. Explicitly, for each non-empty finite set S we are given a cochain complex $A(S)$, we are given functorial chain maps $A(S)\otimes A(T) \to A(S\sqcup T)$ for any pair of non-empty finite sets, and the following two diagrams commute:

$$\begin{array}{ccc} A(S)\otimes A(T)\otimes A(U) & \longrightarrow & A(S\sqcup T)\otimes A(U) \\ \downarrow & & \downarrow \\ A(S)\otimes A(T\sqcup U) & \longrightarrow & A(S\sqcup T\sqcup U) \end{array}$$

and

$$\begin{array}{ccc} A(S)\otimes A(T) & \longrightarrow & A(S\sqcup T) \\ \downarrow\cong & & \downarrow\cong \\ A(T)\otimes A(S) & \longrightarrow & A(T\sqcup S) \end{array}$$

A *morphism* of tcdgas is a symmetric monoidal natural transformation. We say that a morphism $A \to A'$ is a *quasi-isomorphism* if $A(S) \to A'(S)$ is a quasi-isomorphism for all S.

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2.3 If A is a tcdga, then so is its cohomology $H^\bullet(A)$, where we set $H^\bullet(A)(S) = H^\bullet(A(S))$.

2.4 A skeleton of the category Fin_+ is given by the disjoint union of all symmetric groups $\mathbb{S}_n$, $n \geqslant 1$. It follows that a tcdga can also be described as a sequence of $\mathbb{S}_n$-representations $A(n)$ in the category of dg R-modules, together with $\mathbb{S}_n \times \mathbb{S}_m$-equivariant multiplication maps

$$A(n) \otimes A(m) \to A(n+m)$$

which are associative, and for which the following diagram commutes:

$$\begin{array}{ccc} A(n) \otimes A(m) & \longrightarrow & A(n+m) \\ \downarrow & & \downarrow \\ A(m) \otimes A(n) & \longrightarrow & A(m+n) \end{array}$$

The left vertical arrow is the map switching the two tensor factors (and taking into account the Koszul sign rule), and the right vertical arrow is given by acting with the 'box permutation' in $\mathbb{S}_{n+m}$ that moves the first n elements past the last m elements. We refer to $A(n)$ as the *arity-n component* of the tcdga A. We will switch freely between both definitions of a tcdga.

2.5 Let Ω be a not necessarily unital commutative dg algebra over R. We can associate to Ω a 'constant' tcdga $\underline{\Omega}$ by the rule that $\underline{\Omega}(S) = \Omega$ for every non-empty finite set S, any bijection $S \to T$ is mapped to the identity map $\Omega \to \Omega$, and the maps $\underline{\Omega}(S) \otimes \underline{\Omega}(T) \to \underline{\Omega}(S \sqcup T)$ are given by the multiplication in Ω. In this way we can think of the category of tcdgas as an enlargement of the category of cdgas. Whenever we say that we consider a cdga Ω as a tcdga, this is the construction we have in mind.

2.6 Later, we will also briefly consider arbitrary lax monoidal functors $\mathsf{Fin}_+ \to \mathsf{Ch}_R$, not just lax symmetric monoidal functors. A lax monoidal functor $A\colon \mathsf{Fin}_+ \to \mathsf{Ch}_R$ is called a *twisted associative dg algebra*.

2.7 A definition equivalent to § 2.2 is that a tcdga is a reduced left module over the operad Com of non-unital commutative algebras; similarly, a twisted associative dg algebra is a reduced left module over the operad Ass of non-unital associative algebras. More generally, if $\mathcal{P}$ is a dg operad, then left modules over $\mathcal{P}$ are sometimes called *twisted $\mathcal{P}$-algebras*. The terminology goes back to Barratt [Bar78].

Shuffle algebras

2.8 In § 6 we will study the cohomology of configuration spaces of points on i-acyclic spaces, with coefficients in an arbitrary ground ring R. There will be technical complications arising from the fact that the trivial representation of $\mathbb{S}_n$ is not a projective $R[\mathbb{S}_n]$-module. For this reason we will need to work also with a notion of a twisted commutative algebra in which one has 'forgotten' about the actions of the symmetric group – more precisely, we will need to work with *commutative shuffle dg algebras*. Commutative shuffle algebras are significantly less studied than twisted commutative algebras: shuffle algebras (not necessarily commutative) were introduced by Ronco [Ron11, § 2], and to my knowledge, the only explicit mention of their evident commutative analogue in the literature is in [BD16, § 4.6.3]. (A word of caution: there exist several completely unrelated notions of 'shuffle algebra' in the literature.) The reader who is only interested in working rationally may skip this subsection.

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2.9 Consider the functor category $\mathsf{Ch}_R^{\mathsf{Fin}_+}$, equipped with the monoidal structure given by Day convolution:

$$(A \otimes B)(S) = \bigoplus_{S=T\sqcup T'} A(T) \otimes B(T').$$

It is a general fact about Day convolution that commutative monoids in this monoidal category can be canonically identified with lax symmetric monoidal functors $\mathsf{Fin}_+ \to \mathsf{Ch}_R$, which gives yet another reformulation of the definition of a tcdga. Similarly, not necessarily commutative monoids in this category may be identified with twisted associative dg algebras.

2.10 Let Ord_+ denote the category of finite non-empty totally ordered sets and order-preserving bijections. The functor category $\mathsf{Ch}_R^{\mathsf{Ord}_+}$ can be given a symmetric monoidal structure analogous to the one on $\mathsf{Ch}_R^{\mathsf{Fin}_+}$, namely,

$$(A \otimes B)(S) = \bigoplus_{S=T\sqcup T'} A(T) \otimes B(T').$$

In this formula, $S = T \sqcup T'$ should be read as saying that S is the union of two non-empty disjoint subsets T, T', and that the total orders on T and T' are the ones inherited from S. But we do *not* insist that every element in T is smaller than every element of T'; that is, S is not in any sense the coproduct of T and T' in the category Ord_+.

DEFINITION 2.11. Commutative monoids in the monoidal category $\mathsf{Ch}_R^{\mathsf{Ord}_+}$ are called *commutative shuffle dg algebras*. Not necessarily commutative monoids are called *shuffle dg algebras*.

Remark 2.12. One motivation for introducing the above monoidal product is that it makes the forgetful functor $\mathsf{Ch}_R^{\mathsf{Fin}_+} \to \mathsf{Ch}_R^{\mathsf{Ord}_+}$ strong symmetric monoidal. This implies that the forgetful functor from twisted commutative algebras to commutative shuffle algebras preserves certain algebraic properties that the forgetful functor to $\mathbf{N}$-graded associative algebras does not; for example, a presentation of a twisted commutative algebra by generators and relations will also be a presentation of its underlying commutative shuffle algebra. If we had considered the tensor product on $\mathsf{Ch}_R^{\mathsf{Ord}_+}$ given by Day convolution, then the forgetful functor would only be lax symmetric monoidal.

2.13 We can make the notion of a (commutative) shuffle dg algebra more explicit, as we did for tcdgas in § 2.4. A shuffle dg algebra is a collection of dg R-modules $A(n)$, $n \geqslant 1$, and for every shuffle permutation $\pi \in \mathrm{Sh}(n,m) \subset \mathbb{S}_{n+m}$ a composition map

$$\circ_\pi \colon A(n) \otimes A(m) \to A(n+m),$$

satisfying the following associativity law: any shuffle permutation $\pi \in \mathrm{Sh}(n,m,l)$ can be uniquely written as $\pi = (\mathbf{1}_n \times \sigma) \circ \rho$ where $\sigma \in \mathrm{Sh}(m,l)$ and $\rho \in \mathrm{Sh}(n, m+l)$, and also as $\pi = (\tau \times \mathbf{1}_l) \circ \lambda$ with $\tau \in \mathrm{Sh}(n,m)$ and $\lambda \in \mathrm{Sh}(n+m,l)$, and there is an equality

$$x \circ_\rho (y \circ_\sigma z) = (x \circ_\tau y) \circ_\lambda z$$

for any $x \in A(n)$, $y \in A(m)$, $z \in A(l)$. A shuffle dg algebra is said to be commutative if it has the following additional property: if we denote the evident bijection $\mathrm{Sh}(n,m) \to \mathrm{Sh}(m,n)$ by $\pi \mapsto \pi'$, then we should have $x \circ_\pi y = y \circ_{\pi'} x$ for any $\pi \in \mathrm{Sh}(n,m)$, $x \in A(n)$ and $y \in A(m)$.

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2.14 There is an evident diagram of forgetful functors:

$$\begin{array}{ccc} (\text{twisted commutative dg algebras}) & \longrightarrow & (\text{twisted associative dg algebras}) \\ \downarrow & & \downarrow \\ (\text{commutative shuffle dg algebras}) & \longrightarrow & (\text{shuffle dg algebras}) \end{array}$$

Remark 2.15. As noted in § 2.7, twisted commutative algebras are the same thing as left modules over the commutative operad. Although we will not use it in the sequel, let us mention that there is an analogous description of commutative shuffle algebras, using *shuffle operads*. A shuffle operad is essentially a symmetric operad in which one has forgotten the actions of the symmetric group in a minimally destructive manner; they interpolate between symmetric and non-symmetric operads. Shuffle operads were introduced by Dotsenko and Khoroshkin [DK10] for the purposes of developing a Gröbner theory for operads. Let $(-)^f$ denote the forgetful functor from symmetric to shuffle operads. Then commutative shuffle dg algebras are the same thing as left modules over the shuffle operad Com^f, and shuffle dg algebras are the same thing as left modules over Ass^f.

3. Twisted commutative cochains

The functors $R\Gamma^{\otimes}$ and $R\Gamma_c^{\otimes}$

3.1 Before proceeding further it is perhaps useful to quickly recall some of the basics of sheaf cohomology, since this is not always part of the 'standard toolbox' for a working algebraic topologist. Fix a topological space X. We have left exact functors Γ and Γ_c from sheaves of R-modules on X to R-modules, given by taking global sections and global sections with compact support, respectively. Cohomology (respectively, cohomology with compact support) of a sheaf of R-modules is defined as the derived functors of Γ (respectively, Γ_c). These derived functors can be defined using injective resolutions, but in practice they are usually computed by some other acyclic resolution. A standard choice is the Godement resolution, which has several pleasant formal properties: it is functorial, and the monoidal structure of the Godement resolution allows for the construction of functorial cup products on the chain level. The sheaves in the Godement resolution are flabby, meaning that all restriction maps are surjective. This implies that they are acyclic for both Γ and Γ_c. We define $R\Gamma(X,-)$ and $R\Gamma_c(X,-)$ to be the global sections (respectively, global sections with compact support) of the Godement resolution of a sheaf or complex of sheaves. It is somewhat more natural to consider all of the above as taking place in the derived category of sheaves of R-modules, that is, all operations are taken on complexes of sheaves rather than on individual sheaves, and quasi-isomorphisms are formally inverted. We will also need to consider the derived tensor product of sheaves of R-modules, which is instead computed by a flat resolution; a sheaf is said to be flat if all its stalks are flat R-modules.

3.2 Let X be a topological space, and $\mathcal{F}$ a complex of sheaves of R-modules on X. We define two tcdgas $\Gamma^{\otimes}(X,\mathcal{F})$ and $\Gamma_c^{\otimes}(X,\mathcal{F})$ associated to this data: we let

$$\Gamma^{\otimes}(X,\mathcal{F})(S) = \Gamma(X,\mathcal{F}^{\otimes S}) \quad \text{and} \quad \Gamma_c^{\otimes}(X,\mathcal{F})(S) = \Gamma_c(X,\mathcal{F}^{\otimes S}).$$

The multiplication maps are the obvious ones: a section of $\mathcal{F}^{\otimes S}$ and a section of $\mathcal{F}^{\otimes T}$ can be multiplied together, to produce a section of $\mathcal{F}^{\otimes S \sqcup T}$.

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Alternatively, we may think of this construction as first taking a complex of sheaves $\mathcal{F}$, and constructing a sheaf of tcdgas on X, given by $S \mapsto \mathcal{F}^{\otimes S}$ (essentially the tensor algebra on $\mathcal{F}$). Moreover, as the functors $\Gamma(X, -)$ and $\Gamma_c(X, -)$ are lax symmetric monoidal,[4] the global sections (with or without compact support) of a sheaf of tcdgas are themselves a tcdga.

3.3 Even though Γ and Γ_c are lax symmetric monoidal functors, their derived versions $R\Gamma$ and $R\Gamma_c$ are not – if they were, $R\Gamma(X, \mathbf{Z})$ would be a commutative cochain model for the space X, and it is well known that one cannot in general construct strictly commutative cochains unless one works over a field of characteristic 0. Nevertheless we can define derived versions of the functors $\Gamma^{\otimes}$ and $\Gamma_c^{\otimes}$, producing a strictly commutative tcdga. This means that the 'commutative cochain problem' can always be solved in the larger category of *twisted* dg algebras.

Lemma 3.4. *Let X be a topological space, R a ring of finite global dimension. Let $\mathcal{F}$ be a bounded-below complex of sheaves of R-modules on X. Then $\mathcal{F}$ is functorially quasi-isomorphic to a bounded-below complex of flat and flabby sheaves.*

Proof. First of all, the fact that R is of global dimension d implies that every R-module M admits a functorial free resolution of length less than or equal to d: indeed, one can form a functorial free resolution in the usual way and then truncate, and the dimension hypothesis assures us that we still have a free resolution. It follows from this that every bounded-below complex of R-modules has a functorial free resolution which is also bounded below. Globally on X this shows that any bounded-below complex of presheaves of R-modules on X has a functorial bounded-below resolution by representable presheaves. Representable presheaves in particular are flat, so $\mathcal{F}$ can be resolved (as a presheaf) in a functorial manner by a bounded-below complex of flat presheaves. But sheafification is an exact functor and preserves flatness, since flatness is a condition on stalks. So we obtain such a resolution also in the category of sheaves. Now we form the Godement resolution of the resulting bounded-below complex, and use the fact that the Godement resolution of a flat sheaf is again flat [Ive86, VI.1, Proof of Proposition 1.3]. □

Definition 3.5. Let X be a topological space, $\mathcal{F}$ a bounded-below complex of sheaves of R-modules on X, $\mathrm{gldim}(R) < \infty$. Let $\mathcal{L}$ be the flat and flabby resolution of $\mathcal{F}$ constructed in Lemma 3.4. Define

$$R\Gamma^{\otimes}(X, \mathcal{F}) = \Gamma^{\otimes}(X, \mathcal{L}) \quad \text{and} \quad R\Gamma_c^{\otimes}(X, \mathcal{F}) = \Gamma_c^{\otimes}(X, \mathcal{L}).$$

This definition is not the 'right' one, however, unless we impose some point-set hypotheses on X.

Proposition 3.6. *Let X, $\mathcal{F}$ and R be as in Definition 3.5. If X is paracompact and Hausdorff, then*

$$H^{\bullet}(R\Gamma^{\otimes}(X, \mathcal{F}))(S) \cong H^{\bullet}(X, \underbrace{\mathcal{F} \otimes^L \cdots \otimes^L \mathcal{F}}_{S \text{ factors}}).$$

Moreover, if X is locally compact, then

$$H^{\bullet}(R\Gamma_c^{\otimes}(X, \mathcal{F}))(S) \cong H_c^{\bullet}(X, \underbrace{\mathcal{F} \otimes^L \cdots \otimes^L \mathcal{F}}_{S \text{ factors}}).$$

[4] Note, though, that Γ_c is only a non-unital monoidal functor.

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Proof. Let $\mathcal{L}$ be the resolution of $\mathcal{F}$ of Lemma 3.4. Then $\mathcal{L}^{\otimes n} \simeq \mathcal{F} \otimes^L \cdots \otimes^L \mathcal{F}$ (n factors) for all n, so to deduce the result we need to know that $\mathcal{L}^{\otimes n}$ is a Γ-acyclic (respectively, Γ_c-acyclic) resolution. This is not automatic: although $\mathcal{L}$ was a complex of flabby sheaves, the tensor product of two flabby sheaves is not necessarily flabby. However, flabby sheaves are soft, any tensor product with a flat and soft sheaf is again soft, and soft sheaves are Γ-acyclic. Similarly, if X is locally compact then flabby sheaves are c-soft, any tensor product with a flat and c-soft sheaf is c-soft, and c-soft sheaves are Γ_c-acyclic. □

Commutative cochains over Q

3.7 As remarked above, the functors $R\Gamma$ and $R\Gamma_c$ are not lax symmetric monoidal: if they were, one could construct strictly commutative cochains on any space. On the other hand, strictly commutative cochains do exist over $\mathbf{Q}$, as constructed by Quillen [Qui69] and Sullivan [Sul77]. Hence one might expect that when working with sheaves of $\mathbf{Q}$-vector spaces, the functors $R\Gamma$ and $R\Gamma_c$ can be replaced by commutative versions. This is indeed the case, using a construction due to Navarro Aznar [NA87, § 4].

Theorem 3.8 (Navarro Aznar). *Let X be a topological space, $C^+_{\mathbf{Q}}(X)$ the category of bounded-below complexes of sheaves of $\mathbf{Q}$-vector spaces on X. There are exact lax symmetric monoidal functors*

$$R\Gamma^{\mathrm{TW}}, R\Gamma_c^{\mathrm{TW}} \colon C^+_{\mathbf{Q}}(X) \to \mathsf{Ch}^+_{\mathbf{Q}}$$

equipped with monoidal quasi-isomorphisms of functors $R\Gamma \implies R\Gamma^{\mathrm{TW}}$, $R\Gamma_c \implies R\Gamma_c^{\mathrm{TW}}$. In particular, if A is a sheaf of commutative dg $\mathbf{Q}$-algebras on X, then $R\Gamma^{\mathrm{TW}}(X, A)$ and $R\Gamma_c^{\mathrm{TW}}(X, A)$ are themselves commutative dg algebras.

Sketch of proof. Recall how to define the Godement resolution: there is a monad T on the category of sheaves on X given by p_*p^*, where $p \colon X^\delta \to X$ is the projection from X equipped with the discrete topology. With this monad, one attaches to any sheaf $\mathcal{F}$ on X a cosimplicial sheaf $\mathsf{T}(\mathcal{F})$ [Wei94, 8.6.15], whose totalization is the Godement resolution.

Thus the usual functor $R\Gamma$ is the composition of three functors: the cosimplicial Godement construction, the totalization functor on cosimplicial objects, and the functor Γ. Of these, the first and the last are already lax symmetric monoidal. This reduces the problem to finding a commutative version of the totalization functor on cosimplicial objects, which works over $\mathbf{Q}$. This is indeed what Navarro Aznar does [NA87, §§ 2–3], by constructing the so-called 'Thom–Whitney totalization', using Sullivan's polynomial de Rham forms. To be more precise, the usual totalization functor assigns to a cosimplicial object $A \colon \Delta \to \mathsf{Ch}_R$ the end of the bifunctor $\Delta^{\mathrm{op}} \times \Delta \to \mathsf{Ch}_R$ given by $(\alpha, \beta) \mapsto C^\bullet(\Delta^\alpha, R) \otimes A^\beta$, where $C^\bullet(\Delta^\alpha, R)$ denotes the standard cochains on the α-simplex. The reason why totalization is monoidal but not symmetric monoidal is that $C^\bullet(\Delta^*, R)$ is a simplical dg algebra, but it is not commutative. Over $\mathbf{Q}$ we may replace $C^\bullet(\Delta^*, \mathbf{Q})$ with the simplical commutative dg algebra $\Omega(\Delta^*)$ given by Sullivan's polynomial de Rham forms [Sul77], which gives a lax symmetric monoidal totalization functor equipped with a monoidal quasi-isomorphism with the usual totalization.

Hence if $\mathcal{F}$ is a sheaf of commutative dg $\mathbf{Q}$-algebras on X, its Godement resolution, defined using the Thom–Whitney totalization described above, is again a sheaf of commutative dg $\mathbf{Q}$-algebras, and a flabby resolution of the original complex of sheaves.

One argues in the same way for the functor $R\Gamma_c$. □

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THEOREM 3.9. *Let X be a paracompact Hausdorff space. The tcdga $R\Gamma^{\otimes}(X, \mathbf{Q})$ is quasi-isomorphic to the cdga $R\Gamma^{\mathrm{TW}}(X, \mathbf{Q})$, considered as a 'constant' tcdga as in § 2.5. Moreover, if X is locally compact then $R\Gamma_c^{\otimes}(X, \mathbf{Q})$ is quasi-isomorphic to the cdga $R\Gamma_c^{\mathrm{TW}}(X, \mathbf{Q})$.*

Proof. Let $\mathrm{Gode}^{\mathrm{TW}}(\mathbf{Q})$ be the Godement resolution of the sheaf $\mathbf{Q}$ on X, defined using Navarro Aznar's Thom–Whitney totalization, as in the proof of Theorem 3.8. Note that the lax symmetric monoidal structure of the functor $\mathrm{Gode}^{\mathrm{TW}}$ defines for any n an $\mathbb{S}_n$-equivariant quasi-isomorphism

$$\mathrm{Gode}^{\mathrm{TW}}(\mathbf{Q})^{\otimes n} \xrightarrow{\sim} \mathrm{Gode}^{\mathrm{TW}}(\mathbf{Q}).$$

Now we may compute the tcdga $R\Gamma^{\otimes}(X, \mathbf{Q})$ using the resolution $\mathrm{Gode}^{\mathrm{TW}}(\mathbf{Q})$, so that for $S \in \mathrm{ob}\, \mathsf{Fin}_+$ we have

$$R\Gamma^{\otimes}(X, \mathbf{Q})(S) \simeq \Gamma(X, \mathrm{Gode}^{\mathrm{TW}}(\mathbf{Q})^{\otimes S}) \xrightarrow{\sim} \Gamma(X, \mathrm{Gode}^{\mathrm{TW}}(\mathbf{Q})) \simeq R\Gamma^{\mathrm{TW}}(X, \mathbf{Q}),$$

where the quasi-isomorphism above is induced by the symmetric monoidal structure. □

LEMMA 3.10. *Let X be a compact Hausdorff space, $A \subset X$ a closed subset. There is a natural quasi-isomorphism $R\Gamma_c^{\mathrm{TW}}(X \setminus A, \mathbf{Q}) \xrightarrow{\sim} \mathrm{cone}(R\Gamma^{\mathrm{TW}}(X, \mathbf{Q}) \to R\Gamma^{\mathrm{TW}}(A, \mathbf{Q}))$.*

Proof. Let $j\colon (X \setminus A) \to X$ and $i\colon A \to X$ be the inclusions. We get a short exact sequence

$$0 \to j_! \mathbf{Q} \to \mathbf{Q} \to i_* \mathbf{Q} \to 0$$

of sheaves on X. Applying the functor $R\Gamma^{\mathrm{TW}}(X, -)$ gives the result, since $R\Gamma_c^{\mathrm{TW}}(X \setminus A, \mathbf{Q}) = R\Gamma^{\mathrm{TW}}(X, j_! \mathbf{Q})$ (here we use X compact) and $R\Gamma^{\mathrm{TW}}(A, \mathbf{Q}) = R\Gamma^{\mathrm{TW}}(X, i_* \mathbf{Q})$. □

3.11 We claim that the functor $R\Gamma^{\mathrm{TW}}(-, \mathbf{Q})$ is a cochain theory on the category of CW complexes in the sense of Mandell [Man02]. Indeed, the functor $R\Gamma$ is homotopy invariant [Ive86, IV.1], hence a weak homotopy invariant of CW complexes by Whitehead's theorem. The wedge axiom is clear. It is well known that in the presence of the wedge axiom/direct limit axiom, it is enough to verify exactness/excision for *finite* CW complexes. But if (X, A) is a pair of finite CW complexes, then exactness and excision are immediate from the preceding lemma.

THEOREM 3.12. *Let X be a CW complex. The cdga $R\Gamma^{\mathrm{TW}}(X, \mathbf{Q})$ is quasi-isomorphic to Sullivan's cdga $A_{PL}(X)$. If X is the complement of a subcomplex in a finite CW complex then $R\Gamma_c^{\mathrm{TW}}(X, \mathbf{Q})$ is quasi-isomorphic to the augmentation ideal in the algebra $A_{PL}(X^*)$, where X^* denotes the one-point compactification.*

Proof. Both functors A_{PL} and $R\Gamma^{\mathrm{TW}}$ are cochain theories. Then the first claim follows from Mandell's uniqueness theorem [Man02, Corollary, p. 550], which says in particular that there is a unique cochain theory taking values in commutative dg $\mathbf{Q}$-algebras, up to quasi-isomorphism. The second claim follows from the first and Lemma 3.10. □

COROLLARY 3.13. *Let X be a CW complex. The tcdga $R\Gamma^{\otimes}(X, \mathbf{Q})$ is quasi-isomorphic to the constant tcdga associated to a cdga model for the cochains $C^{\bullet}(X, \mathbf{Q})$. If X is the complement of a subcomplex in a finite CW complex then $R\Gamma_c^{\otimes}(X, \mathbf{Q})$ is quasi-isomorphic to a cdga model for the compactly supported cochains $C_c^{\bullet}(X, \mathbf{Q})$.*

Proof. Indeed, $A_{PL}(X)$ is a cdga model for $C^{\bullet}(X, \mathbf{Q})$, and the augmentation ideal in $A_{PL}(X^*)$ is a cdga model for $C_c^{\bullet}(X, \mathbf{Q})$. □

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The relationship with $\mathbb{E}_\infty$-algebras

3.14 There is a close relationship between twisted commutative dg algebras and $\mathbb{E}_\infty$-algebras. We briefly state the main results here; these are principally due to Sagave and Schlichtkrull [SS12] (who only treated $\mathbb{E}_\infty$-algebras in the category of spaces; see Richter and Shipley [RS17] for the analogous results for cochain complexes) and Pavlov and Scholbach [PS18]. The results of this section will not be used in the sequel and are included only to put our constructions into a broader context. We will first state the results in the *non-unital setting*, since the statements are cleaner in this case, and the algebras of interest to us are non-unital.

Definition 3.15. Let I_+ be the category of non-empty finite sets and injections. The category I_+ is a (non-unital) symmetric monoidal category under disjoint union. A *dg I_+-algebra* is a lax symmetric monoidal functor $\mathsf{I}_+ \to \mathsf{Ch}_R$.

3.16 Every dg I_+-algebra has an underlying tcdga, given by restriction along the functor $\mathsf{Fin}_+ \to \mathsf{I}_+$. The forgetful functor from dg I_+-algebras to tcdgas has a left adjoint given by the left Kan extension.

3.17 The following theorem can be deduced as a very special case of general results of Pavlov and Scholbach [PS18], although they do not state it in the precise form we need it. Indeed, their goal is to construct a model structure on operadic algebras in abstract symmetric spectra with respect to the *stable* model structure on spectra, whereas Theorem 3.18 concerns the *unstable* model structure, from which the stable model structure is obtained by localization. But all the properties that they use of the unstable model structure are first verified for the stable model structure. Moreover, they consider *unital* objects throughout, which in this context means that they consider the categories $\mathsf{Ch}_R^{\mathsf{Fin}}$ and $\mathsf{Ch}_R^{\mathsf{I}}$, where the empty set is included in the domain category. This forces them to work with a 'positive' model structure, in which a map $A \to B$ is a weak equivalence (fibration) if $A(S) \to B(S)$ is a weak equivalence (fibration) for *non-empty* S. The categories $\mathsf{Ch}_R^{\mathsf{Fin}_+}$ and $\mathsf{Ch}_R^{\mathsf{I}_+}$ are Quillen equivalent as symmetric monoidal non-unital model categories to $\mathsf{Ch}_R^{\mathsf{Fin}}$ and $\mathsf{Ch}_R^{\mathsf{I}}$, respectively, with their positive model structures, and this implies with only a little bit of work also a Quillen equivalence between categories of P-algebras in the respective categories.

Theorem 3.18. *The category of twisted commutative dg algebras over a ring R and the category of dg I_+-algebras both admit model structures in which the weak equivalences are the quasi-isomorphisms and the fibrations are the degreewise surjections. In particular, the forgetful functor from dg I_+-algebras to tcdgas is a right Quillen functor. More generally, if P is any operad with $P(0) = 0$, then the categories of P-algebras in $\mathsf{Ch}_R^{\mathsf{Fin}_+}$ and in $\mathsf{Ch}_R^{\mathsf{I}_+}$ both admit such model structures.*

Remark 3.19. Since both $\mathsf{Ch}_R^{\mathsf{Fin}_+}$ and $\mathsf{Ch}_R^{\mathsf{I}_+}$ are *non-unital* symmetric monoidal categories, the notion of a P-algebra in these categories does not even make sense unless $P(0) = 0$. Indeed, a P-algebra is defined by structure maps $P(n) \otimes A^{\otimes n} \to A$, and in a non-unital monoidal category the tensor power $A^{\otimes 0}$ is undefined.

Sketch of proof of Theorem 3.18. Let $\mathcal{C}$ denote either of the two categories $\mathsf{Ch}_R^{\mathsf{Fin}_+}$ or $\mathsf{Ch}_R^{\mathsf{I}_+}$. The strategy is to lift the projective model structure on $\mathcal{C}$ along the forgetful-free adjunction

$$U \colon \mathrm{Alg}(P) \leftrightarrows \mathcal{C} \colon F_P.$$

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The non-trivial property that needs to be checked in order for the model structure to lift is that if $M \to N$ is a trivial cofibration in $\mathcal{C}$, A is a P-algebra, and $F_P(M) \to A$ is any P-algebra morphism then the pushout

$$A \to A \amalg_{F_P(M)} F_P(N)$$

is a trivial cofibration. We may in fact take $M \to N$ to be a generating trivial cofibration, so $0 = M = F_P(M)$ and we are computing a coproduct. Now for any P-algebra A in $\mathcal{C}$ there is an operad O_A in $\mathcal{C}$ such that O_A-algebras are equivalent to P-algebras with a morphism from A [GJ94, Lemma 1.18]. The arity-0 component of O_A is isomorphic to A, since it is the initial object in the category of O_A-algebras. The free O_A-algebra on an object N of $\mathcal{C}$ is the pushout $A \amalg F_P(N)$. Now

$$F_{O_A}(N) = A \oplus \bigoplus_{n \geqslant 1} O_A(n) \otimes_{\mathbb{S}_n} N^{\otimes n}$$

and we claim that $O_A(n) \otimes_{\mathbb{S}_n} N^{\otimes n}$ is acyclic for all $n \geqslant 1$. Indeed,

$$N^{\otimes n}(k) = \bigoplus_{\{1,\dots,k\} = S_1 \sqcup \cdots \sqcup S_n} N(S_1) \otimes \cdots \otimes N(S_n) = R[\mathbb{S}_n] \otimes \bigoplus_{\substack{\{1,\dots,k\} = S_1 \sqcup \cdots \sqcup S_n \\ \min(S_1) < \cdots < \min(S_n)}} N(S_1) \otimes \cdots \otimes N(S_n)$$

and each $N(S_i)$ is an acyclic complex of free R-modules. Crucially, we used the fact that $N(0) = 0$ in the last step to ensure that none of the subsets S_i were empty. □

3.20 Theorem 3.18 is perhaps surprising at first, since R is a completely arbitrary ring. Hinich [Hin97] and Harper [Har10] proved analogous results for P-algebras in Ch_R and in $\mathsf{Ch}_R^{\mathsf{Fin}}$ (i.e. left P-modules) only when R contains $\mathbf{Q}$, and without this assumption there is in general no such model structure [GS07, Example 3.7]. Theorem 3.18 shows that the arity-0 component is in fact the only obstruction to transferring the model structure: once we impose the condition that $M(0) = 0$, everything works without a hitch over an arbitrary base ring. Thus twisted commutative dg algebras are 'better behaved' than ordinary commutative algebras in positive characteristic, as evidenced in particular by the fact that the commutative cochain problem can always be solved in the category of tcdgas, as we explained in § 3.3. There are in fact several earlier results in the literature of the following flavor: for a dg operad P, left P-modules M with $M(0) = 0$ exhibit 'homotopically correct' behavior in arbitrary characteristic, even in situations where P-algebras do not. We mention three examples.

(i) Stover [Sto93] proved that reduced left Lie modules satisfy the Milnor–Moore theorem over an arbitrary base ring; the classical Milnor–Moore theorem for Lie algebras requires working over a $\mathbf{Q}$-algebra.

(ii) Richter [Ric15] proved that if A is a reduced left Com-module (i.e. a non-unital tcdga), then the Harrison homology, André–Quillen homology and Gamma homology of A all coincide, and give a summand of the Hochschild homology. For ordinary commutative algebras this is in general only true over $\mathbf{Q}$.

(iii) Fresse [Fre00] introduced the notion of a divided power structure on a P-algebra. The definition admits an evident generalization to a divided power structure on a left P-module M. If $M(0) = 0$, then it follows from [Sto93, 9.10] that M is always *canonically* equipped with a divided power structure.

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3.21 We are now ready to state the relationship between I_+-algebras and $\mathbb{E}_\infty$-algebras, which can be summed up in the following two theorems, which are simple modifications of the statements of [RS18, Theorem 2.13] and [RS18, Theorem 4.10].

Theorem 3.22. *Let A be a dg I_+-algebra. Then the Bousfield–Kan homotopy colimit $\operatorname{hocolim}_{\mathsf{I}_+} A$ admits a natural action by the Barratt–Eccles operad, making it into a non-unital $\mathbb{E}_\infty$-algebra.*

Theorem 3.23. *The category of non-unital $\mathbb{E}_\infty$-algebras can be identified, via a zigzag of Quillen equivalences, with a Bousfield localization of the category of dg I_+-algebras. The functor $\operatorname{hocolim}_{\mathsf{I}_+}$ from dg I_+-algebras to non-unital $\mathbb{E}_\infty$-algebras models the composite of the derived functors arising from this zigzag. The fibrant objects in the Bousfield localization are the I_+-algebras for which $A(S) \to A(T)$ is a quasi-isomorphism for every injection $S \hookrightarrow T$.*

3.24 Concretely, Theorem 3.23 says that we can think of the category of non-unital $\mathbb{E}_\infty$-algebras as a full subcategory of dg I_+-algebras. In particular, there is then also a natural forgetful functor from non-unital $\mathbb{E}_\infty$-algebras to twisted commutative dg algebras, by Theorem 3.18. If Ω is a non-unital $\mathbb{E}_\infty$-algebra, then by Theorem 3.23 we may associate to Ω a fibrant dg I_+-algebra $\underline{\Omega}$, well defined up to quasi-isomorphism. Since all the maps $\underline{\Omega}(S) \to \underline{\Omega}(T)$ are quasi-isomorphisms, each component $\underline{\Omega}(S)$ is in fact quasi-isomorphic to the homotopy colimit Ω, and the multiplication maps $\underline{\Omega}(S) \otimes \underline{\Omega}(T) \to \underline{\Omega}(S \sqcup T)$ are given up to homotopy by the multiplication in Ω. Thus we may think of this as a homotopy coherent version of the construction of § 2.5. Composing with the forgetful functor from dg I_+-algebras to twisted commutative algebras, we may in particular think of a non-unital $\mathbb{E}_\infty$-algebra as defining a 'homotopically constant' tcdga.

3.25 Let us explain how Theorems 3.22 and 3.23, which concern non-unital $\mathbb{E}_\infty$-algebras, follow from analogous theorems for unital $\mathbb{E}_\infty$-algebras. We briefly recall these results here; for a more complete account with detailed references see [RS18, § 4]. Let I denote the unital symmetric monoidal category of all finite sets and injections. The category of functors $\mathsf{I} \to \mathsf{Ch}_R$ can be given a *positive* model structure in which $F \to G$ is a fibration (weak equivalence) if $F(S) \to G(S)$ is so for all non-empty sets S; no assumption is made on $F(\emptyset) \to G(\emptyset)$. We say that $F \to G$ is an I-*equivalence* if $\operatorname{hocolim}_{\mathsf{I}} F \to \operatorname{hocolim}_{\mathsf{I}} G$ is a weak equivalence. The positive model structure admits a Bousfield localization with weak equivalences the I-equivalences, which we call the *positive* I-*model structure*. A functor F is fibrant for this model structure if $F(S) \to F(T)$ is a weak equivalence for all $\emptyset \neq S \hookrightarrow T$. By an I-*algebra* we mean a (unital) lax symmetric monoidal functor $\mathsf{I} \to \mathsf{Ch}_R$, that is, a commutative monoid in $\mathsf{Ch}_R^{\mathsf{I}}$. There is an induced model structure on I-algebras, with $A \to B$ a morphism of I-algebras a fibration (weak equivalence) when $A \to B$ is a fibration (weak equivalence) for the positive I-model structure. Then the category of I-algebras is Quillen equivalent to the category of unital $\mathbb{E}_\infty$-algebras, and if A is an I-algebra then $\operatorname{hocolim}_{\mathsf{I}} A$ is an $\mathbb{E}_\infty$-algebra.

Now the category of non-unital $\mathbb{E}_\infty$-algebras in Ch_R is Quillen equivalent to the category of augmented $\mathbb{E}_\infty$-algebras $A \to R$, where the latter category has its natural model structure as a slice category. By the opposite of [Hov99, Theorem 1.3.17] and the results recalled in § 3.25, non-unital $\mathbb{E}_\infty$-algebras are then Quillen equivalent to the category of I-algebras with an augmentation $A \to \underline{R}$, where $\underline{R}$ denotes the constant I-algebra at R. If A is such an augmented I-algebra then its augmentation ideal $\overline{A}$, defined by $\overline{A}(S) = \operatorname{Ker}(A(S) \to R)$, is a well-defined I_+-algebra. The functor from augmented I-algebras to I_+-algebras has a left adjoint given by adjoining a unit freely in each arity. This is in fact a Quillen adjunction between augmented

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I-algebras (with the positive I-model structure) and I_+-algebras (with the model structure of Theorem 3.18). The category of I_+-algebras admits a Bousfield localization in which the weak equivalences are the maps inducing a weak equivalence of homotopy colimits, and the Quillen adjunction between augmented I-algebras and I_+-algebras induces a Quillen equivalence between augmented I-algebras and this Bousfield localization, proving Theorem 3.23. The homotopy colimit of an I_+-algebra is the augmentation ideal in the homotopy colimit of the I-algebra obtained by freely adjoining a unit in each arity, proving Theorem 3.22.

Proposition 3.26. *Let X be a paracompact Hausdorff space. The tcdga $R\Gamma^{\otimes}(X,R)$ has a natural structure of fibrant dg* I*-algebra. If X is in addition locally compact, then $R\Gamma_c^{\otimes}(X,R)$ is a fibrant dg I_+-algebra.*

Proof. Recall that $R\Gamma^{\otimes}(X,R)$ and $R\Gamma_c^{\otimes}(X,R)$ are obtained by applying the functor Γ (respectively, Γ_c) to the sheaf of tcdgas on X given by $S \mapsto \mathrm{Gode}(R)^{\otimes S}$. We claim that the construction $S \mapsto \mathrm{Gode}(R)^{\otimes S}$ can be naturally given the structure of a sheaf of I-algebras. Indeed, note that there is an augmentation $R \xrightarrow{\sim} \mathrm{Gode}(R)$, and that R is the monoidal unit in the category of sheaves of R-modules on X. By inserting the monoidal unit and applying the augmentation, we obtain natural maps $\mathrm{Gode}(R)^{\otimes S} \xrightarrow{\sim} \mathrm{Gode}(R)^{\otimes T}$ for every injection $S \hookrightarrow T$. Since both sheaves are soft (respectively, c-soft) when S and T are non-empty – see the proof of Proposition 3.6 – the induced map on (compactly supported) global sections is a quasi-isomorphism too, so $R\Gamma^{\otimes}(X,R)$ and $R\Gamma_c^{\otimes}(X,R)$ are fibrant with respect to the resulting dg I-algebra (respectively, dg I_+-algebra) structure. □

Theorem 3.27. *Let X be a CW complex. The tcdga $R\Gamma^{\otimes}(X,R)$ is weakly equivalent to the 'homotopically constant' tcdga given by the $\mathbb{E}_\infty$-algebra of cochains $C^\bullet(X,R)$. If X is the complement of a subcomplex in a finite CW complex then $R\Gamma_c^{\otimes}(X,R)$ is weakly equivalent to the compactly supported cochains $C_c^\bullet(X,R)$ with its non-unital $\mathbb{E}_\infty$-algebra structure.*

Proof. As in Theorem 3.12, this follows from Mandell's uniqueness theorem. □

Remark 3.28. A direct proof is also possible. Let us just sketch the argument. We take X to be paracompact, Hausdorff and locally contractible. First one shows, using local contractibility, that there is a weak equivalence of sheaves of $\mathbb{E}_\infty$-algebras on X between the constant sheaf R and the sheafification of the complex of presheaves $C^\bullet(-,R)$ of singular cochains. Then their derived global sections are also weakly equivalent as $\mathbb{E}_\infty$-algebras. But the global sections of the sheafification of $C^\bullet(-,R)$ are the quotient

$$C^\bullet(X,R)/C_0^\bullet(X,R)$$

by the subcomplex of singular cochains which are zero on some open cover of X, and the quotient map $C^\bullet(X,R) \to C^\bullet(X,R)/C_0^\bullet(X,R)$ is a weak equivalence of $\mathbb{E}_\infty$-algebras. Also, the sheafification of $C^\bullet(-,R)$ is flabby and its derived global sections are just its global sections. On the other hand, we claim that $\mathrm{hocolim}_{\mathsf{I}} R\Gamma^{\otimes}(X,R) \cong R\Gamma(X,R)$ as $\mathbb{E}_\infty$-algebras; indeed, this follows from $R\Gamma^{\otimes}(X,R)$ being a fibrant I-algebra.

Remark 3.29. Richter and Sagave [RS18] have recently constructed a dg I-algebra model for the cochains on a space by completely different methods. Our sheaf-theoretically defined functor $R\Gamma^{\otimes}$ offers an alternative to their construction. After this paper appeared on the arXiv, Chataur and Cirici [CC18] constructed a functor $\mathbb{R}_{\mathcal{E}}\Gamma$ associated to any cofibrant $\mathbb{E}_\infty$-operad $\mathcal{E}$, quasi-isomorphic to $R\Gamma$ and taking sheaves of $\mathcal{E}$-algebras to $\mathcal{E}$-algebras.

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4. Combinatorial preliminaries

Order complex

4.1 Throughout this section we let $(P, \preceq)$ denote a finite poset. We define the *order complex* of P, $\|P\|$, to be the simplicial complex whose p-simplices are chains $x_0 \prec x_1 \prec \cdots \prec x_p$ of comparable elements of P. Equivalently, $\|P\|$ is the geometric realization of the nerve of P, when we think of P as a category.

4.2 If P has either a largest or smallest element, its order complex is contractible. In this case it is common to remove the top and/or bottom element before taking the order complex; however, the following related construction will be more convenient for us. We always use $\hat{0}$ and $\hat{1}$ to denote a smallest and a largest element of P, respectively. Define

$$\begin{aligned} \llbracket P \rrbracket &= \|P\|/\|P \setminus \{\hat{1}\}\|, \\ \lfloor\!\lfloor P \rfloor\!\rfloor &= \|P\|/\|P \setminus \{\hat{0}\}\|, \\ [\![P]\!] &= \|P\|/(\|P \setminus \{\hat{0}\}\| \cup \|P \setminus \{\hat{1}\}\|). \end{aligned}$$

For example, $[\![P]\!]$ is the based CW complex obtained from $\|P\|$ by collapsing to a point the subcomplex consisting of all simplices not containing both $\hat{0}$ and $\hat{1}$. We always follow the standard convention of interpreting the quotient space X/A as the pushout $X \leftarrow A \rightarrow \mathrm{pt}$, so that when A is empty we obtain X with a disjoint basepoint.

Lemma 4.3. *If P has a largest element $\hat{1}$ or a smallest element $\hat{0}$ respectively, then*

$$\begin{aligned} \llbracket P \rrbracket &\cong \Sigma\|P \setminus \{\hat{1}\}\|, \\ \lfloor\!\lfloor P \rfloor\!\rfloor &\cong \Sigma\|P \setminus \{\hat{0}\}\|, \end{aligned}$$

where Σ denotes the unreduced suspension. If P has both a top and bottom element, then

$$[\![P]\!] \cong \begin{cases} S^0 & \text{if } P = \{*\}, \\ \Sigma^2\|P \setminus \{\hat{0}, \hat{1}\}\| & \text{if } |P| \geqslant 2. \end{cases}$$

Proof. If P has a largest element, then $\|P\| \cong C\|P \setminus \{\hat{1}\}\|$, where CX denotes the cone on a space X. Then use the fact that $CX/X \cong \Sigma X$. When P has both a bottom and a top element, apply this argument twice, treating the case when $\hat{0} = \hat{1}$ separately. □

Remark 4.4. In this paper we will only be concerned with the (reduced) cohomology of order complexes. The reader may reasonably wonder why we bother introducing the constructions $\llbracket P \rrbracket, \lfloor\!\lfloor P \rfloor\!\rfloor$ and $[\![P]\!]$, when by the previous lemma their cohomologies can all be expressed in terms of the cohomology of the usual order complex of P with some elements removed. However, in this way we reduce the number of degree shifts involved in the constructions and the main results; moreover, if we did not have $[\![*]\!] \cong S^0$ then we would have to introduce some rather unnatural conventions to deal with the case of a one-element poset (as was done in [Pet17, top of p. 2531]). We also have the following appealing formula.

Lemma 4.5. *Let P and Q be posets with largest and smallest elements. Then*

$$[\![P \times Q]\!] \cong [\![P]\!] \wedge [\![Q]\!],$$

$\wedge$ denoting the smash product.

Proof. The order complex of a product is homeomorphic to the product of the order complexes. Now one checks that both sides are obtained by collapsing the same subspace of $\|P \times Q\|$ to a point. □

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4.6 We say that P is a *lattice* if every subset of P has a unique least upper bound (join) and greatest lower bound (meet). In particular, P must have a largest and smallest element, which are the empty join and meet, respectively. In a finite poset, the existence of joins implies the existence of meets, and vice versa: the meet of a subset can be defined as the join of the set of all its lower bounds.

4.7 Suppose that P has a smallest element $\hat{0}$. The minimal elements of $P \setminus \{\hat{0}\}$ are called *atoms*.

LEMMA 4.8. *Let P be a finite lattice. Let $J_P \subseteq P$ be the sublattice consisting of all elements which are joins of atoms, including the empty join. If $\hat{1} \in P$ does not lie in J_P, then $[\![P]\!]$ is contractible. If $\hat{1}$ lies in J_P, then $[\![P]\!]$ is homotopy equivalent to $[\![J_P]\!]$.*

Proof. It is well known that an adjunction between categories induces a homotopy equivalence of nerves: the unit and counit $\mathbf{1} \Rightarrow GF$ and $FG \Rightarrow \mathbf{1}$ give rise to homotopies between the induced maps between nerves, whence the result. In the case at hand we have such an adjunction: we let $J_P \to P$ be the inclusion, and $P \to J_P$ takes an element x to the join of all atoms below x. If $\hat{1} \in J_P$ then this restricts to an adjunction between $J_P \setminus \{\hat{0}, \hat{1}\}$ and $P \setminus \{\hat{0}, \hat{1}\}$, which together with Lemma 4.3 gives the result. If $\hat{1} \notin J_P$ then we get an adjunction between $J_P \setminus \{\hat{0}\}$ and $P \setminus \{\hat{0}, \hat{1}\}$. But $\|J_P \setminus \{\hat{0}\}\|$ is contractible since J_P has a largest element, so then also $\|P \setminus \{\hat{0}, \hat{1}\}\|$ is contractible, so $[\![P]\!]$ is contractible. □

Sheaf cohomology on posets

4.9 We say that a subset $U \subseteq P$ is *upwards closed* if $x \in U$ and $x \preceq y$ implies $y \in U$. We let $\mathcal{U}(P)$ denote the collection of upwards closed subsets of P. The set $\mathcal{U}(P)$ is itself a finite poset, partially ordered by inclusion. The upwards closed subsets form a topology on P, the *Alexandrov topology*.

4.10 Let $\Phi\colon P \to \mathsf{Ch}_R$ be a functor from P to cochain complexes. Let $U \subseteq P$ be upwards closed. We define a double complex

$$\mathfrak{B}(U, \Phi) = \bigoplus_{\emptyset \neq C \subseteq U \text{ a chain}} \Phi(\max C),$$

where the direct sum is taken over all strictly increasing chains of elements in U. The vertical grading and vertical differentials are given by the internal grading and differential of $\Phi(x)$. The horizontal grading is given by $|C|-1$, and the horizontal differential is an alternating sum over all ways of adding an extra element to the chain; note that if $C \subset C'$, then $\max C \preceq \max C'$, which produces a map $\Phi(\max C) \to \Phi(\max C')$. Equivalently, $\mathfrak{B}(U, \Phi)$ is the bar resolution computing the homotopy limit $\operatorname{holim}_U \Phi$.

Remark 4.11. For our purposes we will never need the interpretation of $\mathfrak{B}(U, \Phi)$ as a homotopy limit. Nevertheless it may be helpful psychologically to know that it is indeed a homotopy limit, and the reader may at this point object that the bar resolution is usually only used to compute homotopy limits in simplicial model categories. But the model category of unbounded cochain complexes is not simplicial. Let us for completeness argue that this is a homotopy limit. If $\Phi\colon P \to \mathsf{Ch}_R$ is a functor, then define

$$R\Phi(x) = \bigoplus_{\substack{\emptyset \neq C \subseteq P \text{ a chain} \\ \min(C) \succeq x}} \Phi(\max(C))$$

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with a differential as above given by adding elements to the chain. If $x \preceq y$ then we obtain $R\Phi(x) \to R\Phi(y)$ by projecting onto the summands for which $\min(C) \succeq y$. We get a natural transformation $\Phi \to R\Phi$, mapping $\Phi(x)$ diagonally into the summands given by singleton chains, which is a quasi-isomorphism pointwise (argue as in Proposition 4.18). Since fibrations in Ch_R are surjections, it follows that $R\Phi$ is Reedy fibrant (i.e. all maps $R\Phi(x) \to R\Phi(y)$ are surjections), so $\Phi \to R\Phi$ is a fibrant replacement. The limit of $R\Phi$ is precisely the bar resolution, and P (being a poset) has cofibrant constants so that the homotopy limit is computed by any Reedy fibrant replacement.

4.12 Assume now that P has a smallest element $\hat{0}$. We define similarly a double complex

$$\widetilde{\mathfrak{B}}(U, \Phi) = \bigoplus_{C \subseteq U \text{ a chain}} \Phi(\max C),$$

where we now allow also the empty chain; $\max(\emptyset) = \hat{0}$ is the minimal element of P. Again the vertical differential and grading come from Φ, the horizontal differential is an alternating sum over all ways of adding an element to a chain, but the horizontal grading is given by the cardinality $|C|$. The two cochain complexes $\mathfrak{B}(U, \Phi)$ and $\widetilde{\mathfrak{B}}(U, \Phi)$ differ only by an 'augmentation' by $\Phi(\hat{0})$ and by a degree shift; there is a short exact sequence

$$0 \to \Phi(\hat{0}) \to \widetilde{\mathfrak{B}}(U, \Phi) \to \mathfrak{B}(U, \Phi)[1] \to 0.$$

4.13 If Φ is the constant functor taking each element to the R-module R, then the complexes $\mathfrak{B}(U, \Phi)$ and $\widetilde{\mathfrak{B}}(U, \Phi)$ compute the cohomology (respectively, the reduced cohomology) of the order complex $\|U\|$. Indeed, there is an evident isomorphism with the complexes of (reduced) cellular chains of $\|U\|$.

Proposition 4.14. *Let $\mathcal{U}(P)$ denote the poset of upwards closed subsets in P. Then $\mathfrak{B}(-, \Phi)$ and $\widetilde{\mathfrak{B}}(-, \Phi)$ define contravariant functors $\mathcal{U}(P) \to \mathsf{Ch}_R$. If Φ and Ψ are quasi-isomorphic as functors $P \to \mathsf{Ch}_R$, then $\mathfrak{B}(-, \Phi)$ is quasi-isomorphic to $\mathfrak{B}(-, \Psi)$, and similarly for $\widetilde{\mathfrak{B}}$.*

Proof. This is clear. □

Remark 4.15. When P is understood as a topological space with the Alexandrov topology, Φ can be seen as defining a complex of sheaves on this topological space. Indeed, a sheaf on a topological space is completely determined by its values on a basis of open sets, and any functor defined on basic open sets which satisfies the sheaf axiom for covers by basic opens sets extends uniquely to a sheaf on arbitrary open sets by Kan extension. Now a basis for the Alexandrov topology is given by open sets of the form $P^{\succeq x} = \{y \in P : y \succeq x\}$, and any functor $P \to \mathsf{Ch}_R$ defines a presheaf of cochain complexes on this basis; moreover, this presheaf will automatically satisfy the sheaf axiom, since any open cover of $P^{\succeq x}$ needs to contain the whole open set $P^{\succeq x}$ as one of its elements. The complex $\mathfrak{B}(U, \Phi)$ computes the cohomology of this complex of sheaves over the open set U – indeed, sheaf cohomology on a space X can be defined as the homotopy limit of the values of the sheaf over the category of all open subsets of X. However, this perspective will not really be used in this paper.

Two spectral sequences

4.16 If $x \in P$, then we denote by $P^{\preceq x}$ the the lower interval $\{y \in P : y \preceq x\}$.

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4.17 Let P, U and Φ be as in §§ 4.10 and 4.12. Suppose, moreover, that we are given a strictly increasing function $\rho\colon P \to \mathbf{Z}$. We may define decreasing filtrations of the complexes $\mathfrak{B}(U,\Phi)$ and $\widetilde{\mathfrak{B}}(U,\Phi)$ as

$$F^p\mathfrak{B}(U,\Phi) = \bigoplus_{\substack{\emptyset\neq C\subseteq U \text{ a chain} \\ \rho(\max(C))\geqslant p}} \Phi(\max C)$$

and

$$F^p\widetilde{\mathfrak{B}}(U,\Phi) = \bigoplus_{\substack{C\subseteq U \text{ a chain} \\ \rho(\max(C))\geqslant p}} \Phi(\max C).$$

PROPOSITION 4.18. *The above filtration on $\mathfrak{B}(U,\Phi)$ has the property that*

$$H^\bullet(\mathrm{Gr}^p\mathfrak{B}(U,\Phi)) = \bigoplus_{\substack{x\in U \\ \rho(x)=p}} \widetilde{H}^\bullet(\llbracket U^{\preceq x}\rrbracket, \Phi(x)).$$

For $U \subsetneq P$, let $U_0 = U \cup \{\hat{0}\}$. Then

$$H^\bullet(\mathrm{Gr}^p\widetilde{\mathfrak{B}}(U,\Phi)) = \bigoplus_{\substack{x\in U_0 \\ \rho(x)=p}} \widetilde{H}^\bullet(\llbracket U_0^{\preceq x}\rrbracket, \Phi(x)).$$

Proof. Clearly we have

$$\mathrm{Gr}^p\mathfrak{B}(U,\Phi) = \bigoplus_{\substack{\emptyset\neq C\subseteq U \text{ a chain} \\ \rho(\max C)=p}} \Phi(\max C).$$

If two chains in the complex have different maximal elements, then the two chains are incomparable. (This is where we use the fact that ρ was assumed strictly increasing.) Thus this complex becomes a direct sum of complexes indexed by the possible maximal elements, that is, the elements $x \in U$ with $\rho(x) = p$. The summand corresponding to x is given by

$$\bigoplus_{\substack{C\subseteq U \text{ a chain} \\ \max C = x}} \Phi(x).$$

But the chains in U whose largest element equal x can be identified with the reduced cellular cochains of $\llbracket U^{\preceq x}\rrbracket$, so this summand is equal to $\widetilde{C}^\bullet(\llbracket U^{\preceq x}\rrbracket) \otimes \Phi(x)$. The cohomology of this complex is $\widetilde{H}^\bullet(\llbracket U^{\preceq x}\rrbracket, \Phi(x))$.

The argument for $\widetilde{\mathfrak{B}}$ is similar. □

PROPOSITION 4.19. *Under the above assumptions, there are spectral sequences*

$$E_1^{pq} = \bigoplus_{\substack{x\in U \\ \rho(x)=p}} \widetilde{H}^{p+q}(\llbracket U^{\preceq x}\rrbracket; \Phi(x)) \implies H^{p+q}(\mathfrak{B}(U,\Phi))$$

and

$$E_1^{pq} = \bigoplus_{\substack{x\in U_0 \\ \rho(x)=p}} \widetilde{H}^{p+q}(\llbracket U_0^{\preceq x}\rrbracket; \Phi(x)) \implies H^{p+q}(\widetilde{\mathfrak{B}}(U,\Phi)).$$

Proof. This is the spectral sequence of a filtered complex, applied to the filtrations on $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$. □

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4.20 These spectral sequences, in the context of sheaf cohomology on posets, were first considered by Bacławski [Bac75, § 4].

Proposition 4.21. *Let P, U and Φ be as in §§ 4.10 and 4.12, and assume that P is a lattice. Let $\rho\colon P \to \mathbf{Z}$ be strictly increasing. Let $U \subsetneq P$ be upwards closed and let J_U be subposet consisting of all joins of minimal elements of U. Let $U_0 = U \cup \{\hat{0}\}$ and $J_{U_0} = J_U \cup \{\hat{0}\}$. The spectral sequences of Proposition 4.19 may be simplified to*

$$E_1^{pq} = \bigoplus_{\substack{x \in J_U \\ \rho(x)=p}} \widetilde{H}^{p+q}(\llbracket J_U^{\preceq x} \rrbracket; \Phi(x)) \implies H^{p+q}(\mathfrak{B}(U, \Phi))$$

and

$$E_1^{pq} = \bigoplus_{\substack{x \in J_{U_0} \\ \rho(x)=p}} \widetilde{H}^{p+q}(\llbracket J_{U_0}^{\preceq x} \rrbracket; \Phi(x)) \implies H^{p+q}(\widetilde{\mathfrak{B}}(U, \Phi)).$$

Proof. Let $x \in U_0$. Lemma 4.8 applied to the lattice $U_0^{\preceq x}$ shows that $\llbracket U_0^{\preceq x} \rrbracket$ is contractible if $x \notin J_{U_0}$, so in the second spectral sequence of Proposition 4.19 we may remove all terms except those with $x \in J_{U_0}$. When $x \in J_{U_0}$, Lemma 4.8 also implies that $\llbracket U_0^{\preceq x} \rrbracket \simeq \llbracket J_{U_0}^{\preceq x} \rrbracket$. The result follows. The argument for the first spectral sequence is analogous. □

Semidirect product

4.22 Let G be a group acting on P. Define the *semidirect product* $G \ltimes P$ as the following category: the objects of $G \ltimes P$ are the elements of P, and a morphism $x \to y$ is an element $g \in G$ such that $g \cdot x \preceq y$. Composition of morphisms is given by group multiplication. This is an instance of the Grothendieck construction, when we think of P as a category and the action of G on P as defining a functor $G \to \mathsf{Cat}$. Note that there is a functor $G \ltimes P \to G$, and that P sits inside $G \ltimes P$ as the subcategory consisting of all morphisms given by the identity element in G. Hence we get a 'short exact sequence' of sorts,

$$P \to G \ltimes P \to G.$$

4.23 Suppose that Φ is a functor $G \ltimes P \to \mathsf{Ch}_R$. We may in particular restrict Φ to P, which allows us to make sense of $\mathfrak{B}(U, \Phi)$ and $\widetilde{\mathfrak{B}}(U, \Phi)$ for $U \subseteq P$ upwards closed. However, the action of G furnishes an additional functoriality: if $g \cdot U = V$, for $U, V \subseteq P$ upwards closed, then multiplication by g induces a map $\mathfrak{B}(U, \Phi) \to \mathfrak{B}(V, \Phi)$. Indeed, if $g \cdot x = y$ then g gives a morphism $x \to y$ in $G \ltimes P$, so $\Phi(g)$ is a morphism $\Phi(x) \to \Phi(y)$; we use this to map each summand of $\mathfrak{B}(U, \Phi)$ to a corresponding summand of $\mathfrak{B}(V, \Phi)$. We record this as a proposition.

Proposition 4.24. *If we are given $\Phi\colon G \ltimes P \to \mathsf{Ch}_R$, then $\mathfrak{B}(-, \Phi)$ and $\widetilde{\mathfrak{B}}(-, \Phi)$ define contravariant functors $G \ltimes \mathcal{U}(P) \to \mathsf{Ch}_R$. In particular, if $G_U \subseteq G$ is the subgroup of elements g such that $g \cdot U = U$, then the cochain complexes $\mathfrak{B}(U, \Phi)$ and $\widetilde{\mathfrak{B}}(U, \Phi)$ are naturally representations of G_U. The spectral sequences of Proposition 4.21 are similarly equivariant.*

5. Cohomology of configuration spaces

Sheaves on the partition lattice

5.1 Let S be a finite set. Let Π_S denote the poset consisting of all equivalence relations on S, ordered by refinement: we say that $\sim \preceq \sim'$ if $x \sim y$ implies $x \sim' y$. The poset Π_S is a lattice: the

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join of $\sim$ and $\sim'$ is the relation $\sim''$ defined by $x \sim'' y \iff x \sim y$ or $x \sim' y$; the meet of $\sim$ and $\sim'$ is given by $x \sim'' y \iff x \sim y$ and $x \sim' y$. We call Π_S the *partition lattice*. When $S = \{1, \dots, n\}$ we denote it simply by Π_n.

5.2 We usually prefer to think of the elements of Π_S as partitions of S into non-empty disjoint subsets. The smallest element is the partition of S into one-element blocks, and the largest element is the partition of S into a single block. In an abuse of notation we denote a typical partition $S = \coprod_{i \in I} T_i$ by the symbol T.

5.3 Let A be a tcdga (§ 2.2). Then we obtain a functor $\Phi_A \colon \Pi_S \to \mathsf{Ch}_R$, for any finite set S. Specifically, if T denotes the partition $S = \coprod_{i \in I} T_i$, then

$$\Phi_A(T) = \bigotimes_{i \in I} A(T_i).$$

If $T \preceq T'$ then T' is obtained from T by merging some blocks of the partition. The map $\Phi_A(T) \to \Phi_A(T')$ is given by applying the maps $A(T_i) \otimes A(T_j) \to A(T_i \sqcup T_j)$ furnished by the tcdga structure.

5.4 In fact to define a functor $\Pi_S \to \mathsf{Ch}_R$ we do not need the full data of a tcdga. Let us instead suppose that S is totally ordered. Then every subset of S inherits a natural total order as well, so that for each partition $\coprod_{i \in I} T_i$ of S we get an induced total ordering on each of the blocks T_i. If A is a commutative dg shuffle algebra (§ 2.11) then we may similarly define

$$\Phi_A(T) = \bigotimes_{i \in I} A(T_i),$$

where the maps $\Phi_A(T) \to \Phi_A(T')$ are now given by merging blocks using the shuffle algebra structure on A.

5.5 Note that if A and A' are quasi-isomorphic as tcdgas (or as commutative dg shuffle algebras), then Φ_A and $\Phi_{A'}$ are quasi-isomorphic as functors.

5.6 The difference between the constructions of §§ 5.3 and 5.4 is that when A is a tcdga we will not only get a functor $\Pi_n \to \mathsf{Ch}_R$ for all n, but in fact a functor

$$\mathbb{S}_n \ltimes \Pi_n \to \mathsf{Ch}_R$$

from the semidirect product (§ 4.22) of Π_n with $\mathbb{S}_n$, where $\mathbb{S}_n$ acts in the obvious way on the partition lattice. This will have the consequence that the various spectral sequences we write down for the cohomology of configuration spaces are $\mathbb{S}_n$-equivariant.

The functors $\mathcal{CF}$ and $\mathcal{CD}$, and the connection with configuration spaces

DEFINITION 5.7. Let $U \subset \Pi_n$ be upwards closed, and let A be a tcdga. We define

$$\mathcal{CD}(U, A) = \mathfrak{B}(U, \Phi_A),$$
$$\mathcal{CF}(U, A) = \widetilde{\mathfrak{B}}(U, \Phi_A),$$

where $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ are defined in §§ 4.10 and 4.12, and Φ_A is defined in § 5.3.

Remark 5.8. We observe that $\mathcal{CD}$ and $\mathcal{CF}$ are functors

$$(\mathbb{S}_n \ltimes \mathcal{U}(\Pi_n)) \times \mathsf{TCDGA}_R \to \mathsf{Ch}_R.$$

Note also that quasi-isomorphic tcdgas give rise to quasi-isomorphic functors. Finally, we remark that if A is only a commutative shuffle dg algebra, then we can define $\mathcal{CD}$ and $\mathcal{CF}$ similarly; the resulting functors are just not defined on the semidirect product with $\mathbb{S}_n$.

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5.9 Let X be a locally compact Hausdorff space space and let $\mathcal{F}$ be a complex of sheaves on X^n. For $T \in \Pi_n$, let $j(T)\colon X(T) \to X^n$ be the corresponding locally closed subset of X^n, and let $i(T)\colon \overline{X(T)} \to X^n$ be the inclusion of its closure. Note that each closed subset $\overline{X(T)}$ is a cartesian product of X with itself: if T has blocks $T_1, \dots, T_k$, then $\overline{X(T)} \cong X^k$. Let $U \subseteq \Pi_n$ be upwards closed, $j\colon F(X,U) \to X^n$ the open inclusion, and $i\colon D(X,U) \to X^n$ its closed complement. The following result is a special case of [Pet17, Proposition 3.1]; see also the results of [Get99b] when $F(X,U) = F(X,n)$.

Proposition 5.10. *There are quasi-isomorphisms*

$$j_! j^* \mathcal{F} \simeq \bigoplus_{C \subseteq U \text{ a chain}} i(\max C)_* i(\max C)^* \mathcal{F}$$

and

$$i_* i^* \mathcal{F} \simeq \bigoplus_{\emptyset \neq C \subseteq U \text{ a chain}} i(\max C)_* i(\max C)^* \mathcal{F},$$

where the right-hand sides are considered as double complexes of sheaves: the bigradings and the differentials on the right-hand side are given by exactly the same formulas as for the complexes $\mathcal{CF}$ and $\mathcal{CD}$; see §§ 4.10 and 4.12.

Proof. Consider X^n as a stratified space with a single open stratum given by $F(X,U)$; the remaining strata are the locally closed subsets $X(T) \subset X^n$ where $T \in U$. Then the poset of strata is given by $U \cup \{\hat{0}\}$. In [Pet17, § 3], a general construction is explained for resolving sheaves of the form $j_! j^* \mathcal{F}$ in this situation, and the double complex $\bigoplus_{C \subseteq U \text{ a chain}} i(\max C)_* i(\max C)^* \mathcal{F}$ is exactly equal to the complex $L^\bullet(\mathcal{F})$ defined in [Pet17, § 3].

Now there is also a distinguished triangle

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F},$$

and the double complex $\bigoplus_{C \subseteq U \text{ a chain}} (i^{\max C})_* (i^{\max C})^* \mathcal{F}$ differs only by a shift in grading and a coaugmentation by the additional summand $\mathcal{F}$, corresponding to the empty chain. The result follows. □

5.11 Since the proof is short, let us outline the proof of the quasi-isomorphism $j_! j^* \mathcal{F} \to L^\bullet(\mathcal{F})$ from [Pet17]. Consider the summand of $\bigoplus_{C \subseteq U \text{ a chain}} i(\max C)_* i(\max C)^* \mathcal{F}$ corresponding to the empty chain; this gives a copy of $\mathcal{F}$. The natural map $j_! j^* \mathcal{F} \to \mathcal{F}$ provides a map of complexes

$$j_! j^* \mathcal{F} \to \bigoplus_{C \subseteq U \text{ a chain}} i(\max C)_* i(\max C)^* \mathcal{F}.$$

We may consider this as an augmented complex and it is enough to prove that it is acyclic. This can be checked on stalks. If $x \in F(X,U)$ then the stalk of both sides at x is just $\mathcal{F}_x$; note that all summands except the empty chain on the right-hand side have support outside $F(X,U)$. If x lies in some non-open stratum $X(T)$, then the stalk of the left-hand side is zero and the stalk of the right-hand side is given by the tensor product

$$\mathcal{F}_x \otimes \widetilde{C}^\bullet(\llbracket U_0^{\preceq T} \rrbracket),$$

where $U_0 = U \cup \{\hat{0}\}$. But this order complex is contractible since $U_0^{\preceq T}$ has a largest element.

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THEOREM 5.12. *Let X be a paracompact Hausdorff space, $\mathcal{F}$ a complex of sheaves of R-modules on X, R a ring of finite global dimension. Let $U \subset \Pi_n$ be upwards closed. There are quasi-isomorphisms of cochain complexes*

$$\mathcal{CF}(U, R\Gamma^{\otimes}(X, \mathcal{F})) \simeq C^{\bullet}(X^n, D(X,U); \mathcal{F}^{\boxtimes n}),$$
$$\mathcal{CD}(U, R\Gamma^{\otimes}(X, \mathcal{F})) \simeq C^{\bullet}(D(X,U), \mathcal{F}^{\boxtimes n}),$$

which are natural in X, $\mathcal{F}$ and U. If X is in additional locally compact, then

$$\mathcal{CF}(U, R\Gamma_c^{\otimes}(X, \mathcal{F})) \simeq C_c^{\bullet}(F(X,U), \mathcal{F}^{\boxtimes n}),$$
$$\mathcal{CD}(U, R\Gamma_c^{\otimes}(X, \mathcal{F})) \simeq C_c^{\bullet}(D(X,U), \mathcal{F}^{\boxtimes n}).$$

Proof. Let $\mathcal{L}$ be the flat and flabby resolution of $\mathcal{F}$ constructed in Lemma 3.4. Apply Proposition 5.10 to get quasi-isomorphisms

$$j_! j^* \mathcal{F}^{\boxtimes n} \simeq \bigoplus_{C \subseteq U \text{ a chain}} i(\max C)_* i(\max C)^* \mathcal{L}^{\boxtimes n}$$

and

$$i_* i^* \mathcal{F}^{\boxtimes n} \simeq \bigoplus_{\emptyset \neq C \subseteq U \text{ a chain}} i(\max C)_* i(\max C)^* \mathcal{L}^{\boxtimes n}.$$

Note now that the right-hand sides are complexes of soft sheaves (respectively, c-soft if X is locally compact), so these resolutions can be used to compute the cohomology of $j_! j^* \mathcal{F}$ and $i_* i^*$.

Now the key observation is that the complex of global sections of the complex of sheaves

$$\bigoplus_{C \subseteq U \text{ a chain}} i(\max C)_* i(\max C)^* \mathcal{L}^{\boxtimes n}$$

is *isomorphic* to $\mathcal{CF}(U, R\Gamma^{\otimes}(X, \mathcal{F}))$. Indeed, the summands in the double complex given by $\mathcal{CF}(U, R\Gamma^{\otimes}(X, \mathcal{F}))$ correspond in exactly the same way to chains in the upwards closed subset U, and each summand is given precisely by global sections of tensor powers of the resolution $\mathcal{L}$. Moreover, general sheaf theory tells us that there is a quasi-isomorphism $R\Gamma(X^n, j_! j^* \mathcal{F}^{\boxtimes n}) \simeq C^{\bullet}(X^n, D(X,U); \mathcal{F}^{\boxtimes n})$; this proves the first quasi-isomorphism of the theorem. Similarly the compactly supported global sections of the same complex of sheaves are isomorphic to $\mathcal{CF}(U, R\Gamma_c^{\otimes}(X, \mathcal{F}))$, and general sheaf theory also tells us that there is a quasi-isomorphism $R\Gamma_c(X^n, j_! j^* \mathcal{F}^{\boxtimes n}) \simeq C_c^{\bullet}(F(X,U), \mathcal{F}^{\boxtimes n})$, etc. This finishes the proof. □

5.13 Let us record some special cases of Theorem 5.12.

COROLLARY 5.14. *Let X be a paracompact and locally compact Hausdorff space. Let A be a cdga model for the compactly supported cochains $C_c^{\bullet}(X, \mathbf{Q})$. If A is considered as a constant tcdga, then $\mathcal{CF}(U, A) \simeq H_c^{\bullet}(F(X,U), \mathbf{Q})$. The functor*

$$\mathbb{S}_n \ltimes \mathcal{U}(\Pi_n) \to \mathsf{Ch}_R$$

which assigns to $U \subset \Pi_n$ upwards closed the compactly supported cohomology $H_c^{\bullet}(F(X,U), \mathbf{Q})$ depends up to quasi-isomorphism only on a cdga model of algebra $C_c^{\bullet}(X, \mathbf{Q})$.

COROLLARY 5.15. *Let X be a paracompact and locally compact Hausdorff space. Suppose that the compactly supported cochain algebra $C_c^{\bullet}(X, Q)$ is formal. Then the functor $U \mapsto H_c^{\bullet}(F(X,U), \mathbf{Q})$ depends only on the cohomology ring $H_c^{\bullet}(X, \mathbf{Q})$.*

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Corollary 5.16. *Let X be an oriented manifold. The functor $U \mapsto H_\bullet(F(X,U),\mathbf{Z})$ depends up to quasi-isomorphism only on the non-unital $\mathbb{E}_\infty$-algebra structure on $C_c^\bullet(X,\mathbf{Z})$; equivalently, it depends only on the intersection product on the singular chains of X.*

Remark 5.17. The fact that our methods are purely sheaf-theoretic means in particular that the results can be applied equally well in algebro-geometric settings, with only minor modifications. For example, if X is an algebraic variety and $\mathcal{F}$ is a complex of ℓ-adic étale sheaves on X, then we can define tcdgas $R\Gamma^\otimes(X,\mathcal{F})$ and $R\Gamma_c^\otimes(X,\mathcal{F})$ by the same procedure as in § 3. Namely, flat resolutions exist in any ringed topos, and the Godement resolution exists in any topos with enough points; the appropriate topos in this case is the pro-étale site [BS15]. With $\mathbf{Q}_\ell$-coefficients one can also define the functors $R\Gamma^{\mathrm{TW}}$ and $R\Gamma_c^{\mathrm{TW}}$ as in Theorem 3.8. The proofs of Proposition 5.10 and Theorem 5.12 go through with no changes. The spectral sequences obtained from our constructions (as we will explain shortly) will in the algebraic case be spectral sequences of ℓ-adic Galois representations.

However, let us point out that in defining the 'resolution' $L^\bullet(\mathcal{F})$ used in the proof of Proposition 5.10 we do crucially use the fact that the functors i^* and i_* are t-exact. So we do need slightly more than just a naked six-functor formalism; we also need a t-structure with expected properties.

Spectral sequences

Theorem 5.18. *Let X, $\mathcal{F}$ and U be as in Theorem 5.12. Each of the four cases of Theorem 5.12 produces a spectral sequence, which can be written as*

$$E_1^{pq} = \bigoplus_{\substack{T\in J_{U_0}\\ |T|=n-p}} \bigoplus_{i+j=p+q} \widetilde{H}^i([\![J_{\overline{U}_0}^{\preceq T}]\!]; H^j(\overline{X(T)}, i(T)^*\mathcal{F}^{\boxtimes n})) \implies H^{p+q}(X^n, D(X,U); \mathcal{F}^{\boxtimes n}),$$

$$E_1^{pq} = \bigoplus_{\substack{T\in J_{U}\\ |T|=n-p}} \bigoplus_{i+j=p+q} \widetilde{H}^i([\![J_{\overline{U}}^{\preceq T}]\!]; H^j(\overline{X(T)}, i(T)^*\mathcal{F}^{\boxtimes n})) \implies H^{p+q}(D(X,U), \mathcal{F}^{\boxtimes n}),$$

$$E_1^{pq} = \bigoplus_{\substack{T\in J_{U_0}\\ |T|=n-p}} \bigoplus_{i+j=p+q} \widetilde{H}^i([\![J_{\overline{U}_0}^{\preceq T}]\!]; H_c^j(\overline{X(T)}, i(T)^*\mathcal{F}^{\boxtimes n})) \implies H_c^{p+q}(F(X,U), \mathcal{F}^{\boxtimes n}),$$

$$E_1^{pq} = \bigoplus_{\substack{T\in J_{U}\\ |T|=n-p}} \bigoplus_{i+j=p+q} \widetilde{H}^i([\![J_{\overline{U}}^{\preceq T}]\!]; H_c^j(\overline{X(T)}, i(T)^*\mathcal{F}^{\boxtimes n})) \implies H_c^{p+q}(D(X,U), \mathcal{F}^{\boxtimes n}),$$

respectively. Here $|T|$ denotes the number of blocks in the partition T.

Proof. This is Proposition 4.21 specialized to the current situation. □

Remark 5.19. Suppose that the partition T has blocks $T_1,\dots,T_k$ of size $n_1,\dots,n_k$, respectively. Then $\overline{X(T)} \cong X^k$, and the sheaf $i(T)^*\mathcal{F}^{\boxtimes n}$ is given by

$$\mathcal{F}^{\otimes n_1} \boxtimes \cdots \boxtimes \mathcal{F}^{\otimes n_k}.$$

Hence in all cases the E_1-differential of the spectral sequence depends only on: (a) the reduced cohomology of lower intervals in the poset J_U; (b) the (compactly supported) cohomology of X with coefficients in the tensor powers of the sheaf $\mathcal{F}$. The E_1-differential is given by multiplication in the twisted commutative algebra given by the direct sum

$$\bigoplus_{n\geqslant 1} H^\bullet(X, \mathcal{F}^{\otimes n}),$$

that is, the cohomology of the tcdga $R\Gamma^\otimes(X,\mathcal{F})$.

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Comparison with the Cohen–Taylor–Totaro spectral sequence

5.20 Suppose that M is an oriented d-dimensional manifold. By Poincaré duality there is a natural isomorphism

$$H_{d-k}(M, \mathbf{Z}) \cong H^k_c(M, \mathbf{Z})$$

relating homology to cohomology with compact supports. Thus if X is an oriented manifold, the third spectral sequence of Theorem 5.18 yields a spectral sequence computing the homology of the configuration space $F(X, n)$. We claim that the resulting spectral sequence is the *dual* of the Cohen–Taylor–Totaro spectral sequence [CT78, Tot96] computing the *cohomology* of $F(X, n)$. More precisely, Totaro constructed this spectral sequence simply as the Leray spectral sequence in sheaf cohomology associated to the inclusion $F(X, n) \hookrightarrow X^n$, and our claim is that the third spectral sequence of Theorem 5.18 coincides with the *homological* Leray spectral sequence of $F(X, n) \hookrightarrow X^n$, up to reindexing and a degree shift.

5.21 Before proving this claim, let us recall how the Leray spectral sequence is constructed. Let Y be a space, $f\colon Y \to \mathrm{pt}$ the map to a point. Then $H^\bullet(Y, \mathbf{Z})$ is the cohomology of the complex $Rf_*\mathbf{Z}$ in the derived category $D^+(\mathrm{pt})$, where $\mathbf{Z}$ denotes the constant sheaf on Y (which might be more naturally denoted by $f^*\mathbf{Z}$). Any map from Y to another space Z gives us a factorization

$$Y \xrightarrow{g} Z \xrightarrow{h} \mathrm{pt},$$

so that we can write $Rf_*\mathbf{Z} \simeq Rh_*Rg_*\mathbf{Z}$. Now the object $Rg_*\mathbf{Z}$ of $D^+(Z)$ has a canonical filtration induced by the t-structure on $D^+(Z)$, which is uniquely determined in the filtered derived category by the fact that its qth graded summand is isomorphic to $R^q g_*\mathbf{Z}[-q]$. After applying Rh_* we also obtain a filtration of $Rf_*\mathbf{Z}$, and the spectral sequence of this filtered complex is the Leray spectral sequence $E_2^{pq} = H^p(Z, R^q g_*\mathbf{Z}) \implies H^{p+q}(Y, \mathbf{Z})$. If the spaces involved are of finite type then the homology of Y is given by the Verdier dual $\mathbb{D}Rf_*\mathbf{Z} \simeq Rf_!\mathbb{D}\mathbf{Z}$, and the filtration of $\mathbb{D}Rf_*\mathbf{Z} \simeq \mathbb{D}Rh_*Rg_*\mathbf{Z}$ induced by the filtration of $Rg_*\mathbf{Z}$ is called the homological Leray spectral sequence.

5.22 We will also need to recall the relationship between the cohomology of the partition lattice and the cohomology of configuration spaces of points in Euclidean space. Consider the configuration space $F(\mathbf{R}^d, n)$. Its cohomology is non-zero only in degrees divisible by $d-1$, and there is an isomorphism

$$H^{q(d-1)}(F(\mathbf{R}^d, n), \mathbf{Z}) \cong \bigoplus_{\substack{T \in \Pi_n \\ |T| = n-q}} H^q([\![\Pi_n^{\preceq T}]\!], \mathbf{Z}),$$

which for example follows from the Goresky–MacPherson formula [GM88]; see also [Tot96, Lemma 3].

5.23 Let us now assume that X is an oriented d-dimensional manifold. Verdier duality on X furnishes an isomorphism $\mathbb{D}\mathbf{Z} \cong \mathbf{Z}[d]$ in the derived category, where $\mathbf{Z}$ denotes the constant sheaf on X. Note that this implies in particular the Poincaré duality isomorphism between the homology and the compact support cohomology of X, since

$$\mathbb{D}Rf_*\mathbf{Z} \simeq Rf_!\mathbb{D}\mathbf{Z} \simeq Rf_!\mathbf{Z}[d],$$

where f now denotes the map from X to a point: indeed, the cohomology of the left-hand complex is $H_{-\bullet}(X, \mathbf{Z})$, and the cohomology of the right-hand complex is $H_c^{\bullet+d}(X, \mathbf{Z})$.

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Let $j\colon F(X,n) \hookrightarrow X^n$ be the inclusion, and $a\colon X^n \to \mathrm{pt}$ the projection. The homological Leray spectral sequence associated to j is given by filtering $\mathbb{D}Ra_*Rj_*\mathbf{Z} \simeq Ra_!\mathbb{D}Rj_*\mathbf{Z}$ via the canonical filtration of $Rj_*\mathbf{Z}$. The spectral sequence of Theorem 5.18, in the case $\mathcal{F} = \mathbf{Z}$, is given by filtering $Ra_!Rj_!\mathbf{Z}$ by equipping the 'resolution' of $Rj_!\mathbf{Z}$ from Proposition 5.10 with the filtration described in § 4.17. Verdier duality gives $\mathbb{D}Rj_*\mathbf{Z} \simeq Rj_!\mathbb{D}\mathbf{Z} \simeq Rj_!\mathbf{Z}[nd]$, so our claim will follow once we have verified that the two filtrations of $Rj_!\mathbf{Z}$ coincide up to reindexing; equivalently, that the filtration on $Rj_*\mathbf{Z}$ obtained by dualizing the filtration on $Rj_!\mathbf{Z}$ given by the resolution of Proposition 5.10 coincides with the canonical filtration coming from the t-structure of $D^+(X^n)$. We see from Proposition 4.18 that

$$\mathrm{Gr}^q\, Rj_!\mathbf{Z} \cong \bigoplus_{\substack{T\in\Pi_n\\ |T|=n-q}} \widetilde{H}^q([\![\Pi_n^{\preceq T}]\!],\mathbf{Z}) \otimes i(T)_*\mathbf{Z}[-q],$$

where we have used the fact that the partition lattice is Cohen–Macaulay, which implies that $[\![\Pi_n^{\preceq T}]\!]$ is a wedge of spheres of dimension q. Hence

$$\mathbb{D}\,\mathrm{Gr}^q\, Rj_!\mathbf{Z}[nd] \cong \mathrm{Gr}^q\, \mathbb{D}Rj_!\mathbf{Z}[nd] \cong \bigoplus_{\substack{T\in\Pi_n\\ |T|=n-q}} \widetilde{H}^q([\![\Pi_n^{\preceq T}]\!],\mathbf{Z}) \otimes i(T)_*\mathbf{Z}[-q(d-1)],$$

where in the last step we used $\mathbb{D}i(T)_*\mathbf{Z}[nd] \cong i(T)_*\mathbf{Z}[-qd]$, since $\overline{X(T)}$ is an oriented manifold of dimension $(n-q)d$. But it also follows from [Tot96, Proof of Theorem 1] that there is an isomorphism

$$R^{q(d-1)}j_*\mathbf{Z} \cong \bigoplus_{\substack{T\in\Pi_n\\ |T|=n-q}} \widetilde{H}^q([\![\Pi_n^{\preceq T}]\!],\mathbf{Z}) \otimes i(T)_*\mathbf{Z}.$$

It follows that the two filtrations on $Rj_*\mathbf{Z} \simeq \mathbb{D}Rj_!\mathbf{Z}[nd]$ are isomorphic in the filtered derived category up to a reindexing given by $q \leftrightarrow q(d-1)$, since the canonical filtration of a complex with respect to a t-structure is uniquely determined by its associated graded object. Hence our spectral sequence is the homological version of Totaro's spectral sequence. In particular, this argument does not just identify the first non-trivial pages of the two spectral sequences; it implies an isomorphism of all pages of the two spectral sequences.

6. Configuration spaces of points on i-acyclic spaces

Recovering results of Arabia

6.1 Let us begin by recalling the notion of i-acyclicity, which was recently introduced by Arabia [Ara16]. When a space X is i-acyclic, the compact support cohomology of the configuration space of points on X depends on the cohomology of X itself in the simplest way possible. Although Arabia did not phrase the result in this way, we will see that i-acyclic spaces have the property that the two compactly supported spectral sequences of Theorem 5.18 degenerate immediately.

Definition 6.2. A topological space X is said to be *i-acyclic over R* if $H_c^\bullet(X,R) \to H^\bullet(X,R)$ is the zero map.

Example 6.3. Let us give some examples of i-acyclic spaces [Ara16, Proposition 1.2.4].

– Euclidean space is i-acyclic over any ring. This can be seen as the special case of either of the next two examples.

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- If X is an oriented manifold, then X is i-acyclic over R if and only if the intersection product on $H_\bullet(X,R)$ vanishes.
- If X is non-compact and has the cohomology of a point over R, then X is i-acyclic.
- Open subsets of i-acyclic spaces are i-acyclic.
- If X is i-acyclic, and either $H_c^\bullet(X,R)$ or $H_c^\bullet(Y,R)$ is a projective R-module, then $X\times Y$ is i-acyclic. For example, a non-compact Lie group is topologically the product of a maximal compact subgroup with Euclidean space, and is therefore i-acyclic over any ring.
- If X is i-acyclic over R and G is a finite group of automorphisms of X whose order is invertible in R, then X/G is i-acyclic over R.

THEOREM 6.4 (Arabia). *Let X be an i-acyclic metrizable σ-compact and locally compact space over a field k. The groups $H_c^k(F(X,n),k)$ depend only on the graded vector space $H_c^\bullet(X,k)$. If $k=\mathbf{Q}$, the same is true for the decomposition of $H_c^k(F(X,n),k)$ into irreducible representations.*

Remark 6.5. In fact Arabia did more: he studied certain generalized configuration spaces denoted by $\Delta_{\leqslant\ell}X^n$ and $\Delta_\ell X^n$, where $\Delta_n X^n = F(X,n)$, and gave explicit formulas for their compact support cohomology in terms of the compact support cohomology of X in the i-acyclic case. The spaces $\Delta_{\leqslant\ell}X^n$ and $\Delta_\ell X^n$ also fit into our framework: $\Delta_{\leqslant\ell}X^n$ can be written as $D(X,U)$, where $U\subset\Pi_n$ is the subposet consisting of partitions with at most ℓ blocks, and $\Delta_\ell X^n$ is in general a disjoint union of configuration spaces $F(X,\ell)$.

6.6 We have already seen in Corollary 5.14 that $H_c^k(F(X,n),\mathbf{Q})$ depends only on a cdga model for the compactly supported cochains $C_c^\bullet(X,\mathbf{Q})$. One might therefore hope for the following to be true, which would re-prove and reinterpret Arabia's result: if a space X is i-acyclic over $\mathbf{Q}$, then the cdga $C_c^\bullet(X,\mathbf{Q})$ is formal, and the cup product on $H_c^\bullet(X,\mathbf{Q})$ vanishes. Equivalently, there is a quasi-isomorphism between $C_c^\bullet(X,\mathbf{Q})$ and $H_c^\bullet(X,\mathbf{Q})$, where the cohomology is considered as a cdga with zero differential and zero multiplication. This turns out to be the case. Moreover, our method of proof will work with minor modifications to prove a result over an arbitrary coefficient ring. Let us state the main theorems of this section.

THEOREM 6.7. *Let X be a paracompact Hausdorff locally compact space which is i-acyclic over $\mathbf{Q}$. Then a cdga model for $C_c^\bullet(X,\mathbf{Q})$ is given by $H_c^\bullet(X,\mathbf{Q})$, considered as a cdga with identically zero differential and multiplication.*

Remark 6.8. Theorem 6.7 has the following easy consequence. Let X be a space whose one-point compactification $\bar{X}$ is nilpotent. If X is i-acyclic then $\bar{X}$ has the based rational homotopy type of a wedge of spheres. Indeed, (a cdga model of) $C_c^\bullet(X,\mathbf{Q})$ being formal is equivalent to $C^\bullet(\bar{X},\mathbf{Q})$ being formal as an augmented algebra, since $C_c^\bullet(X,\mathbf{Q})$ is just the augmentation ideal. For example, we have seen that if Y is arbitrary, then $Y\times\mathbf{R}$ is i-acyclic, and so the claim is that the one-point compactification of $Y\times\mathbf{R}$ should have the rational homotopy type of a wedge of spheres. But the one-point compactification of $Y\times\mathbf{R}$ is the suspension of the one-point compactification of Y, so this recovers the familiar fact that any suspension is a wedge of spheres rationally.

COROLLARY 6.9. *Let X be a paracompact Hausdorff locally compact space which is i-acyclic over $\mathbf{Q}$. Let $U\subset\Pi_n$ be upwards closed. Then the cohomology groups $H_c^\bullet(F(X,U),\mathbf{Q})$ and $H_c^\bullet(D(X,U),\mathbf{Q})$ depend only on the graded vector space $H_c^\bullet(X,\mathbf{Q})$. In fact we have*

$$H_c^k(D(X,U),\mathbf{Q}) \cong \bigoplus_{T\in J_U}\bigoplus_{p+q=k} H_c^p(X^{|T|},\mathbf{Q})\otimes \widetilde{H}^q(\llbracket J_U^{\preceq T}\rrbracket,\mathbf{Q})$$

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and

$$H_c^k(F(X,U),\mathbf{Q}) \cong \bigoplus_{T\in J_{U_0}} \bigoplus_{p+q=k} H_c^p(X^{|T|},\mathbf{Q}) \otimes \widetilde{H}^q([\![J_{U_0}^{\preceq T}]\!],\mathbf{Q}).$$

These isomorphisms are equivariant in the sense that if $G \subseteq S_n$ is the subgroup preserving $U \subset \Pi_n$ (meaning that $G \cdot U \subseteq U$), then the isomorphisms are equivariant with respect to the action of G on both sides.

Theorem 6.10. *Let X be a paracompact Hausdorff locally compact space which is i-acyclic over the ring R, and assume that $H_c^\bullet(X,R)$ is a projective R-module. Consider the tcdga given by $R\Gamma_c^\otimes(X,R)$. Its cohomology is the 'constant' tcdga (§ 2.5) given by $H_c^\bullet(X,R)$ with identically zero multiplication and differential. Moreover, $R\Gamma_c^\otimes(X,R)$ is quasi-isomorphic to its cohomology in the category of commutative shuffle dg algebras.*

Corollary 6.11. *Let X be a paracompact Hausdorff locally compact space which is i-acyclic over the ring R, and assume that $H_c^\bullet(X,R)$ is a projective R-module. Let $U \subset \Pi_n$ be upwards closed. Then the cohomology groups $H_c^\bullet(F(X,U),R)$ and $H_c^\bullet(D(X,U),R)$ depend only on the graded R-module $H_c^\bullet(X,R)$. In fact we have*

$$H_c^k(D(X,U),R) \cong \bigoplus_{T\in J_U} \bigoplus_{p+q=k} H_c^p(X^{|T|}, \widetilde{H}^q([\![J_U^{\preceq T}]\!],R))$$

and

$$H_c^k(F(X,U),R) \cong \bigoplus_{T\in J_{U_0}} \bigoplus_{p+q=k} H_c^p(X^{|T|}, \widetilde{H}^q([\![J_{U_0}^{\preceq T}]\!],R)).$$

These isomorphisms are equivariant in a weak sense: if $G \subseteq S_n$ is the subgroup preserving $U \subset \Pi_n$, then there is a filtration on the left-hand side for which the associated graded is isomorphic to the right-hand side as a G-module.

Example 6.12. Suppose that $X = \mathbf{R}$. Then $F(X,U)$ is the complement of a hypergraph subspace arrangement in $\mathbf{R}^n$ (also known as a diagonal arrangement), and any hypergraph subspace arrangement arises for an appropriate choice of U. In this case $F(X,U)$ is an oriented manifold and its compact support cohomology equals its homology, so Corollary 6.11 gives a formula for the integral homology of the complement of the arrangement in terms of the cohomology of the posets $J_U^{\preceq T}$. The resulting formula is exactly the Goresky–MacPherson formula [GM88], in the special case of a hypergraph arrangement. What is perhaps surprising is that we obtain a formula of exactly the same 'shape' for any i-acyclic space whatsoever; it is not at all clear from existing proofs of the Goresky–MacPherson formula that such a formula should exist for (say) $X = \mathbf{R} \times \Sigma$, where Σ is a compact surface. Note also that the Goresky–MacPherson formula is not obviously equivariant: the equivariant Goresky–MacPherson formula (for $\mathbf{Q}$-coefficients) is a theorem of Sundaram and Welker [SW97b].

Example 6.13. Another simple application of Corollary 6.11 is that if the compact support cohomology of an i-acyclic space X is torsion-free, and the posets $J_{U_0}^{\preceq T}$ have torsion-free cohomology (e.g. U is Cohen–Macaulay), then all the spaces $F(X,U)$ have torsion-free compact support cohomology. The property that $\widetilde{H}^\bullet([\![J_{U_0}^{\preceq T}]\!],\mathbf{Z})$ is torsion-free is well studied in the subject of the topology of arrangements; it is equivalent to the corresponding complement of a subspace arrangement in $\mathbf{R}^n$ having torsion-free homology.

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Recollections on A_∞- and C_∞-algebras

6.14 In this subsection we very briefly recall some definitions from the theory of A_∞- and C_∞-algebras. The reader who is not familiar with these notions may consult [LV12, ch. 10] for a detailed account.

6.15 We define an A_∞-*algebra structure* on a graded R-module A to be a degree-1 square-zero coderivation of the tensor coalgebra $\mathsf{coAss}(A[1])$. A degree-1 coderivation of $\mathsf{coAss}(A[1])$ determines and is uniquely determined by linear maps of degree $2-n$,

$$\mu_n \colon A^{\otimes n} \to A,$$

for $n \geqslant 1$. That the coderivation squares to zero implies, for example, that μ_1 is a differential making A into a cochain complex and that μ_2 is a multiplication satisfying the Leibniz rule. Hence there is an induced multiplication on $H^\bullet(A)$, which turns out to be associative. However, μ_2 is not itself associative. Informally, the idea of an A_∞-algebra is that it is an algebra in which the multiplication μ_2 satisfies the associativity laws up to coherent homotopy. For example, the next equation one obtains from the condition that the coderivation squares to zero is that the associator of μ_2, considered as a map $A^{\otimes 3} \to A$, is the differential of the map μ_3. When taking the cohomology one then gets associativity on the nose. In general one gets an infinite sequence of quadratic equations expressing relations between the maps μ_n.

6.16 If A is an A_∞-algebra, then we denote $\mathsf{coAss}(A[1])$, with its differential given by the A_∞-structure, by BA, and we call it the *bar construction* on A. The classical bar construction on an associative dg algebra A of Eilenberg and MacLane is a codifferential on the tensor coalgebra on the suspension of A, as above; in this way, every dg algebra may be considered as an A_∞-algebra in a natural manner.

6.17 We define an A_∞-*morphism* $A \to A'$ to be a morphism of dg coalgebras $f \colon BA \to BA'$. Such a morphism determines and is determined by a collection of linear maps of degree $1-n$,

$$f_n \colon A^{\otimes n} \to A'.$$

The condition that f is a chain map then results in an infinite sequence of quadratic equations relating the maps f_n and the A_∞-structures on A and A'. We say that f is an A_∞-*quasi-isomorphism* if f_1 is a quasi-isomorphism. If $A = A'$ and f_1 is the identity map, then we say that f is an A_∞-*isotopy.*

6.18 We define similarly a C_∞-algebra structure on A to be a degree-1 square-zero coderivation of the cofree conilpotent Lie coalgebra $\mathsf{coLie}(A[1])$. Morphisms, quasi-isomorphisms, and isotopies of C_∞-algebras are defined in a completely analogous manner. Via the projection $\mathsf{coAss}(A[1]) \to \mathsf{coLie}(A[1])$ we may consider every C_∞-algebra to be an A_∞-algebra, and every C_∞-morphism to be an A_∞-morphism. Every commutative dg algebra may be considered as a C_∞-algebra. Informally, a C_∞-algebra is a dg algebra in which associativity only holds up to coherent higher homotopy, but whose multiplication is strictly commutative. Moreover, the higher homotopies are themselves commutative in a certain weak sense.

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6.19 Assume now that the ground ring R contains $\mathbf{Q}$. We will need only the following basic properties of the categories of A_∞- and C_∞-algebras,

(i) If A and A' are dg algebras connected by an A_∞-quasi-isomorphism $A \to A'$, then there is a zigzag of quasi-isomorphisms of dg algebras connecting A and A'.

(ii) If A and A' are commutative dg algebras connected by a C_∞-quasi-isomorphism $A \to A'$, then there is a zigzag of quasi-isomorphisms of commutative dg algebras connecting A and A'.

(iii) If A is a dg algebra, then there exist an A_∞-algebra structure on $H(A)$ and an A_∞-quasi-isomorphism $H(A) \to A$. The induced A_∞-structure on $H(A)$ is unique up to non-canonical A_∞-isotopy.

(iv) If A is a commutative dg algebra, then there exist a C_∞-algebra structure on $H(A)$ and a C_∞-quasi-isomorphism $H(A) \to A$. The induced C_∞-structure on V is unique up to non-canonical C_∞-isotopy.

For this, see [LV12, Theorem 11.4.9] for the first two points and [LV12, Theorem 10.3.1] for the third and fourth. The third and fourth properties are theorems of Kadeishvili; see [Kad80] for the A_∞-case.

Proof of Theorem 6.7

Lemma 6.20. *Let A be a differential graded algebra over $\mathbf{Q}$, and let $I \subset A$ be an ideal. Suppose that the induced map $H(I) \to H(A)$ vanishes. Then I is formal as a dg algebra, and the multiplication on $H(I)$ is identically zero.*

Proof. We construct an A_∞-quasi-isomorphism from $H(I)$, considered as a dg algebra with vanishing multiplication and differential, to I. This is enough to conclude the result by § 6.19(1). Unwinding the definitions, this means that we have to find maps

$$f_n \colon H(I)^{\otimes n} \to I, \quad n \geqslant 1$$

of degree $1 - n$, such that

$$d \circ f_n = \sum_{i+j=n} (-1)^i f_i \cdot f_j.$$

Let $f \colon H(I) \to I$ be a map taking every class to a representing cocycle. Let $g \colon H(I) \to A$ be a map such that $d \circ g = -f$. Such maps exist because we work over a field, and since every cocycle in I is a coboundary in A. Now define for all $n \geqslant 1$,

$$f_n(x) = f(x) \cdot \underbrace{g(x) \cdots g(x)}_{n-1 \text{ times}}.$$

Since I is an ideal in A, this product is well defined as an element of I. One checks, using the Leibniz rule, that the collection $\{f_n\}$ defines an A_∞-morphism; it is clearly a quasi-isomorphism. □

Lemma 6.21. *Let A be a cdga over $\mathbf{Q}$, and let $I \subset A$ be an ideal. Suppose that the induced map $H(I) \to H(A)$ vanishes. Then I is formal as a cdga, and the multiplication on $H(I)$ is identically zero.*

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Proof. By § 6.19(4) we may give $H(I)$ some C_∞-algebra structure, making it C_∞-quasi-isomorphic to I. On the other hand, we may give $H(I)$ an A_∞-algebra structure by setting all operations identically to zero, and by the previous lemma this structure is A_∞-quasi-isomorphic to I. By the uniqueness part of § 6.19(3) this means that the transferred C_∞-structure on $H(I)$ must be A_∞-isotopic to the degenerate A_∞-structure with all operations zero. While it is hard in general to determine whether two A_∞-structures are A_∞-isotopic, the case when one of them is identically zero is easy: the only thing isotopic to the identically zero A_∞-structure is the identically zero A_∞-structure. □

Remark 6.22. One can also deduce Lemma 6.21 from Lemma 6.20 by appealing to [Sal17, Theorem 1.3].

THEOREM 6.23 (Theorem 6.7 restated). *Let X be a paracompact Hausdorff locally compact space which is i-acyclic over $\mathbf{Q}$. Then a cdga model for $C_c^\bullet(X, \mathbf{Q})$ is given by $H_c^\bullet(X, \mathbf{Q})$, considered as a cdga with identically zero differential and multiplication.*

Proof. We may take for our cdga model for $C_c^\bullet(X, \mathbf{Q})$ the algebra $R\Gamma_c^{\mathrm{TW}}(X, \mathbf{Q})$. It is an ideal inside the cdga $R\Gamma^{\mathrm{TW}}(X, \mathbf{Q})$ and the induced map $H_c^\bullet(X, \mathbf{Q}) \to H^\bullet(X, \mathbf{Q})$ is zero by assumption. The result follows from Lemma 6.21. □

Proof of Theorem 6.10

6.24 The proof of Theorem 6.10 is nearly identical to the proof of Theorem 6.7. We will, however, need to work with C_∞- and A_∞-algebras in more general categories than Ch_R, which serves to add an additional layer of abstraction; moreover, since we do not work over $\mathbf{Q}$ we should argue that the statements of § 6.19 remain valid.

6.25 We will need to consider A_∞-algebras and C_∞-algebras in the category of functors $\mathsf{Mod}_R^{\mathsf{Ord}_+}$, with the 'shuffle' monoidal structure introduced in § 2.10:

$$(A \otimes B)(S) = \bigoplus_{S = T \sqcup T'} A(T) \otimes B(T').$$

That is, we repeat verbatim the definitions of A_∞-algebra and C_∞-algebra, but the tensor coalgebra and cofree conilpotent Lie coalgebra are now computed using the above tensor product. An associative dg algebra in this category of functors is nothing but a a shuffle dg algebra, and a commutative dg algebra is nothing but a commutative shuffle dg algebra, as defined in § 2.11. We shall refer to A_∞-algebras in this category as *shuffle* A_∞*-algebras*. They form a natural enlargement of the category of shuffle dg algebras. Similarly, we may consider shuffle C_∞-algebras as an enlargement of the category of commutative shuffle dg algebras.

PROPOSITION 6.26. *Let A and A' be shuffle dg algebras. If there exists an A_∞-quasi-isomorphism $f\colon A \to A'$, then there exists a zigzag of quasi-isomorphisms of shuffle dg algebras connecting A and A'. Similarly for commutative shuffle dg algebras and shuffle C_∞-algebras.*

6.27 Before giving the proof of Proposition 6.26, let us recall how one proves this for the usual A_∞-algebras and C_∞-algebras over $\mathbf{Q}$. The idea is to show that each morphism in the zigzag

$$A \leftarrow \Omega BA \xrightarrow{\Omega f} \Omega BA' \to A'$$

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is a quasi-isomorphism, where the first and last arrows are given by the counit of the bar–cobar adjunction [LV12, § 11.3] and the second arrow is given by applying the functor cobar to the map $BA \to BA'$ corresponding to the A_∞-morphism f. Disregarding the differential, we have

$$\Omega BA \cong \mathsf{Ass}(s\mathsf{coAss}(s^{-1}A)) \cong \mathsf{Ass}(\Lambda^{-1}\mathsf{coAss}(A)) \cong \bigoplus_{k\geqslant 1}(\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(k)\otimes_{\mathbb{S}_k} A^{\otimes k},$$

where Ass denotes the associative operad, coAss the coassociative cooperad, Λ^{-1} the operadic desuspension (i.e. $\Lambda^{-1}\mathsf{coAss}$ is the Koszul dual cooperad of Ass); see [LV12, ch. 11]. We can define an increasing filtration L on ΩBA by

$$L_m\Omega BA = \bigoplus_{k=1}^{m}(\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(k)\otimes_{\mathbb{S}_k} A^{\otimes k}.$$

We also give A the trivial filtration with $L_0A = \{0\}$ and $L_1A = A$, and we define similarly filtrations on $\Omega BA'$ and A'. We will argue that all maps in the zigzag $A \leftarrow \Omega BA \to \Omega BA' \to A'$ are filtered quasi-isomorphisms, which is enough to conclude the result since a filtered quasi-isomorphism of complexes with bounded-below exhaustive filtrations is a quasi-isomorphism.

Now the induced differential on $\mathrm{Gr}^L_m\,\Omega BA \cong (\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(m)\otimes_{\mathbb{S}_m} A^{\otimes m}$ is given only by the Koszul differential on $(\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(m)$. But this Koszul complex is acyclic for $m > 1$; indeed, this is the definition of Koszul duality. So certainly each morphism in the zigzag is a quasi-isomorphism on Gr^L_m for $m > 1$. For $m = 1$ we have an identification $\mathrm{Gr}^L_1\,\Omega BA = A$, and applying Gr^L_1 to the zigzag gives simply

$$A \xleftarrow{=} A \xrightarrow{f_1} A' \xrightarrow{=} A',$$

and the result follows. The proof for C_∞-algebras over $\mathbf{Q}$ works the same, except that we replace $\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss}$ with the Koszul complex $\mathsf{Com}\circ\Lambda^{-1}\mathsf{coLie}$. The only place where we use the fact that the ground ring is $\mathbf{Q}$ is when we deduce that $(\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(m)\otimes_{\mathbb{S}_m} A^{\otimes m}$ is acyclic from the fact that $(\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(m)$ is acyclic – over a general ground ring, the functor $-\otimes_{\mathbb{S}_m} A^{\otimes m}$ has no reason to be exact in general. (In fact for A_∞-algebras the result is valid over $\mathbf{Z}$, since $(\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(m)$ turns out to be a complex of free $\mathbf{Z}[\mathbb{S}_m]$-modules. But for C_∞-algebras we do need to work over $\mathbf{Q}$ in general.)

Proof of Proposition 6.26. As in the previous paragraph, we write down the zigzag

$$A \longleftarrow \Omega BA \longrightarrow \Omega BA' \longrightarrow A',$$

and we define the same filtration on each of the terms. We get in the same way an isomorphism

$$\mathrm{Gr}^L_m\,\Omega BA \cong (\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(m)\otimes_{\mathbb{S}_m} A^{\otimes m},$$

where we now use the fact that the category $\mathsf{Ch}_R^{\mathsf{Ord}_+}$ is tensored over Ch_R. Now we need to use an observation already made in Theorem 3.18: that $A^{\otimes m}$ is a free $R[\mathbb{S}_m]$-module for any $A \in \mathrm{ob}\,\mathsf{Ch}_R^{\mathsf{Ord}_+}$, as follows from the calculation

$$A^{\otimes m}(k) = \bigoplus_{\{1,\dots,k\}=S_1\sqcup\cdots\sqcup S_m} A(S_1)\otimes\cdots\otimes A(S_m) = R[\mathbb{S}_m]\otimes \bigoplus_{\substack{\{1,\dots,k\}=S_1\sqcup\cdots\sqcup S_m\\ \min(S_1)<\cdots<\min(S_m)}} A(S_1)\otimes\cdots\otimes A(S_m).$$

In particular, since $(\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss})(m)$ is acyclic for $m > 1$, we deduce that $\mathrm{Gr}^L_m\,\Omega BA$ is acyclic for $m > 1$. Thus the proof goes through as in the preceding paragraph without changes. For C_∞-algebras we employ the same argument except that we replace $\mathsf{Ass}\circ\Lambda^{-1}\mathsf{coAss}$ with the Koszul complex $\mathsf{Com}\circ\Lambda^{-1}\mathsf{coLie}$. □

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PROPOSITION 6.28. *Let A be a shuffle dg algebra such that $H(A)$ is a projective R-module in arity n for all n. There exist a shuffle A_∞-algebra structure on $H(A)$ and an A_∞-quasi-isomorphism $H(A) \to A$. The induced shuffle A_∞-structure on $H(A)$ is unique up to non-canonical A_∞-isotopy. Similarly, the cohomology of a commutative shuffle dg algebra can be given a shuffle C_∞-algebra structure with a C_∞-quasi-isomorphism $H(A) \to A$.*

Proof. The projectivity assumption implies the existence of a quasi-isomorphism $f\colon H(A) \to A$ in $\mathsf{Ch}_R^{\mathsf{Ord}+}$. According to [Pet19, Theorem 2], the conclusion will follow once we have verified that f induces a quasi-isomorphism

$$\underline{\mathrm{Hom}}_{\mathsf{Ch}_R^{\mathsf{Ord}+}}(\mathsf{coAss}(H(A)), H(A)) \longrightarrow \underline{\mathrm{Hom}}_{\mathsf{Ch}_R^{\mathsf{Ord}+}}(\mathsf{coAss}(H(A)), A),$$

where $\underline{\mathrm{Hom}}$ denotes the internal Hom; for C_∞-algebras we should instead consider

$$\underline{\mathrm{Hom}}_{\mathsf{Ch}_R^{\mathsf{Ord}+}}(\mathsf{coLie}(H(A)), H(A)) \longrightarrow \underline{\mathrm{Hom}}_{\mathsf{Ch}_R^{\mathsf{Ord}+}}(\mathsf{coLie}(H(A)), A).$$

In both cases, the result is immediate from the fact that $H(A)$ is projective, and then so are $\mathsf{coAss}(H(A))$ and $\mathsf{coLie}(H(A))$, so $\underline{\mathrm{Hom}}_{\mathsf{Ch}_R^{\mathsf{Ord}+}}(\mathsf{coAss}(H(A)), -)$ and $\underline{\mathrm{Hom}}_{\mathsf{Ch}_R^{\mathsf{Ord}+}}(\mathsf{coLie}(H(A)), -)$ are both exact functors. The proof in [Pet19] is formulated in Ch_R but works with no modifications in $\mathsf{Ch}_R^{\mathsf{Ord}+}$. □

Remark 6.29. There are very many ways of proving that one can transfer A_∞- or C_∞-algebra structure to cohomology, for example using sums over trees [LV12, § 10.3], homological perturbation theory [Ber14], or abstract model-categorical arguments [BM03]. In the preceding proof we refer to the paper [Pet19] since the other approaches mentioned above all require stronger assumptions on the ground ring or on the operads or complexes involved; we do not want to assume, for example, that $H(A) \to A$ is a homotopy equivalence. For A_∞-algebras the proof in [Pet19] is exactly the same as that of Kadeishvili [Kad80]. According to [Kad08], Kadeishvili proved the analogous transfer result for C_∞-algebras over a general ground ring as well in the 1980s, but these references are not easily available, and Kadeishvili's definition of a C_∞-algebra is slightly different from the one used here; the two notions are equivalent only over a ground ring containing $\mathbf{Q}$.

LEMMA 6.30. *Let A be a commutative shuffle dg algebra, $I \subset A$ an ideal. Suppose that the induced map $H(I) \to H(A)$ vanishes, and that $H(I)$ is projective – that is, $H(I)(n)$ is degreewise a projective R-module, for every $n \geqslant 1$. Then I is formal as a commutative shuffle dg algebra, and the multiplication on $H(I)$ is identically zero.*

Proof. Repeat the proof of Lemma 6.21, with the tacit understanding that all C_∞-algebras are now shuffle C_∞-algebras, all A_∞-algebras are now shuffle A_∞-algebras, etc., and where a quasi-isomorphism $H(I) \to I$ in $\mathsf{Ch}_R^{\mathsf{Ord}+}$ now exists since we assumed $H(I)(n)$ to be a projective R-module for all n. □

Remark 6.31. The preceding proof shows why we need to work with commutative shuffle algebras instead of twisted commutative algebras when we are not over a field of characteristic 0: in order to construct a quasi-isomorphism $H(I) \to I$ in $\mathsf{Ch}_R^{\mathsf{Ord}+}$ we only need $H(I)(n)$ to be a projective R-module, but if we wanted a quasi-isomorphism $H(I) \to I$ in $\mathsf{Ch}_R^{\mathsf{Fin}+}$ we would need $H(I)(n)$ to be a projective $R[\mathbb{S}_n]$-module for all n, which will practically never be true.

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Theorem 6.32 (Theorem 6.10 restated). *Let X be a paracompact Hausdorff locally compact space which is i-acyclic over the ring R, such that $H_c^\bullet(X,R)$ is degreewise a projective R-module. Then $R\Gamma_c^\otimes(X,R)$ is formal as a commutative shuffle dg algebra, and the multiplication on its cohomology $H_c^\bullet(X,R)$ is identically zero.*

Proof. Apply the preceding lemma to the ideal $R\Gamma_c^\otimes(X,R)$ inside the commutative shuffle dg algebra $R\Gamma^\otimes(X,R)$. □

Remark 6.33. If $H_c^\bullet(X,R)$ is not a projective R-module there is still something one can say. By a modification of the above arguments one can show that if X is 'derived i-acyclic', meaning that the map $C_c^\bullet(X,R) \to C^\bullet(X,R)$ is zero in the derived category of dg R-modules, then $R\Gamma_c^\otimes(X,R)$ is quasi-isomorphic as a commutative dg shuffle algebra to a projective resolution of $H_c^\bullet(X,R)$, considered as a commutative dg shuffle algebra with identically zero multiplication. When the cohomology is projective as an R-module, i-acyclicity is equivalent to derived i-acyclicity. We omit the details.

7. Example application: k-equals configuration spaces

7.1 In this section we will illustrate how our results may be applied. Let X be an i-acyclic space, $k \geqslant 2$ an integer, and let $F_{<k}(X,n)$ denote the *k-equals configuration space* – the space parametrizing n ordered points on X such that no subset of k points coincide. We write down explicit formulae for the cohomology groups $H_c^i(F_{<k}(X,n),\mathbf{Q})$ and their decompositions into irreducible $\mathbb{S}_n$-representations, in terms of the compactly supported Poincaré polynomial of X. We assume throughout that X is paracompact locally compact Hausdorff and that all of the cohomology groups $H_c^i(X,\mathbf{Q})$ are finite-dimensional $\mathbf{Q}$-vector spaces.

7.2 Our calculation is formulated in terms of the algebra of symmetric functions, and we will use the correspondence between symmetric functions and symmetric sequences. Let us briefly recall the relevant terminology, although part of it has already been used (in greater generality) in previous sections.

Symmetric sequences and a reformulation of the calculation

Definition 7.3. To avoid repeating hypotheses, we will for the remainder of this section use the term *symmetric sequence* to mean a sequence $\mathbf{A} = \{\mathbf{A}(n)\}_{n=0}^\infty$ of representations of the symmetric groups $\mathbb{S}_n$ in the category of graded, degreewise finite-dimensional $\mathbf{Q}$-vector spaces. We have already used the notion of symmetric sequence in greater generality: we considered the tensor product of symmetric sequences in § 2.9; moreover, we have freely used the language of operads, and an operad is nothing but a monoid in the category of symmetric sequences with respect to the monoidal structure given by composition product.

Definition 7.4. Let $\mathbf{A}$ and $\mathbf{B}$ be symmetric sequences. Their *tensor product* is defined by

$$(\mathbf{A}\otimes\mathbf{B})(n) = \bigoplus_{n=k_1+k_2} \mathrm{Ind}_{\mathbb{S}_{k_1}\times\mathbb{S}_{k_2}}^{\mathbb{S}_n} \mathbf{A}(k_1)\otimes\mathbf{B}(k_2).$$

If $\mathbf{B}(0)=0$, then their *composition product* is the symmetric sequence defined by

$$(\mathbf{A}\circ\mathbf{B})(n) = \bigoplus_{k=0}^\infty \mathbf{A}(k) \otimes_{\mathbb{S}_k} \mathbf{B}^{\otimes k}(n),$$

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where $\mathbf{B}^{\otimes k}$ denotes the k-fold tensor product of $\mathbf{B}$ with itself. (The definition makes sense also when $\mathbf{B}(0) \neq 0$, but $\mathbf{A} \circ \mathbf{B}$ might no longer be degreewise finite-dimensional as a vector space in each arity.)

7.5 Let $\Pi_{(k,1^{n-k})} \subset \Pi_n$ be the subposet of partitions all of whose blocks are either singletons or have size at least k. Let $\mathbf{P}_k$ denote the symmetric sequence with

$$\mathbf{P}_k(n) = \widetilde{H}^\bullet([\![\Pi_{(k,1^{n-k})}]\!], \mathbf{Q})$$

for $n > 0$. Let $\mathbf{E}$ denote the symmetric sequence with $\mathbf{E}(n) = \mathbf{Q}$, the trivial representation concentrated in degree 0, for all $n \geqslant 0$. Finally, we consider $H_c^\bullet(X, \mathbf{Q})$ as a symmetric sequence concentrated in arity 0.

Remark 7.6. Note that the composition product $H_c^\bullet(X, \mathbf{Q}) \circ \mathbf{P}_k$ is just the aritywise tensor product:

$$(H_c^\bullet(X, \mathbf{Q}) \circ \mathbf{P}_k)(n) = H_c^\bullet(X, \mathbf{Q}) \otimes \mathbf{P}_k(n).$$

Remark 7.7. The composition product $\mathbf{E} \circ \mathbf{A}$ can be understood as an 'exponential' of $\mathbf{A}$, and can be explicitly written as

$$(\mathbf{E} \circ \mathbf{A})(n) = \bigoplus_{\substack{T \in \Pi_n \\ |T_i| = t_i,\ i=1,\dots,\ell}} \bigotimes_{i=1}^{\ell} \mathbf{A}(t_i),$$

where the summation indicates that we sum over all partitions T, and that the partition T has ℓ blocks of size $t_1, \dots, t_\ell$.

PROPOSITION 7.8. *Let X be i-acyclic over $\mathbf{Q}$. There is an isomorphism of $\mathbb{S}_n$-representations*

$$(\mathbf{E} \circ H_c^\bullet(X, \mathbf{Q}) \circ \mathbf{P}_k)(n) = H_c^\bullet(F_{<k}(X, n), \mathbf{Q}).$$

Proof. Let $U \subset \Pi_n$ be the upwards closed set consisting of all partitions containing a block of size at least k, so that $F_{<k}(X, n) = F(X, U)$. As in § 4.8, we denote by J_{U_0} the set of joins of atoms in U, including the empty join $\hat{0}$. Note that $J_{U_0} = \Pi_{(k,1^{n-k})}$. If $T \in J_{U_0}$ has blocks of size $t_1, \dots, t_\ell$, then the poset $J_{U_0}^{\preccurlyeq T}$ decomposes as a cartesian product

$$J_{U_0}^{\preccurlyeq T} \cong \prod_{i=1}^{\ell} \Pi_{(k,1^{t_i-k})},$$

and hence Lemma 4.5 says that there is a homeomorphism

$$[\![J_{U_0}^{\preccurlyeq T}]\!] \cong [\![\Pi_{(k,1^{t_1-k})}]\!] \wedge [\![\Pi_{(k,1^{t_2-k})}]\!] \wedge \cdots \wedge [\![\Pi_{(k,1^{t_\ell-k})}]\!].$$

But according to Corollary 6.9 there is an $\mathbb{S}_n$-equivariant isomorphism

$$H_c^\bullet(F_{\leqslant k}(X, n), \mathbf{Q}) \cong \bigoplus_{T \in J_{U_0}} \widetilde{H}^\bullet([\![J_{U_0}^{\preccurlyeq T}]\!], \mathbf{Q}) \otimes H_c^\bullet(X^{|T|}, \mathbf{Q})$$

and the above expression for $[\![J_{U_0}^{\preccurlyeq T}]\!]$ implies that the right-hand side can be rewritten as

$$\bigoplus_{\substack{T \in J_{U_0} \\ |T_i| = t_i,\ i=1,\dots,\ell}} \bigotimes_{i=1}^{\ell} \widetilde{H}^\bullet([\![\Pi_{(k,1^{t_i-k})}]\!], \mathbf{Q}) \otimes H_c^\bullet(X, \mathbf{Q})$$

which is now nothing but the arity-n component of $\mathbf{E} \circ H_c^\bullet(X, \mathbf{Q}) \circ \mathbf{P}_k$, where we use Remarks 7.6 and 7.7. □

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The formalism of symmetric functions

7.9 Let Λ denote the ring of symmetric functions over $\mathbf{Q}$. It is graded: $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda_n$. We let $\widehat{\Lambda}$ denote the completion of Λ with respect to the filtration by degree. Explicitly, $\widehat{\Lambda} = \prod_{n=0}^{\infty} \Lambda_n$.

7.10 For each n there is an isomorphism $\Lambda_n \cong R(\mathbb{S}_n)$ between Λ_n and the Grothendieck group of representations of $\mathbb{S}_n$. Here it is crucial that we work over a field of characteristic 0. If M is a representation of $\mathbb{S}_n$, then we denote by $\mathrm{ch}_n M$ the corresponding symmetric function.

7.11 Let $\mathcal{R}$ denote the ring

$$\mathcal{R} = \prod_{n=0}^{\infty} \Lambda_n \otimes \mathbf{Q}[t, t^{-1}],$$

considered as a subring of $\widehat{\Lambda}[\![t, t^{-1}]\!]$. If $\mathbf{M}$ is a symmetric sequence, then we denote

$$\mathrm{ch}\, \mathbf{M} = \sum_{n=0}^{\infty} \sum_{i \in \mathbf{Z}} (-t)^i \,\mathrm{ch}_n\, \mathbf{M}(n)^i \in \mathcal{R},$$

where $\mathbf{M}(n)^i$ denotes the component of $\mathbf{M}(n)$ in cohomological degree i. Two symmetric sequences $\mathbf{M}$ and $\mathbf{M}'$ are isomorphic if and only if $\mathrm{ch}\, \mathbf{M} = \mathrm{ch}\, \mathbf{M}'$. This is a consequence of the fact that the category of symmetric sequences is semisimple. The assignment $\mathbf{M} \mapsto \mathrm{ch}\, \mathbf{M}$ induces an isomorphism between $K_0(\mathsf{Mod}_{\mathbb{S}})$ (the Grothendieck group of symmetric sequences) and $\mathcal{R}$. This is in fact an isomorphism of *commutative rings*, where $K_0(\mathsf{Mod}_{\mathbb{S}})$ carries the ring structure induced from the tensor product of symmetric sequences.

7.12 We will consider two particular bases for the ring Λ. Recall that irreducible representations of $\mathbb{S}_n$ are parametrized in a standard way by partitions $\lambda \vdash n$; we let V_λ denote the irreducible representation (Specht module) corresponding to λ. The elements $\mathrm{ch}_n V_\lambda \in \Lambda_n$ are called *Schur polynomials* and will be denoted by s_λ. For example, s_n corresponds to the trivial representation of $\mathbb{S}_n$, and s_{1^n} the sign representation. Schur polynomials form a basis for Λ as a vector space. We will also use the *power sums* p_n, $n \geqslant 1$, which are uniquely determined by the equality of formal power series

$$\sum_{n \geqslant 0} s_n t^n = \exp \sum_{n \geqslant 1} p_n t^n / n.$$

The power sums freely generate Λ as a $\mathbf{Q}$-algebra.

7.13 The ring Λ carries an operation called *plethysm*, denoted $f \circ g$. Plethysm is uniquely determined by the following properties.

- For all $g \in \Lambda$, the map $f \mapsto f \circ g$ is a ring homomorphism.
- For all $n \geqslant 1$, the map $g \mapsto p_n \circ g$ is a ring homomorphism.
- $p_n \circ p_m = p_{nm}$.

We extend plethysm to an operation on $\Lambda[t, t^{-1}]$ by imposing two additional rules.

- Plethysm is $\mathbf{Q}[t, t^{-1}]$-linear in the first variable.
- $p_n \circ t = t^n$.

Plethysm does not extend to a well-defined operation on $\mathcal{R}$, but if g has no constant term then we may define $f \circ g$ by the above rules.

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7.14 The isomorphism $K_0(\mathsf{Mod}_{\mathbb{S}}) \to \mathcal{R}$ carries the composition product of symmetric sequences to the plethysm of symmetric functions. Note that we defined the composition product $\mathbf{M} \circ \mathbf{N}$ of symmetric sequences only when $\mathbf{N}(0) = 0$, and the plethysm $f \circ g$ only when g has no constant term, so the claim should be understood as saying that the isomorphism is compatible with these partially defined functions.

The calculation for k-equals configuration spaces

7.15 Consider the following three elements of $\mathcal{R}$:

$$\mathsf{S}_k = -\sum_{n \geqslant k} (-t)^{n-k+2} s_{n-k+1,1^{k-1}},$$
$$\mathsf{L} = \sum_{n \geqslant 1} \frac{1}{n} \sum_{d|n} (-1)^{n/d-1} \mu(d) p_d^{n/d},$$
$$\mathsf{E} = \sum_{n \geqslant 0} s_n.$$

Here μ denotes the usual number-theoretic Möbius function. The following is a theorem of Sundaram and Wachs [SW97a, Theorem 3.5], slightly reformulated. Their result was previously applied to the computations of the $\mathbb{S}_n$-equivariant rational cohomologies of the spaces $F_{\leqslant k}(\mathbf{R}^d, n)$ by Sundaram and Welker [SW97b]. The non-equivariant version of the calculation was done previously by Björner and Welker [BW95], who also proved that the cohomology is torsion-free.

THEOREM 7.16 (Sundaram–Wachs). *Let $\mathbf{P}_k$ denote the symmetric sequence defined in § 7.5. There is an equality*

$$s_1 + t^{-1}\mathsf{L} \circ \mathsf{S}_k = \operatorname{ch} \mathbf{P}_k$$

in $\mathcal{R}$.

7.17 Let $\mathbf{F}_{<k}(X)$ denote the symmetric sequence given by

$$\mathbf{F}_{<k}(X)(n) = H_c^\bullet(F_{<k}(X, n), \mathbf{Q}).$$

By combining Proposition 7.8 with Theorem 7.16, we can give an explicit formula for $\operatorname{ch} \mathbf{F}_{\leqslant k}(X)$ when X is i-acyclic. In other words, we obtain a complete calculation of all cohomology groups $H_c^i(F_{<k}(X, n), \mathbf{Q})$ with their decompositions into irreducible representations of $\mathbb{S}_n$.

COROLLARY 7.18. *Let X be an i-acyclic space over $\mathbf{Q}$, and let $h(t) = \sum_i (-t)^i \dim_{\mathbf{Q}} H_c^i(X, \mathbf{Q})$. Then there is an equality*

$$\mathsf{E} \circ (h(t) \cdot (s_1 + t^{-1}\mathsf{L} \circ \mathsf{S}_k)) = \operatorname{ch} \mathbf{F}_{<k}(X)$$

in $\mathcal{R}$.

Proof. Apply the operator ch to Proposition 7.8, and use the fact that the composition product of symmetric sequences is mapped to plethysm of symmetric functions. Then plug in the calculation of $\operatorname{ch} \mathbf{P}_k$ given by Theorem 7.16, and the obvious equalities $\operatorname{ch} \mathbf{E} = \mathsf{E}$ and $\operatorname{ch} H_c^\bullet(X, \mathbf{Q}) = h(t)$. □

7.19 When $k = 2$ (i.e. in the case of the usual configuration spaces of points) the above formula can be simplified, since there is a simpler formula for $\operatorname{ch} \mathbf{P}_2$. Namely, one has

$$s_1 + t^{-1}\mathsf{L} \circ \mathsf{S}_2 = \sum_{n \geqslant 1} \frac{t^{n-1}}{n} \sum_{d|n} (-1)^{n/d-1} \mu(d) p_d^{n/d}.$$

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Remark 7.20. If X is not assumed to be i-acyclic, then the formula $\mathsf{E}\circ(h(t)\cdot(s_1+t^{-1}\mathsf{L}\circ\mathsf{S}_k))$ still computes the characteristic of the E_1 page of the spectral sequence calculating $H_c^\bullet(F_{<k}(X,n),\mathbf{Q})$. If we suppose that $H_c^\bullet(X,\mathbf{Q})$ is finite-dimensional (not just degreewise finite-dimensional), then we may take the Euler characteristic of both sides (i.e. set $t=1$) to obtain a generating function for the $\mathbb{S}_n$-equivariant compactly supported Euler characteristic of $F_{<k}(X,n)$:

$$\mathsf{E}\circ\left(\chi_c(X)\cdot\left(s_1+\mathsf{L}\circ\sum_{n\geqslant k}(-1)^{n-k+1}s_{n-k+1,1^{k-1}}\right)\right)=\sum_{n\geqslant 0}\chi_c^{\mathbb{S}_n}(F_{<k}(X,n)),$$

where this is now an equality in $\widehat{\Lambda}$. If X is a complex algebraic variety we may replace $\chi_c(X)$ with the Hodge–Deligne polynomial of X, or even better the class $\sum_i(-1)^i[H_c^i(X,\mathbf{Q})]$ in the Grothendieck ring $K_0(\mathsf{MHS}_{\mathbf{Q}})$ of rational mixed Hodge structures. In that case the above formula may be considered as an equality in the completion of $\Lambda\otimes K_0(\mathsf{MHS}_{\mathbf{Q}})$, valid for any complex algebraic variety. When $k=2$ these generating series for Hodge–Deligne polynomials were previously obtained by Getzler [Get99b].

8. Chevalley–Eilenberg homology of twisted Lie algebras

8.1 The goal of this section is to explain that in the case of the usual configuration space $F(X,n)$, our functor $\mathcal{CF}$ can be reinterpreted in terms of operadic cohomology of twisted Lie algebras (left modules over the Lie operad).

8.2 Let A be a commutative algebra and $\mathfrak{g}$ a Lie algebra. Then the tensor product $A\otimes\mathfrak{g}$ is again a Lie algebra. If $\mathfrak{g}$ is instead a twisted dg Lie algebra (a left module over the Lie operad) and A is a twisted commutative dg algebra, then their so-called Hadamard tensor product, defined by

$$(A\otimes_H\mathfrak{g})(n)=A(n)\otimes\mathfrak{g}(n),$$

is again a twisted Lie algebra. Equivalently, it can be considered as a Lie algebra in the symmetric monoidal category of symmetric sequences. As such, we can consider its Chevalley–Eilenberg (co)homology, which will itself be a symmetric sequence (equivalently, a sequence of representations of $\mathbb{S}_n$). We will find it more natural to work with Lie algebra *homology*, so in this section we will change to homological grading via usual the convention $C^i=C_{-i}$ for $i\in\mathbf{Z}$, where $C^\bullet$ is any cochain complex. We denote the complex of Chevalley–Eilenberg chains by $C_\bullet^{\mathrm{CE}}(-)$.

8.3 We will also need the *suspension* operation on twisted commutative algebras. If A is a twisted commutative dg algebra, then we define

$$\mathsf{S}A(n)=A(n)[-n]\otimes\mathrm{sgn}_n.$$

Then $\mathsf{S}A$ is itself in a natural way a tcdga (indeed, S is a symmetric monoidal endofunctor on the category of symmetric sequences).

Theorem 8.4. *Let A be a tcdga, and $\mathsf{S}A$ its suspension. Let $U=\Pi_n\setminus\{\hat{0}\}$. Then $\mathcal{CF}(U,A)\simeq C_\bullet^{\mathrm{CE}}(\mathsf{S}A\otimes_H\mathsf{Lie})(n)$, where the operad Lie is considered as a left module over itself.*

Proof. We need to carefully piece together several observations.

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(i) The complex of cellular cochains $\widetilde{C}^\bullet([\![\Pi_n]\!], \mathbf{Z})$ is isomorphic to the arity-n component of $\Omega\mathsf{coCom}$, the cobar construction on the cocommutative co-operad. See [Fre04, Prolog and Observation 6.1] for a large number of bibliographical references concerning variations of this result.

(ii) The cobar construction $\Omega\mathsf{coCom}$ is quasi-isomorphic to the operadic suspension $\Lambda\mathsf{Lie}$ of the Lie operad, by Koszul duality between the commutative and Lie operads.

(iii) The Chevalley–Eilenberg chains on a twisted Lie algebra $\mathfrak{g}$ are given by the cofree left comodule over the cocommutative cooperad on $\mathfrak{g}[1]$, equipped with the Chevalley–Eilenberg differential. In particular the underlying symmetric sequence of $C^{\mathrm{CE}}_\bullet(\mathfrak{g})$ is given by $\mathsf{coCom} \circ \mathfrak{g}[1]$. Since $\mathsf{coCom}(n)$ is the trivial representation of $\mathbb{S}_n$, the composition product $\mathsf{coCom} \circ \mathfrak{g}[1]$ is exactly the 'exponential' considered in Remark 7.7.

(iv) The complex $\mathcal{CF}(U, A)$ can be written as a direct sum indexed by partitions $T \in \Pi_n$, where the summand corresponding to a partition $\{1, \dots, n\} = \coprod_{i \in I} T_i$ is given by the tensor product $\bigotimes_{i \in I} A(T_i) \otimes \widetilde{C}^\bullet([\![\Pi_{T_i}]\!])$. Indeed, $\mathcal{CF}$ is a sum over chains in the partition lattice, and we can decompose the complex $\mathcal{CF}$ according to the maximal element of the chain (which is what we denoted by T in the previous sentence). Comparing again with Remark 7.7, we see that this is the arity-n component of $\mathsf{coCom} \circ (A \otimes_H \Omega\mathsf{coCom}) \simeq \mathsf{coCom} \circ (A \otimes_H \Lambda\mathsf{Lie})$.

We should get the suspensions right. We have $\Lambda\mathsf{Lie}(n) = \mathsf{Lie}(n)[-n+1] \otimes \mathrm{sgn}_n$. It follows that $\mathsf{coCom} \circ (A \otimes_H \Lambda\mathsf{Lie}) = \mathsf{coCom} \circ (\mathsf{S}A \otimes_H \mathsf{Lie})[1] = C^{\mathrm{CE}}_\bullet(\mathsf{S}A \otimes_H \mathsf{Lie})$. We omit the verification that the differential in the complex $\mathcal{CF}$ can be identified with the Chevalley–Eilenberg differential (it is harder to write down than to derive). □

8.5 Let us now take $A = R\Gamma^\otimes_c(X, \mathcal{F})$, in which case $\mathsf{S}A = R\Gamma^\otimes_c(X, \mathcal{F}[-1])$. Theorem 8.4 combined with Theorem 5.12 gives our next result.

COROLLARY 8.6. *Let X be a paracompact and locally compact Hausdorff space, and $\mathcal{F}$ a bounded-below complex of sheaves of R-modules on X, where R is a ring of finite global dimension. Then*

$$H^{\mathrm{CE}}_\bullet(R\Gamma^\otimes_c(X, \mathcal{F}[-1]) \otimes_H \mathsf{Lie}) \cong \bigoplus_{n \geqslant 0} H_c^{-\bullet}(F(X, n), \mathcal{F}^{\boxtimes n}).$$

Remark 8.7. The proof actually gives a slightly stronger chain-level result. Consider on the one hand the twisted dg Lie algebra $R\Gamma^\otimes_c(X, \mathcal{F}[-1]) \otimes_H \mathsf{Lie}$, and on the other hand the twisted dg cocommutative coalgebra whose arity-n component is given by $C^\bullet_c(F(X, n), \mathcal{F}^{\boxtimes n})$. Its comultiplication is the map

$$C^\bullet_c(F(X, n+m), \mathcal{F}^{\boxtimes(n+m)}) \longrightarrow C^\bullet_c(F(X, n), \mathcal{F}^{\boxtimes n}) \otimes C^\bullet_c(F(X, m), \mathcal{F}^{\boxtimes m})$$

given by extension by zero, noting that $F(X, n+m)$ is an open subspace of $F(X, n) \times F(X, m)$. The claim is that these algebras are dual to each other under Koszul duality.

COROLLARY 8.8. *Let X be a paracompact and locally compact Hausdorff space. If A denotes a cdga model for the compactly supported cochains on X with $\mathbf{Q}$-coefficients, then*

$$H^{\mathrm{CE}}_\bullet(A \otimes \mathsf{SLie}) \cong \bigoplus_{n \geqslant 0} H_c^{-\bullet}(F(X, n), \mathbf{Q}).$$

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Corollary 8.9. *Let X be a manifold of dimension d (possibly with boundary), $\mathcal{L}$ the orientation sheaf on X. Then*

$$H_\bullet^{\mathrm{CE}}(R\Gamma_c^{\otimes}(X, \mathcal{L}[d-1]) \otimes_H \mathsf{Lie}) \cong \bigoplus_{n \geqslant 0} H_\bullet(F(X,n), \mathbf{Z}).$$

Proof. The homology of any space X is canonically isomorphic to the compactly supported cohomology of its dualizing complex $\mathbb{D}\mathbf{Z}$. When X is a manifold, its dualizing complex is $\mathcal{L}[\dim X]$, the orientation sheaf shifted into degree $\dim X$. □

8.10 Corollary 8.6 gives a generalization of results of Knudsen [Knu17] and Hô [Hô17]. Namely, if we specialize Corollary 8.6 to the case where X is a manifold, $\mathcal{F}$ is the dualizing complex of X (i.e. the orientation sheaf of X placed in cohomological degree $\dim X$) tensored with $\mathbf{Q}$, and we take $\mathbb{S}_n$-invariants in each arity, then we recover exactly the main theorem of [Knu17]. The main result of [Hô17] is the analogous statement in the étale cohomology of an algebraic variety X with $\mathbf{Q}_\ell$-coefficients, and the sheaf-theoretic methods used here can treat this case as well, as indicated in Remark 5.17. Some statements closely related to Corollary 8.6 can already be found in the literature. In an unpublished preprint, Getzler [Get99a, p. 4] states an isomorphism $H_\bullet^{\mathrm{CE}}(\Omega_c^\bullet(M) \otimes \mathsf{Lie}) \cong \bigoplus_{n \geqslant 0} H_c^\bullet(F(M,n), \mathbf{R})[-n] \otimes \mathrm{sgn}_n$, where M is an oriented manifold and $\Omega_c^\bullet(M)$ is the compactly supported de Rham complex. (The formula in [Get99a, p. 4] is stated without compact supports on both sides, but the above formula is presumably what is intended.)

One can also extract a version of Corollary 8.6 from computations in Goodwillie calculus. If X is a based space, let $\bar{F}(X,n) = X^{\wedge n}/D_n$, where D_n is the 'big diagonal' inside the smash product. Now note that if X is the one-point compactification of a space Y, then $\bar{F}(X,n)$ is the one-point compactification of $F(Y,n)$. In particular, the reduced cohomology of $\bar{F}(X,n)$ is the compactly supported cohomology of $F(Y,n)$. We now need the following inputs from Goodwillie calculus (see [Aro99, Chi05] and in particular [AC11, Example 17.28]).

- Let $F\colon \mathsf{Top}_* \to \mathsf{Top}_*$ be a pointed homotopy functor. Then its derivatives $\partial_* F$ form a left module over $\partial_* I_{\mathsf{Top}_*}$, the 'spectral Lie operad'.
- If we instead consider $G\colon \mathsf{Top}_* \to \mathsf{Spectra}$, then $\partial_* G$ is a left module over $\partial_*(\Sigma^\infty \Omega^\infty)$, a spectral version of the commutative operad.
- If $G = \Sigma^\infty F$, then $\partial_* G$ and $\partial_* F$ are Koszul dual to each other.
- If $F = \mathrm{map}(X, -)$, then its nth derivative is given by $X \wedge \partial_n I_{\mathsf{Top}_*}$.
- Let X be a finite complex. The nth derivative of $\Sigma^\infty \mathrm{map}(X, -)$ is given by $\mathbb{D}\bar{F}(X,n)$, where $\mathbb{D}$ denotes the Spanier–Whitehead dual.

Putting it all together, we see that we obtain a 'spectral' version of the Koszul duality statement of Remark 8.7. Taking cohomology recovers Remark 8.7, but only in the case of the constant sheaf $\mathbf{Z}$. The statement obtained from Goodwillie calculus is neither more nor less general than the one here: the spectral version allows one to work with an arbitrary cohomology theory, but the sheafy version allows one, for example, to plug in the dualizing complex of X. It seems likely that there exists a six-functor formalism for sheaves of spectra over spaces and that the methods of this paper would work equally well in such a setting to prove a statement which specializes to both formulae, but no such formalism exists in the literature.

Remark 8.11. Here is a simple example of the kind of result one could prove using a hypothetical formalism of six functors for sheaves of spectra. If we take for $\mathcal{F}$ in Corollary 8.6 the dualizing complex then on the right-hand side we should get the stable homotopy types of the configuration

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spaces of points on X. The theorem would therefore answer the question of how much more information than the homotopy type of X one needs to determine the stable homotopy type of the configuration space of points on X: one needs also to know the dualizing complex. For example, when X is a manifold, then the dualizing complex should be the parametrized Thom spectrum given by the tangent bundle (this would be a form of Atiyah duality), and since the stable tangent bundle is a homotopy invariant of compact manifolds one would deduce that the stable homotopy type of configuration spaces of points on a compact manifold M depends only on the homotopy type of M, recovering a theorem of Aouina and Klein [AK04]. Knudsen [Knu18, Theorem C] gave a proof of the Aouina–Klein result in a slightly stronger form (for one thing, Knudsen's result is equivariant) using factorization homology, which is at least morally the 'same' proof as the one suggested in this remark (in the manifold case).

8.12 Let us explain how to recover the results of Knudsen and Hô from Corollary 8.6. Suppose we work over $\mathbf{Q}$. Then there is an *exact* symmetric monoidal functor

$$(\text{symmetric sequences of chain complexes}) \xrightarrow{\text{inv}} (\text{chain complexes with an extra } \mathbf{N}\text{-grading})$$

given by taking $\mathbb{S}_n$-invariants in each arity. If $\mathfrak{g}$ is a twisted dg Lie algebra, then $\operatorname{inv}\mathfrak{g}$ is a Lie algebra in the category of $\mathbf{N}$-graded dg $\mathbf{Q}$-vector spaces, and since inv is exact it commutes with taking Lie algebra cohomology: $\operatorname{inv} H^{\mathrm{CE}}_\bullet(\mathfrak{g}) \cong H^{\mathrm{CE}}_\bullet(\operatorname{inv}\mathfrak{g})$. In particular we get the following proposition.

Proposition 8.13. *Let X be a locally compact Hausdorff space, and let $\mathbb{D}\mathbf{Q}$ be the dualizing complex of X with rational coefficients. Then*

$$H^{\mathrm{CE}}_\bullet(\operatorname{inv} R\Gamma^\otimes_c(X, \mathbb{D}\mathbf{Q}[-1]) \otimes_H \mathsf{Lie}) \cong \bigoplus_{n\geqslant 0} H_\bullet(F(X,n)/\mathbb{S}_n, \mathbf{Q}).$$

8.14 Now the main result of [Knu17] is the description of an explicit $\mathbf{N}$-graded Lie algebra over $\mathbf{Q}$ associated to a manifold X, whose homology is given by the direct sum $\bigoplus_{n\geqslant 0} H_\bullet(F(X,n)/\mathbb{S}_n, \mathbf{Q})$. What we must show is therefore that $\operatorname{inv} R\Gamma^\otimes_c(X, \mathbb{D}\mathbf{Q}[-1]) \otimes_H \mathsf{Lie}$ is isomorphic to this $\mathbf{N}$-graded Lie algebra when X is a manifold. We will need two observations.

- Suppose the complex $\mathcal{F}$ consists of a single sheaf placed in even degree. Then the cohomology of $R\Gamma^\otimes_c(X, \mathcal{F})(n)$ consists only of the trivial representation of $\mathbb{S}_n$. If instead $\mathcal{F}$ is concentrated in odd degree, then the cohomology of $R\Gamma^\otimes_c(X, \mathcal{F})(n)$ transforms via the sign representation of $\mathbb{S}_n$.
- The trivial representation occurs in $\mathsf{Lie}(n)$ only for $n = 1$, where $\mathsf{Lie}(1) = \mathbf{Q}$ is the trivial representation. The sign representation of $\mathbb{S}_n$ occurs only for $n = 1$ and $n = 2$, where $\mathsf{Lie}(1)$ and $\mathsf{Lie}(2)$ are both given by a copy of the sign representation.

So let X be a d-dimensional manifold. We see from the above considerations that if d is odd then $\operatorname{inv} R\Gamma^\otimes_c(X, \mathcal{L}[d-1]) \otimes_H \mathsf{Lie}$ is concentrated in degree 1, where it is given by $H^\bullet_c(X, \mathcal{L}[d-1]) \cong H_{\bullet+1}(X, \mathbf{Q})$ (with identically zero Lie bracket). If d is even then $\operatorname{inv} R\Gamma^\otimes_c(X, \mathcal{L}[d-1]) \otimes_H \mathsf{Lie}$ is concentrated in degrees 1 and 2. Its degree-1 component is $C^\bullet_c(X, \mathcal{L}[d-1])$, and its degree-2 component is $C^\bullet_c(X, \mathcal{L}[d-1]^{\otimes 2}) \cong C^{\bullet+2d-2}_c(X, \mathbf{Q})$, which has an evident two-step nilpotent Lie bracket given by the cup product. Moreover, this Lie algebra is formal, by the argument of [Knu17, Remark 7.7], and hence we can take the components to be given by the cohomologies $H^\bullet_c(X, \mathcal{L})$ and $H^\bullet_c(X, \mathbf{Q})$ (appropriately shifted). We have recovered exactly the Lie algebra defined by Knudsen.

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