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TRANSFINITE NUMBERS IN PARACONSISTENT SET THEORY

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Abstract. This paper begins an axiomatic development of naive set theory—the consequences of a full comprehension principle—in a paraconsistent logic. Results divide into two sorts. There is classical recapture, where the main theorems of ordinal and Peano arithmetic are proved, showing that naive set theory can provide a foundation for standard mathematics. Then there are major extensions, including proofs of the famous paradoxes and the axiom of choice (in the form of the well-ordering principle). At the end I indicate how later developments of cardinal numbers will lead to Cantor's theorem, the existence of large cardinals, and a counterexample to the continuum hypothesis.

§1. Introduction. The axioms of *naive set theory* define sets through existence and uniqueness conditions. The first axiom is the principle of comprehension, that any collection of objects is a set, and is itself an object. The second is the principle of extensionality, that the members of a set completely determine the identity of that set. On the basis of these principles alone, the mathematics of set theory can be developed, as well as some novel inconsistent mathematics, as this article begins to demonstrate.

The comprehension principle is inconsistent. In the presence of classical logic, it is outright incoherent. Since Russell told this to Frege, it has been the cause of some surprise and consternation. The most prevalent response has been to maintain classical logic, and to adopt Zermelo's (1908) selection of some instances of the comprehension principle. This is not the only possible response. Classical logic makes comprehension absurd; the consternation, often tracked by using the word 'paradox', is due to the deep sense that comprehension is not absurd. Others have tried to maintain comprehension, and adopt, say, Routley and Meyer's (1976) selection of logical axioms with which to reason about it.

The resulting theory has several important features. First, it contains the basic theorems of standard ZFC set theory, initially proving the ZF axioms and building to the theory of ordinals and Peano postulates. This has been called the *classical recapture*, and has proved startlingly elusive until now. The demands placed on the underlying logic are simply very stringent. Speaking to this point, the usually optimistic authors of Meyer *et al.* (1978) conclude in dramatic fashion that "naive set theory is untenable" (p. 128). Priest summarizes the situation:

> Since the early days of paraconsistent logic, it has been clear that the rejection of *ex contradictione [quodlibet]* is not possible without the rejection of other things which appear to be much more integral to classical reasoning.... Several logicians (including Brady, Meyer, Mortensen,

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> Priest and Routley) have attempted to reconstruct various fragments of classical reasoning.... While the results are not definitive, they are not terribly encouraging. Though ... a highly important—indeed, essential—enterprise, it would now appear that the aim of reconstructing sensible classical reasoning in this way may not be realizable (Priest, 2006, p. 222).

This paper begins to show that the goal is realizable. Recapture of higher theories goes as far as showing the Peano axioms hold. Not all classical consequences of, for example, Peano's axioms are paraconsistent consequences—at least not automatically—so the issue of what can be proved from here is still open. Classical recapture even to this modest point, though, shows that paraconsistent set theory is recognizably set theory; it shows, a fortiori, that the assumption of consistency in basic set theory is unnecessary.

Second, paraconsistent set theory can settle some disputes of its classical counterpart. For a start, the infamous paradoxes are just theorems. The existence of various intuitive collections, such as the set of all sets and and the set of all ordinals, is immediate; so the theory allows us to work with objects that have elsewhere been called 'proper classes' and have remained rather philosophically and mathematically mysterious. With these, a proof of Cantor's well-ordering principle, and so Zermelo's axiom of choice, can be given. While cardinal numbers are not treated here, I do indicate at the end some straightforward arguments that would establish, again from the set concept alone, the existence of large cardinals and a decision on the generalized continuum hypothesis.

Finally, the theory begins to address new mathematical questions that arise in coherent inconsistent settings, showing that objects beyond the consistent are mathematically rich. I make no attempt to interpret or philosophically defend at any length the results. I make only an uncontroversial claim to rigor: If one draws out consequences of the naive set concept in a first-order paraconsistent logic, then this is some of what one will find.

§2. Logic. A paraconsistent logic is one in which the inference $\Phi, \neg\Phi \vdash \Psi$, called *explosion*, does not hold for every sentence Φ, Ψ. There are many paraconsistent logics; but most of them are not appropriate for treating naive set theory. Comprehension is an extremely powerful principle and can only be channeled by weak logics. Let us briefly consider some of the constraints, leading to the present choice of logic.

Prominently, any logic with contraction $(\Phi \rightarrow (\Phi \rightarrow \Psi)) \rightarrow (\Phi \rightarrow \Psi)$, or its mate, axiomatic or *pseudo*-modus ponens $\Phi \wedge (\Phi \rightarrow \Psi) \rightarrow \Psi$, will explode in the presence of naive comprehension (Meyer *et al.*, 1978). Further, almost no paraconsistent logic can include the disjunctive syllogism; then any logic without a non-extensional implication connective is all but useless, since we are actually interested in proving and detaching theorems. These considerations eliminate da Costa's C systems and Priest's LP as suitable background logics for a robust set theory (although there are some interesting results from both, e.g., da Costa (2000) in Batens *et al.* (2000) and Restall (1992), respectively).

For the purposes of set theory, we also want a logic that respects some natural behaviors of sets. For example, given that two sets with all and only the same members are identical, it should also be the case that sets with different members are not identical. Or again, when $y = \{z : \Phi(z)\}$ and $\neg\Phi(x)$, then it should be that $x \notin y$. These are fairly strong requirements on negation. When formalized (e.g., with a counterexample Axiom VIII

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below), these demands mean that set negation excludes the middle, tertium non datur: $\Phi \vee \neg\Phi$ for any sentence Φ. This is to be expected. An exhaustive universe is explicitly postulated by Cantor and Frege; it is the cause of inconsistency in comprehension to begin with. In light of the noted duality between paraconsistent and paracomplete (i.e., partial or intuitionistic) systems, it is at least reasonable to investigate the mathematics of a gap-free theory.

Essentially, we want the strongest logic possible that does not explode when given a comprehension principle. Once all these are tailored and formalized, we obtain one of the original tools fashioned for the job. Let us have the language of first-order set theory: with primitives $\wedge, \neg, \rightarrow, \forall, =$, and $\in$; variables $x, y, z, \ldots$; names $a, b, c, \ldots$; and formulae $\Phi, \Psi, \Upsilon, \ldots$, built up by standard formation rules. The usual shorthand[1] is used: $\Phi \vee \Psi$ for $\neg(\neg\Phi \wedge \neg\Psi)$; $\Phi \leftrightarrow \Psi$ for $(\Phi \rightarrow \Psi) \wedge (\Psi \rightarrow \Phi)$; $\exists$ is $\neg\forall\neg$.

2.1. Axioms. All instances of the following schemata are theorems:

I	$\Phi \rightarrow \Phi$	
IIa	$\Phi \wedge \Psi \rightarrow \Phi$	
IIb	$\Phi \wedge \Psi \rightarrow \Psi$	
III	$\Phi \wedge (\Psi \vee \Upsilon) \rightarrow (\Phi \wedge \Psi) \vee (\Phi \wedge \Upsilon)$	(*distribution*)
IV	$(\Phi \rightarrow \Psi) \wedge (\Psi \rightarrow \Upsilon) \rightarrow (\Phi \rightarrow \Upsilon)$	*(conjunctive syllogism)*
V	$(\Phi \rightarrow \Psi) \wedge (\Phi \rightarrow \Upsilon) \rightarrow (\Phi \rightarrow \Psi \wedge \Upsilon)$	
VI	$(\Phi \rightarrow \neg\Psi) \rightarrow (\Psi \rightarrow \neg\Phi)$	(*contraposition*)
VII	$\neg\neg\Psi \rightarrow \Psi$	*(double negation elimination)*
VIII	$(\Phi \rightarrow \Psi) \rightarrow \neg(\Phi \wedge \neg\Psi)$	(*counterexample*)
IXa	$(\Phi \rightarrow \Psi) \rightarrow [(\Psi \rightarrow \Upsilon) \rightarrow (\Phi \rightarrow \Upsilon)]$	
IXb	$(\Phi \rightarrow \Psi) \rightarrow [(\Upsilon \rightarrow \Phi) \rightarrow (\Upsilon \rightarrow \Psi)]$	*(hypothetical syllogisms)*
X	$\forall x\Phi \rightarrow \Phi(y/x)$	
XI	$\forall x(\Phi \rightarrow \Psi) \rightarrow (\Phi \rightarrow \forall x\Psi)$	
XII	$\forall x(\Phi \vee \Psi) \rightarrow \Phi \vee \forall x\Psi$.	

In Axiom X, y is free for x in Φ. In Axioms XI and XII, x is not free in Φ.

The hypothetical syllogism pair IXa and IXb are called *suffixing* and *prefixing*, respectively. Without these we have the weak relevant logic DLQ, *dialethic logic with quantifiers*, introduced in Routley & Meyer (1976) as 'dialectical' logic. Adding them amplifies the logic to TLQ.

2.2. Rules. The following rules are valid:

I	$\Phi, \Psi \vdash \Phi \wedge \Psi$	(*adjunction*)
II	$\Phi, \Phi \rightarrow \Psi \vdash \Psi$	*(modus ponens)*
III	$\Phi \rightarrow \Psi, \Upsilon \rightarrow \Delta \vdash (\Psi \rightarrow \Upsilon) \rightarrow (\Phi \rightarrow \Delta)$	
IV	$\Phi \vdash \forall x\Phi$	
V	$x = y \vdash \Phi(x) \rightarrow \Phi(y)$	(*substitution*).

Modus ponens is also valid in the single premise form: $\Phi \wedge (\Phi \rightarrow \Psi) \vdash \Psi$, not to be mistaken for the illegitimate axiom form. Substitution, similarly, is only valid in rule form. Given Axioms IXa and IXb, the hypothetical syllogism pair, then Rule III is actually redundant because it is derivable. There is no harm in keeping it explicit, though.

[1] Taking these as definitions means that , for example, $\Phi \vee \Psi \rightarrow \neg(\neg\Phi \wedge \neg\Psi)$ is no more than an instance of Axiom I below.

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Brady proves that naive set theory in the closely related logics DKQ and TKQ is non-trivial (Brady, 2006, p. 242), in the sense that it has a model in which some sentences fail. Brady's work begins in Brady (1971) and he and Routley collaborated on the proof in Brady & Routley (1989). Routley used DKQ in his **?**, calling the set theory DST, for dialectical set theory, which in present terminology would be *dialethic set theory*. The logic DLQ is the same as DKQ, except that $\Phi \vee \neg\Phi$ is strengthened to the counterexample axiom; and TLQ, again, adds the hypothetical syllogisms. I have found DKQ to be too weak to obtain satisfactory results, namely, the classical recapture. Non-triviality of naive set theory in DLQ and TKQ is an open problem. I conjecture that DLQ is the weakest logic of its kind that can support any robust mathematics.

The following derived facts about the logic will be most helpful.

Double negation introduction follows from Axioms I and VI, and modus ponens: $(\neg\Phi \to \neg\Phi) \to (\Phi \to \neg\neg\Phi)$. Then the contraposition Axiom VI can be rearranged to $(\Phi \to \Psi) \to (\neg\Psi \to \neg\Phi)$. Counterexample gets its name from the contraposed form $\Phi \wedge \neg\Psi \to \neg(\Phi \to \Psi)$. From the instance of counterexample $(\Phi \to \Phi) \to \neg(\Phi \wedge \neg\Phi)$, by Axiom I and modus ponens we have the law of non-contradiction: $\neg(\Phi \wedge \neg\Phi)$. Then by the definition of disjunction follows the law of excluded middle, $\Phi \vee \neg\Phi$. Contraposition on Axiom V gives a schema for $\vee$-elimination: $(\Phi \to \Psi) \wedge (\Upsilon \to \Psi) \to (\Phi \vee \Upsilon \to \Psi)$. Then the pair $(\Phi \to \neg\Phi) \to \neg\Phi$ and $(\neg\Phi \to \Phi) \to \Phi$, reductio and consequentia mirabilis, are theorems. We have the scheme for existential instantiation, $\forall x(\Phi \to \Psi) \to \exists x\Phi \to \Psi$, from Axiom XI, contraposition, and the definition of $\exists$, and subject to the condition that x is not free in Ψ.

A useful derived rule is

$$\Phi \to (\Psi \to \Upsilon), \Upsilon \to \Delta \vdash \Phi \to (\Psi \to \Delta).$$

The rule holds because $\Upsilon \to \Delta$ implies $[\Phi \to (\Psi \to \Upsilon)] \to [\Phi \to (\Psi \to \Delta)]$ by two applications of prefixing; and then the conclusion follows by modus ponens. Similarly, an extra application of prefixing gives the extended

$$\Phi \to (\Psi \to (\Upsilon \to \Gamma), \Gamma \to \Delta \vdash \Phi \to (\Psi \to (\Upsilon \to \Delta)),$$

and so forth. These are called *derived prefixing rules*.

Like other relevant logics in its area, the deduction theorem fails, exactly because a demonstration that Ψ is provable from Φ is not sufficient to show that Φ relevantly implies Ψ; probably a good deal more information than Φ alone is needed to get Ψ. So the difference between deductions marked by $\vdash$ and implications marked by $\to$ are stark. While the move from $\Phi \to \Psi$ to $\Phi \vdash \Psi$ is modus ponens, the other direction does not follow. In general, $\to$ carries the burden of relevance and theorems containing it are harder to obtain. To this end we have one.

2.3. Meta-rule. If $\Phi \vdash \Psi$, then $\Phi \vee \Upsilon \vdash \Psi \vee \Upsilon$.

The idea of meta-rules is Meyer's (Brady, 2006, p. 6). Importantly, the meta-rule softens the loss of the deduction theorem; in particular, it validates the following argument:

$$\text{If } \Phi \vdash \Psi, \text{ and } \Upsilon \vdash \Psi, \text{ then } \Phi \vee \Upsilon \vdash \Psi.$$

This is useful for disjunction eliminations when reasoning under hypothesis. We also obtain the useful substitution form, $a = b, \Phi(a) \vdash \Phi(b)$. Another simple derived rule is: $\Phi \wedge$

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$(\Phi \rightarrow \Psi) \vdash \Psi$, and its quantificational counterpart, $\Phi(a) \wedge \forall x(\Phi(x) \rightarrow \Psi(x)) \vdash \Psi(a)$. So although pseudo-modus ponens is not included, using the meta-rule we can still reason fairly naturally even under hypothesis.

Perhaps the most difficult challenge to set theory in relevant logic is formalizing *restricted quantification*.[2] There seem to be many cases in which we want to say that all Φs are Ψs, without meaning or being able to prove the entailment that $\forall x(\Phi(x) \rightarrow \Psi(x))$. A weaker statement is the inference $\Phi(x) \vdash \Psi(x)$, but this is not part of the object language, so it cannot be used to define predicates. Formalizing restricted quantification in a useful way is a major challenge and is, ultimately, not addressed in the present work. For now, following again Routley's original presentation of dialethic set theory, we recognize the problem by use of the t constant, to be thought of as the conjunction of all theorems, and added conservatively through a two-way rule[3]:

2.4. t-rule. $\Phi \dashv\vdash t \rightarrow \Phi$.

Then the following enthymematic conditional is defined:

$$\Phi \mapsto \Psi := \Phi \wedge t \rightarrow \Psi.$$

The main technical behavior of $\mapsto$ is to validate $\Phi \vdash \Psi \mapsto \Phi$, and its quantificational analogue. Thus $\mapsto$ is obtainable sometimes when $\rightarrow$ is not (and always obtainable when $\rightarrow$ is), but does not contrapose.[4]

§3. Axioms. Our first-order formal language is now augmented with a variable binding term forming operator $\{\cdot : -\}$; it remains open how to conservatively add term-forming symbols in relevance contexts (Brady, 2006, p. 177), and is not a problem addressed here. The set concept is now characterized by two axioms.

AXIOM 3.1 (**Abstraction**). $x \in \{z : \Phi(z)\} \leftrightarrow \Phi(x)$.

AXIOM 3.2 (**Extensionality**). $\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y$.

Existential generalization on abstraction immediately yields the comprehension principle[5]:

THEOREM 3.3 (**Comprehension**). $\exists y \forall x(x \in y \leftrightarrow \Phi(x))$.

Under abstraction, the substitution rule is $x = y \vdash \forall z(x \in z \rightarrow y \in z)$.

Abstraction and extensionality can be reconnected, as in Frege's axiom:

THEOREM 3.4 (**Basic law V**). $\{x : \Phi\} = \{x : \Psi\} \leftrightarrow \forall x(\Phi \leftrightarrow \Psi)$.

Peano chose the $\in$ sign to denote predication, from the Greek verb $\epsilon\sigma\tau\iota\nu$, 'to be'. Sets are predicates in extension. Since arbitrary predicates determine sets, then, in the comprehension principle the occurrence of y in the predicate Φ is not ruled out. Following Routley, this is completely unrestricted comprehension; without this, some sets would not be obtainable, for example, the limiting case of diagonal sets, $Z = \{x : x \notin Z\}$.

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[2] One natural approach is outlined in Brady (2003, p. 327).

[3] The constant is added conservatively to the *logic*. Whether it is also conservative over the set theory is unknown.

[4] See Mares (2004, pp. 198–200), Priest (2006, p. 255), Beall *et al.* (2006), Asmus (2009), Weber (forthcoming-a).

[5] The terminological distinction between abstraction and comprehension I take from Libert (2005).

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To guarantee that such instances are valid abstractions—to ensure that every predicate, even groundless ones, determines a set—we have abstraction instances of the form

$$x \in \{zu : \Phi(z, u)\} \leftrightarrow \Phi(z/x, u/\{zu : \Phi(z, u)\})$$

where the right-hand side indicates a simultaneous substitution in Φ of z by x, and u by the term $\{zu : \Phi(z, u)\}$. (At first, in Brady & Routley (1989, p. 419), a new quantifier, formation rule, and reflection axiom were added to handle circular predicates; but by Brady (2006, pp. 177, 200), the idea is streamlined as above.) Axiom 3.1 in this way includes cases

$$x \in \{zu : \Phi(z, u)\} \leftrightarrow \Phi(x, \{zu : \Phi(z, u)\}).$$

We can then say 'a set y of all x such that $\ldots x \ldots y \ldots$', as opposed to the usual 'the set of all x such that $\ldots x \ldots$'.

Naive set theory is studied here with the full comprehension scheme for three reasons. The first has already been stated, namely, the philosophical conviction that all predicates, even baroque predicates, determine sets.[6] Note that the liar paradox has a similar kind of self-reference. Second is a pragmatic motive: Since weak relevant logics are so terribly weak, the proving power for theorems must come from somewhere. Unrestricted comprehension will allow easier construction of functions, diagonal sets, and other useful objects. The third motive, then, is to model easily some venerable fixed-point phenomena in set theory; see , for example, our recursive characterization of the ordinals and natural numbers below. Uniqueness is not essential to these cases, as there are non-standard models in the classical account anyway.

It is frequently convenient to use names for sets, for example, for a set a there is a set of all the subsets of a, called the powerset of a. The symbol $P(a)$ is used to denote this set in the same way that $\emptyset$ is used to denote a particular set below; it is a name. So, to use a bit of notation to be introduced below, $x \in P(a)$ is just $x \in \{y : y \subseteq a\}$. Similar comments apply to complementation $\overline{a}$, intersection, union, and so forth. This notation can be thought of as governed by instances of comprehension:

$$\langle x_0, \ldots, x_n \rangle \in y \leftrightarrow \Phi(x_0, \ldots, x_n).$$

Since in later sections we will officially develop ordered pairs and the natural numbers to act as indices, their use as primitives here is only heuristic.

§4. Basics.

PROPOSITION 4.1. $y = \{z : \Phi(z)\} \leftrightarrow \forall x(x \in y \leftrightarrow \Phi(x))$.

Proof. By extensionality, $y = \{z : \Phi(z)\} \leftrightarrow \forall x(x \in y \leftrightarrow x \in \{z : \Phi(z)\}$. By abstraction, $\forall x(x \in \{z : \Phi(z)\} \leftrightarrow \Phi(x))$. Then by transitivity of the conditional, $\forall x(x \in y \leftrightarrow \Phi(x))$. For the converse, we again invoke the abstraction scheme, where $\forall x(\Phi(x) \leftrightarrow x \in \{z : \Phi(z)\}$, so by transitivity $\forall x(x \in y \leftrightarrow x \in \{z : \Phi(z)\}$. And this with the extensionality axiom completes the proof. □

[6] Priest and Routley: "The naive notion of set is that of the extension of an arbitrary predicate.... This is as tight an account as can be expected from any fundamental notion. It was thought to be problematical only because it was assumed (under the ideology of consistency) that 'arbitrary' could not mean arbitrary. However, it does" (Priest *et al.*, 1989, p. 499).

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PROPOSITION 4.2. *Identity is an equivalence relation:*

$$x = x,$$

$$x = y \rightarrow y = x,$$

$$x = y \wedge y = z \rightarrow x = z.$$

Proof. These follow directly from the properties of $\rightarrow$ and $\wedge$ (Axioms I, II, IV, and V), and extensionality. □

As with Proposition 4.8 below, identity also obeys an alternate form of transitivity due to the hypothetical syllogism, namely

$$x = y \rightarrow (y = z \rightarrow x = z).$$

PROPOSITION 4.3. $\forall x \forall y (x \in y \vee x \notin y)$.

Proof. With Axiom I, $a \in y \rightarrow a \in y$. Then by Axiom VIII, $a \notin y \vee a \in y$. Rule IV generalizes to the result. □

PROPOSITION 4.4. *Sets that differ with respect to membership are not identical. In particular,* $\exists x (x \in a \wedge x \notin a) \rightarrow a \neq a$.

Proof. We prove the contrapositive.

1	$a = a \rightarrow \forall z (z \in a \leftrightarrow z \in a)$	$Ext.$
2	$\forall z (z \in a \leftrightarrow z \in a) \rightarrow (b \in a \leftrightarrow b \in a)$	$1, Ax.X$
3	$(b \in a \leftrightarrow b \in a) \rightarrow b \notin a \vee b \in a$	$Ax.II, VIII$
4	$b \notin a \vee b \in a \rightarrow \neg (b \in a \wedge b \notin a)$	$3, Ax.I$
5	$a = a \rightarrow \neg (b \in a \wedge b \notin a)$	$Ax.IV$
6	$(b \in a \wedge b \notin a) \rightarrow a \neq a$	$Ax.VI.$

Existential generalization completes the result. □

When a set a is such that its membership is inconsistent, some $b \in a$ and $b \notin a$, then a is *inconsistent*.

PROPOSITION 4.5. $\exists x (x \neq x)$.

Proof. By comprehension we have Russell's set, $R = \{x : x \notin x\}$.

1	$\forall x (x \in R \leftrightarrow x \notin x)$	$Comp.$
2	$R \in R \leftrightarrow R \notin R$	$1, Ax.X$
3	$R \in R \rightarrow R \notin R$	$2, Ax.II$
4	$R \notin R \vee R \notin R$	$3, Ax.VIII$
5	$R \notin R$	$4, Ax.V$
6	$R \in R$	$2, 5, RuleII$
7	$R \in R \wedge R \notin R$	$5, 6, RuleI.$

Since R differs from itself with respect to membership, by Proposition 4.4, $R \neq R$. □

From this novelty of inconsistent set theory, Restall (1992, p. 427) infers by generalization the

COROLLARY 4.6 (**Restall**). *There are at least two objects,* $\exists x \exists y (x \neq y)$.

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These few facts already show us that the present set theory will have as theorems some propositions not contained by classical theory, and that classical theorems may be recaptured by distinctively non-classical means.

DEFINITION 4.7 (**Subsets**). $x \subseteq y := \forall z(z \in x \to z \in y)$;
a proper subset is $x \subset y := x \subseteq y \wedge \exists z(z \in y \wedge z \notin x)$.

This leads to a natural partial order; the converse of antisymmetry actually holds, too, since this is just the axiom of extensionality rewritten.

PROPOSITION 4.8. *Subsets are reflexive and antisymmetric,*

$$x \subseteq x,$$

$$x \subseteq y \wedge y \subseteq x \to x = y.$$

They are also transitive:

$$x \subseteq y \wedge y \subseteq z \to x \subseteq z,$$

$$x \subseteq y \to (y \subseteq z \to x \subseteq z),$$

and $$x \subseteq y \to (z \subseteq x \to z \subseteq y).$$

Proof. Reflexivity and antisymmetry come from extensionality, Axiom I, and the commutativity of conjunction. The forms of transitivity are direct results of conjunctive syllogism and the hypothetical syllogism pair, respectively. □

PROPOSITION 4.9. *Every set has a* complement, $\overline{x} = \{y : y \notin x\}$, *and* $x = \overline{\overline{x}}$.

Proof. Existence is by comprehension; its behavior is checked with double negation introduction and elimination: If $y \in \overline{x} \leftrightarrow y \notin x$, then $y \in \overline{\overline{x}} \leftrightarrow y \in x$. □

Moving from these facts, we fix a top and bottom of the universe of sets, an empty set and a universal set, noting that von Neumann's definition of set is an object that is a member of something. V and $\emptyset$ will exhibit most of the expected structure of a boolean lattice, except that $x \cap \overline{x} = \emptyset$ and $x \cup \overline{x} = V$ do not generally hold. Note that the characterizations we are about to use are not those given by Russell and Whitehead, which have since become standard: $\{x : x = x\}$ for the universe and its complement, $\{x : x \neq x\}$ for the empty set. We have already seen in $R \neq R$ that there are objects not in, and in, these sets respectively. Nor, in our logic, would it be the case that $y \subseteq \{x : x = x\}$ or $\{x : x \neq x\} \subseteq y$ for every y, since $\Phi \to x = x$ is irrelevant. Our universal and empty sets, by contrast, are better behaved.

PROPOSITION 4.10. *The* universe,

$$V = \{x : \exists y(x \in y)\},$$

exists, and $\forall x(x \in V)$. *The* empty set,

$$\emptyset = \{x : \forall y(x \in y)\},$$

exists, too, and is empty: $\forall x(x \notin \emptyset)$.

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Proof. Existence of both sets is by comprehension. To prove that $\forall x(x \in V)$,

1	$\forall x(x \in V \vee x \notin V)$	$prop.4.3$
2	$\forall x(x \notin V \to x \in \overline{V})$	$abs.$
3	$\forall x(x \in \overline{V} \to \exists y x \in y)$	$Ax.X$
4	$\forall x(\exists y x \in y \to x \in V$	$abs.$
5	$\forall x(x \notin V \to x \in V)$	2, 3, 4.

The conclusion follows by $\vee$-elimination on Line 1. To show that the empty set is empty, similarly, if $x \in \emptyset$ then $\forall y x \in y$; then $x \in \overline{\emptyset}$ and therefore $x \notin \emptyset$. □

PROPOSITION 4.11. *$\forall x(x \subseteq V)$ and $\forall x(\emptyset \subseteq x)$.*

Proof. By existential generalization, $z \in x \to \exists y z \in y$. Then $z \in x \to z \in V$ by abstraction, and therefore $x \subseteq V$. Similarly, $z \in \emptyset \to \forall y z \in y$, and then $z \in x$ by $\forall$-elimination. □

What these theorems do *not* show—and indeed what probably *cannot* be shown, suggesting that this set theory is not trivial—is the uniqueness of a universal or empty set. The extensionality of naive set theory in light of 'dopplegängers' is discussed in Weber (forthcoming-a).

PROPOSITION 4.12. *$V = \overline{\emptyset}$ and $\emptyset = \overline{V}$.*

Proof. One direction of each is already proved. For $V \subseteq \overline{\emptyset}$ we argue the contrapositive: $x \in \emptyset \to \forall y x \in y$, and $\forall y x \in y \to x \in \overline{V}$. Then $x \notin V$. For $\overline{V} \subseteq \emptyset$, $x \notin V \to \neg\exists y x \in y$, so $\forall y x \notin y$; in particular, $x \notin \overline{\emptyset}$. Double negation elimination completes the proof. □

As this proof makes clear, the empty set is explosive: $x \in \emptyset \to x \in \{z : \Phi\}$ for any Φ. One way to read this is that $\emptyset$ is *really* empty, and V is *really* the universe. That they must, on pain of triviality, behave consistently is often a useful fallback.

§5. ZF. Now it is time to produce all the axioms of Zermelo–Fraenkel set theory, except *Fundierung*. Since general comprehension induces sets that are not well founded, for example, $V \in V$, the foundation axiom is not a part of the present theory, as is expected from Restall's results in Restall (1992). That the other axioms are forthcoming is not especially surprising, since Zermelo in 1908 (Zermelo, 1967) saw them as a consistent fragment of the naive theory. The axiom of infinity will be provable, too, showing that the naive theory does not fail a reductive program in the same way as Russell and Whitehead's system did.

In each proposition, the step of universally generalizing on free a, b is omitted.

PROPOSITION 5.1 (**Aussonderung**). $\exists y \forall x(x \in y \leftrightarrow x \in a \wedge \Phi(x))$.

PROPOSITION 5.2 (**Powerset**). $\exists y \forall x(x \in y \leftrightarrow x \subseteq a)$.

For any a, we use the name $P(a) := \{x : x \subseteq a\}$.

PROPOSITION 5.3 (**Pairing**). $\exists y \forall x(x \in y \leftrightarrow x = a \vee x = b)$.

For any a, b, $\{a, b\} := \{x : x = a \vee x = b\}$. A special case is the *singleton* $\{a\} := \{x : x = a\}$. For relevance purposes, sometimes singletons are relativized to a particular set, which is a key to defining ordinal and numerical successor below, and so to which we call special attention.

DEFINITION 5.4 (**Relevant singleton**). $\{a\}_b := \{x : x = a \wedge x \in b\}$.

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The essential property is that $\{a\}_b \subseteq b$ when $a \in b$, as is the case classically. Actually we have the stronger fact that $\{a\}_b \subseteq b$ even without $a \in b$.

PROPOSITION 5.5 (**Union**). $\exists y \forall x (x \in y \leftrightarrow \exists z (z \in a \wedge x \in z))$.

PROPOSITION 5.6 (**Intersection**). $\exists y \forall x (x \in y \leftrightarrow \forall z (z \in a \rightarrow x \in z))$.

The standard names are adopted:
$\bigcup a := \{x : \exists z (z \in a \wedge x \in z)\}$ and
$\bigcap a := \{x : \forall z (z \in a \rightarrow x \in z)\}$ for the union and intersection of a, respectively. Note that, because the conditional is not material, these are not necessarily duals. On the other hand,
$a \cup b := \{x : x \in a \vee x \in b\}$ and
$a \cap b := \{x : x \in a \wedge x \in b\}$ obey their usual algebra—save the explosive $a \cap \overline{a} \subseteq \emptyset$. The complement of b in a is now easily construed as $a - b := a \cap \overline{b}$.

The next axiom is the axiom of infinity. Note that V is already a set containing $\emptyset$ and its successors (since every set whatever is in V), and that we will be properly introducing the natural numbers later. So V is infinite[7] and the following is only for curiosity; compare Petersen's characterization of the natural numbers in (Petersen, 2000, p. 386).

PROPOSITION 5.7 (**Infinity**). *There is a non-empty set i such that when $x \in i$, also $\{x\}_i \in i$.*

Proof. Consider $i = \{x : x \subseteq i\}$. Both $i \in i$ and $\emptyset \in i$, so i is not empty. For $x \in i$, also $\{x\}_i \subseteq i$, and so $\{x\}_i \in i$. □

In the next definition and beyond, we take to writing some sets in a commonplace way, as , for example, $\{\langle x, y\rangle : x \in a \wedge y \in b\}$. As noted above, this is to be controlled with appropriate instances of abstraction. While our language does not, at the moment, contain function symbols, when $\langle x, y\rangle \in f$ and f is a function, then we will write $f(x) = y$, having in mind something like a two-place predicate: $\forall x \exists y [\Phi(x, y) \wedge \forall z (\Phi(x, z) \rightarrow y = z)]$.

DEFINITION 5.8. *An* ordered pair *is* $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$. *The* cartesian product *is* $a \times b = \{\langle x, y\rangle : x \in a \wedge y \in b\}$. *A* relation *is* $r \subseteq a \times b$. *A* function *is a relation* $f : a \longrightarrow b$ *with* domain $dom(f) = a$ *and* range $rng(f) = b$, *such that*

$$\langle x, u\rangle \in f \wedge \langle x, v\rangle \in f \mapsto u = v.$$

The composition *of two functions f, g is* $f \circ g = \{\langle x, z\rangle : \exists y (\langle x, y\rangle \in g \wedge \langle y, z\rangle \in f\}$. *The* image of a under f *is* $f''a = \{f(x) : x \in a\}$. *The* restriction *of f to x is* $f|x = f \cap (x \times a)$.

PROPOSITION 5.9 (**Replacement**). *Let f be a function with domain a. Then $f''a$ exists.*

PROPOSITION 5.10 (**Ordered pairs**). $\langle a, b\rangle = \langle c, d\rangle \dashv\vdash a = c \wedge b = d$.

Proof. Right to left is substitution, $a = c \vdash a \in \langle a, b\rangle \rightarrow c \in \langle a, b\rangle$, and similarly for $b = d$. We prove left to right. For $a = c$,

[7] A set X is *dedekind infinite* if there is an injection from X to some $Y \subset X$. Then it is provable that some sets are dedekind infinite, more or less in the same way that Dedekind proved it in 1888. Consider $f = \{\langle x, \{x\}\rangle : x \in V\}$. This is an injection of V into a proper part of itself. Therefore V is dedekind infinite.

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1	$\langle a, b\rangle = \langle c, d\rangle$	*premise*
2	$\forall x(x \in \{\{a\}, \{a, b\}\} \leftrightarrow x \in \{\{c\}, \{c, d\}\})$	*def*.5.8
3	$\forall x)(x = \{a\} \vee x = \{a, b\} \leftrightarrow x = \{c\} \vee x = \{c, d\}$	2, *Abs*.
4	$\{a\} = \{a\} \rightarrow \{a\} = \{c\} \vee \{a\} = \{c, d\}$	3, *Ax*.*X*
5	$\{a\} = \{c\} \vee \{a\} = \{c, d\} \rightarrow$ $\forall x(x \in \{a\} \leftrightarrow x \in \{c\}) \vee \forall x(x \in \{a\} \leftrightarrow x \in \{c, d\})$	4, *Ext*.
6	$\forall x(x = a \leftrightarrow x = c) \vee \forall x(x = a \leftrightarrow x = c \vee x = d)$	5, *Abs*.
7	$\forall x(x = a \leftrightarrow x = c) \rightarrow (a = a \rightarrow a = c)$ $\wedge\forall x(x = a \leftrightarrow x = c \vee x = d) \rightarrow (c = c \vee c = d \rightarrow c = a)$	6, *Ax*.*IX*
8	$a = a \vee c = c \vee c = d \rightarrow a = c$	7, *Ax*.*V*
9	$a = c$	8, *prop*.4.2, *mp*.

To prove that $b = d$, first we prove that $b = d \vee b = a$. Then we prove that $b = d \vee d = a$. And these together will give the result.

1	$\{a, b\} = \{a, b\} \rightarrow \{a, b\} = \{c\} \vee \{a, b\} = \{c, d\}$	*def*.5.10
2	$\forall x(x = a \vee x = b \leftrightarrow x = c)$ $\vee\forall x(x = a \vee x = b \leftrightarrow x = c \vee x = d)$	1, *Abs*
3	$(b = a \vee b = b \leftrightarrow b = c)$ $\vee(b = a \vee b = b \leftrightarrow b = c \vee b = d)$	2, *Ax*.*X*
4	$b = c \rightarrow b = c \vee b = d$	*Ax*.*II*
5	$b = a \vee b = b \rightarrow b = c \vee b = d$	3, 4, *Ax*.*IV*, *V*
6	$b = b$	*prop*4.2
7	$b = c \vee b = d$	5, *mp*.

Then using $a = c$, proven at Line 9 above,

8	$a = c$	
9	$a = c \wedge (b = c \vee b = d)$	7, 8, *RuleI*
10	$(a = c \wedge b = c) \vee (a = c \wedge b = d)$	9, *Ax*.*III*
11	$a = c \wedge b = c \rightarrow a = b$	*prop*.4.2
12	$b = a \vee b = d$	10, *Ax*.*V*

completing the first step of the argument. Now, again with $a = c$,

1	$a = c$	
2	$\langle a, b\rangle = \langle c, d\rangle$	*premise*
3	$\langle a, b\rangle = \langle a, d\rangle$	1, 2, *RuleV*
4	$\forall x(x = \{a\} \vee x = \{a, b\} \leftrightarrow x = \{a\} \vee x = \{a, d\})$	3, *def*.5.8
5	$\{a\} = \{a\} \rightarrow \{a\} = \{a, d\} \vee \{a, d\} = \{a, b\}$	4, *Ax*.*X*
6	$\{a\} = \{a, d\} \vee \{a, d\} = \{a, b\} \rightarrow$ $\forall x(x \in \{a\} \leftrightarrow x \in \{a, d\}) \vee \forall x(x \in \{a, d\} \leftrightarrow x \in \{a, b\})$	5, *Ext*.
7	$d = d \rightarrow d = a \vee d = b$	6, *Ax*.*X*, *V*
8	$d = a \vee d = b$	*prop*.4.2, 7, *mp*

completing the second step of the argument. To conclude, we adjoin the two facts $(b = a \vee b = d) \wedge (d = a \vee d = b)$; distributing possibilities,

$$(b = a \wedge a = d) \vee (b = a \wedge d = b) \vee (b = d \wedge d = a) \vee (b = d \wedge d = b),$$

each of which implies that $b = d$, as required. □

Writing proofs out in this way is tedious; as we go along the more usual informal presentation in mathematical english is used. But it is important to be clear that the present logic, weak as it as, is sufficient.

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Most set theories include the axiom of choice; proving choice would recapture the axioms of ZFC. It has been a vexed question as to whether or not a choice principle is or should be a fact about set theory (though it is now mostly regarded without suspicion). For our naive theory, we will see that choice is a consequence of a deeper Cantorian principle: that every set can be put into a well order. To show this, though, we must first do a lot of work on the notion of order itself. Useful classical references for the next section are Drake (1974), Levy (1979), and Kunen (1980).

§6. Ordinals.

DEFINITION 6.1. *A set a with respect to $\in$ is:*
strictly ordered *iff*

$$\begin{aligned} x, y, z \in a \rightarrow \quad & x \notin x \\ \wedge \quad & (x \in y \wedge x \notin x \rightarrow y \notin x) \\ \wedge \quad & (y \in z \rightarrow (x \in y \rightarrow x \in z)); \end{aligned}$$

totally *or* linearly ordered *by* $\subseteq$ *iff a is strictly ordered and*

$$x \in a \mapsto (y \in a \mapsto x \subseteq y \vee y \subseteq x),$$

that is, $\subseteq$-trichotomy *holds;*
well founded*,* $Wf(a)$*, iff*

$$y \subseteq a \wedge \exists z z \in y \mapsto \exists z (z \in y \wedge \neg \exists x (x \in z \wedge x \in y))];$$

well ordered*,* $Wo(a)$*, iff totally ordered and well founded;*
transitive*,* $Tr(a)$*, iff* $x \in a \rightarrow x \subseteq a$*;*
an ordinal *iff a is a well-ordered and transitive set of ordinals,* $\subseteq$*-connected to all other ordinals.*

In summary, by (full) comprehension

PROPOSITION 6.2. *There is a set of all ordinals,* $On = \{x : x \text{ is an ordinal }\}$*, such that*

$$\begin{aligned} x \in On \leftrightarrow \quad & Wo(x) \\ \wedge \quad & y \in x \rightarrow y \subseteq x \\ \wedge \quad & x \subseteq On \\ \wedge \quad & y \in On \mapsto (x \subseteq y \vee y \subseteq x). \end{aligned}$$

The definition of ordinal is adapted from von Neumann. We have added impredicative clauses, to capture the recursive idea that the ordinals are the same inside and out. The ordinals are an analysis of the concept of induction, generalized to the transfinite and reified. The hard work in the theory of ordinals is in locating the right definition. Then the properties of the ordinals, culminating in the Burali–Forti theorem, should all follow, as it were, by logic alone.

A few more detailed comments. Our extra clause in the antisymmetry condition for strict order, $x \notin x$, is due to the relevance constraint on implication, as we will see in the proofs below. In the well-founding clause, the rendering of a set having a least member is material; the intensional alternative would have been that for any non-empty $y \subseteq a$, there is a $z \in y$ such that $\forall x (x \in z \rightarrow z \notin y)$. Using this definition, however, makes it

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too difficult to prove that anything is well founded. Similarly, to gloss z as a least member if $z \cap y = \emptyset$ would be almost impossible to confirm, since one would have to show that $x \in z \cap y$ not only fails, but is absurd. It may be contradictory, but it is not absurd. For linear order, we are using the $\subseteq$ relation instead of the $\in$ relation, and have built in an added clause into the definition of On based on this choice. Finally, notice that $\rightarrow$ can be replaced with $\mapsto$.

On to the mathematics. Ordinals are written with lowercase Greek letters.

PROPOSITION 6.3. $\emptyset \in On$.

Proof. This is because $\emptyset$ is explosive. First, $\emptyset \subseteq On$, by Proposition 4.11. Similarly, $x \in \emptyset \rightarrow x \subseteq \emptyset$, transitivity, and $x \in \emptyset \rightarrow (y \in \emptyset \rightarrow x \subseteq y \vee y \subseteq x)$, a linear order. To show that $\emptyset$ is $\subseteq$-connected to all ordinals it again suffices that $\emptyset \subseteq x$ for any x at all. Finally, to show well foundedness, we have $a \in y \wedge (x \in y \rightarrow x \in \emptyset) \rightarrow a \in y \wedge (x \in \emptyset \vee x \notin y)$, by counterexample. Again since $\emptyset$ is explosive we get

$$a \in y \wedge (x \in y \rightarrow x \in \emptyset) \rightarrow a \in y \wedge (x \notin a \vee x \notin y).$$

Generalization completes the proof. □

PROPOSITION 6.4. *Subsets of a well-ordered set are well ordered:*

$$Wo(\alpha), \beta \subseteq \alpha \vdash Wo(\beta).$$

Proof. Let α be well ordered and $\beta \subseteq \alpha$. By Proposition 4.8, $\beta \subseteq \alpha \rightarrow (y \subseteq \beta \rightarrow y \subseteq \alpha)$. And $(y \subseteq \beta \rightarrow y \subseteq \alpha)$ means that members of β behave just as members of α, giving well order. For example, linearity is proved using hypothetical syllogism and (the rarely seen) Rule III, with $\mapsto$ replacing $\rightarrow$ as appropriate:

1	$x \in \beta \rightarrow x \in \alpha$	
2	$y \in \beta \rightarrow y \in \alpha$	
3	$(y \in \alpha \rightarrow x \subseteq y \vee y \subseteq x) \rightarrow (y \in \beta \rightarrow x \subseteq y \vee y \subseteq x)$	2, *hyp. syl.*
4	$(x \in \alpha \mapsto (y \in \alpha \mapsto x \subseteq y \vee y \subseteq x))$ $\rightarrow (x \in \beta \mapsto (y \in \beta \mapsto x \subseteq y \vee y \subseteq x))$	1, 3*Rule III*
5	$x \in \beta \mapsto (y \in \beta \mapsto x \subseteq y \vee y \subseteq x)$	4, *def.*6.1, *m.p.*

For well foundedness, notice that

(*) $\quad \Phi \rightarrow \Psi \vdash \Phi \wedge \Upsilon \rightarrow \Psi \wedge \Upsilon.$

This allows the following derivation.

1	$\beta \subseteq \alpha.$	
2	$\beta \subseteq \alpha \rightarrow (y \subseteq \beta \rightarrow y \subseteq \alpha).$	*prop.*4.8
3	$y \subseteq \beta \rightarrow y \subseteq \alpha.$	1, 2, *m.p.*
4	$y \subseteq \beta \wedge \exists z z \in y \rightarrow y \subseteq \alpha \wedge \exists z z \in y.$	3, (*)
5	$y \subseteq \alpha \wedge \exists z z \in y \mapsto \exists z(z \in y \wedge \forall x(x \notin z \vee x \notin y)).$	*def.*6.1
6	$y \subseteq \beta \wedge \exists z z \in y \mapsto \exists z(z \in y \wedge \forall x(x \notin z \vee x \notin y)).$	4, 5*Ax.IV*

as required. □

Proposition 6.4 extends immediately to ordinals, as these are well-ordered sets.

PROPOSITION 6.5. *On is transitive,* $\alpha \in On \rightarrow \alpha \subseteq On$.

Proof. This is a clause in the definition of being an ordinal. □

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PROPOSITION 6.6. *An ordinal α is the set of all preceding ordinals,*

$$\alpha = \{x : x \in On \wedge x \in \alpha\}.$$

Proof. Let $\alpha \in On$. The previous theorem and the axiomatic instance $\beta \in \alpha \rightarrow \beta \in \alpha$ show that $\beta \in \alpha \rightarrow \beta \in On \wedge \beta \in \alpha$. The other direction is immediate. Therefore $\alpha = \alpha \cap On$. □

PROPOSITION 6.7. $\alpha \in On \rightarrow \alpha \notin \alpha$.

Proof. The idea is that, were $\alpha \in \alpha$, then still $\alpha \notin \alpha$.

1	$\alpha \in On \rightarrow \forall x(x \in \alpha \rightarrow x \notin x)$.	*def*.6.1
2	$\forall x(x \in \alpha \rightarrow x \notin x) \rightarrow (\alpha \in \alpha \rightarrow \alpha \notin \alpha)$.	*Ax*.*X*.
3	$(\alpha \in \alpha \rightarrow \alpha \notin \alpha) \rightarrow \alpha \notin \alpha \vee \alpha \notin \alpha$.	*Ax*.*VIII*
4	$\alpha \notin \alpha \vee \alpha \notin \alpha \rightarrow \alpha \notin \alpha$.	*Ax*.*V*.

Conjunctive syllogism completes the proof. □

PROPOSITION 6.8. $\alpha \in \beta \wedge \alpha \notin \alpha \rightarrow \beta \notin \alpha$.

Proof. $\alpha \in \beta \wedge \alpha \notin \alpha \rightarrow \beta \not\subseteq \alpha$ by counterexample, and $\beta \not\subseteq \alpha \rightarrow \beta \notin \alpha$. □

PROPOSITION 6.9. $\beta \in \gamma \rightarrow (\alpha \in \beta \rightarrow \alpha \in \gamma)$.

Proof. By the definition of transitivity on γ. □

The last few propositions have showed, by virtue of self-similarity, that the ordinals are strictly ordered. Now we need total and well ordering.

THEOREM 6.10 (**Trichotomy of ordinals**). *Any two ordinals are $\subseteq$-connected,*

$$\alpha \in On \rightarrow (\beta \in On \mapsto \alpha \subseteq \beta \vee \beta \subseteq \alpha).$$

Proof. This is a clause in the definition of ordinal. □

This delivers, for a start, some miscellany like the following.

PROPOSITION 6.11. *$\alpha \cap \beta \in On$, and $\alpha \cup \beta \in On$.*

Proof. We can show $\alpha \cap \beta$ is well ordered and transitive independently of trichotomy. Let $x \in \alpha \cap \beta$. Then $x \in \alpha \wedge x \in \beta$; then $x \subseteq \alpha \wedge x \subseteq \beta$; then $x \subseteq \alpha \cap \beta$, showing transitivity. Meanwhile, $\alpha \cap \beta \subseteq \alpha$; since α is well ordered, $\alpha \cap \beta$ is well ordered, too.

Easier still is to notice that if $\alpha \subseteq \beta$, then $\alpha \cap \beta = \alpha$; and if $\beta \subseteq \alpha$ then $\alpha \cap \beta = \beta$, ordinals both. The case for $\alpha \cup \beta$ is just like this. □

More weakly, we see that certain intervals of On must be empty. Since $\alpha \cap \beta \in On$, we have $\alpha \cap \beta \notin \alpha \cap \beta$. Therefore either $\alpha \cap \beta \notin \alpha$ or $\alpha \cap \beta \notin \beta$. Therefore, there cannot be any ordinals intervening between both α and $\alpha \cap \beta$, and β and $\alpha \cap \beta$. Observations like these are the spur for the classical proof that On is linearly ordered.

PROPOSITION 6.12. *The ordinals are well founded.*

Proof. We have to show that a non-empty $\theta \subseteq On$ has a least member. Let $\beta \in \theta$. The idea is that β is $\in$-least in θ, or else the least member of β is the least member of θ. Note that $\beta \cap \theta \subseteq \beta$; since β is well ordered, $\beta \cap \theta$ is well ordered (by Proposition 6.4). Either $\beta \cap \theta$ is empty, or not.

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Suppose $\forall y(y \notin \beta \cap \theta)$. We infer by t-introduction that

$$\theta \subseteq On \wedge \beta \in \theta \mapsto \beta \in \theta \wedge \forall y(y \notin \beta \vee y \notin \theta).$$

This says that β is a least member of θ, and generalizes:

$$\theta \subseteq On \wedge \exists z z \in \theta \mapsto \exists z(z \in \theta \wedge \forall y(y \notin z \vee y \notin \theta)).$$

So far this shows $\forall y(y \notin \beta \cap \theta) \vdash Wf(On)$. On the other hand, suppose $\exists y(y \in \beta \cap \theta)$. Since $\beta \cap \theta$ is well founded, trading $\exists$ for γ we have

$$\gamma \in \beta \wedge \gamma \in \theta \wedge \forall y(y \notin \gamma \vee y \notin \beta \vee y \notin \theta).$$

As $\gamma \subseteq \beta$ (since $\gamma \in \beta$), by contraposition $y \notin \beta \rightarrow y \notin \gamma$. Therefore the above formula reduces to

$$\gamma \in \theta \wedge \forall y(y \notin \gamma \vee y \notin \theta),$$

which says that γ is $\in$-minimal in θ. Again with t-introduction this generalizes to

$$\theta \subseteq On \wedge \exists z z \in \theta \mapsto \exists z(z \in \theta \wedge \forall y(y \notin z \vee y \notin \theta)).$$

This shows $\neg\forall y(y \notin \beta \cap \theta) \vdash Wf(On)$.

Both directions of an excluded middle lead to $Wf(On)$. The derived meta-rule,

if $\Phi \vdash \Upsilon$ and $\neg\Phi \vdash \Upsilon$, then $\Phi \vee \neg\Phi \vdash \Upsilon$,

is used to incorporate the law of excluded middle. We have a theorem. Any arbitrary non-empty subset of ordinals has a least member. □

PROPOSITION 6.13. *Any transitive set of ordinals, connected to all other ordinals by* $\subseteq$*, is an ordinal.*

Proof. Any set of ordinals is well ordered, by the previous theorems. Then the definition of being an ordinal is satisfied. □

THEOREM 6.14 (**Burali–Forti 1897**). $On \in On$.

Proof. By the transitivity of On, we have $y \in On \rightarrow y \subseteq On$. By $\vee$-introduction, $y \subseteq On \rightarrow y \subseteq On \vee On \subseteq y$. With conjunctive syllogism, $y \in On \mapsto y \subseteq On \vee On \subseteq y$ as required. On is a well-ordered, transitive set of ordinals connected to all other ordinals. □

COROLLARY 6.15. *$On \notin On$, and then $On \neq On$.*

Proof. From Proposition 6.7, and then Proposition 4.4. □

The Burali–Forti contradiction is a key reason to think naive set theory is inconsistent on an interesting mathematical level (as opposed to Russell's contradiction, which can look artificial). It would have been perverse if a logic for naive set theory could not reproduce the paradoxes of naive set theory. With Theorem 6.14, we have shown that, with a paraconisistent logic, we can still prove the very theorems that moved us in the first place.

At this point, nevertheless, the limitations of our logic leave us with an unsolved problem of basic relevant set theory. To see that On is closed under some expected operations, and that certain objects are ordinals, we need the following lemma:

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CONJECTURE 6.16. *Let $\alpha, \beta \in On$, let θ be a well-ordered, transitive set of ordinals, and let $\alpha \subseteq \theta \subseteq \beta$. Then $\theta \in On$.*

To prove this, we need to show under the assumptions of the conjecture that $\forall y(y \in On \mapsto y \subseteq \theta \vee \theta \subseteq y)$; then all the conditions for θ being an ordinal are satisfied. There are no counterexamples to the conjecture, due to the facts about intervals cited after Proposition 6.11 above—which show that any stretches of ordinals failing the conjecture would have to be empty. Nevertheless, I do not have a proof that is adequate from a paraconsistent perspective. (*Solatium miseris, socios habuisse malorum.*) Meanwhile, any set of ordinals θ is such that $\emptyset \subseteq \theta \subseteq On$; so if this conjecture were proved, then for any θ that is transitive and well ordered, θ is an ordinal. We will assume the conjecture for the duration.

DEFINITION 6.17. *$\alpha^+ = \alpha \cup \{\alpha\}_{On}$ is the* successor *of α. Ordinals α with no predecessor, $\neg\exists\beta(\beta \in On \wedge \beta^+ = \alpha)$, are called* limits.

In the definition, $\{\alpha\}$ is relativized to On, that is, as $\{x \in On : x = \alpha\}$. This is harmless, because the relevance-inducing conjunct $x \in On$ can always be eliminated going right. And the definition makes it easy to prove, with $\rightarrow$s, that every member of α^+ is an ordinal.

PROPOSITION 6.18. $\alpha^+ \subseteq On$.

Proof. Well, $x \in \alpha^+ \rightarrow (x \in \alpha \vee x = \alpha) \wedge x \in On$ by the definitions of relevant singleton and ordinal successor. And $(x \in \alpha \vee x = \alpha) \wedge x \in On \rightarrow x \in On$ by $\wedge$ elimination, so the result holds by conjunctive syllogism. □

PROPOSITION 6.19. $\alpha \in On \vdash \alpha^+ \in On$.

Proof. All the members of α^+ are ordinals, therefore α^+ is well ordered. For transitivity,

1 $x \in \alpha^+ \rightarrow x \in \alpha \vee (x \subseteq \alpha \wedge x \in On)$.
2 $x \in \alpha \rightarrow x \subseteq \alpha$.
3 $x \subseteq \alpha \rightarrow x \subseteq \alpha \cup \{\alpha\}_{On}$.
4 $x \in \alpha^+ \rightarrow x \subseteq \alpha^+$.

To show that α is connected to all other ordinals, it suffices to notice that $\alpha \subseteq \alpha^+ \subseteq On$. □

PROPOSITION 6.20. *On is a successor, and On is a limit.*

Proof. To show successor: Since On is an ordinal, On^+ is an ordinal. So $On^+ \in On$, and by transitivity, $On^+ \subseteq On$. And $On \in On^+$ like all other ordinals, so $On \subseteq On^+$ by transitivity again. Therefore $On = On^+$. To show limit, we need $\beta^+ = On \vdash \beta \notin On$ for all β. We can then apply this rule disjunctively to excluded middle, to obtain $\forall\beta(\beta \notin On \vee \beta^+ \neq On)$, which is equivalent to On being a limit. We prove $\beta^+ = On \vdash \beta = On$ from which $\beta \notin On$ follows, since $On \notin On$ by Corollary 6.15. So let $\beta^+ = On$; then by Definition 6.17, $\beta \subseteq On$. Conversely, since $On \in On$ by Theorem 6.14, also $On \in \beta^+$. Then by Definition 6.17 $On \in \beta \vee On = \beta$ and in either case, $On \subseteq \beta$ by the transitivity of β. So $\beta = On$ as required.[8] □

[8] Thanks here to Brady.

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§7. Arithmetic. Full comprehension induces the set

$$\omega = \{x \in On : [x = \emptyset \vee \exists y(y^+ = x)] \wedge \forall y(y \in x \rightarrow y \in \omega)\}.$$

Members $n, m, \ldots$ of ω are called *natural numbers*, or just numbers. By definition, $\omega \subseteq On$; so it is a well-ordered set.

PROPOSITION 7.1. $\omega \in On$.

Proof. Let $n \in \omega$. Then by definition, $n \subseteq \omega$, showing transitivity. As a subset of On, ω is well ordered. And $n \subseteq \omega \subseteq On$, showing $\subseteq$-connection to all other ordinals. So ω is an ordinal. □

To prove the Peano postulates, we will again consider a restriction of the successor function to ω, using a relevant singleton. That is, for every $n \in \omega$, n^+ means $n \cup \{x \in \omega : x = n\}$.

THEOREM 7.2. $\emptyset \in \omega$.

Proof. Since $\emptyset \in On \wedge \emptyset = \emptyset \wedge \forall y(y \in \emptyset \rightarrow y \in \omega)$, zero is a number. □

THEOREM 7.3. $n^+ \neq \emptyset$.

Proof. Since $n \in n^+$ but no $n \in \emptyset$, zero is not the successor of any number. That is, if $n^+ = \emptyset$ then $n \in \emptyset$, which implies everything, including the desired theorem. □

THEOREM 7.4. $n \in \omega \vdash n^+ \in \omega$.

Proof. If $n \in \omega$, then n^+ meets all the requirements to be a number: $n^+ \in On$, and has a predecessor, and all its members are in n, in which case they are numbers, or else a number identical to n, a number again. □

THEOREM 7.5. $n^+ = m^+ \vdash n = m$.

Proof. If $n^+ = m^+$, then $\forall z(z \in n \vee (z \in \omega \wedge z = n) \leftrightarrow z \in m \vee (z \in \omega \wedge z = m))$. Then picking n and m for z respectively,

$$(n \in m \vee (n \in \omega \wedge n = m)) \wedge (m \in n \vee (m \in \omega \wedge m = n)),$$

which, once the extraneous conjuncts are dropped, distributes to

$$(n \in m \wedge m \in n) \vee (n \in m \wedge m = n) \vee (n = m \wedge m \in n) \vee (n = m \wedge m = n).$$

Since $n \in m \wedge m \in n$ implies $n = m$ by two applications of transitivity, each above conjunct implies that $n = m$. So the successor of every number is unique. □

Names of the first few natural numbers are

$$\begin{aligned} \emptyset &= 0 \\ \emptyset^+ &= 1 \\ \emptyset^{++} &= 2 \\ &\vdots \end{aligned}$$

The fifth postulate is induction, proved in the general, transfinite case over On.

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THEOREM 7.6 (**Transfinite induction**). *Let $\theta \subseteq On$. Suppose*

$$\forall\beta(\beta \notin \alpha \vee \beta \in \theta) \rightarrow \alpha \in \theta.$$

Then $\neg\exists\alpha(\alpha \in On \wedge \alpha \notin \theta)$.

Proof. Suppose $\exists\alpha(\alpha \in On \wedge \alpha \notin \theta)$. Then there is a least such,

$$\exists\alpha[\alpha \notin \theta \wedge \forall\beta(\beta \notin \alpha \vee \beta \in \theta)].$$

But the hypothesis implies $\exists\beta(\beta \in \alpha \wedge \beta \notin \theta) \vee \alpha \in \theta$, for every α, which negates this claim. Therefore there is no least, and so no ordinal not in θ, as required. □

Transfinite induction will hold for any well-ordered set, including ω. The base case, 0, is covered automatically by the induction hypothesis: If $\neg\Phi(0) \rightarrow \exists x(x \in 0 \wedge \neg\Phi(x))$, then $\Phi(0)$ by the explosiveness of $\emptyset$.

A close mate of induction is definition by recursion. There is something appropriate in proving the recursion theorem, as we are about to, with a set containing itself in its defining condition.

THEOREM 7.7 (**Transfinite recursion**). *Let h be a function from V to V. There is a function f from On to V such that*

$$f(\alpha) = h(f|\alpha).$$

Proof. Take the set

$$\langle x, y\rangle \in f \leftrightarrow y = h(f|x).$$

Existence is immediate from full comprehension. This is a function because h is; if $\langle x, y\rangle, \langle x, z\rangle \in f$ then $y = h(f|x) = z$. □

The recursion scheme is used to define ordinal arithmetic. To deal with limit ordinals, we look to least upper bounds.[9]

DEFINITION 7.8. *Let X be a set of ordinals. The* supremum *of X, $sup(X)$, is the least ordinal δ such that every $x \in X$ is either in δ or identical to δ.*

Addition, taking h to be $h(\alpha) = \alpha^+$, is

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha + \beta &= sup\{\alpha + \gamma : \gamma \in \beta\} \qquad \textit{for limit } \beta; \end{aligned}$$

[9] The usual identification of the $sup(\alpha)$ with $\bigcup\alpha$ does not seem to follow in the logic here, or at least not with ordinals as they are here defined, because the existential quantifier resists a proof that $\bigcup\alpha$ is an ordinal. Nevertheless, where X is a set of ordinals, then ordinals including every member of X certainly exist—On, for example—and so, by well foundedness, a least exists, too. If there is any question about the *uniqueness* of a supremum, a choice function—developed in the next section independently of ordinal arithmetic—can be applied to make a functional selection.

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multiplication, taking h to be $h_\beta(\alpha) = \alpha + \beta$, is

$$\begin{aligned} \alpha \cdot 0 &= 0 \\ \alpha \cdot (\beta + 1) &= \alpha \cdot \beta + \alpha \\ \alpha \cdot \beta &= sup\{\alpha \cdot \gamma : \gamma \in \beta\} \qquad \textit{for limit } \beta; \end{aligned}$$

exponentation, with $h_\beta(\alpha) = \alpha \cdot \beta$, is

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \cdot \alpha \\ \alpha^\beta &= sup\{\alpha^\gamma : \gamma \in \beta\} \qquad \textit{for limit } \beta. \end{aligned}$$

§8. Global choice. In 1977, Routley produced an argument for the axiom of global choice from full comprehension (Routley, 1980), (Priest *et al.*, 1989, p. 374). Since then the claim that choice is a theorem of naive set theory has become part of paraconsistent folklore. There are some problems with Routley's proofs (Weber, forthcoming-b), but his general idea is correct. Answering a fundamental question about the set concept, we here surpass classical theory and derive a global choice theorem without need of any further assumptions. In fact what is proved is a weak version of Cantor's well-ordering principle, which he took to be a *Denkgesetz*, a law of thought (Hallett, 1984). It was in service of proving this principle, essential to the theory of transfinite cardinals, that Zermelo formulated his choice principle in 1904. The well ordering here is produced by injecting V into a particular subset of On, thereby giving a well order for every subset of V, that is, every set.

A function $f : a \longrightarrow b$ is *injective* iff $\forall x \forall y(x \neq y \mapsto f(x) \neq f(y))$.

THEOREM 8.1. *The universe can be well ordered.*

Proof. An injection $f : V \longrightarrow On$ is required. Consider

$$\langle x, y \rangle \in f \leftrightarrow x \in V \wedge y \in On \wedge y = On.$$

That is, for each $x \in V$, $f(x) = On$. This is clearly a function. Now,

$$\{\alpha \in On : \alpha = On\} \subseteq On,$$

showing that the range of f is a segment of the ordinals and therefore well ordered. Intuitively, the Burali–Forti paradox indicates that the members of the range of f are discrete (Corollary 6.15), of the form

$$\ldots \in On \in On \in \ldots,$$

so $\{On\}$ may be injected into by arbitrarily large sets, inducing a well order on them. Formally, because $On \neq On$, $\forall x \forall y(x \neq y \mapsto On \neq On)$, so $\forall x \forall y(x \neq y \mapsto f(x) \neq f(y))$. Therefore f is an injection. Thus

$$\{x_{f(x)} : f(x) \in On\}$$

is a well order on V. □

The proof is clearly not constructive; given the ordering $\{a_{On}, b_{On}, c_{On}, \ldots\}$ on V with each On distinct, it is not said how to determine a first member. This is exactly the case

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with Zermelo's choice principle, which is a pure existence claim. A proof of a Cantorian "law of thought" will inevitably be by demonstration of a bare existence of an ordering; and so here, it has been established that for any well-ordered set there is a least member, and $\{On\}$ is well ordered. The difference between Zermelo's proof and our own is that no extra assumptions are needed to produce the existence claim. It comes directly from the set concept.

Since subsets of a well-ordered set are well ordered.

COROLLARY 8.2. *(Zermelo 1904) Every set can be well ordered.*

The following familiar steps lead to choice (Rubin & Rubin, 1985). The details are not at all trivial, but are here omitted. A *chain* is a set connected by $\subseteq$.

PROPOSITION 8.3 (**Hausdorff's maximal principle**). *Every set has a maximal chain.*

LEMMA 8.4 (**Zorn**). *If every chain of some non-empty a has an upper bound, then a has a maximal element.*

A function f is called a *choice function* on a iff $f(x) \in x$ when $x \in a$ and $\exists z z \in x$.

THEOREM 8.5 (**Choice**). *There is a choice function on every set.*

COROLLARY 8.6 (**Global choice**). *There is a choice function on V. A fortiori, for every non-empty a there is a choice function f such that $f(a) \in a$.*

§9. Conclusions. The results in this paper are a beginning. For future work, a more definite approach to restricted quantification, once one is known, will be invaluable. A proof of the open conjecture at the end of the section on ordinals is obviously desirable. And a more effective transfinite induction principle, with $\rightarrow$s instead of extensional connectives in key places, is desirable, too. Nevertheless, I hope these omissions can be forgiven for now, in light of the significant increase in detail and insight provided.

To conclude on a forward-looking note, let me indicate some very striking results awaiting in the theory of cardinal numbers. With the cardinals, we see clearly why all the work on ordinals has been worthwhile, and also just how powerful the theory is. The construction follows von Neumann's assignment of least ordinals in an equinumerous class as cardinals. Let $<$ denote strict ordering by cardinality.

First, On should itself be a cardinal number, being evidently the least of its size. It would be the cardinal of V. All cardinals are ordinals, so all cardinals are less than On. This shows that

$$\exists x \forall y(|y| < |x|).$$

Then a simple quantifier swap proves the essence of *Cantor's theorem*, that there are distinctive orders of infinity:

$$\forall y \exists x(|y| < |x|).$$

For example, if $|\omega| := \aleph_0$ by definition, then $\exists y(\aleph_0 < y)$, so there are uncountable cardinals. Note that this transcendence result holds even for V or On themselves—Cantor's paradoxes.

In a similar vein, a cardinal λ is said to be *inaccessible* if for every $\kappa < \lambda$, also $2^\kappa < \lambda$. The existence of inaccessible cardinals is unprovable in ZFC. But, almost trivially, On is such a cardinal: For $\kappa < On$, also $2^\kappa < On$, because On is the biggest. Therefore inaccessible cardinals exist.

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Finally, the *generalized continuum hypothesis* GCH conjectures for all cardinals κ, λ that $\neg(\kappa < \lambda < |P(\kappa)|)$. The GCH evidently fails at On, as the cardinal On provides a counterexample. Assuming that $On < |P(On)|$, still we would have (by the Burali–Forti contradiction) that $On < On < |P(On)|$. Then again, this does not rule out the GCH in general; in fact, an instance of GCH holds at On. Let λ be a cardinal $On < \lambda < |P(On)|$. Being a cardinal, $\neg(On < \lambda)$, since all cardinals are members of, and less than, On. Thus $\neg(On < \lambda < |P(On)|)$. In fact, further assuming that $On = On + 1$ obtains via the Schröder–Bernstein theorem, and that $On = \aleph_{On}$, then by existential generalization,

$$\exists\alpha(2^{\aleph_\alpha} = \aleph_{\alpha+1}).$$

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