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On mixing and the local central limit theorem for hyperbolic flows

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Abstract. We formulate abstract conditions under which a suspension flow satisfies the local central limit theorem. We check the validity of these conditions for several systems including reward renewal processes, Axiom A flows, as well as the systems admitting Young's tower, such as Sinai's billiard with finite horizon, suspensions over Pomeau–Manneville maps, and geometric Lorenz attractors.

1. *Introduction*

Various stochastic aspects of deterministic, chaotic dynamical systems have been extensively studied in the last few decades. The central limit theorem (CLT) is a famous example. However, its local version (LCLT) has been studied much less for maps and especially for flows. In many cases, it is useful to view a chaotic flow as a suspension over a base map, whose chaotic properties are easier to prove. Then one tries to lift these statements from the map to the flow. This approach has been applied several times in the dynamics literature (see e.g. [**R73**, **DP84**, **MT04**]), but there are few studies specifically investigating the LCLT (we are only aware of [**DN16**, **AN17**]). Therefore, this is the focus of our paper.

Some special cases, where the LCLT has been proven for hyperbolic flows, are as follows:

(i) Axiom A flows under a non-arithmeticity condition for observables [**W96**];
(ii) the LCLT was obtained in [**I08**] for a class of flows whose transfer operator has a spectral gap on a suitable space;
(iii) the free path for Sinai's billiard flows with finite horizon [**DN16**].

In the present paper, we formulate a set of abstract conditions which imply the LCLT for suspension flows. The most important assumption is the LCLT for the base map. We also discuss connections of the LCLT to mixing. Namely, we give a necessary and sufficient condition for mixing of suspension flows satisfying LCLT and also prove a joint extension

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of mixing and the LCLT (which we abbreviate MLCLT). We check that the conditions imposed on the base map are satisfied by a large class of systems, where a Young tower with a sufficiently fast return time can be constructed. In particular, we generalize results (i) and (iii) mentioned above. Our approach is different from the methods of [**W96**] and [**I08**]. In fact we extend the method of [**DN16**] to a more general setting.

More precisely, we will work with a metric space (X, d) with Borel algebra $\mathcal{G}$, a probability measure ν and T, a self-map of X that preserves ν. Let $\tau : X \to \mathbb{R}_+$ be an integrable roof function, bounded away from zero. Let Φ^t be the corresponding suspension (semi-)flow on the phase space $\Omega = \{(y, s) : y \in X, s \in [0, \tau(y)]\}/\sim$, where $(y, \tau(y)) \sim (Ty, 0)$. The induced invariant measure is $\mu = \nu \otimes \mathrm{Leb}/\nu(\tau)$. Let $\boldsymbol{\varphi} : \Omega \to \mathbb{R}$ be a function.

Setting aside several technical conditions, the LCLT for flows can be informally stated as follows:

$$t^{1/2}\mu\left(\boldsymbol{x} : \int_0^t \boldsymbol{\varphi}(\Phi^s(\boldsymbol{x}))\, ds \in w\sqrt{t} + I\right) \sim \mathfrak{g}(w)u(I) \tag{1}$$

as $t \gg 1$, where I is a bounded subinterval of $\mathbb{R}$, $\mathfrak{g}$ is a Gaussian density and u is a measure having a large symmetry group. The case $u = \mathrm{Leb}$ is called continuous. The case where u is invariant under a sublattice $a\mathbb{Z}$ is called discrete. Several previous papers on the LCLT for dynamical systems presented conditions for u to be equal to the counting measure on $a\mathbb{Z}$. We allow more general measures since it increases the domain of validity of our results. On the other hand, the conditions presented in our paper do not guarantee that $a\mathbb{Z}$ is the largest group of translations preserving u since the factor measure on $\mathbb{R}/a\mathbb{Z}$ could 'accidentally' have some extra symmetries.

We note that the LCLT clearly implies the CLT (assuming that the convergence is uniform for w in compact intervals).

MLCLT is a joint generalization of the LCLT and the mixing of the flow (Ω, ν, Φ^t). Recall that mixing means that, for all measurable sets A, B,

$$\mu(\boldsymbol{x} \in A,\, \Phi^t\boldsymbol{x} \in B) \sim \mu(\boldsymbol{x} \in A)\mu(\Phi^t\boldsymbol{x} \in B) = \mu(A)\mu(B) \quad \text{as } t \to \infty.$$

Thus, the natural definition of MLCLT would be

$$\mu\left(x \in A,\, \Phi^t x \in B,\, \int_0^t \boldsymbol{\varphi}(\Phi^s(\boldsymbol{x}))\, ds \in w\sqrt{t} + I\right)$$
$$\sim t^{-1/2}\mu(A)\mu(B)\mathfrak{g}(w)u(I) \quad \text{as } t \to \infty. \tag{2}$$

In the continuous case we indeed define MLCLT by (2) while in the discrete case we require that (2) holds after a certain change of variables which straightens u in the fibers, see Definition 3.1 for details.

In the continuous case, under an assumption that $\boldsymbol{\varphi}$ has zero mean, (2) can be interpreted as mixing property of flow $\bar{\Phi}$ acting on $\Omega \times \mathbb{R}$ by $\bar{\Phi}^t(x, z) = (\Phi^t x, z + \int_0^t \boldsymbol{\varphi}(\Phi^s(\boldsymbol{x}))\, ds)$ with respect to an infinite invariant measure $\mu \times \mathrm{Leb}$. Also, in the discrete case, MLCLT has an interpretation as mixing of a certain $\mathbb{Z}$ extension of Φ (see [**AN17**]).

The result of our analysis is that (1) may in general fail for some arithmetic reasons (see §6.1 for an explicit example). However, all limit points of the left-hand side of (1) are of the form given by the right-hand side of (1) with, possibly, different measures u. We also provide sufficient conditions for (1) as well as for MLCLT to hold.

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The remaining part of the paper consists of five sections. Sections 2 and 3 study suspension flows over abstract spaces under the assumptions that the ergodic sums satisfy the MLCLT. Section 2 provides a characterization of mixing in this setting while §3 contains some abstract assumptions implying various versions of the MLCLT. In §§4 and 5 we verify our abstract assumptions for certain systems admitting Young towers. Section 4 deals with expanding Young towers and allows MLCLT to be obtained for suspension flows over non-invertible systems. In §5 we extend the results of §4 to invertible systems admitting a Young tower. In §6 we present several specific examples satisfying our assumptions: reward renewal processes, Axiom A flows, Sinai's billiard with finite horizon, suspensions over Pomeau–Manneville maps and geometric Lorenz attractors.

2. *Local limit theorem and mixing*

2.1. *Definitions.* We use the following notation.

(1) $S_f(n, x) = \sum_{k=0}^{n-1} f(T^k x)$ for some function $f : X \to \mathbb{R}^d$.
(2) Leb_d denotes the d-dimensional Lebesgue measure.
(3) $\mathfrak{g}_\Sigma$ denotes the centered Gaussian density with covariance matrix Σ.
(4) Given a closed subgroup V of $\mathbb{R}^d$, u_V denotes the Haar measure on V (Lebesgue times counting measure), normalized so that

$$u_V(\{z : \|z\| \le R\}) \sim \mathrm{Leb}_d(\{z : \|z\| \le R\}) \quad \text{as } R \to \infty.$$

We need several definitions.

Definition 2.1. Let ρ_n and ρ be locally finite measures on a locally compact Polish space E. Then ρ_n converges vaguely to ρ if $\rho_n(f) \to \rho(f)$ for every compactly supported continuous f (or equivalently $\rho_n(\mathcal{H}) \to \rho(\mathcal{H})$ for every compact $\mathcal{H} \subset E$ with $\rho(\partial\mathcal{H}) = 0$). If f is only assumed to be bounded and continuous, then the convergence is called weak convergence.

Definition 2.2. We say that (T, f) satisfies the MLCLT ($f : X \to \mathbb{R}^d$ is square integrable) if there are some functions g and h, where h is bounded and ν-almost everywhere continuous, such that

$$f = g - h + h \circ T, \tag{3}$$

a closed subgroup M of $\mathbb{R}^d$, a translation $r \in \mathbb{R}^d/M$ and a positive definite matrix Σ such that, as $n \to \infty$, the following holds for any $w \in \mathbb{R}^d$, for any bounded and continuous $\mathfrak{x}, \mathfrak{y} : X \to \mathbb{R}$, and for any continuous and compactly supported $\mathfrak{z} : \mathbb{R}^d \to \mathbb{R}$. For any sequence w_n satisfying

$$w_n \in M + nr,\ \|w_n - w\sqrt{n}\| \le K,$$

we have

$$\begin{aligned} n^{d/2} \int_X \mathfrak{x}(x)\mathfrak{y}(T^n x)\mathfrak{z}(S_g(n, x) - n\nu(g) - w_n)\, d\nu(x) \\ \to \mathfrak{g}_\Sigma(w) \int_X \mathfrak{x}\, d\nu \int_X \mathfrak{y}\, d\nu \int_M \mathfrak{z}\, du_M \end{aligned} \tag{4}$$

and the convergence is uniform once K is fixed and w is chosen from a compact set.

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Definition 2.3. We say that $f : X \to \mathbb{R}^d$ is non-degenerate if for any g cohomologous to f (i.e. satisfying (3) with some measurable h), the minimal translated subgroup supporting g is d-dimensional. That is, if L is a subgroup of $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ is a vector such that $g \in a + L$, then L is d-dimensional.

We remark that a slight generalization of the MLCLT would allow $\dim(M) < d$ and would naturally accommodate degenerate observables. We do not consider this case here.

If f satisfies the MLCLT with $h = 0$, we say that f is *minimal* (this is the case e.g. if $M = \mathbb{R}^d$). For non-minimal functions, the limit may not be a product measure, but we have instead the following lemma which is a consequence of the continuous mapping theorem (recall that h is bounded and almost everywhere continuous).

LEMMA 2.1. *Assume that (T, f) satisfies the MLCLT. Then with the notation of Definition 2.2,*

$$\lim_{n\to\infty} n^{d/2} \int_X \mathfrak{x}(x)\mathfrak{y}(T^n x)\mathfrak{z}(S_f(n, x) - n\nu(f) - w_n)\, d\nu(x)$$
$$= \mathfrak{g}_\Sigma(w) \int_{X\times M\times X} \mathfrak{x}(x)\mathfrak{y}(y)\mathfrak{z}(z - h(x) + h(y))\, d(\nu(x) \times u_M(z) \times \nu(y)).$$

The following lemma is a consequence of Lemma 2.1.

LEMMA 2.2. *If (T, f) satisfies the MLCLT, then $M \subset \mathbb{R}^d$ and $r \in \mathbb{R}^d/M$ are uniquely defined.*

Proof. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous compactly supported function, e.g. $\psi(r) = \max(1 - r, 0)$. Fix an arbitrary point $\bar{x} \in X$ and let $\mathbf{m}_{n,\varepsilon,R,\bar{x}}$ denote the following measure on $\mathbb{R}^d$

$$\mathbf{m}_{n,\varepsilon,R,\bar{x}}(\mathfrak{z}) = n^d \left[\int_X \psi\left(\frac{d(x, \bar{x})}{\varepsilon}\right) d\nu(x)\right]^{-4} \iint_{X\times X} \psi\left(\frac{d(x_1, \bar{x})}{\varepsilon}\right)\psi\left(\frac{d(T^n x_1, \bar{x})}{\varepsilon}\right)$$
$$\times \psi\left(\frac{d(x_2, \bar{x})}{\varepsilon}\right)\psi\left(\frac{d(T^{n+1} x_2, \bar{x})}{\varepsilon}\right)\psi\left(\frac{d(S_f(x_1, n), 0)}{R}\right)\psi\left(\frac{d(S_f(x_2, n+1), 0)}{R}\right)$$
$$\times \mathfrak{z}(S_f(x_2, n+1) - S_f(x_1, n))\, d\nu(x_1)\, d\nu(x_2).$$

Call a point $z \in \mathbb{R}^d$ $\bar{x}$-*inessential* if there exists a constant $\eta > 0$ such that for all R

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{m}_{n,\varepsilon,R,\bar{x}}(B(z, \eta)) = 0$$

and call z $\bar{x}$-*essential* otherwise. We claim that if $\bar{x}$ is a continuity point of h, then z is $\bar{x}$-essential if and only if $z \in M + r$. In particular, for ν almost every point $\bar{x} \in X$, the set of $\bar{x}$-essential points equals to $M + r$. This clearly implies the lemma.

To prove the claim, assume first that $z \notin M + r$. Let $\eta = \operatorname{dist}(z, M + r)/2$, $\delta = \eta/7$ and choose ε so that the oscillation of h on $B(\bar{x}, \varepsilon)$ is less than δ. Writing $C_\varepsilon = \int_X \psi(d(x, \bar{x})/\varepsilon)\, d\nu(x)$ and $\rho_{\varepsilon,n}(x) = \psi(d(x, \bar{x})/\varepsilon)\psi(d(T^n x, \bar{x})/\varepsilon)\psi(d(S_f(x, n), 0)/R)$, we have

$$\mathbf{m}_{n,\varepsilon,R,\bar{x}}(B(z, \eta))$$
$$= n^{d/2} C_\varepsilon^{-2} \int_X \rho_{\varepsilon,n}(x_1) n^{d/2} C_\varepsilon^{-2}$$
$$\times \int_X \rho_{\varepsilon,n+1}(x_2) 1_{S_f(x_2,n+1)-S_f(x_1,n)\in B(z,\eta)}\, d\nu(x_2)\, d\nu(x_1).$$

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We can restrict the second integral in the above expression to $\{x_2 \in X : \operatorname{dist}(S_f(x_2, n+1), (n+1)(M+r)) < 3\delta\}$ by making an error that vanishes in the limit $n \to \infty$. Indeed, Lemma 2.1 implies that

$$\limsup_{n\to\infty} n^{d/2}\nu(x_2 : \operatorname{dist}(S_f(x_2, n+1) + h(x_2) - h(T^{n+1}x_2), (n+1)(M+r)) > \delta) = 0$$

and by the choice of ε, $\|h(x_1) - h(T^n x_1)\| < 2\delta$. Similarly, we can restrict the first integral to $\{x_1 \in X : \operatorname{dist}(S_f(x_1, n), n(M+r)) < 3\delta\}$. After these restrictions, the double integral becomes zero since $\eta = 7\delta$. This shows that z is $\bar{x}$-inessential.

Next, assume that $z \in M + r$. Choose R sufficiently large so that for every n there exists $z_1 = z_1(n)$, $z_2 = z_2(n) \in B(0, R)$ so that $z_1 \in n(M+r)$, $z_2 = z_1 + z$ and $\|z_i\| + 1 < R$. Let $u \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ be such that u is constant 1 on $B(0, \eta/4)$ and vanishes outside of $B(0, \eta/2)$. We have

$$\mathbf{m}_{n,\varepsilon,R,\bar{x}}(B(z, \eta)) \geq \prod_{i=1,2} n^{d/2} C_\varepsilon^{-2} \int_X \rho_{\varepsilon,n+i-1}(x_i) u(S_f(x_i, n+i-1) - z_i)\, d\nu(x_i).$$

Lemma 2.1 shows that this product is bounded away from zero as $n \to \infty$ and then $\varepsilon \to 0$. Thus, z is $\bar{x}$-essential. □

In accordance with Lemma 2.2, we will use the notation $M(f)$ and $r(f)$.

Remark 2.1. By standard arguments concerning vague convergence (sometimes called the Portmanteau theorem) one can give equivalent formulations of Definition 2.2 and Lemma 2.1. We will use these versions for convenience.

Definition 2.2 remains unchanged if we choose $\mathfrak{x}$, $\mathfrak{y}$ to be indicators of sets A, B whose boundary has ν-measure zero and $\mathfrak{z}$ to be an indicator of a bounded set $\mathcal{H}$ whose boundary, with respect to the topology on M has u_M-measure zero. For fixed A, B, we can think about the MLCLT as vague convergence of measures.

Similarly, Lemma 2.1 remains valid if we consider the indicator test functions as above, now also satisfying

$$\nu(\partial A) = \nu(\partial B) = (\nu \times \nu)((x, y) : u_M(\partial(\mathcal{H} + h(x) - h(y) \cap M)) > 0) = 0.$$

2.2. *Characterization of mixing.* We impose the following hypotheses.

(H1) (T, τ) satisfies the MLCLT.

(H2) (*Moderate deviation bounds*) For some (and hence for all) R large enough, we have

$$\lim_{K\to\infty} \limsup_{w\to\infty} \sum_{n:|n-w|>K\sqrt{w}} \nu(x : S_\tau(n, x) \in [w\nu(\tau) - R, w\nu(\tau) + R]) = 0,$$

PROPOSITION 2.1. *Assume* **(H1)** *and* **(H2)**. *Then the following are equivalent:*

(a) Φ *is weakly mixing;*

(b) Φ *is mixing;*

(c) *either* $M(\tau) = \mathbb{R}$ *or* $M(\tau) = \alpha\mathbb{Z}$ *and* $r(\tau)/\alpha \notin \mathbb{Q}$.

Clearly (b) implies (a). To prove the proposition we first show in §2.3 that if (c) fails then the flow is not weak mixing, and then we show in §2.4 that (c) implies (b).

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2.3. *Non-mixing case.* Assume that either τ is a coboundary or $M(\tau) = \alpha\mathbb{Z}$ and $r/\alpha \in \mathbb{Q}$, where $r = r(\tau)$. In both cases, there is some h_τ, some rationally related numbers α, r and a function $g : X \to \mathbb{Z}$ such that

$$\tau(x) = h_\tau(x) - h_\tau(Tx) + r + \alpha g(x). \tag{5}$$

(If τ is a coboundary, then $\alpha = 0$.) Note that h_τ is defined up to an additive constant. Choose this constant in such a way that $\nu(Y) > 0$, where $Y = \{x : 0 < h_\tau(x) < \tau(x)\}$ and define $C = \{(x, t) : t = h_\tau(x) \in [0, \tau(x)]\}$. Let $\varsigma : Y \to \mathbb{R}_+$ be defined by

$$\varsigma(x) = \min_{s>0}\{\Phi^s(x, h_\tau(x)) \in C\}.$$

Then (5) gives

$$\varsigma(x) = -nr + \alpha \sum_{i=1}^{n} g(T^{i-1}x)$$

where $n = n(x)$ is the number of hits of the roof before time $\varsigma(x)$. Thus, $\varsigma(x) \in G$, the group generated by r and α. Since α and r are rationally related, there is some $b > 0$ such that $G = b\mathbb{Z}$. Let us fix some $\delta \in (0, b/2)$ and write $c_\delta = \{(x, t) : h_\tau(x) \in [0, \tau(x)], |t - h_\tau(x)| < \delta\}$. By construction, $\mu(C_\delta \cap \Phi^{-t}C_\delta) = 0$ whenever $t \in kb + (\delta, b - \delta)$. This shows that Φ is not weakly mixing.

2.4. *Mixing case.* Assume that $M(\tau) = \mathbb{R}$ or $M(\tau) = \alpha\mathbb{Z}$ and the shift $r = r(\tau)$ associated with τ satisfies $r \notin \mathbb{Q}(\alpha)$. We use the formulation of LCLT given in Remark 2.1.

It is enough to show that

$$\lim_{t\to\infty} \mu(\mathcal{A} \cap \Phi^{-t}\mathcal{B}) = \mu(\mathcal{A})\mu(\mathcal{B}) \tag{6}$$

in the case where $\mathcal{A} = A \times I$, $\mathcal{B} = B \times J$. Decompose

$$\tau(x) = \hat{\tau}(x) - h_\tau(x) + h_\tau(Tx)$$

where h_τ is given in (3). Let us fix some $s \in I$ and write

$$N_u = N_u(x) = \max\{n : S_\tau(n, x) \le u\}. \tag{7}$$

Then

$$\mu(\mathcal{A} \cap \Phi^{-t}\mathcal{B}) = \frac{1}{\nu(\tau)} \int_I \nu\big(x \in A : S_\tau(N_{t+s}, x) - t \in -J + s,\, T^{N_{t+s}(x)}(x) \in B\big)\, ds.$$

For a fixed s, let $C(s)$ be the set of points $(x, z, y) \in X \times \mathbb{R} \times X$ that satisfy:
(1) $x \in A$;
(2) $y \in B$;
(3) $z \in -J + s + h_\tau(x) - h_\tau(y)$.
Then we have

$$\mu(\mathcal{A} \cap \Phi^{-t}\mathcal{B}) = \int_I \frac{1}{\nu(\tau)} \nu(x : (x, S_{\hat{\tau}}(N_{t+s}, x) - t, T^{N_{t+s}}(x)) \in C(s))\, ds.$$

Let

$$C_n(s) = \{x : (x, S_{\hat{\tau}}(n, x) - t, T^n(x)) \in C(s)\}.$$

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Observe that $x \in C_n(s)$ implies $t+s-S_\tau(n,x) \in J$ and consequently $N_{t+s}(x) = n$. Thus,

$$\mu(\mathcal{A} \cap \Phi^{-t}\mathcal{B}) = \int_I \frac{1}{\nu(\tau)} \nu(x : x \in C_{N_{t+s}}(s))\, ds = \int_I \sum_{n=1}^{t/\inf\tau} \frac{1}{\nu(\tau)} \nu(C_n(s))\, ds.$$

We write the above sum as $S_1 + S_2$, where

$$S_1 = \sum_{n:|n-t/\nu(\tau)|<K\sqrt{t}} \int \cdots$$

with $K \gg 1$ and S_2 is an error term (which is small by **(H2)**). Now let us apply **(H1)** (and Lemma 2.1) to compute S_1.

Assume first that $M(\tau) = \mathbb{R}$. Observe that $(\nu \times \mathrm{Leb}_1 \times \nu)(C(s)) = \nu(A)|J|\nu(B)$. Now **(H1)** implies that for any fixed $s \in I$, for any n with $|n - t/\nu(\tau)| < K\sqrt{t}$,

$$\nu(C_n(s)) = \frac{1}{\sqrt{n}} \left[\mathfrak{g}_\sigma \left(\frac{m}{\sqrt{n}} \nu(\tau) \right) + o(1) \right] \nu(A)|J|\nu(B), \tag{8}$$

where σ is the variance of τ and $m = \lfloor t/\nu(\tau) \rfloor - n$ and the $o(1)$ term converges to 0 uniformly for all n in the above range and $s \in I$. Integrating with respect to s we get

$$S_1 = \left[\sum_{n:|n-t/\nu(\tau)|<K\sqrt{t}} \frac{1}{\nu(\tau)\sqrt{n}} \mathfrak{g}_\sigma \left(\frac{m}{\sqrt{n}} \nu(\tau) \right) \right] \nu(A)|J|\nu(B)|I| + o(1). \tag{9}$$

Substituting a Riemann sum with the Riemann integral, we find that

$$S_1 \sim \mu(\mathcal{A})\mu(\mathcal{B})(1 + o_{K\to\infty}(1)). \tag{10}$$

On the other hand, by **(H2)** $S_2 \to 0$ as $K \to \infty$. This proves (6).

In the case $M(\tau) = \alpha\mathbb{Z}$, $r \notin \mathbb{Q}(\alpha)$ we apply a similar approach with the following differences. Equation (9) is replaced by

$$\sum_{n:|n-t/\nu(\tau)|<K\sqrt{t}} \frac{1}{\nu(\tau)\sqrt{n}} \mathfrak{g}_\sigma \left(\frac{m}{\sqrt{n}} \nu(\tau) \right) \int_I (\nu \times u_{\alpha\mathbb{Z}} \times \nu)(C(s) + \varkappa_n)\, ds \tag{11}$$

where $\varkappa_n \in \mathbb{R}/(\alpha\mathbb{Z})$ is defined by

$$\varkappa_n = t - nr \pmod{\alpha}$$

and $C(s) + \varkappa$ is defined as

$$\{(x, z + \check{\varkappa}, y) : (x, z, y) \in C(s)\},$$

with some $\check{\varkappa} \in \mathbb{R}$, $\check{\varkappa} + \alpha\mathbb{Z} = \varkappa$ (since u is the Haar measure, (11) does not depend on the choice of the representative). Now writing the sum in (11) as

$$\sum_{j=0}^{2K\sqrt{t}/N-1} \sum_{m=-K\sqrt{t}+jN}^{-K\sqrt{t}+(j+1)N-1},$$

for some large N and using Weyl's theorem, we conclude that (10) still holds. Then we can complete the proof as before.

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3. *From LLT for base map to LLT for flows*

3.1. *Definitions, assumptions.* Given an observable $\boldsymbol{\varphi} : \Omega \to \mathbb{R}$ let

$$\check{\boldsymbol{\varphi}}(x) = \int_0^{\tau(y)} \boldsymbol{\varphi}(x, s)\, ds. \tag{12}$$

We impose the following hypotheses.

(H3) Φ is mixing.

(H4) $\mu(\boldsymbol{\varphi}) = 0$.

(H5) $\boldsymbol{\varphi}$ is bounded and μ-almost everywhere continuous.

(H6) $(T, (\check{\boldsymbol{\varphi}}, \tau))$ satisfies the MLCLT.

(H7) (*Moderate deviation bounds*) For $f = (\tau, \check{\boldsymbol{\varphi}})$ and for some (and hence for all) R large enough, we have

$$\lim_{K\to\infty} \limsup_{w\to\infty} w^{1/2} \sum_{n:|n-w|>K\sqrt{w}} \nu(x : S_f(n, x) \in B(w\nu(f), R)) = 0,$$

where $B(v, R)$ is the ball of radius R centered at $v \in \mathbb{R}^d$.

Now we define the MLCLT for the flow for d-dimensional observables (most of this paper investigates the case $d = 1$).

Definition 3.1. We say that $(\Phi, \boldsymbol{\varphi})$ ($\boldsymbol{\varphi} : \Omega \to \mathbb{R}^d$ square integrable) satisfies the MLCLT if there exists some closed subgroup $\mathcal{V} \subset \mathbb{R}^d$, $R \in \mathbb{R}^d/\mathcal{V}$, a μ-almost everywhere continuous function $H : \Omega \to \mathbb{R}^d$, bounded on $\{(x, s) \in \Omega, s \le M\}$ for all $M > 0$ and a positive definite matrix $\Sigma = \Sigma(\boldsymbol{\varphi})$ such that, as $t \to \infty$, the following holds for any $M > 0$, any bounded and continuous $\mathfrak{X}, \mathfrak{Y} : \{(x, s) \in \Omega, s \le M\} \to \mathbb{R}$ and for any continuous and compactly supported $\mathfrak{Z} : \mathbb{R}^d \to \mathbb{R}$:

$$t^{d/2} \int_\Omega \mathfrak{X}(x, s)\mathfrak{Y}(\Phi^t(x, s))\mathfrak{Z}(\mathcal{F}(x, s))\, d\mu(x, s) \to \mathfrak{g}_\Sigma(W) \int_\Omega \mathfrak{X}\, d\mu \int_\Omega \mathfrak{Y}\, d\mu \int_{\mathcal{V}} \mathfrak{Z}\, du_{\mathcal{V}}.$$

Here,

$$\mathcal{F}(x, s) = \int_0^t \boldsymbol{\varphi}(\Phi^{s'}(x, s))\, ds' + H(x, s) - H(\Phi^t(x, s)) - W(t), \tag{13}$$

and $W(t)$ is assumed to satisfy

$$W(t) \in \mathcal{V} + Rt, |W(t) - W\sqrt{t}| \le K. \tag{14}$$

We also require that the convergence is uniform once K is fixed and W is chosen from a compact set.

We can omit the coboundary term similarly to the case of the map (cf. Lemma 2.1).

LEMMA 3.1. *Assume that* $(\Phi, \boldsymbol{\varphi})$ *satisfies the MLCLT. Then with the notation of Definition 3.1,*

$$\begin{aligned}\lim_{t\to\infty} t^{d/2} \int_\Omega \mathfrak{X}(x, s)\mathfrak{Y}(\Phi^t(x, s))\mathfrak{Z}\left(\int_0^t \boldsymbol{\varphi}(\Phi^{s'}(x, s))\, ds' - W(t)\right) d\mu(x, s) \\ = \mathfrak{g}_\Sigma(W) \int_{\Omega\times\mathcal{V}\times\Omega} \mathfrak{X}(x, s)\mathfrak{Y}(y, s')\mathfrak{Z}(z - H(x, s) + H(y, s')) \\ \times d(\mu(x, s) \times u_{\mathcal{V}}(z) \times \mu(y, s')).\end{aligned} \tag{15}$$

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Before proceeding to the MLCLT for the flow, we make some important remarks.

Remark 3.1. As Lemma 3.1 shows, we can assume that H take values in $\mathbb{R}^d/\mathcal{V}$.

Remark 3.2. Definition 3.1 becomes simpler if τ is bounded. Indeed, in this case $\mathfrak{X}$ and $\mathfrak{Y}$ are just any bounded and continuous functions on Ω. (In general, X is not compact.) If τ is unbounded, we only consider the test functions given in Definition 3.1 in the first (abstract) part of the paper. However, later we will extend the convergence to any bounded and continuous $\mathfrak{X}$ and $\mathfrak{Y}$ in important applications (namely, first return map to the base in Young towers, see Proposition 4.3).

Remark 3.3. Similarly to Remark 2.1, we have the following reformulations of Definition 3.1 and Lemma 3.1.

Definition 3.1 remains unchanged if we choose $\mathfrak{X}$, $\mathfrak{Y}$ to be indicators of sets $\mathcal{A}$, $\mathcal{B}$ whose boundary has μ-measure zero and $\mathfrak{Z}$ to be an indicator of a bounded set $\mathcal{H}$ whose boundary, with respect to the topology on $\mathcal{V}$ has $u_{\mathcal{V}}$-measure zero. Furthermore, it suffices to consider indicators of product sets $\mathfrak{X} = 1_{\mathcal{A}}$, where $\mathcal{A} = A \times I$, $\nu(\partial A) = 0$ and I is a subinterval of $[0, \inf\{\tau(x), x \in A\}]$. Similarly, we can choose $\mathfrak{Y} = 1_{\mathcal{B}}$, where $\mathcal{B} = B \times J$, $\nu(\partial B) = 0$ and J is a subinterval of $[0, \inf\{\tau(x), x \in B\}]$.

Also Lemma 3.1 remains valid if we consider the indicator test functions as above, satisfying

$$\nu(\partial A) = \nu(\partial B) = \mu \times \mu((x, s, y, s') : u_{\mathcal{V}}(\partial(\mathcal{H} + H(x, s) - H(y, s') \cap \mathcal{V})) > 0) = 0.$$

3.2. *The linearized group.* Consider $M = M(\tau, \check{\boldsymbol{\varphi}})$ and $r = r(\tau, \check{\boldsymbol{\varphi}})$. The *linearized group* of $(\tau, \check{\boldsymbol{\varphi}})$ is the closure of the group generated by M and r. We denote this group by $\hat{\mathcal{V}}$. Define $\varkappa_n \in \mathbb{R}^2/M$ by

$$\varkappa_n = -nr \pmod{M}$$

and write $M = Y \times L$, where Y is a subspace of dimension d_1 and L is a lattice of dimension d_2 with $d_1, d_2 \in \{0, 1, 2\}$, $d_1 + d_2 = 2$. The self map of $\mathbb{R}^2/M$, defined by $\varkappa \mapsto \varkappa - r$ is linearly conjugate to a translation of the d_2-dimensional torus. Consequently, the closure of any orbit is a subtorus. Furthermore, the orbit is uniformly distributed on this subtorus by Weyl's theorem. We make the following conclusion.

LEMMA 3.2. *$(1/N)\sum_{n=1}^{N} u_M(\cdot + \check{\varkappa}_n)$ converges weakly to $u_{\hat{\mathcal{V}}}$ as $N \to \infty$, where $\check{\varkappa}_n \in \mathbb{R}^2$ satisfies $\check{\varkappa}_n + M = \varkappa_n$.*

Note that by Proposition 2.1 and by the definition of the linearized group, the projection of $\hat{\mathcal{V}}$ to the second coordinate is dense. Consequently, under assumptions **(H1)**–**(H7)**, one of the following cases holds:

(A) $\hat{\mathcal{V}} = \mathbb{R}^2$;

(B) $\hat{\mathcal{V}} = a\mathbb{Z} \times \mathbb{R}$;

(C) $\hat{\mathcal{V}} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 - \alpha x_1 \in \beta\mathbb{Z}\}$ with some $\beta \neq 0$;

(D) $\hat{\mathcal{V}}$ is generated by (a, b) and $(0, d)$, where b/d is irrational. We can assume $a, d > 0$;

(E) $\hat{\mathcal{V}}$ is generated by (a', b') and (c', d'), where a'/c' and b'/d' are irrational. We can assume $d' > 0$ and $a'd' - b'c' > 0$.

We can interpret cases **(A)** and **(B)** as arithmetic independence between $\check{\boldsymbol{\varphi}}$ and τ, as $\hat{\mathcal{V}} = \pi_1\hat{\mathcal{V}} \times \pi_2\hat{\mathcal{V}}$, where π_i is the projection to the ith coordinate.

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3.3. *MLCLT for* Φ. To fix notation, we write the decomposition (3) for the functions $\check{\boldsymbol{\varphi}}$ and τ as

$$\check{\boldsymbol{\varphi}}(x) = \psi(x) - h(x) + h(Tx), \quad \text{and} \tag{16}$$

$$\tau(x) = \hat{\tau}(x) - h_\tau(x) + h_\tau(Tx). \tag{17}$$

THEOREM 3.1. *Assume hypotheses* **(H1)–(H7)**. *In cases* **(A)–(C)**, $(\Phi, \boldsymbol{\varphi})$ *satisfies the MLCLT with* $\mathcal{V}$, R *and* H *given by the following:*

(A) $\mathcal{V} = \mathbb{R}$, $R = 0$, $H(x, s) = 0$;

(B) $\mathcal{V} = a\mathbb{Z}$, $R = 0$; $H_B(x, s) = \int_0^s \boldsymbol{\varphi}(x, s')\, ds' + h(x) \pmod{\mathcal{V}}$,

(C) $\mathcal{V} = (\beta/\alpha)\mathbb{Z}$, $R = 1/\alpha$,

$$H_C(x, s) = \int_0^s \boldsymbol{\varphi}(x, s')\, ds' + h(x) - \frac{1}{\alpha}(s + h_\tau(x)) \pmod{\mathcal{V}}.$$

In all the above cases, $\Sigma(\boldsymbol{\varphi}) = (1/\nu(\tau))(\Sigma(\check{\boldsymbol{\varphi}}, \tau))_{11}$.

In Case **(D)**, a weaker version of Theorem 3.1 holds. Namely, the limiting measure depends on $t \pmod d$ and is not a product over $\Omega \times a\mathbb{Z} \times \Omega$. Let us denote by $\Xi_t = \Xi_{t, W(t), \boldsymbol{\varphi}, H}$ the push forward of the measure μ by the map

$$(x, s) \mapsto \left((x, s), \int_0^t \boldsymbol{\varphi}(\Phi^{s'}(x, s))\, ds' + H(x, s) - H(\Phi^t(x, s)) - W(t), \Phi^t(x, s)\right).$$

For simplicity, we only consider the case when:

(H8) τ is bounded.

PROPOSITION 3.1. *Assume conditions* **(H1)–(H8)** *and Case* **(D)**. *Recall the notation introduced in Remark 2.1. Then for any* $l \in \mathbb{Z}$ *and any* $W(t) \in a\mathbb{Z}$ *with* $W(t) \sim W\sqrt{t}$, *we have*

$$\lim_{t\to\infty} |\sqrt{t}\, \Xi_{t, W(t), \boldsymbol{\varphi}, H}(\mathcal{A} \times \{la\} \times \mathcal{B}) - \mathcal{I}_t| = 0,$$

where $H = \int_0^s \boldsymbol{\varphi}(x, s')\, ds' + h(x) \pmod a$,

$$\mathcal{I}_t = \frac{ad\mathfrak{g}_\Sigma(W)}{\nu(\tau)} \sum_{|k| \le (\|\tau\|_\infty + 2\|h_\tau\|_\infty + 1)/d} \int_{(x,s)\in\mathcal{A}} \int_{y\in B} 1_{\{\rho + kd + h_\tau(x) - h_\tau(y) \in J\}} \times d\nu(y)\, d\mu(x, s), \tag{18}$$

and $\rho \in [0, d)$ *satisfies*

$$\rho \equiv s + t - \left(\frac{W(t)}{a} + l\right) b \pmod d. \tag{19}$$

Note that in the special case when τ is minimal and thus h_τ can be chosen to be zero, the formula (18) reduces to

$$\mathcal{I}_t = \frac{\nu(A)}{\nu(\tau)} \mathfrak{g}_\Sigma(W) ad \int_I \mathrm{Card}(m : \rho + md \in J)\, ds \frac{\nu(B)}{\nu(\tau)}. \tag{20}$$

This formula is consistent with Theorem 3.1**(B)** in the sense that for I and J fixed and $d \ll 1$, $d \int_I \mathrm{Card}(m : \rho + md \in J)\, ds \approx |I||J|$ and $u_{a\mathbb{Z}}$ is a times the counting measure. Thus, recalling Remark 2.1, we recover Theorem 3.1**(B)** in the limit $d \searrow 0$.

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PROPOSITION 3.2. *Assume conditions* **(H1)–(H8)** *and Case* **(E)**. *Then the statement of Proposition 3.1 remains valid with the following changes:*

(1) $a := a' - c'd'/b'$, $b := b'$, $d := d'$;
(2) $W(t) \in a\mathbb{Z} + (c'/d')t$;
(3) $H_E(x, s) = \int_0^s \boldsymbol{\varphi}(x, s')\, ds' + h(x) - (c'/d')(s + h_\tau(x)) \pmod a$;
(4) *equation (19) has to be replaced by* $\rho \equiv s + t - ((W(t) - c't/d')/a + l)b \pmod d$.

Remark 3.4. Case **(C)** can be reduced to Case **(B)** and Case **(E)** can be reduced to Case **(D)** by applying the shear $\left[\begin{smallmatrix} 1 & -v \\ 0 & 1 \end{smallmatrix}\right]$ to $\hat{\mathcal{V}}$ with $v = 1/\alpha$ in Case **(C)** and with $v = c'/d'$ in Case **(E)** (this will produce a non-zero R in the MLCLT and in its weaker form as in Propositions 3.1 and 3.2). We note that while the shear is uniquely determined in Case **(C)**, it is not unique in Case **(E)**.

3.4. *Higher dimensions.* Here we state the high-dimensional generalization of Theorem 3.1. We omit the proof as it is analogous to the proof of the one-dimensional case. We need to replace $w^{1/2}$ by $w^{d/2}$ in **(H7)**. Observe that we can construct the group $\hat{\mathcal{V}}$ exactly as before as Weyl's theorem holds in any dimension. Now we have the following.

THEOREM 3.2. *Assume* **(H1)–(H8)**. *Assume furthermore that there is a closed subgroup* $\mathcal{V} \subset \mathbb{R}^d$, *a* $(d+1) \times (d+1)$ *matrix of the form* $A = \left[\begin{smallmatrix} I_d & -\boldsymbol{v} \\ \mathbf{0}^T & 1 \end{smallmatrix}\right]$, *with* $\boldsymbol{v} \in \mathbb{R}^d$ *such that* $A\hat{\mathcal{V}} = \mathcal{V} \times \mathbb{R}$ *(we assume that* $\boldsymbol{v}$ *is orthogonal to the linear subspace contained in* $\mathcal{V}$, *the choice of such* $\boldsymbol{v}$ *is unique). Then* $(\Phi, \boldsymbol{\varphi})$ *satisfies the MLCLT with* $\mathcal{V}$, $R = \boldsymbol{v} + \mathcal{V}$ *and*

$$H(x, s) = \int_0^s \boldsymbol{\varphi}(x, s')\, ds' + h(x) - v(s + h_\tau(x)) \pmod{\mathcal{V}}.$$

3.5. *Proof of Theorem 3.1.* The proof of Theorem 3.1 is similar to the proof that (c) implies (b) in Proposition 2.1. The main difference is that we apply **(H6)** instead of **(H1)**.

Recall the notation introduced in (16) and (17) and write

$$\hat{H}(x, s) = \int_0^s \boldsymbol{\varphi}(x, s')\, ds' + h(x). \tag{21}$$

Recall (7). By construction, for any $(x, s) \in \Omega$ and any $t > 0$, we have

$$\int_0^t \boldsymbol{\varphi}(\Phi^{s'}(x, s))\, ds' + \hat{H}(x, s) - \hat{H}(\Phi^t(x, s)) \tag{22}$$

$$= S_{\check{\boldsymbol{\varphi}}}(N_{t+s}, x) + h(x) - h(T^{N_{t+s}}x)$$

$$= S_\psi(N_{t+s}, x). \tag{23}$$

According to Remark 3.3, we choose $\mathfrak{X} = 1_{\mathcal{A}}$, $\mathfrak{Y} = 1_{\mathcal{B}}$ where $\mathcal{A} = A \times I$, $\mathcal{B} = B \times J$, and $\mathfrak{Z} = 1_{\mathcal{H}}$, where $\mathcal{H} \subset \mathcal{V}$. Without loss of generality, we can assume that $\mathcal{H}$ is a compact interval in Cases **(A)** and **(B)** and $|\mathcal{H}| = 1$ in Case **(C)**.

Recall the definition of Ξ_t and write $\mathcal{C} = \mathcal{A} \times \mathcal{H} \times \mathcal{B}$. We have

$$\Xi_t(\mathcal{C}) = \mu((x, s) \in \mathcal{A} : S_\psi(N_{t+s}, x) - W(t) \in \mathcal{H},\ \Phi^t(x, s) \in \mathcal{B}) \tag{24}$$

$$= \frac{1}{\nu(\tau)} \int_I \nu(x \in A : S_\psi(N_{t+s}, x) - W(t) \in \mathcal{H}, \tag{25}$$

$$S_\tau(N_{t+s}, x) - t \in -J + s,\ T^{N_{t+s}(x)}(x) \in B)\, ds.$$

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Now for some fixed $s \in I$, we define $C(s)$ to be the set of points (x, z, y), $x, y \in X$, $z = (z_1, z_2) \in \mathbb{R}^2$ that satisfy

$$x \in A; \quad y \in B; \quad z_1 \in \mathcal{H}; \quad z_2 \in -J + s + h_\tau(x) - h_\tau(y).$$

Then we have

$$\Xi_t(\mathcal{C}) = \int_I \frac{1}{\nu(\tau)} \nu(x : (x, S_\psi(N_{t+s}, x) - W(t), S_{\hat{\tau}}(N_{t+s}, x) - t, T^{N_{t+s}}(x)) \in C(s))\, ds.$$

Let

$$C_n(s) = \{x : (x, S_\psi(n, x) - W(t), S_{\hat{\tau}}(n, x) - t, T^n(x)) \in C(s)\}.$$

Observe that $x \in C_n(s)$ implies $t + s - S_\tau(n, x) \in J$. Hence, $N_{t+s}(x) = n$. Therefore,

$$\Xi_t(\mathcal{C}) = \int_I \frac{1}{\nu(\tau)} \nu(x : x \in C_{N_{t+s}}(s))\, ds = \int_I \sum_{n=1}^{t/\inf\tau} \frac{1}{\nu(\tau)} \nu(C_n(s))\, ds. \tag{26}$$

By **(H4)**, $\nu(\check{\boldsymbol{\varphi}}, \tau) = (0, \nu(\tau))$. We write the above sum as $S_1 + S_2$, where

$$S_1 = \sum_{n:|n - t/\nu(\tau)| < K\sqrt{t}} \int \cdots$$

with $K \gg 1$ and S_2 is an error term which is small by **(H7)**. It suffices to compute S_1.

Let us first study the special case when $M(\check{\boldsymbol{\varphi}}, \tau) = \mathbb{R}^2$. We will refer to this case as *non-arithmetic*. Clearly, the non-arithmeticity implies Case **(A)**.

3.5.1. *The non-arithmetic case.* We assume $M(\check{\boldsymbol{\varphi}}, \tau) = \mathbb{R}^2$. Clearly, $r = 0$, $\mathcal{V} = \mathbb{R}$ and $R = 0$.

For a fixed K, **(H6)** implies that

$$S_1 \sim \sum_{n:|n - t/\nu(\tau)| < K\sqrt{t}} \frac{1}{\nu(\tau) n} \mathfrak{g}^* \int_I (\nu \times \mathrm{Leb}_2 \times \nu)(C(s))\, ds, \tag{27}$$

where $m = \lfloor t/\nu(\tau) \rfloor - n$ and $\mathfrak{g}^* = \mathfrak{g}_\sigma(-W\sqrt{\nu(\tau)}, (m/\sqrt{n})\nu(\tau))$.

To compute $\int_I (\nu \times \mathrm{Leb}_2 \times \nu)(C(s))\, ds$, let us evaluate the integral with respect to z_2 first. For any s, x, y fixed, this integral is equal to $|J|$. Consequently,

$$\int_I (\nu \times \mathrm{Leb}_2 \times \nu)(C(s))\, ds = |I| \nu(A) \mathrm{Leb}_1(\mathcal{H}) |J| \nu(B) = (\nu(\tau))^2 \mu(\mathcal{A}) \mathrm{Leb}_1(\mathcal{H}) \mu(\mathcal{B}).$$

Substituting the Riemann sum in (27) with a Riemann integral, we obtain

$$S_1 \sim \left(\frac{\nu(\tau)}{t}\right)^{1/2} \left[\int_{-K\sqrt{\nu(\tau)}}^{K\sqrt{\nu(\tau)}} \mathfrak{g}_\sigma(-W\sqrt{\nu(\tau)}, \mathfrak{z}\nu(\tau))\, d\mathfrak{z}\right] \nu(\tau) \mu(\mathcal{A}) \mathrm{Leb}_1(\mathcal{H}) \mu(\mathcal{B}), \tag{28}$$

where $\sigma \in GL(2, \mathbb{R})$ is the covariance matrix of $(\check{\boldsymbol{\varphi}}, \tau)$. Let us write $\sigma' = \sigma_{11}$. By the properties of the Gaussian distribution,

$$\int_{-K\sqrt{\nu(\tau)}}^{K\sqrt{\nu(\tau)}} \mathfrak{g}_\sigma(-W\sqrt{\nu(\tau)}, \mathfrak{z}\nu(\tau))\, d\mathfrak{z} = \frac{1}{\nu(\tau)} \mathfrak{g}_{\sigma'}(-W\sqrt{\nu(\tau)})(1 + o_{K\to\infty}(1)),$$

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and

$$\mathfrak{g}_{\sigma'}(-W\sqrt{\nu(\tau)}) = \frac{1}{\sqrt{\nu(\tau)}}\mathfrak{g}_{1/\nu(\tau)\sigma'}(W).$$

Substituting the last two displays into (28) gives

$$S_1 \sim \frac{1}{t^{1/2}}\mathfrak{g}_{(1/\nu(\tau))\sigma'}(W)\mu(\mathcal{A})\mathrm{Leb}_1(\mathcal{H})\mu(\mathcal{B})(1+o_{K\to\infty}(1)).$$

This implies

$$\Xi_t(\mathcal{C}) \sim \frac{1}{t^{1/2}}\mathfrak{g}_{(1/\nu(\tau))\sigma'}(W)\mu(\mathcal{A})\mathrm{Leb}_1(\mathcal{H})\mu(\mathcal{B}).$$

Noting that $H = \hat{H} \pmod{\mathcal{V}}$, we obtain the MLCLT with variance $\Sigma = (1/\nu(\tau))\sigma'$.

3.5.2. *Cases (A) and (B).* We follow the strategy of §3.5.1. The main difference is that now (27) is replaced by

$$\sum_{n:|n-t/\nu(\tau)|<K\sqrt{t}} \frac{1}{\nu(\tau)n}\mathfrak{g}^*\int_I (\nu\times u_M\times\nu)(C(s)+\varkappa_n)\,ds \tag{29}$$

where $M = M(\check{\boldsymbol{\varphi}},\tau)$, $r = r(\check{\boldsymbol{\varphi}},\tau)$, $\varkappa_n \in \mathbb{R}^2/M$ is defined by

$$\varkappa_n = (W(t),t) - nr \pmod{M}$$

and $C(s)+\varkappa$ is defined as

$$\{(x,z+\check{\varkappa},y) : (x,z,y)\in C(s)\},$$

with some $\check{\varkappa}\in\mathbb{R}^2$, $\check{\varkappa}+M=\varkappa$ (since u is the Haar measure, (29) does not depend on the choice of the representative).

Recall the definition of the linearized group $\hat{\mathcal{V}}$ and the notation $m = \lfloor t/\nu(\tau)\rfloor - n$. Writing the sum in (29) as

$$\sum_{j=0}^{2K\sqrt{t}/N-1}\Bigg(\sum_{m=-K\sqrt{t}+jN}^{-K\sqrt{t}+(j+1)N-1}\cdots\Bigg),$$

and using Lemma 3.2, we conclude that

$$S_1 \sim \sum_{n:|n-t/\nu(\tau)|<K\sqrt{t}} \frac{1}{\nu(\tau)n}\mathfrak{g}^*\int_I (\nu\times u_{\hat{\mathcal{V}}}\times\nu)(C(s)+(W(t),t))\,ds. \tag{30}$$

In Cases **(A)** and **(B)**, $\hat{\mathcal{V}} = \mathcal{V}\times\mathbb{R}$ and $W(t)\in\mathcal{V}$. Consequently,

$$(W(t),t)\in\hat{\mathcal{V}}. \tag{31}$$

We need to compute

$$\int_I (\nu\times u_{\hat{\mathcal{V}}}\times\nu)(C(s)+(W(t),t))\,ds = \int_I (\nu\times u_{\hat{\mathcal{V}}}\times\nu)(C(s))\,ds. \tag{32}$$

Integrating with respect to z_2 we conclude that (32) is equal to

$$|I|\nu(A)u_{\mathcal{V}}(\mathcal{H})|J|\nu(B).$$

The rest of the proof is identical to §3.5.1.

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3.5.3. *Case (C).* We need to adjust the above proof to cover Case **(C)** as the measure $u_{\hat{\mathcal{V}}}$ is not a product in the coordinates z_1, z_2. Since the proof is similar, we just list the required modifications. First, $\hat{H}$ is replaced by

$$\int_0^s \boldsymbol{\varphi}(x, s')\, ds' + h(x) - \frac{1}{\alpha}(s + h_\tau(x)).$$

Then we need to replace $S_\psi(N_{t+s}, x)$ by

$$S_\psi(N_{t+s}, x) - \frac{1}{\alpha}[S_{\hat{\tau}}(N_{t+s}, x) - t]$$

in formulas (24) and (25). Also, we replace $C(s)$ by the set of points (x, z, y), such that $x, y \in X$, $z = (z_1, z_2)$ and

$$x \in A; \quad y \in B; \quad z_2 \in -J + s + h_\tau(x) - h_\tau(y); \quad z_1 \in \mathcal{H} + \frac{1}{\alpha} z_2.$$

With this modification, we repeat the previous proof up to the derivation of the formula (30). Note that (31) and (32) hold as well since $W(t) \in (\beta/\alpha)\mathbb{Z} + t(1/\alpha)$. Now we can easily compute (32). Indeed, for any s, x, y fixed, we have

$$u_{\hat{\mathcal{V}}}\left((z_1, z_2) : z_2 \in -J + s + h_\tau(x) - h_\tau(y),\, z_1 \in \mathcal{H} + \frac{1}{\alpha} z_2\right) = u_{\mathcal{V}}(\mathcal{H})|J|$$

and we can complete the proof as before.

3.6. *Cases (D) and (E).*

Proof of Proposition 3.1. The proof is similar to the one in §3.5. That is, we define N_{t+s} by (7) as before and analogously to (25), we have

$$\begin{aligned}
&\Xi_t(\mathcal{A} \times \{la\} \times \mathcal{B}) \\
&\quad = \frac{1}{\nu(\tau)} \int_I \nu(x \in A : S_\psi(N_{t+s}, x) - W(t) = la, \\
&\qquad S_{\hat{\tau}}(N_{t+s}, x) - t \in -J + s + h_\tau(x) - h_\tau(T^{N_{t+s}}(x)),\, T^{N_{t+s}}(x) \in B)\, ds. \qquad (33)
\end{aligned}$$

The main difference from §3.5 is that the set of z_2's such that $(W(t) + la, z_2) \in \hat{\mathcal{V}}$ is discrete (and not an interval). We will index this set by k. Fix some $s \in I$. Given $x \in A$ and k with $|k| \le (\|\tau\|_\infty + 2\|h_\tau\|_\infty + 1)/d$, let

$$B_{k,x} = \{y \in B : \rho + kd + h_\tau(x) - h_\tau(y) \in J\}$$

and

$$C_{n,k}(s) = \{x \in A : S_\psi(n, x) - W(t) = la,\, s + t - S_{\hat{\tau}}(n, x) = \rho + kd,\, T^n x \in B_{k,x}\}.$$

Observe that $x \in C_{n,k}(s)$ implies

$$0 \le \rho + kd + h_\tau(x) - h_\tau(T^n x) = s + t - S_\tau(n, x) < \tau(T^n x),$$

and consequently $n = N_{t+s}(x)$. It follows that

$$C_{n,k}(s) \cap C_{n',k'}(s) = \emptyset \quad \text{unless } n = n' \quad \text{and} \quad k = k'. \qquad (34)$$

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Let $C^{(k)} = \{(x, la, s - \rho - kd, y) : x \in A, y \in B_{k,x}\}$. Then,

$$C_{n,k}(s) = \{x : (x, S_\psi(n, x) - W(t), S_{\hat{\tau}}(n, x) - t, T^n(x)) \in C^{(k)}\}. \tag{35}$$

Note that by the definition of ρ, $(W(t) + la, s + t - \rho - kd) \in \hat{\mathcal{V}}$. Thus, we can use **(H6)** similarly to the proof of Theorem 3.1, to deduce

$$\nu(x : x \in C_{N_{t+s},k}(s)) \sim \frac{\mathfrak{g}_\Sigma(W)}{\nu(\tau)\sqrt{t}}(\nu \times \nu)(x \in A, y \in B_{k,x}) \operatorname{vol}(\mathbb{R}^2/\hat{\mathcal{V}}).$$

Next, we have by (34) that

$$\nu\left(x : x \in \bigcup_k C_{N_{t+s},k}(s)\right) = \sum_k \nu(x : x \in C_{N_{t+s},k}(s)).$$

We obtain the result by integrating with respect to $s \in I$ and using that $\operatorname{vol}(\mathbb{R}^2/\hat{\mathcal{V}}) = ad$ and $\mu = \nu \otimes \mathrm{Leb}_1/\nu(\tau)$. □

Proof of Proposition 3.2. Case **(E)** can be reduced to Case **(D)** similarly to the reduction of Case **(C)** to Case **(B)**. We only list the adjustments needed to the proof of Proposition 3.1. First, we replace $S_\psi(N_{t+s}, x) - W(t)$ by

$$S_\psi(N_{t+s}, x) - W(t) - \frac{c'}{d'}[S_{\hat{\tau}}(N_{t+s}, x) - t]$$

in (33) and in the definition of $C_{n,k}(s)$. Then, we replace $C^{(k)}$ by

$$\left\{\left(x, la + \frac{c'}{d'}(s - \rho - kd), s - \rho - kd, y\right) : x \in A, y \in B_{k,x}\right\}.$$

Note that (35) is unchanged. Finally, instead of $(W(t) + la, s + t - \rho - kd) \in \hat{\mathcal{V}}$ we have

$$\left(W(t) + la + \frac{c'}{d'}(s - \rho - kd), s + t - \rho - kd\right) \in \hat{\mathcal{V}}.$$

The last inclusion holds by conditions (2) and (4) of Proposition 3.2. Now we can apply **(H6)** as before. □

4. *Expanding Young towers*

4.1. *Setup and results.* Let $(\Delta, \tilde{\nu})$ be a probability space with a partition $(\Delta_{k,l})_{k\in I, l\le r_k}$ into positive measure subsets, where I is either finite or countable and $r_k = r(\Delta_{0,k})$ is a positive integer. Let $F : \Delta \to \Delta$ be a map satisfying the following.

(A1) For every $i \in I$ and $0 \le j < r_i - 1$, F is a measure-preserving isomorphism between $\Delta_{i,j}$ and $\Delta_{i,j+1}$.

(A2) For every $i \in I$, F is an isomorphism between Δ_{i,r_i-1} and

$$X := \Delta_0 := \bigcup_{i\in I} \Delta_{i,0}.$$

(A3) Let $r(x) = r(\Delta_{0,k})$ if $x \in \Delta_{0,k}$ and $T : X \to X$ be the first return map to the base, i.e. $T(x) = F^{r(x)}(x)$. Let $s(x, y)$, the separation time of $x, y \in X$, be defined as the smallest integer n such that $T^n x \in \Delta_{0,i}$, $T^n y \in \Delta_{0,j}$ with $i \ne j$. As $T : \Delta_{0,i} \to X$ is an isomorphism, it has an inverse. Denote by g the Jacobian of this inverse (with respect to the measure $\tilde{\nu}$). Then there are constants $\vartheta < 1$ and $C > 0$ such that for every $x, y \in \Delta_{0,i}$, $|\log g(x) - \log g(y)| \le C\vartheta^{s(x,y)}$.

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(A4) Extend s to Δ by setting $s(x, y) = 0$ if x, y do not belong to the same $\Delta_{i,j}$ and $s(x, y) = s(F^{-j}x, F^{-j}y) + 1$ if $x, y \in \Delta_{i,j}$. Thus, $(\Delta, \tilde{\nu}, F)$ is exact (hence, ergodic and mixing) with respect to the metric

$$d(x, y) = \vartheta^{s(x,y)}. \tag{36}$$

See [**Y99**] for the introduction and several examples of such maps.

The measure defined by $\nu(A) = \tilde{\nu}(A)/\tilde{\nu}(X)$ for $A \subset X$ is invariant for T. Note that $\nu(A) = \nu(r)\tilde{\nu}(A)$. We assume that

$$\nu(x : r(x) > n) = \nu(r)\tilde{\nu}(x \in X : r(x) > n) = O(n^{-\beta}) \tag{37}$$

with $\beta > 2$.

We consider the space of dynamically Hölder functions on Δ:

$$C_\varkappa(\Delta, \mathbb{R}^d) = \{f : \Delta \to \mathbb{R}^d \text{ bounded and } \exists C : |f(x) - f(y)| \le C\varkappa^{s(x,y)}\},$$

where $\varkappa < 1$ is fixed and $s(x, y)$ is the separation time of x and y (there is no major difference between Hölder and Lipschitz terminologies as one can increase $\vartheta < 1$ in the definition of the metric (36)). We will use the notation $C_\varkappa(X, \mathbb{C})$ for the space of functions with domain X and range $\mathbb{C}$, defined analogously to $C_\varkappa(\Delta, \mathbb{R}^d)$. The corresponding norm is

$$\|f\|_\varkappa = \inf\{C : |f(x) - f(y)| \le C\varkappa^{s(x,y)} \forall x, y\} + \|f\|_\infty.$$

To given $f \in C_\varkappa(\Delta, \mathbb{R}^d)$, we associate the function $f_X : X \to \mathbb{R}^d$ where

$$f_X(x) = \sum_{i=0}^{r(x)-1} f(F^i x). \tag{38}$$

THEOREM 4.1. *Let $T : X \to X$ be as above. Assume that $(\check{\boldsymbol{\varphi}}, \tau) = f_X$ for some $f = (\check{\boldsymbol{\varphi}}, \tilde{\tau}) \in C_\varkappa(\Delta, \mathbb{R}^2)$, where f_X is non-degenerate and τ is bounded away from zero. Then* **(H1)**, **(H2)**, **(H6)**, *and* **(H7)** *hold.*

Before proving Theorem 4.1, we make several remarks and derive several corollaries from that theorem.

Note that $\beta = \alpha + 1$ in the notation of [**Y99**]. By the results of [**Y99**], $C_\varkappa(\Delta, \mathbb{R})$ observables decorrelate at the speed $O(n^{-\beta+1})$ and satisfy the CLT as long as $\beta > 2$.

We would like to conclude the MLCLT for suspensions over some maps which can be modeled by a tower with polynomial tails. Let $\mathbb{F} : \mathbb{M} \to \mathbb{M}$ be a map on a compact Riemannian manifold $\mathbb{M}$ with invariant measure λ that satisfies assumptions 1–4 in [**G05**, §1.3.1] (in [**G05**], $\mathbb{M}$, $\mathbb{F}$, λ are denoted by X, T, ν, respectively). We also assume that λ is the unique Sinai–Ruelle–Bowen (SRB) measure for $\mathbb{F}$. Let υ be a Hölder roof function on $\mathbb{M}$ and Ψ be the corresponding suspension (semi-)flow on the phase space $\aleph$. Let $\boldsymbol{\chi} : \aleph \to \mathbb{R}$ be a zero mean continuous observable such that $\check{\boldsymbol{\chi}}$ is Hölder. Then, as explained in [**G05**, §1.3.1], we can construct a tower $(\Delta, F, \tilde{\nu})$, with $X = \Delta_0 \subset \mathbb{M}$ satisfying assumptions (A1)–(A4) above and a Hölder mapping $\rho : \Delta \to \mathbb{M}$ so that $\rho \circ F = \mathbb{F} \circ \rho$ and $\rho_* \tilde{\nu} = \lambda$. Define $\tilde{\tau} : \Delta \to \mathbb{R}_+$ by $\tilde{\tau}(x) = \upsilon(\rho(x))$. Let $\tilde{\Omega}$ be the phase space of the suspension (semi-)flow over $(\Delta, F, \tilde{\nu})$ with roof function $\tilde{\tau}$ and let $\tilde{\boldsymbol{\varphi}} : \tilde{\Omega} \to \mathbb{R}$ be defined by

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$\tilde{\boldsymbol{\varphi}}(x, s) = \boldsymbol{\chi}(\rho(x), s)$. As before we let (X, T, ν) be the first return to the base of the tower. Let Ω be the phase space of the suspension (semi-)flow over (X, T, ν) with roof function $\tau = \tilde{\tau}_X$. Let $\boldsymbol{\varphi} : \Omega \to \mathbb{R}$ be defined by $\boldsymbol{\varphi}(x, s) = \tilde{\boldsymbol{\varphi}}(x, s)$ (here, $s \in [0, \tau(x))$ and thus $\tilde{\boldsymbol{\varphi}}(x, s)$ is to be interpreted with the usual identification $(x, \tilde{\tau}(x) + s) = (Fx, s) \in \tilde{\Omega}$). Note that $(\aleph, \kappa = (1/\lambda(\upsilon))\lambda \otimes \text{Leb}, \Psi^t)$ is a factor of $(\Omega, \mu = (1/\nu(\tau))\nu \otimes \text{Leb}, \Phi^t)$. Indeed, the mapping $\iota : \Omega \to \aleph$, $\iota(x, s) = (x, s)$ is a homomorphism (mind the identification $(x, \upsilon(x) + s) = (\mathbb{F}(x), s) \in \aleph$) which is, in general, not invertible. We can lift up the test function $\mathfrak{X} : \aleph \to \mathbb{R}$, to $\mathfrak{V} : \Omega \to \mathbb{R}$ by $\mathfrak{V} = \mathfrak{X} \circ \iota$ (similarly, let $\mathfrak{W} = \mathfrak{Y} \circ \iota$). Now we have by definition

$$\int_{\aleph} \mathfrak{X}(x, s)\mathfrak{Y}(\Psi^t(x, s))\mathfrak{Z}\left(\int_0^t \boldsymbol{\chi}(\Psi^{s'}(x, s))\, ds' - W(t)\right) d\kappa(x, s)$$
$$= \int_{\Omega} \mathfrak{V}(x, s)\mathfrak{W}(\Phi^t(x, s))\mathfrak{Z}\left(\int_0^t \boldsymbol{\varphi}(\Phi^{s'}(x, s))\, ds' - W(t)\right) d\mu(x, s). \quad (39)$$

This identity, combined with Theorem 4.1 readily gives the following.

PROPOSITION 4.1. *In the setup of the previous paragraph, let us assume that* $(\Delta, F, \tilde{\nu})$ *satisfies (37) with* $\beta > 2$. *Furthermore, assume that* $(\check{\boldsymbol{\varphi}}, \tau)$ *is minimal and its linearized group falls into Cases* **(A)**, **(B)**, *or* **(C)**. *Then the conclusion of Theorem 3.1 holds for* $(\Phi, \boldsymbol{\varphi})$ *with* $h(x) = 0$, $h_\tau(x) = 0$.

Sometimes the flow Ψ^t does not have a canonical coding as a suspension of a Young tower map. For example, Ψ^t can be given by an ordinary differential equation. In this case it is important to reformulate Proposition 4.1 as an MLCLT for $(\Psi, \boldsymbol{\chi})$. However, if we want to do so, the following two difficulties arise.

(D1) $H : \Omega \to \mathbb{R}$, given by Theorem 3.1 may not be the lift-up of a function on $\aleph$.

(D2) For given bounded and continuous test functions $\mathfrak{X}, \mathfrak{Y} : \aleph \to \mathbb{R}$, the corresponding lift-ups $\mathfrak{V}, \mathfrak{W} : \Omega \to \mathbb{R}$ may not be supported on $\{(x, s) \in \Omega : s \le M\}$ for some finite M.

It is easy to overcome (D1) in Case **(A)** since $H = 0$. In Cases **(B)** and **(C)** we do not know how to overcome (D1) in general. However, at least we can distinguish between Cases **(A)**, **(B)**, and **(C)** by only looking at the manifold itself (rather than at the tower).

PROPOSITION 4.2. $\hat{\mathcal{V}}_{(\check{\boldsymbol{\chi}}, \upsilon)}$ *and* $\hat{\mathcal{V}}_{(\check{\boldsymbol{\varphi}}, \tau)}$ *fall into the same cases* **(A)**–**(E)**.

Next, we introduce the notion of a minimal group of an observable for an abstract dynamical system.

Definition 4.1. Given any dynamical system $\mathcal{T}$ on a measurable space $(\mathcal{X}, \zeta)$ and an observable $u : \mathcal{X} \to \mathbb{R}^d$, we define $S(u)$ as the minimal closed group, a translate of which supports the values of u. Let us define the minimal closed group $M(u)$ by $M(u) = \bigcap_{v \sim u} S(v)$, where $v \sim u$ means that $u + h - h \circ \mathcal{T}$ holds with some $h = h_u : \mathcal{X} \to \mathbb{R}^d$ measurable. Let $r(u) \in \mathbb{R}^d / M(u)$ be the translate, i.e. range$(v) \subset M(u) + r(u)$ with some $v \sim u$. A function u is called minimal if $M(u) = S(u)$ and is called non-arithmetic if $M(u) = \mathbb{R}^d$.

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The notation $M(u), r(u)$ was used in §§2 and 3 where M was the symmetry group of the local distribution in the MLCLT and r was the shift. We will see in the proof of Theorem 4.1, that for Young towers the local limit theorem holds with M, r given by Definition 4.1.

Our next result lifts the MLCLT to Ψ in a special case.

PROPOSITION 4.3. *In the setup of Proposition 4.1, let us also assume that* $(\check{\boldsymbol{\chi}}, \upsilon)$ *is non-arithmetic. Then for any* $W \in \mathbb{R}$*, any continuous* $\mathfrak{X}, \mathfrak{Y} : \aleph \to \mathbb{R}$*, any continuous and compactly supported* $\mathfrak{Z} : \mathbb{R} \to \mathbb{R}$*, and any* $W(t) \in \mathbb{R}$ *with* $W(t)/\sqrt{t} \to W$*,*

$$\lim_{t\to\infty} t^{1/2} \int_{\aleph} \mathfrak{X}(x,s)\mathfrak{Y}(\Psi^t(x,s))\mathfrak{Z}\left(\int_0^t \boldsymbol{\chi}(\Psi^{s'}(x,s))\,ds' - W(t)\right) d\kappa(x,s)$$
$$= \mathfrak{g}_\Sigma(W) \int_{\aleph} \mathfrak{X}\,d\kappa \int_{\aleph} \mathfrak{Y}\,d\kappa \int_{\mathbb{R}} \mathfrak{Z}\,d\mathrm{Leb}.$$

4.2. *Proof of Proposition 4.2.* Let $f_{\mathbb{M}} : \mathbb{M} \to \mathbb{R}^d$ be an observable. Let $f_\Delta : \Delta \to \mathbb{R}^d$ be its lift-up, i.e. $f_\Delta(x) = f_{\mathbb{M}}(\rho(x))$ and $f_X : X \to \mathbb{R}^d$ be the corresponding observable on X, i.e. $f_X(x) = \sum_{i=0}^{r(x)-1} f_\Delta(F^i x)$.

LEMMA 4.1. *$M(f_\Delta)$ is a finite index subgroup of $M(f_{\mathbb{M}})$ and $r(f_{\mathbb{M}}) = \iota'(r(f_\Delta))$, where ι' is the natural surjective homomorphism from $\mathbb{R}^d/M(f_\Delta)$ to $\mathbb{R}^d/M(f_{\mathbb{M}})$.*

Proof. The one-dimensional case is proved in [**G05**, Theorem 1.4].

We can assume $r(f_\Delta) = 0$ by possibly adding a constant to both $f_{\mathbb{M}}$ and f_Δ. Clearly $M(f_\Delta)$ is a subgroup of $M(f_{\mathbb{M}})$, as we can lift up any cohomological equation from $\mathbb{M}$ to Δ. Thus, $M(f_\Delta)$ is isomorphic to $\mathbb{R}^{\tilde{d}_1} \times \tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}}$ is a $\tilde{d}_2$-dimensional lattice and $\tilde{d}_3 := d - \tilde{d}_1 - \tilde{d}_2 \geq 0$. Similarly, $M(f_{\mathbb{M}})$ is isomorphic to $\mathbb{R}^{d_1} \times \mathcal{L}$, where $\mathcal{L}$ is a d_2-dimensional lattice and $d_3 = d - d_1 - d_2 \geq 0$. Furthermore, by the subgroup property, $\tilde{d}_1 \leq d_1$ and $d_3 \leq \tilde{d}_3$. It remains to show that $\tilde{d}_1 = d_1$ and $\tilde{d}_3 = d_3$. Replacing $f_{\mathbb{M}}$ and f_Δ by $Af_{\mathbb{M}}$ and Af_Δ with some invertible matrix A, we can assume that $M(f_\Delta) = \mathbb{R}^{\tilde{d}_1} \times \mathbb{Z}^{\tilde{d}_2}$. Next we observe that for an $\mathbb{R}^d$-valued function u,

$$\mathrm{Cl}(\pi_V)M(u) = M(\pi_V u), \tag{40}$$

where Cl means closure and π_V is the orthogonal projection to the (coordinate) subspace V. Similarly, let π_k be the projection to the kth coordinate subspace.

If $\tilde{d}_3 > d_3$, applying (40) to the $(\tilde{d}_1 + \tilde{d}_2 + 1)$st coordinate direction, we obtain $M(\pi_{\tilde{d}_1+\tilde{d}_2+1} f_\Delta) = \mathbb{Z}$ while $M(\pi_{\tilde{d}_1+\tilde{d}_2+1} f_{\mathbb{M}}) = 0$, which is a contradiction to the one-dimensional case. Thus, $\tilde{d}_3 = d_3$. Similarly, if $d_1 > \tilde{d}_1$, then there exists some d' with $\tilde{d}_1 < d' \leq \tilde{d}_1 + \tilde{d}_2$ such that $\mathbb{R} = \mathrm{Cl}(\pi_{d'} M(f_{\mathbb{M}})) = M(\pi_{d'} f_{\mathbb{M}})$. On the other hand, $M(\pi_{d'} f_\Delta) = \mathbb{Z}$, which is again a contradiction to the one-dimensional case. □

LEMMA 4.2. *The linearized groups of $f : \Delta \to \mathbb{R}^d$ and $f_X : X \to \mathbb{R}^d$ are the same, i.e. $\hat{\mathcal{V}}_f = \hat{\mathcal{V}}_{f_X}$.*

Proof. Assume that $f = g + h - h \circ F$, where $g : \Delta \to \hat{\mathcal{V}}_f$. Then $f_X = g_X + h - h \circ T$, where $g_X : X \to \hat{\mathcal{V}}_f$. Thus, $\hat{\mathcal{V}}_{f_X} \subseteq \hat{\mathcal{V}}_f$. Next, assume that for some $f : \Delta \to \mathbb{R}^d$,

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we have $f_X = \underline{g} + h - h \circ T$, with $\underline{g} : X \to \hat{\mathcal{V}}_{f_X}$. Let us define $\tilde{f} : \Delta \to \mathbb{R}^d$ by $\tilde{f}(x, l) = \underline{g}(x)1_{\{l=0\}} + h_l(x) - h_{l+1}(x)$, where $h_0(x) = h(x)$, $h_{r(x)}(x) = h(Tx)$ and $h_l(x) = 0$ if $l \notin \{0, r(x)\}$. Thus, $\hat{\mathcal{V}}_{\tilde{f}} \subseteq \hat{\mathcal{V}}_{f_X}$. By construction, $(\tilde{f})_X = f_X$ and thus f and $\tilde{f}$ are cohomologous, $\hat{\mathcal{V}}_{\tilde{f}} = \hat{\mathcal{V}}_f$. □

Proof of Proposition 4.2. Observe that $\hat{\mathcal{V}}_{(\check{\boldsymbol{\chi}},\upsilon)}$ is a finite index subgroup of $\hat{\mathcal{V}}_{(\tilde{\check{\boldsymbol{\varphi}}},\tilde{\tau})}$ by Lemma 4.1. Thus, by Lemma 4.2, $\hat{\mathcal{V}}_{(\check{\boldsymbol{\chi}},\upsilon)}$ is a finite index subgroup of $\hat{\mathcal{V}}_{(\check{\boldsymbol{\varphi}},\tau)}$. In particular, $\hat{\mathcal{V}}_{(\check{\boldsymbol{\chi}},\upsilon)}$ and $\hat{\mathcal{V}}_{(\check{\boldsymbol{\varphi}},\tau)}$ fall into the same cases **(A)**–**(E)**. □

4.3. *Proof of Theorem 4.1.* The main ingredients of the proof are as follows:

(i) MLCLT with an error estimate; and

(ii) moderate and large deviation estimates.

We start with establishing MLCLT with rates in parts (A) and (B) of Lemma 4.3. Lemma 4.3(C) provides a useful generalization of Lemma 4.3(B). The required moderate and large deviation bounds are contained in Lemmas 4.4–4.6.

LEMMA 4.3. *Consider the setup of Theorem 4.1 with either $d = 1$ and $f_X = \tau$ or $d = 2$ and $f_X = (\check{\boldsymbol{\varphi}}, \tau)$ with $\nu(\check{\boldsymbol{\varphi}}) = 0$. Then*

$$\begin{aligned}
&\text{(A)} \quad (T, f_X) \text{ satisfies the MLCLT.}\\
&\text{(B)} \quad \nu(x : S_{f_X}(n, x) - n\nu(f_X) \in B(v, R))\\
&\qquad\qquad \le C(n^{-d/2}\mathfrak{g}_\sigma(v/\sqrt{n}) + n^{-(d+\beta)/2+1} + n^{-(d+1)/2}).\\
&\text{(C)} \quad \nu(x : S_{f_X}(n, x) - n\nu(f_X) \in B(v, R),\ T^n x \in \Delta_{0,l})\\
&\qquad\qquad \le C\nu(\Delta_{0,l})(n^{-d/2}\mathfrak{g}_\sigma(v/\sqrt{n}) + n^{-(d+\beta)/2+1} + n^{-(d+1)/2}).
\end{aligned}$$

Proof. Lemma 4.3 can be proved by the Fourier method, cf. similar results in [**AD01**, **G05**, **GH88**, **R83**, **SzV04**]. We briefly sketch the proof here only highlighting the differences from the analogous arguments in the above list.

Let us assume first that f_X is minimal in the sense of Definition 4.1. The proof is based on P, the Perron–Frobenius operator associated to T acting on $C_\varkappa(X, \mathbb{C})$ by the formula

$$\nu(f(g \circ T)) = \nu(gPf).$$

and the twisted operators $P_t u = P(e^{i\langle t, f_X\rangle}u)$ for $t \in \mathbb{R}^d$. These operators satisfy the Lasotha–Yorke (also known as the Doeblin–Fortet) inequality:

$$\|P_t^n u\|_{\varkappa'} \le C(1 + |t|)\eta^n \|u\|_{\varkappa'} + C\|u\|_{L_1} \tag{41}$$

(with some $\varkappa' \in (\varkappa, 1)$ and $\eta < 1$). In the one-dimensional case, (41) is included in Lemma 4.1(2) of [**G05**] and its proof is sketched based on [**AD01**, Proposition 2.1]. Note that [**AD01**, Proposition 2.1] is valid in higher dimensions, so the adjustments described in [**G05**] give (41). By classical results of [**I-TM50**], P is quasicompact with a simple eigenvalue at 1 (eigenfunction identically 1 as we took the Jacobian with respect to ν in the definition) and finitely many eigenvalues with modulus in $(\rho, 1]$ for some $\rho < 1$. By perturbation theory, a similar picture holds for P_t for t in a small neighborhood of $0 \in \mathbb{R}^d$.

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We need to understand the asymptotics of λ_t, the eigenvalue of P_t close to 1, and the other eigenvalues on the unit circle.

If $d = 2$, let us fix a vector $s = (s_1, s_2) \in \mathbb{R}^2$ of unit length, and for a moment, only consider $t \in \mathbb{R}^2$ in the direction of s, i.e. $t = |t|s$. Then [**G05**, Proposotion 4.5] applied to the function $\langle s, f_X\rangle$ tells us that

$$\lambda_{|t|s} = \nu(e^{i|t|\langle s, f_X\rangle}) + (c_1 s_1^2 + c_{12} s_1 s_2 + c_2 s_2^2)t^2 + O(|t|^3).$$

On the other hand, the assumption (37) implies that $\nu(\langle s, f_X\rangle > R) = O(R^{-\beta})$ uniformly in s. Now we use [**F68**, §XV.4, Lemma 4] if $\beta \geq 3$ and [**W73**, Theorem 2] if $\beta < 3$ to conclude

$$\nu(e^{i|t|\langle s, f_X\rangle}) = 1 + i\nu(\tau)s_2|t| - (\tilde{c}_1 s_1^2 + \tilde{c}_{12} s_1 s_2 + \tilde{c}_2 s_2^2)t^2 + O(|t|^\beta + |t|^3).$$

Combining the last two displayed equations we obtain

$$\lambda_{|t|s} = 1 + i\nu(\tau)s_2|t| - (\hat{c}_1 s_1^2 + \hat{c}_{12} s_1 s_2 + \hat{c}_2 s_2^2)t^2 + O(|t|^\beta + |t|^3),$$

with some $\hat{c}_1, \hat{c}_2, \hat{c}_{12}$, which are independent of s. The proof of [**G05**, Proposition 4.5] shows that the implied constant in $O(\cdot)$ only depends on $\|\langle s, f_X\rangle\|_{C_\varkappa(X,\mathbb{R})}$, and hence it is uniform in s. We conclude that there is a matrix $\boldsymbol{m}$ such that

$$\lambda_t = 1 + i(0, \nu(\tau))t - t^T \boldsymbol{m} t + O(|t|^\beta + |t|^3). \tag{42}$$

In fact, the coefficients of $\boldsymbol{m}$ are given by the Green–Kubo formula (see the expression for a in [**G05**, Proposition 4.5]) but we will not need this fact.

If $d = 1$, then [**G05**, Proposition 4.5] directly implies that there is a constant $m > 0$ such that $\lambda_t = 1 + i\nu(\tau)t - t^2 m + O(|t|^\beta + |t|^3)$.

The characterization of other eigenvalues of P_t on the complex unit circle $\mathcal{S}^1$ is again analogous to similar computations in [**AD01**, **G05**]: $\lambda \in \mathcal{S}^1$ is an eigenvalue (and in this case, $g = g_t$ is an eigenfunction with $|g_t| = 1$) if and only if $e^{i\langle t, f_X\rangle} g_t = \lambda g_t \circ T$. Then we can finish the proof for f_X minimal as in the above references.

Now assume f_X is not minimal, i.e., $f_X = \varsigma + h - h \circ T$ for some measurable ς, h and $M = M(f_X) = S(\varsigma) \subsetneq S(f_X)$. *A priori* we only know that h and ς are measurable. In order to prove the MLCLT, we need to show that we can choose h and ς so that h is bounded and almost everywhere continuous. In fact, we will show that it is Lipschitz and consequently ς is Lipschitz as well. Then we can repeat the previous argument with f_X replaced by ς to conclude the MLCLT.

Let G be defined by $M = \widehat{\mathbb{R}^2/G}$, where $\widehat{L}$ is the group of characters of L. By [**AD01**], Proposition 3.7, for any $t \in G$, there is some $\lambda \in \mathcal{S}^1$ and a Lipschitz function $g_t : X \to \mathcal{S}^1$ so that $e^{i\langle t, f_X(x)\rangle} = \lambda g_t(x)/g_t(T(x))$. Next, we show that there is a function $h_t : X \to \mathbb{R}$ which is Lipschitz and satisfies $e^{ih_t} = g_t$. Since g_t is Lipschitz, there is some K so that the oscillation of g_t on K-cylinders is less than $\sqrt{2}$ (we call a cylinder of length K a set of the type $\Delta^0_{i_1,\dots,i_K} = \bigcap_{j=1}^K T^{-j+1}\Delta_{0,i_j}$). Fix a cylinder $\Delta^0_{i_1,\dots,i_K}$ and pick an arbitrary element $x_{i_1,\dots,i_K} \in \Delta^0_{i_1,\dots,i_K}$. Let $h_t(x_{i_1,\dots,i_K})$ be the unique real number in $[0, 2\pi)$ that satisfies $e^{ih_t(x_{i_1,\dots,i_K})} = g_t(x_{i_1,\dots,i_K})$. By the choice of K, for any $y \in \Delta^0_{i_1,\dots,i_K}$ there is a unique

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$h_t(y) \in (h_t(x_{i_1,\dots,i_K}) - \pi, h_t(x_{i_1,\dots,i_K}) + \pi)$ satisfying $e^{ih_t(y)} = g_t(y)$. By construction, h_t is Lipschitz. Observe that

$$\langle t, f_X(x)\rangle = \rho + h_t(x) - h_t(T(x)) + \varsigma_t(x),$$

where $\lambda = e^{i\rho}$ and $\varsigma_t : X \to 2\pi\mathbb{Z}$ is some function. Since f_X is non-degenerate, [**AD01**, Proposition 3.9] implies that G is a discrete group. Now we define $h : X \to \mathrm{span}(G)$ via $\langle t, h\rangle = h_t$ for all t in a fixed generator of G. By construction, h is Lipschitz. Lemma 4.3(A) follows.

Next, we prove part (B). Consider the non-negative function

$$h_1(z) = \frac{1 - \cos(\varepsilon z)}{\pi\varepsilon^2 z^2}$$

(with some small $\varepsilon > 0$). Its Fourier transform equals

$$\hat{h}_1(t) = \int_{\mathbb{R}} e^{itz} h_1(z)\, dz = 1_{|t|<\varepsilon}\left(\frac{1}{\varepsilon} - \frac{|t|}{\varepsilon^2}\right).$$

Next, consider $h_2(z_1, z_2) = h_1(z_1)h_1(z_2)$ and its Fourier transform $\hat{h}_2(t_1, t_2) = \hat{h}_1(t_1)\hat{h}_1(t_2)$. We will prove that

$$\int_X h_d(S_{f_X}(n, x) - n\nu(f_X) - v)\nu(\mathrm{d}x) \le C(n^{-d/2}\mathfrak{g}_\sigma(v/\sqrt{n}) + n^{-(d+\beta)/2+1} + n^{-(d+1)/2}). \tag{43}$$

This will imply Lemma 4.3(B) (with some different C) as $h_d(z) \ge c > 0$ for $\|z\| < \varepsilon/2$ and we can cover a ball of radius R with balls of radius $\varepsilon/2$.

The proof of (43) is standard. Namely, we rewrite the left-hand side as

$$\left(\frac{1}{2\pi}\right)^d \int_{[-\varepsilon,\varepsilon]^d} \hat{h}_d(t)\nu(P_t^n 1)e^{-it(n\nu(f_X)+v)}\, dt \tag{44}$$

and estimate $|\nu(P_t^n 1)|$ taking the second order Taylor expansion of P_t at zero. The main contribution comes from the leading eigenvalue which is controlled by (42). A similar computation can be found in [**P09**, §§A.2–A.4]. The setting of [**P09**] is different since only lattice distributions are considered there but the lattice assumption is only used to ensure that the integration in (44) is over a compact set. In our case the compactness comes from the fact that $\hat{h}_d$ has compact support. Thus, the proof of (43) is similar to [**P09**], so we leave it to the reader.

The proof of part (C) is the same as the proof of part (B) except that in (44) $\nu(P_t^n 1)$ has to be replaced by $\nu(1_{\Delta_{0,l}} P_t^n 1)$ providing an additional improvement by the factor $\nu(\Delta_{0,l})$. We refer the reader to the penultimate formula on [**P09**, p. 834] for a similar argument. □

Next, we proceed to the desired moderate and large deviation estimates. We first prove global bounds and then derive local bounds from the global ones.

LEMMA 4.4. (Global moderate deviations) *Consider the setup of Theorem 4.1 with* $f_X = \check{\boldsymbol{\varphi}}$ *or* $f_X = \tau$. *For for any* $\xi \in (1/2, 1]$ *and for any* $\varepsilon > 0$ *there is some constant* $C = C_{\xi,\varepsilon}$ *such that*

$$\nu(x \in X : |S_{f_X}(n, x) - n\nu(f_X)| > n^\xi) \le C_{\xi,\varepsilon} n^{-(\beta-1)(2\xi-1)+\varepsilon}.$$

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Proof. Let us write $\tilde{S}_f(n,x)=\sum_{i=0}^{n-1} f(F^i(x))$ for $x\in\Delta$ where $f\in C_{\mathscr{H}}(\Delta,\mathbb{R}^d)$ is related to f_X via (38). [**M09**, Theorem 1.3] states that for $\xi\in(1/2,1]$,

$$\tilde{\nu}(x\in\Delta:|\tilde{S}_f(n,x)-n\tilde{\nu}(f)|>n^{\xi})\le C_\varepsilon(\ln n)^{\beta-1}n^{-(\beta-1)(2\xi-1)}. \tag{45}$$

We will deduce Lemma 4.4 from (45). Let $\chi_{\text{roof}}:\Delta\to\{0,1\}$ be the indicator function of the top floor of Δ. With the notation

$$t_n:X\to\mathbb{Z},\quad t_n(x):=\min\{m:\tilde{S}_{\chi_{\text{roof}}}(m,x)=n\}$$

we have $S_{f_X}(n,x)=\tilde{S}_f(t_n(x),x)$. Next, we fix some $\xi''<\xi'<\xi$ close to ξ and write $n_\pm=\lfloor n\nu(r)\pm n^{\xi'}\rfloor$. We claim that for n large enough,

$$\{x\in X:|S_{f_X}(n,x)-n\nu(f_X)|>n^{\xi}\}\subset\bigcup_{i=1}^{4}A_i,$$

where

$$\begin{aligned}
A_1&=\{x\in X:|\tilde{S}_f(n\nu(r),x)-n\nu(f_X)|>n^{\xi}/4\},\\
A_2&=\{x\in X:\exists k\in[n_-,n_+]:|\tilde{S}_f(n\nu(r),x)-\tilde{S}_f(k,x)|>n^{\xi}/4\},\\
A_3&=\{x\in X:|\tilde{S}_{\chi_{\text{roof}}}(n_-,x)-n_-\tilde{\nu}(\chi_{\text{roof}})|>n^{\xi''}\},\\
A_4&=\{x\in X:|\tilde{S}_{\chi_{\text{roof}}}(n_+,x)-n_+\tilde{\nu}(\chi_{\text{roof}})|>n^{\xi''}\}.
\end{aligned}$$

To verify the above claim, observe that $\tilde{\nu}(\chi_{\text{roof}})=1/\nu(r)$ implies

$$\{x:t_n(x)<n_-\}\subset A_3\quad\text{and}\quad\{x:t_n(x)>n_+\}\subset A_4.$$

Assuming (as we can) that $\xi''=\xi''(\xi,\varepsilon)$ is sufficiently close to ξ, the identity $\tilde{\nu}(f)=\nu(f_X)/\nu(r)$ and (45) give that $\nu(A_1)+\nu(A_3)+\nu(A_4)\ll n^{-(\beta-1)(2\xi-1)+\varepsilon}$. Since f is bounded, $A_2=\emptyset$ for n sufficiently large. We have finished the proof of Lemma 4.4. □

LEMMA 4.5. (Local moderate deviations) *For any $\varepsilon,\varepsilon'>0$ fixed, there exists a constant $C=C_{\varepsilon,\varepsilon'}$ such that:*
($d=1$) for any L with $|L|\ge n^{1/2+\varepsilon}$

$$\nu(x\in X:S_\tau(n,x)-n\nu(\tau)\in[L,L+1])\le C\frac{n^{\beta-3/2+\varepsilon'}}{(\min\{L,n\})^{2(\beta-1)}};$$

($d=2$) for any $\vec{L}\in\mathbb{R}^2$ with $|\vec{L}|\ge n^{1/2+\varepsilon}$

$$\nu(x\in X:S_{(\check{\varphi},\tau)}(n,x)-n(0,\nu(\tau))\in\vec{L}+[0,1]^2)\le C\frac{n^{\beta-2+\varepsilon'}}{(\min\{|\vec{L}|,n\})^{2(\beta-1)}}.$$

Proof. We prove the case $d=1$ and omit similar proof for the case $d=2$. Recall that a cylinder of length k is a set of the type

$$\Delta^0_{i_1,\dots,i_k}=\bigcap_{j=1}^{k}T^{-j+1}\Delta_{0,i_j}.$$

By bounded distortion, there exists a constant C so that for any k,l, for any length k cylinder $\mathcal{C}_1$ and length l cylinder $\mathcal{C}_2$, and any $m\ge k$,

$$\nu(\mathcal{C}_1\cap T^{-m}\mathcal{C}_2)\le C\nu(\mathcal{C}_1)\nu(\mathcal{C}_2). \tag{46}$$

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Given cylinders $\mathcal{C}_1$ of length $\lfloor n/2 \rfloor$ and $\mathcal{C}_2$ of length $\lceil n/2 \rceil$ (for ease of notation we drop the integer parts) we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are compatible if there is some $x \in \mathcal{C}_1 \cap T^{-n/2}\mathcal{C}_2$ so that $S_\tau(n, x) - n\nu(\tau) \in [L, L+1]$. By (46),

$$\nu(x \in X : S_\tau(n, x) - n\nu(\tau) \in [L, L+1]) \le \sum_{\mathcal{C}_1, \mathcal{C}_2} C\nu(\mathcal{C}_1)\nu(\mathcal{C}_2)$$

where the sum is taken over pairs of compatible cylinders $(\mathcal{C}_1, \mathcal{C}_2)$.

We claim that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are compatible then for some $i = 1, 2$,

$$|S_\tau(n/2, x) - n\nu(\tau)/2| \ge L/2 - \|\tilde{\tau}\|_\kappa/(1-\kappa) \quad \text{for all } x \in \mathcal{C}_i. \tag{47}$$

Indeed, (47) holds if there is some $x \in \mathcal{C}_i$ with $|S_\tau(n/2, x) - n\nu(\tau)/2| \ge L/2$. Let us assume $i = 1$ (the case of $i = 2$ is similar). By Lemma 4.4,

$$\sum_{\mathcal{C}_1 : \mathcal{C}_1 \text{ satisfies (47)}} \nu(\mathcal{C}_1) \le C \frac{n^{\beta-1+\varepsilon'}}{(\min\{L, n\})^{2(\beta-1)}}.$$

By Lemma 4.3(B), for any $\mathcal{C}_1$ fixed,

$$\sum_{\mathcal{C}_2 \text{ compatible with } \mathcal{C}_1} \nu(\mathcal{C}_2) \le Cn^{-1/2}.$$

The lemma follows. □

LEMMA 4.6. (Local superlarge deviations) *There is a constant C so that*
($d = 1$) For any L with $|L| > 2\nu(\tau)n$,

$$\nu(x \in X : S_\tau(n, x) \in [L, L+1]) \le C \frac{n^{1/2}}{L^2}.$$

($d = 2$) For any $\vec{L} \in \mathbb{R}^2$ with $|\vec{L}| > 2\nu(\tau)n$

$$\nu(x \in X : S_{(\check{\varphi}, \tau)}(n, x) \in \vec{L} + [0, 1]^2) \le \frac{C}{L^2}.$$

Proof. By the Chebyshev inequality we have that

$$\nu(x \in X : |S_\tau(n, x)| \ge L) \le C\frac{n}{L^2} \quad \text{and} \quad \nu(x \in X : |S_{(\check{\varphi}, \tau)}(n, x)| \ge |\vec{L}|) \le C\frac{n}{L^2}. \tag{48}$$

The derivation of the local bound from the above global bound is the same as in the proof of Lemma 4.5. □

Proof of Theorem 4.1. **(H1)** and **(H6)** are proved by Lemma 4.3(A). In order to prove **(H2)** and **(H7)**, we decompose the sum as

$$S_1 = \sum_{n=1}^{w/3}, \quad S_2 = \sum_{n=w/3}^{w-w^\gamma}, \quad S_3 = \sum_{n : |n-w| \in [K\sqrt{w}, w^\gamma]}, \quad S_4 = \sum_{n=w+w^\gamma}^{5w/3}, \quad S_5 = \sum_{n=5w/3}^{\infty}$$

where $\gamma = \min\{\beta/4, 3/4\}$. We need to show that for $d = 1$, $f_X = \tau$ and for $d = 2$, $f_X = (\check{\varphi}, \tau)$,

$$\limsup_{w \to \infty} w^{(d-1)/2} \sum_{i=1}^{6} S_i = o_{K \to \infty}(1). \tag{49}$$

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By Lemma 4.6, we have

$$S_1 \le C w^{-2} \sum_{n=1}^{w/3} n^{1-d/2} \le C w^{-d/2} \ll w^{(1-d)/2}.$$

We use Lemma 4.5 with $L \lesssim n$ to conclude

$$w^{(d-1)/2}(S_2 + S_4) \le C w^{(d-1)/2} \sum_{m=w^\gamma}^{2w/3} C w^{\beta-1-d/2+\varepsilon'} m^{-2\beta+2} \le C w^{\beta-3/2+\varepsilon'+\gamma(-2\beta+3)}$$

which is $o(1)$ for sufficiently small $\varepsilon' = \varepsilon'(\beta)$ since $\gamma > 1/2$.

Next, we use Lemma 4.3(B) to estimate S_3. Namely

$$\begin{aligned} w^{(d-1)/2} S_3 &\le C w^{(d-1)/2} \sum_{m=K\sqrt{w}}^{w^\gamma} (w^{-d/2} e^{-cm^2/w} + w^{-(d+\beta)/2+1} + w^{-(d+1)/2}) \\ &\le C' e^{-cK} + C' w^{\gamma-\beta/2+1/2} + C' w^{\gamma-1}. \end{aligned}$$

The above expression is $o_{K\to\infty, w\to\infty}(1)$ by the choice of γ. Finally, we use Lemma 4.5 with $L \gtrsim n$ to estimate S_5:

$$S_5 \le \sum_{n=5w/3}^{\infty} C n^{-\beta+1-d/2+\varepsilon} \le C w^{-\beta+2-d/2+\varepsilon} \ll w^{(1-d)/2}.$$

We have verified (49) and thus finished the proof of Theorem 4.1. □

4.4. *Proof of Proposition 4.3.* As mentioned earlier, under the conditions of Proposition 4.3, Theorem 4.1 implies **(H1)–(H7)** for $(\Phi, \boldsymbol{\varphi})$. Thus, by Theorem 3.1 and by (39), a *weaker* version of the MLCLT follows for $(\Psi, \boldsymbol{\chi})$: namely, when only those bounded and continuous test functions $\mathfrak{X}, \mathfrak{Y} : \aleph \to \mathbb{R}$ are allowed for which the corresponding lift-ups $\mathfrak{V} = \mathfrak{X} \circ \iota$, $\mathfrak{W} = \mathfrak{Y} \circ \iota$ are supported on $\{(x, s) \in \Omega : s \le M\}$ for some finite M.

In order to complete the proof of Proposition 4.3, we need a *stronger* version of the MLCLT for $(\Phi, \boldsymbol{\varphi})$: namely, all bounded and continuous test functions $\mathfrak{V}, \mathfrak{W}$ on Ω are allowed. Recall that $N_{t+s} = \max\{n : S_\tau(n, x) < t + s\}$. Approximating $\mathfrak{V}$ and $\mathfrak{W}$ by $\mathfrak{V}(x, s)1_{s\le M}$ and $\mathfrak{W}(x, s)1_{s\le M}$, we see that it suffices to show that

$$\lim_{M\to\infty} \lim_{t\to\infty} t^{1/2} \mu \Bigg\{ (x, s) : (r(x) > M \text{ or } r(T^{N_{t+s}(x)}(x)) > M) \\ \text{and } \int_0^t \boldsymbol{\varphi}(\Phi^{s'}(x, s))\, ds' - W(t) \in [0, 1] \Bigg\} = 0, \tag{50}$$

uniformly for $W(t)$ satisfying (14). Recall that $r_k = r(\Delta_{0,k})$ is the height of the tower above $\Delta_{0,k}$. Consider the partition $\{\Delta_{0,k}\}_{k\ge1}$ of Δ_0. Note that

$$\begin{aligned} \mu((x, s) : r(x) > t^{1/2-\varepsilon}) &\le \|\tilde\tau\|_\infty \sum_{k : r_k \ge t^{1/2-\varepsilon}} r_k \nu(\Delta_{0,k}) \\ &\le C \Bigg[t^{1/2-\varepsilon} \nu(r \ge t^{1/2-\varepsilon}) + \sum_{m \ge t^{1/2-\varepsilon}} \nu(r \ge m) \Bigg] \\ &\le C t^{(1/2-\varepsilon)(1-\beta)} \ll t^{-1/2} \end{aligned}$$

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assuming that $\epsilon < 1/2 - 1/(2\beta - 2)$. Thus, with the notation

$$\mathcal{I} = \left\{ \int_0^t \boldsymbol{\varphi}(\Phi^{s'}(x, s))\, ds' - W(t) \in [0, 1] \right\} \quad \text{and} \quad \Delta_{0,\le M} = \{x \in \Delta_0 : r(x) \le M\},$$

it suffices to verify that

$$\lim_{M\to\infty} \limsup_{t\to\infty} t^{1/2}(S_1 + S_2 + S_3) = 0,$$

where

$$S_1 = \sum_{k: r_k \in [M, t^{1/2-\varepsilon}]} \mu((x, s) \in \mathcal{I} : x \in \Delta_{0,k},\, T^{N_{t+s}(x)}(x) \in \Delta_{0,\le M}),$$

$$S_2 = \sum_{l: r_l \in [M, t^{1/2-\varepsilon}]} \mu((x, s) \in \mathcal{I} : x \in \Delta_{0,\le M},\, T^{N_{t+s}(x)}(x) \in \Delta_{0,l}),$$

$$S_3 = \sum_{k: r_k \in [M, t^{1/2-\varepsilon}]} \sum_{l: r_l \in [M, t^{1/2-\varepsilon}]} \mu((x, s) \in \mathcal{I} : x \in \Delta_{0,k},\, T^{N_{t+s}(x)}(x) \in \Delta_{0,l}).$$

Next, we have the following.

LEMMA 4.7. *There exists $C > 0$ such that for any $t > 0$, any l with $r_l \le t^{1/2-\varepsilon}$, any $v \in \mathbb{R}$ and any $s \in [0, t^{1/2}]$*

$$\nu\left(x : \int_0^{N_{t+s}(x)} \boldsymbol{\varphi}(\Phi^{s'}(x, 0))\, ds' \in [v, v + 1],\, T^{N_{t+s}(x)}(x) \in \Delta_{0,l}\right) \le C r_l \nu(\Delta_{0,l}) t^{-1/2}.$$

Proof. This lemma can be obtained by a simplified version of the proof of Theorem 3.1 (non-arithmetic case). First observe that the case $s > 0$ can be reduced to the case $s = 0$ by replacing $t + s$ with t'. Then, we proceed as in §3.5, except that we assume $A = X$ and use Lemma 4.3(C) instead of **(H6)** to estimate S_1. In order to estimate S_2, we use the following variant of **(H7)**: there is some $K < \infty$ such that for $f_X = (\tau, \check{\boldsymbol{\varphi}})$,

$$\limsup_{w\to\infty} w^{1/2} \sum_{n: |n-w| > K\sqrt{w}} \frac{\nu(x : S_f(n, x) \in B(w\nu(f), R),\, T^n(x) \in \Delta_{0,l})}{\nu(\Delta_{0,l})} < \infty. \quad (51)$$

We can verify (51) by inserting the term $T^n(x) \in \Delta_{0,l}$ to the events considered in Lemmas 4.4–4.6 and multiplying the right-hand sides by $\nu(\Delta_{0,l})$ and then proceeding as in the proof of Theorem 4.1. For example, we have the following variant of Lemma 4.4 (for $f_X = \tau$ or $f_X = \check{\boldsymbol{\varphi}}$):

$$\nu(x \in X : |S_{f_X}(n, x) - n\nu(f_X)| > n^\xi,\, T^n(x) \in \Delta_{0,l}) \le C_{\xi,\varepsilon} \nu(\Delta_{0,l}) n^{-(\beta-1)(2\xi-1)+\varepsilon}.$$

We can prove this by writing

$$\begin{aligned}
&\nu(x \in X : |S_{f_X}(n, x) - n\nu(f_X)| > n^\xi,\, T^n(x) \in \Delta_{0,l}) \\
&\quad \le \sum_{k: r_k > n^\xi/3\|f\|_\infty} \nu(x : T^{n-1}(x) \in \Delta_{0,k},\, T^n(x) \in \Delta_{0,l}) \\
&\qquad + \nu\Bigg(x : |S_{f_X}(n-1, x) - (n-1)\nu(f_X)| > \frac{n^\xi}{3}, \\
&\qquad\qquad r(T^{n-1}(x)) \le \frac{n^\xi}{3\|f\|_\infty},\, T^n(x) \in \Delta_{0,l}\Bigg) \\
&\quad \le C[n^{-\xi\beta} + n^{-(\beta-1)(2\xi-1)+\varepsilon}]\nu(\Delta_{0,l}) \le C n^{-(\beta-1)(2\xi-1)+\varepsilon} \nu(\Delta_{0,l}),
\end{aligned}$$

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where we used the bounded distortion property and the fact that $\beta > 2, \xi \le 1$ in the last two inequalities. Likewise, we can revisit (48) (considering two cases: when $r(T^{n-1}(x)) < cL$ holds and when it does not hold). The derivation of the local bounds from the global ones and then the proof of (51) is identical to the proofs in §4.3. □

To complete the proof of Proposition 4.3 we verify that

$$\lim_{M\to\infty} \limsup_{t\to\infty} t^{1/2} S_3 = 0, \tag{52}$$

the proofs for S_1 and S_2 are similar and shorter. Pick some k with $r_k \in [M, t^{1/2-\varepsilon}]$. In particular, $|\int_0^{\tau(x)} \boldsymbol{\varphi}(\Phi^{s'}(x,0))\,ds'| \le Ct^{1/2-\varepsilon}$ for $x \in \Delta_{0,k}$. Since the bijection $T : \Delta_{0,k} \to \Delta_0$ has bounded distortion, Lemma 4.7 implies

$$\nu\left(x \in \Delta_{0,k} : \int_{\tau(x)}^{N_{t+s}(x)} \boldsymbol{\varphi}(\Phi^{s'}(x,0))\,ds' \in [v, v+1],\ T^{N_{t+s}(x)}(x) \in \Delta_{0,l}\right) \le Cr_l \nu(\Delta_{0,k})\nu(\Delta_{0,l})t^{-1/2}.$$

Next, observe that

$$x \in \Delta_{0,k}, \quad \int_0^t \boldsymbol{\varphi}(\Phi^{s'}(x,0))\,ds' - W(t) \in [0,1], \quad T^{N_{t+s}(x)}(x) \in \Delta_{0,l}$$

imply

$$\left|\int_{\tau(x)}^{N_{t+s}(x)} \boldsymbol{\varphi}(\Phi^{s'}(x,0))\,ds' - W(t)\right| \le C(r_k + r_l).$$

Thus, S_3 can be bounded from above by

$$\sum_k \sum_l \mu\left[(x,s) : \begin{cases} x \in \Delta_{0,k}, \\ \left|\int_{\tau(x)}^{N_{t+s}(x)} \boldsymbol{\varphi}(\Phi^{s'}(x,0))\,ds' - W(t)\right| \le C(r_k + r_l) \\ T^{N_{t+s}(x)}(x) \in \Delta_{0,l} \end{cases}\right] \le C \sum_{k:r_k\in[M,t^{1/2-\varepsilon}]} \sum_{l:r_l\in[M,t^{1/2-\varepsilon}]} r_k r_l (r_k + r_l)\nu(\Delta_{0,k})\nu(\Delta_{0,l})t^{-1/2}. \tag{53}$$

Next, observe that

$$\sum_{k:r_k\ge M} r_k \nu(\Delta_{0,k}) = \sum_{k:r_k\ge M} \sum_{\ell=1}^{r_k} \nu(\Delta_{0,k}) \le 2 \sum_{k:r_k\ge M} \sum_{\ell=M/2}^{r_k} \nu(\Delta_{0,k}) \le 2 \sum_{\ell\ge M/2} \nu(r \ge \ell) = O(M^{1-\beta})$$

and likewise

$$\sum_{k:r_k\ge M} r_k^2 \nu(\Delta_{0,k}) \le 4 \sum_{\ell\ge M/2} \ell\nu(r \ge \ell) = O(M^{2-\beta}).$$

Using these estimates, we find that (53) is bounded by $CM^{3-2\beta}t^{-1/2}$. We have verified (52) and finished the proof of Proposition 4.3. □

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5. *Hyperbolic Young towers*

Let $\boldsymbol{F}: \boldsymbol{M} \to \boldsymbol{M}$ be a $\mathcal{C}^{1+\varepsilon}$ diffeomorphism of the Riemannian manifold $\boldsymbol{M}$. Assume that $\boldsymbol{F}$ satisfies assumptions (Y1)–(Y5) in [**PSZ17**, §4.1]. These imply that there exists a 'hyperbolic Young tower', namely a dynamical system $(\hat{\Delta}, \hat{\nu}, \hat{F})$ which satisfies the following.

- The base of the tower is the set $\hat{\Delta}_0 = \hat{\Delta}_0^u \times \hat{\Delta}_0^s$. The sets of the form $A \times \hat{\Delta}_0^s$, $A \subset \hat{\Delta}_0^u$ are called u-sets (similarly, sets of the form $\hat{\Delta}_0^u \times B$, $B \subset \hat{\Delta}_0^s$ are called s-sets). Also, sets of the form $\Delta_0^u \times \{x^s\}$ are called unstable manifolds and sets of the form $\{x^u\} \times \Delta_0^s$ are stable manifolds.
- There is a partition of $\hat{\Delta}_0$ into s-sets $\hat{\Delta}_{0,k} = \hat{\Delta}_{0,k}^u \times \hat{\Delta}_0^s$ and positive integers r_k so that $\hat{\Delta} = \bigcup_{k\in\mathbb{Z}_+} \bigcup_{l=0}^{r_k-1} \hat{\Delta}_{l,k}$, where $\hat{\Delta}_{l,k} = \{(x, l) : x \in \hat{\Delta}_{0,k}\}$.
- For all k and all $l = 0, \dots, r_k - 2$, $\hat{F}$ is an isomorphism between $\hat{\Delta}_{l,k}$ and $\hat{\Delta}_{l+1,k}$ and $\hat{F}(x, l) = (x, l + 1)$. Also $\hat{F}$ is an isomorphism between $\hat{\Delta}_{r_k-1,k}$ and $\hat{F}(\hat{\Delta}_{r_k-1,k})$, the latter being a u-set of $\hat{\Delta}_0$. Furthermore, if x_1 and x_2 belong to the same (un)stable manifold, so do $\hat{F}^{r_k}(x_1, 0)$ and $\hat{F}^{r_k}(x_2, 0)$. We write $\hat{T} = \hat{F}^{r_k-l}$ on $\hat{\Delta}_{l,k}$ and $r(x^u, x^s) = r(x^u) = r_k$ for $(x^u, x^s) \in \hat{\Delta}_{0,k}$.
- There is a mapping $\pi : \hat{\Delta} \to \boldsymbol{M}$ with $\pi|_{\hat{\Delta}_0} : \hat{\Delta}_0 \to \Lambda$ being a bijection, where Λ is a set with hyperbolic product structure and $\pi \circ \hat{F} = \boldsymbol{F} \circ \pi$.
- Let Ξ be the function on $\hat{\Delta}$ defined by $\Xi((x^u, x^s), l) = ((\boldsymbol{x}^s, x^u), l)$ with a fixed $\boldsymbol{x}^s$. Let $\tilde{\Delta} = \Xi(\hat{\Delta})$ and $\tilde{\nu} = \Xi_*\hat{\nu}$. By the previously listed properties of $\hat{F}$, there is a well-defined $\tilde{F} : \tilde{\Delta} \to \tilde{\Delta}$ such that $\Xi \circ \hat{F} = \tilde{F} \circ \Xi$. The dynamical system $(\tilde{\Delta}, \tilde{\nu}, \tilde{F})$, is an expanding Young tower, in the sense that it satisfies assumptions (A.1)–(A.4) of §4.1.
- There exists some $a \in (0, 1)$ so that for any x, y on the same stable manifold in $\hat{\Delta}_0$, $d(\pi(\hat{T}(x)), \pi(\hat{T}(y))) < ad(\pi(x), \pi(y))$. Furthermore, for any k and any x, y on the same unstable manifold in $\hat{\Delta}_{0,k}$, $d(\pi(x), \pi(y)) < ad(\pi(\hat{T}(x)), \pi(\hat{T}(y)))$

We also require the following:

(B1) the expanding tower $(\tilde{\Delta}, \tilde{\nu}, \tilde{F})$ satisfies (37) (this is a slightly stronger requirement than the ones in [**PSZ17**, §4.1]);

(B2) there are $K < \infty$ and $\hat{\theta} < 1$ such that for every k, every x, $y \in \hat{\Delta}_{0,k}$ on the same stable manifold and every $0 \le j \le r_k - 1$,

$$d(\pi(\hat{F}^j(x)), \pi(\hat{F}^j(y))) < Kd(\pi(x), \pi(y))\hat{\theta}^j.$$

Consider the dynamical system $(\boldsymbol{M}, \lambda, \boldsymbol{F})$ where $\lambda := \pi_*\hat{\nu}$. Let υ be a positive Hölder roof function, Ψ^t is the corresponding suspension flow on the phase space $\aleph$. Let $\boldsymbol{\chi}$ be a zero mean continuous observable so that $\check{\boldsymbol{\chi}}(x) = \int_0^{\upsilon(x)} \boldsymbol{\chi}(\Psi^s x)\, ds$ is Hölder.

Define $\hat{\zeta} : \hat{\Delta} \to \mathbb{R}_+$ by $\hat{\zeta}(x) = \upsilon(\pi(x))$. Let $\mathcal{T}$ be the phase space of the suspension flow over $(\hat{\Delta}, \hat{\nu}, \hat{F})$ with roof function $\hat{\zeta}$ and let $\hat{\boldsymbol{\eta}} : \mathcal{T} \to \mathbb{R}$ be defined by $\hat{\boldsymbol{\eta}}(x, s) = \boldsymbol{\chi}(\pi(x), s)$. Now we regard this flow as a suspension over the first return to the base of the hyperbolic tower: let $\hat{\tau}(x) = \sum_{j=0}^{r(x)-1} \hat{\zeta}(\hat{F}^j(x))$, $\hat{\Phi}^t$ is the suspension over $(\hat{\Delta}_0, \hat{\nu}|_{\hat{\Delta}_0}, \hat{T})$ with roof function $\hat{\tau}$. Let the phase space of $\hat{\Phi}^t$ be denoted by $\hat{\Omega}$ and its invariant measure be $\hat{\mu} = \hat{\nu}|_{\hat{\Delta}_0} \otimes \mathrm{Leb}$. We consider the observable $\hat{\boldsymbol{\varphi}} : \hat{\Omega} \to \mathbb{R}_+$ defined by $\hat{\boldsymbol{\varphi}}(x, s) = \hat{\eta}(x, s)$ (mind the identification $(x, \hat{\zeta}(x) + s) = (\hat{F}(x), s) \in \mathcal{T}$).

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We also introduce the suspension over the first return to the base of the expanding tower $(\tilde{\Delta}, \tilde{\nu}, \tilde{F})$. A fixed unstable manifold γ^u is identified with the base of the expanding tower, i.e. γ^u is fully crossing Λ in $\boldsymbol{M}$ and $\gamma^u \cap \Lambda = \pi\tilde{\Delta}_0$. Accordingly, any point $(x^u, l) \in \tilde{\Delta}$ is identified with $\boldsymbol{F}^l(\pi(x^u, \boldsymbol{x}^s))$ and with $((x^u, \boldsymbol{x}^s), l) \in \hat{\Delta}$, where $\hat{\Delta}_0^u \times \{\boldsymbol{x}^s\}$ is the lift-up of the unstable manifold γ^u from Λ to $\hat{\Delta}_0$. Let T be the first return to the base, i.e. $T(x^u) = \sum_{j=0}^{r(x^u)-1} \tilde{F}(x^u)$. The first return dynamics is (X, ν, T) where $X = \tilde{\Delta}_0$ and $\nu = \tilde{\nu}|_X$. We consider the suspension flow over (X, ν, T) with roof function $\tau(x^u) = \sum_{j=0}^{r(x^u)-1} \upsilon(\boldsymbol{F}^j(\pi(x^u, \boldsymbol{x}^s)))$. Denote this flow by Φ, its phase space by Ω and consider the observable $\boldsymbol{\varphi}(x^u, s) = \boldsymbol{\chi}(\pi(x^u, \boldsymbol{x}^s), s)$ on Ω (with the identification $(x, \upsilon(x) + s) = (\boldsymbol{F}(x), s) \in \aleph$).

By the above constructions and by (B1), we can apply Theorem 4.1 and Proposition 4.1 to $(\Phi, \boldsymbol{\varphi})$. However, if we want to conclude some result corresponding to Proposition 4.3, we need to extend the MLCLT and the moderate deviation estimates from (X, ν, T) to $(\hat{X} = \hat{\Delta}_0, \hat{\nu}|_{\hat{\Delta}_0}, \hat{T})$. In order to do so, we introduce the metric $d_{\hat{X}}$ on $\hat{X}$. Let $d_{\hat{X}}(x, y) = 1$ if x and y belong to different partition elements $\hat{\Delta}_{0,k}$. If $x = (x^u, x^s)$, $y = (y^u, y^s) \in \hat{\Delta}_{0,k}$, then $d_{\hat{X}}(x, y) = d(\pi(\boldsymbol{x}^u, x^s), \pi(\boldsymbol{x}^u, y^s)) + \beta^{s(x^u, y^u)}$ with a fixed $\boldsymbol{x}^u$, where s is the separation time as in (A3) and $\beta < 1$. Let $\phi : \boldsymbol{F} \to \mathbb{R}^d$ be a piecewise Hölder function, $\hat{\phi} : \hat{\Delta} \to \mathbb{R}^d$ is its lift-up (i.e. $\hat{\phi}(x) = \phi(\pi(x))$). We also define $\tilde{\phi} : \tilde{\Delta} \to \mathbb{R}^d$ by $\tilde{\phi}(x^u, l) = \hat{\phi}((x^u, \boldsymbol{x}^s), l)$, $\hat{\psi} : \hat{X} \to \mathbb{R}^d$ by $\hat{\psi}(x) = \sum_{j=0}^{r(x)-1} \hat{\phi}(x, j)$ and $\tilde{\psi} : X \to \mathbb{R}^d$, $\tilde{\psi}(x^u) = \sum_{j=0}^{r(x^u)-1} \tilde{\phi}(x^u, j)$. We shall use the following standard fact.

LEMMA 5.1. *There is a Hölder function $h : \hat{X} \to \mathbb{R}^d$ such that*

$$\hat{\psi}(x^u, x^s) = \tilde{\psi}(x^u) + h(x^u, x^s) - h(\hat{T}(x^u, x^s)). \tag{54}$$

Proof. Let $h(x^u, x^s) = \sum_{m=0}^{\infty} \hat{\psi}(\hat{T}^m(x^u, x^s)) - \hat{\psi}(\hat{T}^m(x^u, \boldsymbol{x}^s))$. It is straightforward to verify that h is Hölder using (B2). Equation (54) also follows by a direct computation. □

Theorem 4.1 implies that (X, ν, T) satisfies MLCLT. Lemma 5.1 and the continuous mapping theorem allow the MLCLT to be lifted to $(\hat{\Delta}_0, \hat{\nu}|_{\hat{\Delta}_0}, \hat{T})$. Also, since the ergodic sums of $\hat{\psi}$ and $\tilde{\psi}$ differ by $O(1)$, the moderate and large deviation estimates for $\hat{\psi}$ follow from the corresponding estimates for $\tilde{\psi}$. Proceeding as in §4 we obtain the following.

PROPOSITION 5.1. *In the above setup let us also assume that $(\check{\boldsymbol{\chi}}, \upsilon)$ is non-arithmetic. Then for any $W \in \mathbb{R}$, any continuous $\mathfrak{X}, \mathfrak{Y} : \aleph \to \mathbb{R}$, any continuous and compactly supported $\mathfrak{Z} : \mathbb{R} \to \mathbb{R}$, and any $W(t) \in \mathbb{R}$ with $W(t)/\sqrt{t} \to W$,*

$$\lim_{t\to\infty} t^{1/2} \int_{\aleph} \mathfrak{X}(x, s)\mathfrak{Y}(\Psi^t(x, s))\mathfrak{Z}\left(\int_0^t \boldsymbol{\chi}(\Psi^{s'}(x, s))\, ds' - W(t)\right) d\kappa(x, s)$$
$$= \mathfrak{g}_\Sigma(W) \int_{\aleph} \mathfrak{X}\, d\kappa \int_{\aleph} \mathfrak{Y}\, d\kappa \int_{\mathbb{R}} \mathfrak{Z}\, d\mathrm{Leb}.$$

6. *Examples*

6.1. *Independent and identically distributed random variables (reward renewal processes).* Let $\mathbb{P}$ be a probability measure supported on a compact subset of $\mathbb{R} \times \mathbb{R}_+$. Assume that $\int x_1\, d\mathbb{P}(x_1, y_1) = 0$ and that the minimal translated groups supporting

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the measure $\mathbb{P}_2$ defined by $\mathbb{P}_2(A) := \mathbb{P}(\mathbb{R} \times A)$ is either $\mathbb{R}$ or $r + \alpha\mathbb{Z}$ with $r/\alpha \notin \mathbb{Q}$. Let $X = (\mathbb{R} \times \mathbb{R}_+)^{\mathbb{Z}_+}$ equipped with the product topology, $T : X \to X$ is the left shift and $\nu = \mathbb{P}^{\otimes \mathbb{Z}_+}$. For any $\underline{x} = ((x_1, y_1), (x_2, y_2), \ldots) \in X$, let $\tau(\underline{x}) = y_1$ and $\boldsymbol{\varphi}(\underline{x}, t) = \varphi((x_1, y_1), t)$ be a continuous function (where $t \in [0, y_1]$), which also satisfies $\breve{\boldsymbol{\varphi}}(\underline{x}) = x_1$. Then **(H1)**–**(H7)** can be obtained by a much simplified version of our proof of Theorem 4.1 based on classical results in probability theory. Namely, **(H1)** and **(H6)** hold by [**R62**] and [**S65**]. The proof of Lemma 4.3 also extends since [**R62**] and [**S65**] use the Fourier method. Finally, the moderate deviation estimates follow from e.g. [**P95**, Theorem 5.23, Ch. 5]. Consequently the results of §3.3 apply to independent and identically distributed (iid) random variables.

This example could be used to illustrate that the MLCLT does not always hold for suspension flows.

Let (X_1, Y_1) be a random vector that can take the following three values, all with probability 1/3: $(-1, 2 - \sqrt{2})$, $(0, 1)$, $(1, \sqrt{2} - 1)$. Let (X_i, Y_i) be iid. Let $t_n = \sum_{i=1}^n Y_i$, $N_t = \max\{n : \sum_{i=1}^n Y_i \le t\}$ and $S_n = \sum_{i=1}^n X_i$. Then it is well known that S_{N_t} satisfies the CLT as well as many other limit theorems (see e.g. [**GW93**]) but we claim that it does not satisfy the LCLT. Indeed, it is easy to check that $S_{N_t} = 0$ implies that the last renewal time before t is $\lfloor t \rfloor$ and $S_{N_{\lfloor t \rfloor}} = 0$. Thus, for $K \gg 1$ positive integers, $\mathbb{P}(S_{N_{K+a}} = 0) \sim \mathbb{P}(S_{N_{K+b}} = 0)$ if and only if $\lfloor a \rfloor$ and $\lfloor b \rfloor$ fall into the same partition element of

$$\{[0, \sqrt{2} - 1), [\sqrt{2} - 1, 2 - \sqrt{2}), [2 - \sqrt{2}, 1)\}.$$

This shows that $\lim_{t\to\infty} t^{1/2}\mathbb{P}(S_{N_t} = 0)$ only exists along subsequences.

The above example fits into the abstract framework at the beginning of Example 6.1. Namely, $\mathbb{P}$ is the uniform measure supported on the three points $(-1, 2 - \sqrt{2})$, $(0, 1)$, $(1, \sqrt{2} - 1)$. Then the minimal group supporting $\mathbb{P}$, i.e. supporting the values of $(\breve{\boldsymbol{\varphi}}, \tau)$ is the subgroup generated by $(0, 1)$ and $(1, \sqrt{2})$; and the translation is zero. Consequently, the linearized group is the same as the minimal group and we are in Case **(D)**. Note that this is not a generic example among probability measures on three atoms. Indeed, in the generic case, the translation of the minimal lattice would not be rationally related to the lattice and hence the linearization would give $\mathbb{R}^2$, i.e. Case **(A)**. Thus, Case **(A)** is generic even among discrete distributions.

6.2. *Axiom A flows.* Let Ψ^t be a $\mathcal{C}^2$ Axiom A flow, which is topologically transitive on a locally maximal hyperbolic set Λ. Then Bowen [**B73**] and Bowen and Ruelle [**BR75**] proved that there exists a topologically mixing subshift of finite type (Σ_A, σ) and a positive Hölder roof function $\tau : \Sigma_A \to \mathbb{R}_+$ such that for the corresponding suspension flow $\Phi^t : \Omega \to \Omega$ and for a suitable Lipschitz continuous surjection $\rho : \Sigma_A \to \Lambda$, the following diagram commutes:

$$\begin{array}{ccc} \Omega & \xrightarrow{\Phi^t} & \Omega \\ \downarrow \rho & & \downarrow \rho \\ \Lambda & \xrightarrow{\Psi^t} & \Lambda \end{array}$$

Let λ be an equilibrium measure on Λ with Hölder potential G. Then ρ is a measure theoretic isomorphism between (Ω, μ, Φ^t) and $(\Lambda, \lambda, \Psi^t)$, where μ is a Gibbs measure

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with potential $\bar{G} = G \circ \rho$. By general theory, $\mu = \nu \otimes \mathrm{Leb}/\nu(\tau)$, where ν is a measure on Σ_A, invariant under σ (the equilibrium state of $\check{\bar{G}} - P(G)\tau$, where P is the pressure). Thus, the MLCLT for (Ψ^t, λ) is implied by the MLCLT for (Φ^t, μ). For the proof of the latter one, a simplified version of §4 applies (e.g. **(H2)** and **(H6)** follow from [**GH88**] and [**G89**]). Thus, we have the following analog of Corollary 4.1.

PROPOSITION 6.1. *Consider $\Psi^t : \Lambda \to \Lambda$ be Axiom A flow, λ be a Gibbs measure and $\psi : \Lambda \to \mathbb{R}$ be a Hölder observable. Denote $\boldsymbol{\varphi} = \psi \circ \rho$. Assume that $(\check{\boldsymbol{\varphi}}, \tau)$ is minimal and its linearized group falls into cases* **(A)**, **(B)**, *or* **(C)**. *Then the conclusion of Theorem 3.1 holds for (Ψ, ψ) with $h(x) = 0$ and $h_\tau(x) = 0$.*

We mention that [**W96**] essentially proves Proposition 6.1, Case **(A)** (note that the flow-independence condition of [**W96**] Theorem 2 implies Case **(A)** by [**W96**, Proposition 3]).

6.3. *Suspensions over Pomeau–Manneville maps.* Consider next Pomeau–Manneville maps (also known as Liverani–Saussol–Vaienti maps). Namely, let $\mathbb{M} = [0, 1]$ and $\mathbb{F} : \mathbb{M} \to \mathbb{M}$ be defined by

$$\mathbb{F}(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \le x \le 1/2, \\ 2x - 1 & \text{if } 1/2 < x \le 1. \end{cases}$$

Suppose that $\alpha < 1$. Then the first return map to $[\frac{1}{2}, 1]$ gives an expanding Young tower satisfying the assumptions of §4 (see e.g. the discussion in [**G05**, §1.3]). In particular, $\mathbb{F}$ has a unique absolutely continuous invariant probability measure λ (with Lipschitz density on any compact subinterval of $(0, 1]$, see [**LSV99**, Lemma 2.3]). In addition (37) is satisfied with $\beta = 1/\alpha$. In particular, $\beta > 2$ if $\alpha < \frac{1}{2}$. Consider a Hölder roof function υ on $\mathbb{M}$, and let Ψ be the corresponding suspension semi-flow on the phase space $\aleph$. Let $\boldsymbol{\chi} : \aleph \to \mathbb{R}$ be a zero mean continuous observable so that $\check{\boldsymbol{\chi}}$ is Hölder. Applying Proposition 4.3 we get the following.

PROPOSITION 6.2. *MLCLT is valid for suspension semi-flows over Pomeau–Manneville maps with Hölder roof functions provided that $\alpha < \frac{1}{2}$ and the pair $(\check{\boldsymbol{\chi}}, \upsilon)$ is non-arithmetic.*

We note that in the case $\alpha > 1$, the invariant measure λ is infinite. This case is discussed in [**DN17**]. The approach of [**DN17**] is somewhat similar to that of the present paper.

6.4. *Sinai's billiard flows with finite horizon.* Let $\mathcal{D} = \mathbb{T}^2 \setminus \bigcup_{i=1}^I B_i$, where $\mathbb{T}^2$ is the 2-torus and $B_1, \dots, B_I$ are disjoint strictly convex subsets of $\mathbb{T}^2$, whose boundaries are $\mathcal{C}^3$ smooth with curvature bounded away from zero. The Sinai billiard flow Ψ^t describes a point particle moving with unit speed in the interior of $\mathcal{D}$ and having specular reflection on $\partial\mathcal{D}$ (i.e. the angle of incidence equals the angle of reflection). The phase space of Ψ^t is thus $\aleph = \mathcal{D} \times \mathcal{S}^1$ (pre- and post-collisional points on $\partial\mathcal{D} \times \mathcal{S}^1$ are identified). Consider the measure $\kappa = c\mathrm{Leb}_{\mathcal{D}} \otimes \mathrm{Leb}_{\mathcal{S}^1}$, where c is a normalizing constant. We assume the *finite horizon* condition, i.e. that the time in between two collisions with $\partial\mathcal{D}$ is bounded. (This assumption is natural since if the horizon is infinite, then the return time has infinite second moment and a non-standard normalization is needed in the CLT [**SzV07**, **CD09**].) We can

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regard Ψ^t as a suspension flow over the *billiard ball map*: the Poincaré section on the boundary of the scatterers. Namely, the billiard ball map is $\mathbb{F} : \mathbb{M} \to \mathbb{M}$, where

$$\mathbb{M} = \{x = (q, \boldsymbol{v}) \in \partial\mathcal{D} \times \mathcal{S}^1, \langle \boldsymbol{v}, n\rangle \geq 0\},$$

where n is the normal vector of $\partial\mathcal{D}$ at the point q pointing inside $\mathcal{D}$ (post-collisional point) and $\mathbb{F}(x) = \Psi^{\upsilon(x)}(x)$, υ being the time needed until the next collision. Let the projection of κ to $\mathbb{M}$ be denoted by λ (then λ is the SRB measure, and it has density $c\cos(\boldsymbol{v})$ with respect to the Lebesgue measure on $\mathbb{M}$). Let $\boldsymbol{\chi} : \aleph \to \mathbb{R}$ be a dynamically Hölder observable and $\check{\boldsymbol{\chi}} : \mathbb{M} \to \mathbb{R}$ be defined by $\check{\boldsymbol{\chi}}(x) = \int_0^{\upsilon(x)} \boldsymbol{\chi}(x, s)\, ds$ as before. For the dynamical system $(\mathbb{M}, \lambda, \mathbb{F})$, a tower with exponential tails was constructed in [**Y98**]. Thus, Propositions 4.1 and 5.1 imply the following.

PROPOSITION 6.3. *Assume that* $(\check{\boldsymbol{\chi}}, \upsilon)$ *is minimal and its linearized group falls into cases* **(A)**, **(B)**, *or* **(C)**. *Then the conclusion of Theorem 3.1 holds for* $(\Phi, \boldsymbol{\varphi})$ *with* $h(x) = 0$, $h_\tau(x) = 0$. *Assume furthermore that* $(\check{\boldsymbol{\chi}}, \upsilon)$ *is non-arithmetic. Then for any* $W \in \mathbb{R}$, *any continuous* $\mathfrak{X}, \mathfrak{Y} : \aleph \to \mathbb{R}$, *any continuous and compactly supported* $\mathfrak{Z} : \mathbb{R} \to \mathbb{R}$, *and any* $W(t) \in \mathbb{R}$ *with* $W(t)/\sqrt{t} \to W$,

$$\lim_{t\to\infty} t^{1/2} \int_\aleph \mathfrak{X}(x, s)\mathfrak{Y}(\Psi^t(x, s))\mathfrak{Z}\Big(\int_0^t \boldsymbol{\chi}(\Psi^{s'}(x, s))\, ds' - W(t)\Big) d\kappa(x, s)$$
$$= \mathfrak{g}_\Sigma(W) \int_\aleph \mathfrak{X}\, d\kappa \int_\aleph \mathfrak{Y}\, d\kappa \int_{\mathbb{R}} \mathfrak{Z}\, d\mathrm{Leb}.$$

One special case of Proposition 6.3 (Case **(C)**) for the Sinai billiard flow (namely, when $\boldsymbol{\chi}$ is the horizontal coordinate of the free flight function) is analyzed in [**DN16**, §A.2]. We remark that although finding the minimal group of $(\check{\boldsymbol{\varphi}}, \tau)$ in general is not easy, it is possible in some special cases such as the one studied in [**DN16**] (cf. [**DN16**, Lemma A.3]).

6.5. *Geometric Lorenz flows.* Let $T^t : \mathbb{R}^3 \to \mathbb{R}^3$ be a geometric Lorenz flow as defined in [**HM06**]. Then T^t has an SRB measure $\hat{\mu}$. In [**HM06**], T^t is represented as a suspension over a Poincaré map $P : X \to X$ with roof function v ($\hat{\mu}$ is denoted by μ and v is denoted by h in [**HM06**]). Given a Hölder observable $\boldsymbol{\chi} : \mathbb{R}^3 \to \mathbb{R}$, let $\check{\boldsymbol{\chi}} = \int_0^{\upsilon(x)} \boldsymbol{\chi}(x, s)\, ds$. Furthermore, in [**HM06**], a tower $(\Delta, F, \tilde{\nu})$ satisfying assumptions (A1)–(A4) of §4.1 is constructed. (More precisely, as usual, the measure $\tilde{\nu}$ is not *a* priori given, but it exists by [**Y99**] and then it is pulled back to $\mathbb{R}^3$. We note that slightly different earlier constructions are also available in the literature. However, we follow [**HM06**] for simplicity, so we do not review the earlier work here.) They also show that the roof function $f = \tilde{\tau}$ as well as any function $f = \tilde{\check{\boldsymbol{\chi}}}$ corresponding to a given Hölder observable $\boldsymbol{\chi} : \mathbb{R}^3 \to \mathbb{R}$ satisfy

$$|f(x, l)| \leq Cr(x), \quad |f(x, l) - f(y, l)| \leq Cr(x)\varkappa^{s(x,y)}$$

(here, as usual, $(x, l) \in \Delta$). Since these functions are not in $C_\varkappa(\Delta, \mathbb{R})$, we 'stretch' the tower in the following way. Let $\Delta_0 = \bigcup_k \Delta_{0,k}$ be the base of Δ and the height above $\Delta_{0,k}$ is r_k. Then we define $\Delta'_0 = \bigcup_k \Delta'_{0,k}$ with $\Delta'_{0,k} = \Delta_{0,k}$, $r'_k = r_k^2$ and $\Delta' = \bigcup_k \bigcup_{l=0}^{r'_k-1} \Delta'_{0,k} \times \{l\}$. F' is defined by $F'(x, l) = (x, l + 1)$ if $l < r'_k - 1$ and $F'(x, r'_k - 1) = F(x, r_k - 1)$. Finally, $\tilde{\nu}'(A \times \{l\}) = (1/Z)\nu(A \times \{0\})$, where Z is chosen so as $\tilde{\nu}'$

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is a probability measure. By construction, $(\Delta', F', \tilde{\nu}')$ satisfies assumptions (A1)–(A4) of §4.1. Furthermore, since $\tilde{\nu}(r > n)$ decays superpolynomially, (37) holds with any $\beta > 2$ for Δ'. If f is a function on Δ, we denote by f' the function on Δ' defined by $f'(x, l) = f(x, \lfloor l/r(x) \rfloor)/r(x)$. By construction, T, the first return map to Δ_0, and T', the first return map to Δ_0' are isomorphic. Similarly, the suspension flow with base Δ and roof function $\tilde{\tau}$ is isomorphic to the suspension flow with base Δ' and roof function $\tilde{\tau}'$. Furthermore, the functions $\tilde{\tilde{\boldsymbol{\varphi}}}', \tilde{\tau}'$ are elements of $C_{\varkappa}(\Delta', \mathbb{R})$. Thus, Proposition 5.1 gives the following.

PROPOSITION 6.4. *Let T^t be the geometric Lorenz flow. Assume that $(\check{\boldsymbol{\chi}}, \upsilon)$ is non-arithmetic. Then for any $W \in \mathbb{R}$, any continuous $\mathfrak{X}, \mathfrak{Y} : \mathbb{R}^3 \to \mathbb{R}$, any continuous and compactly supported $\mathfrak{Z} : \mathbb{R} \to \mathbb{R}$, and any $W(t) \in \mathbb{R}$ with $W(t)/\sqrt{t} \to W$,*

$$\lim_{t\to\infty} t^{1/2} \int_{\mathbb{R}^3} \mathfrak{X}(x)\mathfrak{Y}(T^t(x))\mathfrak{Z}\left(\int_0^t \boldsymbol{\chi}(T^s(x))\, ds - W(t)\right) d\hat{\mu}(x)$$
$$= \mathfrak{g}_\Sigma(W) \int_{\mathbb{R}^3} \mathfrak{X}\, d\hat{\mu} \int_{\mathbb{R}^3} \mathfrak{Y}\, d\hat{\mu} \int_{\mathbb{R}} \mathfrak{Z}\, d\mathrm{Leb}.$$

Note that υ is non-arithmetic by [**LMP05**], but the non-arithmeticity of the pair $(\check{\boldsymbol{\chi}}, \upsilon)$ is a non-trivial assumption.

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