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Feldman–Katok pseudometric and the GIKN non-hyperbolic ergodic measures

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Abstract. We introduce Feldman–Katok convergence for invariant measures of a topological dynamical system. This can be seen as a counterpart to the convergence with respect to the $\bar{f}$-metric for finite-state stationary processes (shift-invariant measures on a symbolic space). Feldman–Katok convergence is based on a dynamically defined Feldman–Katok pseudometric. This convergence is stronger than weak* convergence. We prove that Feldman–Katok convergence preserves ergodicity and makes the Kolmogorov–Sinai entropy lower semicontinuous, thereby preserving zero entropy. We apply our findings to non-hyperbolic (having at least one vanishing Lyapunov exponent) ergodic measures constructed using the GIKN method as axiomatized by Bonatti, Díaz and Gorodetski [*Nonlinearity*, **23** (2010), 687–705]. The GIKN method, originally introduced by Gorodetski, Ilyashenko, Kleptsyn and Nalsky [*Functional Analysis and its Applications*, **39** (2005), 21–30], has been widely adapted to produce non-hyperbolic ergodic measures for diffeomorphisms of compact manifolds. We prove that an ergodic measure satisfying the conditions provided by the axiomatized GIKN method is the Feldman–Katok limit of a sequence of periodic measures, which implies that it is either a periodic measure or a loosely Kronecker measure (a measure Kakutani equivalent to an aperiodic ergodic rotation on a compact group) and has zero entropy. This classifies all these measures up to Kakutani equivalence and confirms that geometric constructions of non-hyperbolic measures via periodic approximations based on the axiomatized GIKN method presented in Bonatti *et al.* [*op. cit.*] systematically produce zero-entropy systems.

Key words: classical ergodic theory, smooth dynamics, symbolic dynamics, topological dynamics
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1. *Introduction*

A topological dynamical system is a pair (X, T), where X is a compact metrizable space and $T\colon X \to X$ is a continuous map. The weak* topology is the usual choice for the topology on the space of invariant measures $\mathcal{M}_T(X)$ of (X, T). While this topology is compact and metrizable, it is incompatible with several dynamical properties. For many dynamical systems, the entropy function on $\mathcal{M}_T(X)$ endowed with the weak* topology fails to be continuous: low-entropy measures can approximate high-entropy measures arbitrarily closely, and zero-entropy measures are often dense even when non-zero-entropy measures exist. Similarly, mixing and ergodicity are not preserved under weak* convergence. For shift-invariant measures on a shift (symbolic) space $\mathscr{A}^\infty$ over a finite alphabet $\mathscr{A}$, Feldman's $\bar{f}$-metric provides a stronger topology that is better adapted to dynamical behaviour. An even stronger topology is given by Ornstein's $\bar{d}$-metric, which is complete but non-compact (even non-separable). Both metrics, $\bar{d}$ and $\bar{f}$, work only for finite-state stationary processes (shift-invariant measures on symbolic space $\mathscr{A}^\infty$).

Here, we introduce *Feldman–Katok convergence* for invariant measures of topological dynamical systems. This serves as a counterpart to convergence with respect to the $\bar{f}$-metric for not necessarily symbolic systems and measures. This convergence is based on the *Feldman–Katok pseudometric* $\overline{fk}$ measuring the similarity between orbits of a topological dynamical system. It is inspired by the $\bar{f}$-pseudometric for infinite sequences of symbols, which is in turn based on edit distance. The edit distance between two strings u and v of length n with entries coming from a finite set of symbols is $\bar{f}_n(u, v) = k/n$, where k is the minimum number of symbols that must be removed from each string to make the remaining parts identical. Thus, two strings (words) are close in the edit metric if you can make them match by removing only a small fraction of their symbols. For infinite sequences of symbols $x = x_0x_1x_2\cdots$ and $y = y_0y_1y_2\cdots$ we set

$$\bar{f}(x, y) = \limsup_{n\to\infty} \bar{f}_n(x_0x_1\cdots x_{n-1}, y_0y_1\cdots y_{n-1}).$$

We extend this idea to orbits of arbitrary dynamical systems. The Feldman–Katok pseudometric $\overline{fk}$ captures a similar idea of approximate matching with allowed discrepancies. We may ignore some portions of the orbits while requiring the remaining points on each orbit to be matched into pairs close in the usual metric on the space. We discuss the details below. It is not clear whether $\overline{fk}$ corresponds to a metric on the space of T-invariant measures $\mathcal{M}_T(X)$, as it does for the pseudometric $\bar{f}$ on $\mathscr{A}^\infty$ given by (2), which corresponds to a metric on the space of shift-invariant measures $\mathcal{M}_\sigma(\mathscr{A}^\infty)$ given by (5). Nevertheless, we may use $\overline{fk}$ to define a certain notion of 'convergence' for $\mathcal{M}_T(X)$.

Roughly speaking, we say that a sequence $(\mu_n)_{n=1}^\infty$ of T-invariant measures converges in the Feldman–Katok sense if there is a sequence $(x_n)_{n=1}^\infty$ that is a Cauchy sequence with respect to the $\overline{fk}$-pseudometric, such that for each $n \in \mathbb{N}$, the point x_n is generic for μ_n.

Our definition of $\overline{fk}$ and Feldman–Katok convergence draws inspiration from the way Ornstein's $\bar{d}$-metric and the corresponding pseudometric $\bar{d}$ for infinite sequences of symbols were generalized to Besicovitch pseudometrics and a metric on the set of invariant measures [**1**, **36**, **43**, **47**].

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An $\overline{fk}$-Cauchy sequence of generic points naturally determines a measure, which we call the $\overline{fk}$-*limit* of the sequence. We also prove that Feldman–Katok convergence is stronger than the standard weak* convergence of measures because the $\overline{fk}$-limit μ must also be a limit with respect to the weak* topology of the sequence of measures corresponding to the generic points forming an $\overline{fk}$-Cauchy sequence (Theorem 5.7). The $\overline{fk}$-limit of a sequence of ergodic points (that is, points generic for ergodic measures) is ergodic (Theorem 7.3), thus ergodicity is closed with respect to the Feldman–Katok convergence. We show that the Feldman–Katok convergence also preserves the loosely Kronecker property: if a sequence of invariant measures converges in the Feldman–Katok sense and each measure in the sequence is loosely Kronecker, then the measure which is an $\overline{fk}$-limit of a corresponding sequence of generic points is also loosely Kronecker (Theorem 9.1). Moreover, we establish that the Kolmogorov–Sinai entropy function is lower semicontinuous with respect to Feldman–Katok convergence (Theorem 8.1), which immediately implies that this convergence preserves zero entropy.

As an application, we resolve a question about the entropy of non-hyperbolic ergodic invariant measures arising from the original GIKN construction proposed by Gorodetski *et al* [**29**], which was later axiomatized in [**8**]; see also [**16**]. We call such measures *axiomatized GIKN measures* or *GIKN measures* for short (see Definition 3.4). We note that the term 'GIKN construction' can also be applied to describe other variants of the original method (see [**16**]), but to the best of our knowledge, the name GIKN measures has not been used before. This approach was subsequently adapted by many authors, allowing them to find non-hyperbolic measures for systems with partially hyperbolic dynamics (see [**5**, **8**, **10**, **15**, **17**, **18**, **32**], just to name a few). In particular, it is proved in [**15**] that for a C^1-generic diffeomorphism, every non-hyperbolic homoclinic class carries an ergodic GIKN non-hyperbolic measure. Furthermore, under mild assumptions, the support of that measure is the whole class. For more on the role of the GIKN construction in finding non-hyperbolic ergodic measures, we refer the reader to Díaz's survey [**16**]. Note that another method for constructing non-hyperbolic ergodic measures with uncountable support has recently emerged [**6**, **7**]. It uses the so-called *flip-flop families* and provides a set of positive entropy supporting only non-hyperbolic measures. There are also 'mixed' methods combining the GIKN method and flip-flop families [**9**, **37**]. Again, see [**16**] for a survey and [**3**, **17**] for the latest developments.

The GIKN method in general, and its variant described in [**8**] in particular, is tailored to control Lyapunov exponents and ergodicity; up to now, little has been known about other properties of the resulting GIKN measures. The question of whether all GIKN measures necessarily have zero entropy has circulated in the dynamical systems community for some time (according to Lorenzo J. Díaz, it was raised by Jérôme Buzzi in Orsay, and the first-named author learnt about it from Díaz's lecture at the 'Global dynamics beyond uniform hyperbolicity' conference in Olmué, Chile, in September 2015). Although the repetitive structure inherent in the construction of GIKN measures suggests zero entropy [**6**], this heuristic argument has not convinced all researchers [**10**].

Our main result establishes that GIKN measures (that is, measures fulfilling Definition 3.4) always have zero entropy. The proof relies on demonstrating that this variant

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of the GIKN construction produces sequences of periodic points that are $\overline{fk}$-Cauchy (Theorem 6.2). Hence, the resulting GIKN measures always emerge as $\overline{fk}$-limits of zero-entropy periodic measures; furthermore, they are $\overline{fk}$-limits of measures that are ergodic and loosely Kronecker (therefore with zero entropy). It then follows from the properties of the Feldman–Katok convergence that GIKN measures necessarily inherit ergodicity (Theorem 7.3), zero entropy (by the lower semicontinuity of entropy, Theorem 8.1), and the loosely Kronecker property (by Theorem 9.1), which also implies zero entropy.

In fact, proving that GIKN measures are loosely Kronecker, we provide a full characterization of the former class of measures up to Kakutani equivalence. Kakutani equivalence is a natural equivalence relation between transformations preserving an ergodic non-atomic measure that is weaker than the usual notion of isomorphism; nevertheless, the problem of deciding whether two measure-preserving systems are Kakutani equivalent is, in some sense, as hard as for the usual isomorphism (see [26]). Note that entropy is not an invariant for Kakutani equivalence, but it follows from Abramov's formula that this relation preserves the classes of zero, positive, finite, and infinite entropy transformations. Loosely Kronecker systems form the Kakutani equivalence class of any aperiodic ergodic rotation of a compact infinite group (the latter systems are known as Kronecker systems). According to Feldman and Nadler [22], the name 'loosely Kronecker systems' was suggested by Marina Ratner. Independently, these systems were studied by Katok [30] (partly in collaboration with Katok and Sataev [31], who contributed to the subject as well; see also [42]). Katok called the Kakutani equivalence *monotone equivalence*, and he called the loosely Kronecker systems *standard automorphisms*. Every loosely Kronecker system has zero entropy and is loosely Bernoulli; that is, it belongs to a class of measure-preserving systems introduced by Feldman [21]. Recently, loosely Bernoulli (in particular, loosely Kronecker) systems have gained renewed attention. Glasner, Thouvenot and Weiss [28] showed that, generically, an extension of an ergodic loosely Bernoulli system is loosely Bernoulli. Gerber and Kunde [25] showed there exist smooth, weakly mixing loosely Kronecker transformations whose Cartesian square is loosely Kronecker, and Trujillo [46] showed there exist such transformations that are mixing.

Showing that all GIKN measures are loosely Kronecker means that the ergodic measure-preserving system obtained from a homeomorphism and its GIKN invariant measure is isomorphic to a measure-preserving system arising from taking an aperiodic ergodic group rotation and the first-return transformation induced by the rotation on an appropriately chosen measurable subset of the group of positive Haar measure. This description applies to GIKN measures defined in [3, 5, 8, 10, 15, 18, 29, 32].

Applications of Feldman–Katok convergence are not restricted to the study of GIKN measures. Since the announcement of the preprint [34] in 2017, the Feldman–Katok pseudometric has been used in various contexts. Using the Feldman–Katok pseudometric, the authors of [24] characterized uniquely ergodic dynamical systems that are loosely Kronecker. Later, Trilles [45] proved an analogue of this characterization for continuous flows. The Feldman–Katok pseudometric has also been applied in studies of entropy, mean dimension, and pressure [13, 48, 49], as well as having spurred investigations of dynamical pseudometrics and related notions [12, 20, 23].

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1.1. *Organization of the paper.* Section 2 establishes the necessary preliminaries and notation, including the Feldman $\bar{f}$-metric for shift-invariant measures, Kakutani equivalence, and the notion of loosely Kronecker systems, which will play a central role in our classification results. Section 3 presents the GIKN construction following the exposition of Bonatti, Díaz and Gorodetski [8]. We also discuss how this construction controls Lyapunov exponents to produce non-hyperbolic ergodic measures. Section 4 introduces the Feldman–Katok pseudometric $\overline{fk}$, extending the ideas from symbolic spaces to general compact metric spaces. We establish the basic properties of the resulting pseudometric. Section 5 develops the theory of $\overline{fk}$-convergence for invariant measures. We prove that $\overline{fk}$-Cauchy sequences of generic points determine unique invariant measures (Corollary 5.8) and that $\overline{fk}$-convergence is stronger than weak* convergence (Theorem 5.7). Section 6 establishes that every GIKN sequence is $\overline{fk}$-Cauchy (Theorem 6.2), providing the crucial link between the geometric construction and our pseudometric framework. Section 7 proves that $\overline{fk}$-limits of ergodic measures are ergodic (Theorem 7.3) by adapting Oxtoby's criterion for ergodicity [40] to quasi-orbits. This immediately yields the ergodicity of GIKN measures (Corollary 7.4). While most of the results in §§4–7 are rather easy and of technical nature, they are essential for building the Feldman–Katok pseudometric into a useful tool for studying invariant measures. This work pays dividends, as noted in the introduction, in the various subsequent applications. The main novelty comes from the 'right' abstraction of $\bar{f}$ to the general case. The main results (Theorems 8.1 and 9.1) appear in §§8–9. Section 8 demonstrates that the Kolmogorov–Sinai entropy is lower semicontinuous with respect to $\overline{fk}$-convergence (Theorem 8.1), implying in particular that zero entropy is preserved under $\overline{fk}$-limits. In §9, we prove that aperiodic $\overline{fk}$-limits of loosely Kronecker measures are loosely Kronecker (Theorem 9.1), using Katok's criterion [30]. Combined with our earlier results, this establishes that all GIKN measures are loosely Kronecker and, therefore, ergodic with zero entropy and Kakutani equivalent to ergodic rotations on compact groups (Theorem 9.4). This completely characterizes GIKN measures up to Kakutani equivalence and definitively answers the question about their entropy.

2. *Basic definitions and notation*

Throughout this paper $\mathbb{N} = \{1, 2, \ldots\}$, $|A|$ is the cardinality of a set A, and χ_A is its characteristic function. Unless otherwise stated, i, j, k, ℓ, m, n denote integers.

Let $\bar{d}(A)$ be the *upper density* of a set $A \subset \mathbb{N} \cup \{0\}$, that is,

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{0, \ldots, n-1\}|}{n}.$$

The *lower density* of $A \subset \mathbb{Z}$ is $\underline{d}(A) = 1 - \bar{d}(\mathbb{Z} \setminus A)$. The set $A \subset \mathbb{Z}$ *has density* α if $\bar{d}(A) = \underline{d}(A) = \alpha$. Given a set Z, we denote by Z^∞ the family of all Z-valued sequences indexed by $\mathbb{N} \cup \{0\}$. Typically, we write $\underline{z} = (z_i)_{i=0}^\infty$ for elements of Z^∞. By σ we denote the shift operator acting on Z^∞ as $\sigma(\underline{z}) = (z_{i+1})_{i=0}^\infty$. Whenever Z is a topological space, we endow Z^∞ with the product topology.

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2.1. *Dynamical systems (standing assumptions).* We assume that X is a compact metric space, ρ is a metric for X, and $T: X \to X$ is a continuous map. All results remain the same if ρ is replaced by another compatible metric. Given $x \in X$, we distinguish between the *orbit* of x, which is a set $\{T^n(x) : n \geq 0\} \subset X$, and the *trajectory* of x, which is a sequence $\underline{x}_T = (T^j(x))_{j=0}^{\infty} \in X^{\infty}$.

2.2. *Note on invertibility.* Observe that by default we consider transformations that are not necessarily invertible, but some of the results we invoke from the literature assume invertibility. It is easy to see that in all such cases, it is enough to apply the theorem we need to use the natural extension of non-invertible transformations.

2.3. *Measure-preserving systems.* Most of the standard texts on ergodic theory work with measure-preserving transformations of standard Lebesgue spaces. The latter are measure spaces arising as completions of probability measures on Polish metric spaces endowed with their Borel σ-algebras. In this approach, it is hard to consider different measures on the same underlying space, since the σ-algebra depends non-trivially on the measure. This is the primary reason why we work in the Borel category. If necessary, when we work with a single invariant Borel measure μ, we can take the completion of our measure space to obtain a Lebesgue space. All properties of measure-preserving systems considered here remain the same for the original system and its completion, that is, a measure-preserving system $(X, \mathscr{X}, \mu, T)$ has one of these properties if and only if the completed system $(X, \tilde{\mathscr{X}}_{\mu}, \tilde{\mu}, T)$ has the property.

2.4. *Invariant measures and generic sequences.* We write $\mathscr{X}$ for the Borel σ-algebra of X and $\mathcal{M}(X)$ for the set of all Borel probability measures on X. By $\mathcal{M}_T(X)$ we denote T-invariant measures in $\mathcal{M}(X)$. We write $\mathcal{M}_T^e(X)$ for the set of ergodic invariant measures. The quadruple $\mathbf{X} = (X, \mathscr{X}, \mu, T)$ is a *measure-preserving system*, which is invertible, whenever T is a homeomorphism. We give $\mathcal{M}(X)$ the weak* topology. Recall that $(\mu_n)_{n=1}^{\infty}$ converges to μ in $\mathcal{M}(X)$ if and only if $\int \varphi \, d\mu_n \to \int \varphi \, d\mu$ for every continuous $\varphi: X \to \mathbb{R}$. The weak* topology on $\mathcal{M}(X)$ is compact and compatible with the Prokhorov metric

$$D_{\mathrm{P}}(\mu, \nu) = \inf\{\varepsilon > 0 \ : \ \mu(B) \leq \nu(B^{\varepsilon}) + \varepsilon \text{ for every Borel set } B \subset X\},$$

where $B^{\varepsilon} = \{y \in X \ : \ \mathrm{dist}(y, B) < \varepsilon\}$ denotes the *ε-hull* of B. For $x \in X$, let $\hat{\delta}(x) \in \mathcal{M}(X)$ be the Dirac measure supported on $\{x\}$. Let $\mathfrak{m}(\underline{x}, n)$ denote the *n-empirical measure of* $\underline{x} = (x_j)_{j=0}^{\infty} \in X^{\infty}$, that is, $\mathfrak{m}(\underline{x}, n) = (1/n)(\hat{\delta}(x_0) + \hat{\delta}(x_1) + \cdots + \hat{\delta}(x_{n-1}))$. Given $x \in X$, we put $\mathfrak{m}_T(x, n) = \mathfrak{m}(\underline{x}_T, n)$. A measure $\mu \in \mathcal{M}(X)$ is *generated by* $\underline{x} \in X^{\infty}$ if μ is the limit of some subsequence of $(\mathfrak{m}(\underline{x}, n))_{n=1}^{\infty}$. The set of all measures generated by $\underline{x}$ is denoted by $V(\underline{x})$. We say that $\underline{x} \in X^{\infty}$ is a *generic sequence* for μ (respectively, an *ergodic sequence*) if $V(\underline{x}) = \{\mu\}$ for some (ergodic) $\mu \in \mathcal{M}(X)$. We write $\mathrm{Gen}(\mu)$ for the set of all sequences in X^{∞} that are generic for μ. For $z \in X$ we define $V_T(z) = V(\underline{z}_T)$, and we call z a *generic point* (*ergodic point*) if $\underline{z}_T$ is a generic sequence (respectively, ergodic) sequence. Note that *every* invariant measure has a generic sequence in X^{∞} [36, 44], while a non-ergodic invariant measure may have no generic points. Furthermore, one can choose

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a generic sequence that is a *quasi-orbit*. A quasi-orbit is built from long pieces of orbits in such a way that the set of positions at which a quasi-orbit switches from one piece of orbit to another has zero asymptotic density.

Definition 2.1. We say that $\underline{z} = (z_n)_{n=0}^{\infty} \in X^{\infty}$ is a *quasi-orbit* for T if $\bar{d}(\{n \geq 0 : z_{n+1} \neq T(z_n)\}) = 0$.

It is easy to see that every measure generated by a quasi-orbit for T must be T-invariant. Furthermore, the GIKN construction yields a quasi-orbit that is generic for the invariant measure it produces. We work with that quasi-orbit to demonstrate the properties of the underlying measure.

2.5. *Symbolic systems.* Let $\mathscr{A}$ be a finite set with the discrete topology. We endow $\mathscr{A}^{\infty}$ with the product topology and call it the *full shift* over the alphabet $\mathscr{A}$. The *shift map* is the map $\sigma\colon \mathscr{A}^{\infty} \to \mathscr{A}^{\infty}$ given by $\sigma((x_n)_{n=0}^{\infty}) = (x_{n+1})_{n=0}^{\infty}$. The set of (ergodic) shift-invariant measures is denoted by $\mathcal{M}_{\sigma}(\mathscr{A}^{\infty})$ ($\mathcal{M}_{\sigma}^{e}(\mathscr{A}^{\infty})$). When $\mathscr{A} = \{0, 1, \ldots, k-1\}$ for some $k \in \mathbb{N}$, we write $\Omega_k = \{0, 1, \ldots, k-1\}^{\infty}$. We call the elements of $\mathscr{A}^n$ *words of length n* over $\mathscr{A}$. Let $\mathscr{A}^{+} = \bigcup_{n\geq 1} \mathscr{A}^n$ and $|u|$ stand for the length of $u \in \mathscr{A}^{+}$. Every word $u \in \mathscr{A}^{+}$ determines a *cylinder set* $[u] \subset \mathscr{A}^{\infty}$ consisting of all sequences in $\mathscr{A}^{\infty}$ whose first $|u|$ symbols coincide with u. Cylinders form a clopen base for the topology of $\mathscr{A}^{\infty}$ and generate the Borel σ-algebra $\mathscr{B}$ of $\mathscr{A}^{\infty}$. Given two n-words $u = u_0u_1 \cdots u_{n-1}$ and $w = w_0w_1 \cdots w_{n-1}$ over $\mathscr{A}$, we define the *Hamming distance* between u and w as

$$\bar{d}_n(u, w) = \frac{1}{n}|\{0 \leq j < n : u_j \neq w_j\}|.$$

The edit metric between two strings (words) of length n equals $1 - k/n$, where k is the minimum number of symbols that must be removed from each string so that the remaining strings are identical. In other words, the *edit distance* between u and w is given by

$$\bar{f}_n(u, w) = 1 - \frac{k}{n},$$

where k is the largest among those integers ℓ such that for some $0 \leq i_1 < i_2 < \cdots < i_\ell < n$ and $0 \leq j_1 < j_2 < \cdots < j_\ell < n$ we have $u_{i_s} = w_{j_s}$ for $s = 1, \ldots, \ell$. For two infinite sequences $\omega = \omega_0\omega_1\omega_2 \cdots$, $\omega' = \omega'_0\omega'_1\omega'_2 \cdots$ in $\mathscr{A}^{\infty}$ we set

$$\begin{aligned}\bar{d}(\omega, \omega') &= \limsup_{n\to\infty} \bar{d}_n(\omega, \omega') = \limsup_{n\to\infty} \bar{d}_n(\omega_0\omega_1 \cdots \omega_{n-1}, \omega'_0\omega'_1 \cdots \omega'_{n-1}) \\ &= \bar{d}(\{j \geq 0 : \omega_j \neq \omega'_j\}), \qquad (1)\\ \bar{f}(\omega, \omega') &= \limsup_{n\to\infty} \bar{f}_n(\omega, \omega') = \limsup_{n\to\infty} \bar{f}_n(\omega_0\omega_1 \cdots \omega_{n-1}, \omega'_0\omega'_1 \cdots \omega'_{n-1}), \qquad (2)\\ \hat{f}(\omega, \omega') &= \inf\{\varepsilon > 0 : \text{ there are increasing sequences } (i_r), (i'_r) \text{ in } \mathbb{N}^{\infty} \\ &\qquad \text{of lower density at least } 1 - \varepsilon \text{ for which } \omega_{i_r} = \omega'_{i'_r} \text{ for all } r \geq 0\}. \qquad (3)\end{aligned}$$

The functions $\bar{d}$, $\bar{f}$, and $\hat{f}$ are pseudometrics on $\mathscr{A}^{\infty}$. Furthermore, $\bar{f}(\omega, \omega') \leq \hat{f}(\omega, \omega')$ and $\bar{f}(\omega, \omega') \leq \bar{d}(\omega, \omega')$ for $\omega, \omega' \in \mathscr{A}^{\infty}$, and $\bar{f}$, $\hat{f}$ are uniformly equivalent pseudometrics on $\mathscr{A}^{\infty}$ (see [39]).

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2.6. *Processes.* We write $\mathbf{P}^m(X)$ for the set of all Borel measurable partitions of X into at most m sets, called *atoms*. For $\mathcal{P} \in \mathbf{P}^k(X)$ we write $\mathcal{P} = \{P_0, \ldots, P_{k-1}\}$ regardless of the actual number of non-empty elements in $\mathcal{P}$ and we agree that $P_j = \emptyset$ for $|\mathcal{P}| \le j < k$. Let $\mathbf{X} = (X, \mathscr{X}, \mu, T)$ be a measure-preserving system and let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\} \in \mathbf{P}^k(X)$. We identify $\mathcal{P}$ with a function $\mathcal{P}\colon X \to \{0, \ldots, k-1\}$ defined by $\mathcal{P}(x) = j$ for $x \in P_j$. The pair $(\mathbf{X}, \mathcal{P})$ is called a process (see [27, pp. 273]). A *coding* of $\underline{x} = (x_j)_{j=0}^{\infty} \in X^{\infty}$ is $\mathcal{P}(\underline{x}) = (\mathcal{P}(x_j))_{j=0}^{\infty} \in \Omega_k$. The map $\underline{\mathcal{P}}\colon X \to \Omega_k$ given by $\underline{\mathcal{P}}(x) = \mathcal{P}(\underline{x}_T)$ defines a homomorphism of $\mathbf{X}$ and $(\Omega_k, \mathscr{B}, \mu_{\mathcal{P}}, \sigma)$, where $\mu_{\mathcal{P}} = \underline{\mathcal{P}}_*(\mu)$ is the push-forward of the measure μ. For $n > 0$ and $\underline{x} \in X^{\infty}$ let $\varphi_{\mathcal{P}}^n(\underline{x}) = \mathcal{P}(x_0)\mathcal{P}(x_1)\cdots\mathcal{P}(x_{n-1}) \in \{0, 1, \ldots, k-1\}^n$. Let $\mathcal{P}^n$ be the *nth join* of $\mathcal{P}$ given by

$$\mathcal{P}^n = \bigvee_{j=0}^{n-1} T^{-j}(\mathcal{P}) = \{P_{i_0} \cap T^{-1}(P_{i_1}) \cap \cdots \cap T^{-n+1}(P_{i_{n-1}}) : P_{i_j} \in \mathcal{P} \text{ for } 0 \le j < n\}.$$

Note that for $x \in X$ we may write $\varphi_{\mathcal{P}}^n(\underline{x}_T) = \mathcal{P}^n(x)$, because $\varphi_{\mathcal{P}}^n(\underline{x}_T) = i_0 i_1 \cdots i_{n-1}$ if and only if $\mathcal{P}^n(x) = P_{i_0} \cap T^{-1}(P_{i_1}) \cap \cdots \cap T^{-n+1}(P_{i_{n-1}})$. The measure-preserving system $(\Omega_k, \mathscr{B}, \mu_{\mathcal{P}}, \sigma)$ is called the *symbolic representation* of $\mathbf{X}$ with respect to the partition $\mathcal{P}$, and $\mu_{\mathcal{P}}$ is the *symbolic representation measure* of μ. We endow $\mathbf{P}^k(X)$ with the distance d_1^{μ} given for $\mathcal{P}, \mathcal{Q} \in \mathbf{P}^k(X)$ by

$$d_1^{\mu}(\mathcal{P}, \mathcal{Q}) = \frac{1}{2}\sum_{j=0}^{k-1} \mu(P_j \div Q_j) = \frac{1}{2}\sum_{j=0}^{k-1} \int_X |\chi_{P_j} - \chi_{Q_j}| \, \mathrm{d}\mu = \mu(\{x \in X : \mathcal{P}(x) \ne \mathcal{Q}(x)\}).$$

Note that the definition of d_1^{μ} takes into account the order of the partition's elements. We tacitly identify Borel partitions $\mathcal{P}, \mathcal{Q} \in \mathbf{P}^k(X)$ with $d_1^{\mu}(\mathcal{P}, \mathcal{Q}) = 0$. With this identification d_1^{μ} is a complete metric for $\mathbf{P}^k(X)$.

2.7. *Entropy.* For a finite measurable partition $\mathcal{P}$ and $\mu \in \mathcal{M}_T(X)$ we denote by $h(\mu, \mathcal{P})$ the *entropy of $\mathcal{P}$with respect to μ and T* and by $h(\mu)$ the *entropy of μ with respect to T*, that is, $h(\mu) = \sup_{\mathcal{P}} h(\mu, \mathcal{P})$, where $h(\mu, \mathcal{P}) = \inf_{n \in \mathbb{N}} - \sum_{P \in \mathcal{P}^n} \mu(P) \log \mu(P)$. The real-valued function $\mathcal{P} \mapsto h(\mu, \mathcal{P})$ is uniformly continuous on $\mathbf{P}^k(X)$ equipped with d_1^{μ} [27, Lemma 15.9 (5)].

2.8. *Faithful coding.* For $\mathcal{P} \in \mathbf{P}^k(X)$ we define $\partial\mathcal{P} = \partial P_0 \cup \cdots \cup \partial P_{k-1}$. A partition $\mathcal{P} \in \mathbf{P}^k(X)$ with $\mu(\partial\mathcal{P}) = 0$ is called *faithful* for $\mathbf{X}$.

LEMMA 2.2. *If $\underline{x} \in X^{\infty}$ is generic for $\mu \in \mathcal{M}_T(X)$ and $\mathcal{P} \in \mathbf{P}^k(X)$ is such that $\mu(\partial\mathcal{P}) = 0$, then $\omega = \mathcal{P}(\underline{x}) \in \Omega_k$ is a generic point for the measure $\mu_{\mathcal{P}}$ on Ω_k.*

Proof. Note the following two properties of the boundary operator ∂: $\partial(Y \cap Z) \subset \partial Y \cup \partial Z$ for $Y, Z \subset X$ and $\partial T^{-1}(U) \subset T^{-1}(\partial U)$ for any $U \subset X$. Using this and $\mu(\partial P_j) = 0$ for every $0 \le j < k$, we see that

$$\mu\left(\partial\left(\bigcap_{i=0}^{m-1} T^{-i}(P_{j_i})\right)\right) = 0 \quad \text{for every } m \ge 1 \text{ and } 0 \le j_0, j_1, \ldots, j_{m-1} < k.$$

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In other words, $\mu(\partial\mathcal{P}^m) = 0$ for all $m \geq 1$. Then for every $m \geq 1$ and $0 \leq j_0, j_1, \ldots, j_{m-1} < k$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \chi_A(x_j) = \mu(A) \quad \text{for } A = \bigcap_{i=0}^{m-1} T^{-i}(P_{j_i}) \in \mathcal{P}^m.$$

Note that $\omega = \mathcal{P}(\underline{x})$ is an orbit for σ and observe that ω is generic for a σ-invariant measure μ' such that

$$\mu'([j_0 j_1 \cdots j_{m-1}]) = \mu\left(\bigcap_{i=0}^{m-1} T^{-i}(P_{j_i})\right)$$
$$\text{for every } m \geq 1 \text{ and } 0 \leq j_0, j_1, \ldots, j_{m-1} < k.$$

Hence, μ' and $\mu_{\mathcal{P}}$ agrees on cylinders in Ω_k. This implies $\mu' = \mu_{\mathcal{P}}$. □

2.9. *Kakutani equivalence.* Kakutani equivalence serves as a natural equivalence relation among transformations that preserve an ergodic non-atomic measure, weaker than the conventional concept of isomorphism (however, in some sense equally hard; cf. [26]). Although entropy does not remain invariant under Kakutani equivalence, Abramov's formula implies that this relation retains the categories of zero, positive and finite, as well as infinite entropy transformations. Let $\mathbf{X} = (X, \mathscr{X}, \mu, T)$ be a measure-preserving system. For a set $E \in \mathscr{X}$ with $\mu(E) > 0$ and $x \in E$ we define the *return time* $n(x) = \inf\{k > 0 : T^k(x) \in E\}$. This function is finite for μ-almost every $x \in E$, and we define the *induced transformation* $T_E \colon E \to E$ by $T_E(x) = T^{n(x)}(x)$. Measure-preserving systems $\mathbf{X} = (X, \mathscr{X}, \mu, T)$ and $\mathbf{Y} = (Y, \mathscr{Y}, \nu, S)$ are *Kakutani equivalent* (recall that Katok calls this relation *monotone equivalence*) if there exist $E \in \mathscr{X}$ with $\mu(E) > 0$ and $F \in \mathscr{Y}$ with $\nu(F) > 0$ such that T_E is isomorphic with S_F.

2.10. *Loosely Bernoulli systems.* Feldman [21] studied the isomorphism problem in ergodic theory. He introduced a new property (called *loose Bernoulliness*) for finite partitions of a measure-preserving system (X, B, μ, T). He used it to construct important examples of K-automorphisms which are loosely Bernoulli but not Bernoulli. In particular, these measure-preserving systems have positive entropy. The definition of loosely Bernoulli partition follows Ornstein's definition [38] of very weak Bernoulli partition with the Hamming distance of strings of symbols of length n replaced by the weaker edit metric $\bar{f}_n$. Feldman's idea was subsequently extended by Ornstein, Rudolph and Weiss [39]. It turns out that, in each of three entropy classes, the Kakutani equivalence class of loosely Bernoulli transformations is the simplest one (see [39] for more details). Positive and finite (respectively, infinite) entropy transformations Kakutani equivalent to a Bernoulli shift coincide with loosely Bernoulli and positive and finite (respectively, infinite) entropy transformations. Zero-entropy loosely Bernoulli transformations form the Kakutani equivalence class of any ergodic rotation of a compact infinite group (Kronecker system). According to Feldman and Nadler [22], members of the latter equivalence class

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are called *loosely Kronecker* following a suggestion of Marina Ratner. We adapt this terminology and provide a characterization of these systems in §2.13 (cf. Theorem 9.2).

2.11. *Feldman's $\bar{f}$ and Ornstein's $\bar{d}$ metrics for shift-invariant measures on $\mathscr{A}^\infty$.* Let μ and μ' be ergodic shift-invariant measures on $\mathscr{A}^\infty$. By μ_n (respectively, μ'_n) we denote the restriction of μ (respectively, μ') to the set of all n-cylinders, that is, these are measures that μ (respectively, μ') defines on $\mathscr{A}^n$ via the projections onto first n coordinates. A *joining* of μ and μ' is any $\sigma \times \sigma$-invariant measure on $\mathscr{A}^\infty \times \mathscr{A}^\infty$ whose marginals are μ and μ'. We write $J(\mu, \mu')$ for the set of all joinings of μ and μ'. Similarly, $J_n(\mu, \mu')$ denotes the set of all measures λ_n on $\mathscr{A}^n \times \mathscr{A}^n$ whose marginals are μ_n and μ'_n.

Define

$$\bar{f}_n(\mu, \mu') = \inf_{\lambda_n \in J_n(\mu,\mu')} \int_{\mathscr{A}^n \times \mathscr{A}^n} \bar{f}_n(u, u')\lambda_n(u, u'), \tag{4}$$

$$\bar{f}(\mu, \mu') = \limsup_{n\to\infty} \bar{f}_n(\mu, \mu'). \tag{5}$$

One can prove (see [39]) that (5) defines a distance between measures μ and μ' on $\mathscr{A}^\infty$ known as the $\bar{f}$-metric. Ornstein's $\bar{d}$-metric on $\mathcal{M}_\sigma(\mathscr{A}^\infty)$ is defined analogously with the Hamming distance $\bar{d}_n$ on $\mathscr{A}^n$ replacing the edit distance $\bar{f}_n$ in (4) (see [43]).

2.12. *Properties of $\bar{f}$.* The following result is a direct corollary of [39].

LEMMA 2.3. *For every $\varepsilon > 0$ there is a $\delta > 0$ such that if μ and μ' are shift-invariant ergodic measures on $\mathscr{A}^\infty$ and there are generic points ω for μ and ω' for μ' with $\bar{f}(\omega, \omega') < \delta$, then $\bar{f}(\mu, \mu') < \varepsilon$.*

Note also that the entropy function $\mu \mapsto h(\mu)$ is uniformly continuous with respect to the $\bar{f}$-metric on the space of shift-invariant measures on $\mathscr{A}^\infty$ (for ergodic measures this is [39, Proposition 3.4]; the assumption of ergodicity can be removed thanks to the main result of [20]).

2.13. *Loosely Kronecker systems.* An ergodic measure-preserving system $\mathbf{X} = (X, \mathscr{X}, \mu, T)$ is *loosely Kronecker* if it has zero entropy and for every finite Borel partition $\mathcal{P}$ of X and every $\varepsilon > 0$ there are $n > 0$ and a set A_n of atoms of $\mathcal{P}^n = \bigvee_{j=0}^{n-1} T^{-j}(\mathcal{P})$ such that $\mu(A_n) > 1 - \varepsilon$ and $\bar{f}_n(u, w) < \varepsilon$ for $u, w \in A_n$ (here, as usual, we identify atoms of the partition $\mathcal{P}^n$ with words of length n over the alphabet $\{0, 1, \ldots, |\mathcal{P}| - 1\}$).

3. *GIKN construction*

We present the original construction of Gorodetski *et al* from [29] following the exposition provided by Bonatti *et al* in [8]. Recall that X is a compact metric space, ρ is a metric for X, and $T\colon X \to X$ is a continuous map. By definition, each GIKN measure is the weak* limit of measures supported on a sequence of periodic points with special properties. Each periodic orbit in this sequence can be divided in two parts: the *shadowing part* and the *tail*. Their key features are as follows.

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(1) The shadowing part takes a large proportion (growing to 1 as n goes to ∞) of each periodic orbit in the sequence. Furthermore, the images of each point on the shadowing part of the nth orbit are γ_n-close to the $(n-1)$th orbit for the number of iterates equal to the primary period of the $(n-1)$th orbit, and the series formed by γ_ns is summable. This is used to show that the limit measure is ergodic.
(2) There is a fixed (centre) direction such that the Lyapunov exponent in that direction along the sequence of periodic orbits decreases to zero. To achieve this, each periodic orbit spends a small proportion of its primary period (this part of the orbit is called the *tail*) in a region that is far from the previous orbits and is chosen so that the Lyapunov exponent in the centre direction along the whole orbit has a smaller absolute value than the same exponent for the previously constructed orbits. This also guarantees that the limit measure is non-atomic.

3.1. *Topological backbone of the GIKN construction.* In this subsection, we sketch the main features of the GIKN construction. Our results about $\overline{fk}$-limits like Theorems 8.1 and 9.1 apply to any measure defined in this way, because the GIKN construction leads to $\overline{fk}$-convergent sequences of periodic orbits (see Theorem 6.2).

Definition 3.1. We say that a T-periodic orbit Γ is a (γ, κ)-*good approximation* of a T-periodic orbit Λ if there are a subset Δ of Γ with $|\Delta|/|\Gamma| \geq \kappa$ and a constant-to-one surjection $\psi\colon \Delta \to \Lambda$ (called a (γ, κ)-*projection*) such that for each $y \in \Delta$ and $0 \leq j < |\Lambda|$ we have

$$\rho(T^j(y), T^j(\psi(y))) < \gamma.$$

Definition 3.2. We call a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of T-periodic orbits with $|\Gamma_n| \nearrow \infty$ as $n \to \infty$ a *GIKN sequence* if there are sequences of positive real numbers $(\gamma_n)_{n=1}^{\infty}$ and $(\kappa_n)_{n=1}^{\infty}$ such that the following conditions hold:
(1) for each $n \geq 1$, Γ_{n+1} is a (γ_n, κ_n)-good approximation of Γ_n;
(2) $\sum_{n=1}^{\infty} \gamma_n < \infty$;
(3) $\prod_{n=1}^{\infty} \kappa_n > 0$.
By an abuse of terminology, we will also refer to the sequence of ergodic measures $(\mu_n)_{n=1}^{\infty}$, where μ_n is supported on Γ_n, as a *GIKN sequence of measures*.

The following is a slightly reformulated version of [8, Lemma 2.5]. The proof that μ is ergodic in [8] invokes [29, Lemma 2].

THEOREM 3.3. *Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a GIKN sequence of T-periodic orbits. If $(\mu_n)_{n=1}^{\infty}$ is the associated sequence of ergodic measures, then $(\mu_n)_{n\in\mathbb{N}}$ weak* converges to an ergodic measure μ supported on the topological limit of $(\Gamma_n)_{n\in\mathbb{N}}$, that is,*

$$\operatorname{supp}\mu = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} \Gamma_n}.$$

Definition 3.4. We say that μ is *an axiomatized GIKN measure* (or *a GIKN measure* for short) if it is a weak* limit of a GIKN sequence.

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With this terminology, Theorem 3.3 states that every GIKN sequence determines a *GIKN measure*.

3.2. *Lyapunov exponents.* We discuss how the GIKN construction is used to find non-hyperbolic measures. This part is logically independent of the rest of the paper.

Let M be a smooth Riemannian manifold with $\dim M = m$. If $f\colon M \to M$ is a diffeomorphism and μ is an ergodic f-invariant measure, then there exist a set $\Lambda \subset M$ of full μ-measure and real numbers $\chi_\mu^1 \le \cdots \le \chi_\mu^m$ such that for every $x \in \Lambda$ and non-zero vector $v \in T_x M$ one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \chi_\mu^i \quad \text{for some } i = 1, \dots, m.$$

The number χ_μ^i is the ith *Lyapunov exponent* of the measure μ. If there exist a closed f-invariant set $\Xi \subset M$ and a continuous Df-invariant direction field $\mathcal{E} = (E_x)_{x\in\Xi} \subset T_\Xi M$ with $\dim E_x = 1$ for $x \in \Xi$, then for every measure $\nu \in \mathcal{M}_f(M)$ with $\operatorname{supp} \nu \subset \Xi$ there is a Lyapunov exponent $\chi^{\mathcal{E}}(\nu)$ of ν associated with $\mathcal{E}$ in the following sense: for ν-almost every $x \in M$ and a non-zero vector $v \in E_x$ one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \chi^{\mathcal{E}}(\nu).$$

Furthermore, if $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}_f^e(\Xi)$, $\mu \in \mathcal{M}_f^e(M)$ and $\mu_n \to \mu$ as $n \to \infty$ in the weak* topology (this implies that $\operatorname{supp} \mu \subset \Xi$ as well), then $\chi^{\mathcal{E}}(\mu_n) \to \chi^{\mathcal{E}}(\mu)$ as $n \to \infty$ [29, Lemma 1].

THEOREM 3.5. *Assume that $f\colon M \to M$ is a diffeomorphism of a smooth Riemannian compact manifold with a closed f-invariant set $\Xi \subset M$ and a continuous Df-invariant direction field $\mathcal{E} = (E_x)_{x\in\Xi} \subset T_\Xi M$ with $\dim E_x = 1$ for $x \in \Xi$. Let $(\Gamma_n)_{n\in\mathbb{N}} \subset \Xi$ be a sequence of f-periodic orbits and suppose that $|\Gamma_n|$ increases to infinity as $n \to \infty$. For each n let μ_n be the ergodic measure supported on Γ_n. Furthermore, assume that the following conditions hold.*

(1) *There exist sequences of positive real numbers $(\gamma_n)_{n=1}^\infty$ and a constant C such that for each n the orbit Γ_{n+1} is a $(\gamma_n, 1 - C|\chi^{\mathcal{E}}(\mu_n)|)$-good approximation of Γ_n.*

(2) *There exists a constant $0 < \alpha < 1$ such that*

$$|\chi^{\mathcal{E}}(\mu_{n+1})| < \alpha |\chi^{\mathcal{E}}(\mu_n)|.$$

(3) *$\gamma_n < (\min_{1\le i\le n} d_i)/(3 \cdot 2^n)$, where d_i denotes the minimal distance between different points in Γ_i.*

Then $(\Gamma_n)_{n\in\mathbb{N}} \subset \Xi$ is a GIKN sequence and $(\mu_n)_{n\in\mathbb{N}}$ weak converges to a GIKN measure μ that is ergodic and has uncountable support equal to the topological limit of $(\Gamma_n)_{n\in\mathbb{N}}$. Furthermore, μ is non-hyperbolic, since $\chi^{\mathcal{E}}(\mu) = 0$.*

4. *Feldman–Katok pseudometric*

In this section, we first recall the definition of the Besicovitch pseudometric, then introduce the Feldman–Katok pseudometric, which extends $\bar{f}$ to general metric spaces.

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For symbolic dynamical systems (subsystems of the full shift $(\mathscr{A}^{\infty}, \sigma)$) the metrics $\bar{d}$ and $\bar{f}$ on $\mathcal{M}_{\sigma}(\mathscr{A}^{\infty})$ defined in §2.11 are related with identically denoted pseudometrics on $\mathscr{A}^{\infty}$ given respectively by (1) and (2). This connection is described in more detail in [39, 43, 47]. From the point of view of ergodic theory there is no need to extend $\bar{d}$ and $\bar{f}$ from $\mathscr{A}^{\infty}$ to more general metric spaces. It turns out, however, that for some geometric applications an extension of $\bar{d}$, called the *Besicovitch pseudometric*, is very useful (see [1, 2, 36] and references therein). However, we are not aware of any analogue of the $\bar{f}$-pseudometric for general metric space in the literature.

4.1. *Besicovitch pseudometric D_{B}.* For $\underline{x} = (x_j)_{j=0}^{\infty}$, $\underline{z} = (z_j)_{j=0}^{\infty} \in X^{\infty}$ we define the *Besicovitch pseudometric* D_{B} on X^{∞} as

$$D_{\mathrm{B}}(\underline{x}, \underline{z}) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \rho(x_j, z_j).$$

It is known [36] that D_{B} is uniformly equivalent to D'_{B} given by

$$D'_{\mathrm{B}}(\underline{x}, \underline{z}) = \inf\{\delta > 0 : \bar{d}(\{n \geq 0 : \rho(x_n, z_n) \geq \delta\}) < \delta\}. \tag{6}$$

Given $T\colon X \to X$, the *Besicovitch pseudometric* D_{B} on X is defined by $D_{\mathrm{B}}(x, y) = D_{\mathrm{B}}(\underline{x}_T, \underline{z}_T)$. If $X = \mathscr{A}^{\infty}$ and ρ is any metric compatible with the topology on $\mathscr{A}^{\infty}$, then the Besicovitch pseudometric D_{B} and the $\bar{d}$-pseudometric are uniformly equivalent on $\mathscr{A}^{\infty}$.

4.2. *Pseudometric $\overline{fk}$.* Fix $\underline{x} = (x_j)_{j=0}^{\infty}, \underline{z} = (z_j)_{j=0}^{\infty} \in X^{\infty}$, $\delta > 0$, and $n \in \mathbb{N}$.

Definition 4.1. An (n, δ)-*match* between $\underline{x}$ and $\underline{z}$ is an order-preserving bijection $\pi\colon \mathcal{D}(\pi) \to \mathcal{R}(\pi)$ such that $\mathcal{D}(\pi), \mathcal{R}(\pi) \subset \{0, 1, \ldots, n-1\}$ and for every $i \in \mathcal{D}(\pi)$ we have $\rho(x_i, z_{\pi(i)}) < \delta$. The *fit* $|\pi|$ of π is the cardinality of $\mathcal{D}(\pi)$.

Definition 4.2. An (n, δ)-match is *maximal* if its fit is the largest possible. If there is no (n, δ)-match π with $|\pi| \geq 1$, then the *empty match* $\pi_{\emptyset}$ with $|\pi_{\emptyset}| = 0$ is the maximal one. The (n, δ)-*gap* between $\underline{x}$ and $\underline{z}$ is given by

$$\bar{f}_{n,\delta}(\underline{x}, \underline{z}) = 1 - \frac{\max\{|\pi| : \pi \text{ is an } (n, \delta)\text{-match of } \underline{x} \text{ with } \underline{z}\}}{n}.$$

Note that if $X = \mathscr{A}^{\infty}$ for some finite alphabet $\mathscr{A}$ and we endow $X = \mathscr{A}^{\infty}$ with the standard metric given by

$$\rho(\underline{\omega}, \underline{\omega}') = \begin{cases} 0 & \text{if } \underline{\omega} = \underline{\omega}', \\ 2^{-\min\{j\geq 0 : \omega_j \neq \omega'_j\}} & \text{otherwise,} \end{cases}$$

then (with a minor abuse of notation) we have $\bar{f}_{n,1}(\underline{x}, \underline{z}) = \bar{f}_n(x_0 x_1 \cdots x_{n-1}, z_0 z_1 \cdots z_{n-1})$ for any $\underline{x}, \underline{z} \in \mathscr{A}^{\infty}$.

We note some properties of the (n, δ)-gap function $\bar{f}_{n,\delta}$ that hold for any $\underline{x}, \underline{z} \in X^{\infty}$, $\varepsilon, \delta > 0$, and $n \in \mathbb{N}$. The proofs are left to the reader.

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Fact 4.3. If $0 < \delta < \delta'$, then $\bar{f}_{n,\delta'}(\underline{x}, \underline{z}) \leq \bar{f}_{n,\delta}(\underline{x}, \underline{z})$.

Fact 4.4. If $q \in \mathbb{N}$, then $\bar{f}_{n,\delta}(\underline{x}, \underline{z}) \leq \bar{f}_{n+q,\delta}(\underline{x}, \underline{z}) + q/n$.

Fact 4.5. If $\bar{f}_{n,\delta}(\underline{x}, \underline{z}) < \varepsilon$, then $D_{\mathrm{P}}(\mathfrak{m}(\underline{x}, n), \mathfrak{m}(\underline{z}, n)) < \max\{\delta, \varepsilon\}$.

Definition 4.6. The *$\bar{f}_\delta$-pseudodistance* between $\underline{x}$ and $\underline{z}$ is given by

$$\bar{f}_\delta(\underline{x}, \underline{z}) = \limsup_{n\to\infty} \bar{f}_{n,\delta}(\underline{x}, \underline{z}).$$

Fact 4.7. If $i \in \mathbb{N} \cup \{0\}$, then $\bar{f}_\delta(\underline{x}, \underline{z}) = \bar{f}_\delta(\underline{x}, \sigma^i(\underline{z}))$. In particular, $\bar{f}_\delta(\underline{x}, \sigma^i(\underline{x})) = 0$.

Fact 4.8. Assume that $\underline{x}$ and $\underline{z}$ are periodic with a common period N. If $p^{(n)}$ denotes the fit of a maximal (nN, δ)-match, then the sequence $(p^{(n)})_{n=1}^{\infty}$ is subadditive, that is, $p^{(\ell)} + p^{(m)} \geq p^{(\ell+m)}$ for every $\ell, m \in \mathbb{N}$.

Fact 4.9. If $\underline{x}$ and $\underline{z}$ are periodic sequences with a common period N, then

$$\bar{f}_\delta(\underline{x}, \underline{z}) = \inf_{n\in\mathbb{N}} \bar{f}_{nN,\delta}(\underline{x}, \underline{z}) = \lim_{n\to\infty} \bar{f}_{nN+q,\delta}(\underline{x}, \underline{z})$$

for every $0 \leq q < N$.

Definition 4.10. The *Feldman–Katok pseudometric on X^∞* is given by

$$\overline{fk}(\underline{x}, \underline{z}) = \inf\{\delta > 0 : \bar{f}_\delta(\underline{x}, \underline{z}) < \delta\}.$$

The *Feldman–Katok pseudometric on X*, also denoted by $\overline{fk}$, is defined for $x, z \in X$ by

$$\overline{fk}(x, z) = \overline{fk}(\underline{x}_T, \underline{z}_T).$$

The symbols $\bar{f}_{n,\delta}(x, z)$ and $\bar{f}_\delta(x, z)$ have the obvious meaning.

Fact 4.11. If $\underline{x}, \underline{z} \in X^\infty$ and $\bar{f}_\delta(\underline{x}, \underline{z}) \leq \varepsilon$, for some $\delta, \varepsilon > 0$, then $\overline{fk}(\underline{x}, \underline{z}) \leq \delta + \varepsilon$.

Proof. If $\bar{f}_\delta(\underline{x}, \underline{z}) \leq \varepsilon$, then $\bar{f}_{\delta+\varepsilon}(\underline{x}, \underline{z}) < \delta + \varepsilon$ by Fact 4.3. Thus, $\overline{fk}(\underline{x}, \underline{z}) \leq \delta + \varepsilon$. □

Remark 4.12. By Fact 4.11 the set $\{\delta > 0 : \bar{f}_\delta(\underline{x}, \underline{z}) < \delta\}$ is non-empty for every $\underline{x}, \underline{z} \in X^\infty$. Furthermore, $0 \leq \bar{f}_\delta(\underline{x}, \underline{z}) \leq 1$ for all $\underline{x}, \underline{z} \in X^\infty$ and $\delta > 0$. Together with Fact 4.11, this implies that $0 \leq \overline{fk}(\underline{x}, \underline{z}) \leq 1$ for every $\underline{x}, \underline{z} \in X^\infty$.

Fact 4.13. The function $\overline{fk}$ is a pseudometric on X^∞, as well as on X.

Definition 4.14. We say that $x, y \in X$ are *orbitally related* and write $x \overset{T}{\sim} y$ if $T^i(x) = T^j(y)$ for some $i, j \geq 0$.

Fact 4.15. If $x \overset{T}{\sim} x'$ and $y \overset{T}{\sim} y'$, then $\overline{fk}(x, y) = \overline{fk}(x', y')$.

Proof. It is enough to prove that $\bar{f}_\delta(x, y) = \bar{f}_\delta(x', y')$ for every $\delta > 0$. Thus, we fix $\delta > 0$. If $x \overset{T}{\sim} x'$ and $y \overset{T}{\sim} y'$, then there are $i, j, m, n \geq 0$ such that $T^i(x) = T^j(x')$ and $T^m(y) = T^n(y')$. Using Fact 4.7 repeatedly, we have

$$\bar{f}_\delta(x, y) = \bar{f}_\delta(T^i(x), y) = \bar{f}_\delta(T^j(x'), y) = \bar{f}_\delta(x', y),$$

and similarly, $\bar{f}_\delta(x', y) = \bar{f}_\delta(x', y')$. □

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By Fact 4.15 the Feldman–Katok pseudometric depends on the separation between forward orbits of given points, rather than on the points alone.

4.3. *Comparison with the Besicovitch pseudometric.* We note that our results about the Feldman–Katok pseudometric generalize those known for Besicovitch pseudometric (see [1, 11, 36] for more details). In particular, any sequence of generic points converging in the Besicovitch pseudometric provides an example of an $\overline{fk}$-convergent sequence of measures.

LEMMA 4.16. *If $\underline{x}, \underline{z} \in X^\infty$, then $\overline{fk}(\underline{x}, \underline{z}) \le D'_{\mathrm{B}}(\underline{x}, \underline{z})$, where D'_{B} is given by (6).*

Proof. Define

$$\bar{d}_{n,\delta}(\underline{x}, \underline{z}) = \frac{1}{n}|\{0 \le j < n : \rho(x_j, z_j) \ge \delta\}|.$$

If $D'_{\mathrm{B}}(\underline{x}, \underline{z}) < \delta$ for some $\delta > 0$ then for all n large enough $\bar{d}_{n,\delta}(\underline{x}, \underline{z}) < \delta$. It follows that there exists an (n, δ)-match of $\underline{x}$ with $\underline{z}$. Therefore, $\overline{fk}(\underline{x}, \underline{z}) \le D'_{\mathrm{B}}(\underline{x}, \underline{z})$. □

5. $\overline{fk}$-convergence of measures and its properties

The relationship between $\overline{fk}$ and some metric on either $\mathcal{M}^e_T(X)$ or $\mathcal{M}_T(X)$ is not clear, while the pseudometric $\bar{f}$ defined on $\mathscr{A}^\infty$ in (2) relates to a metric on $\mathcal{M}_\sigma(\mathscr{A}^\infty)$ given by (5). Nonetheless, using $\overline{fk}$, we can still introduce a form of 'convergence' in $\mathcal{M}_T(X)$.

Definition 5.1. We say that a sequence of measures $(\mu_n)_{n=1}^\infty \subset \mathcal{M}_T(X)$ *converges in $\overline{fk}$* or *$\overline{fk}$-converges* to $\mu \in \mathcal{M}_T(X)$ if there exists a sequence of quasi-orbits $(\underline{x}^{(n)})_{n=1}^\infty \subset X^\infty$ with $V(\underline{x}^{(n)}) = \{\mu_n\}$ such that for some μ-generic quasi-orbit $\underline{z} \in X^\infty$ we have $\overline{fk}(\underline{z}, \underline{x}^{(n)}) \to 0$ as $n \to \infty$.

Keeping in mind that we are interested in the GIKN construction, we will examine properties of the $\overline{fk}$-convergence in a special case, where the quasi-orbits $(\underline{x}^{(n)})_{n=1}^\infty \subset X^\infty$ are actually orbits (for each $n \in \mathbb{N}$, $\underline{x}^{(n)}$ is the orbit of a μ_n-generic point).

5.1. *'Completeness' of $\overline{fk}$-convergence.* Since $\overline{fk}$ is a pseudometric on X^∞, given $T : X \to X$, such notions as *$\overline{fk}$-Cauchy sequence* or *$\overline{fk}$-limit* have obvious meaning. The first important feature of the $\overline{fk}$-convergence is that it has enough 'completeness' for our purposes: an $\overline{fk}$-Cauchy sequence of orbits defines a quasi-orbit which is its $\overline{fk}$-limit.

Definition 5.2. We say that a sequence of quasi-orbits $(\underline{x}^{(n)})_{n=1}^\infty \subset X^\infty$ is $\overline{fk}$-Cauchy if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\overline{fk}(\underline{x}^{(k)}, \underline{x}^{(\ell)}) < \varepsilon$ for $k, \ell \ge N$.

Repeating the proof of [4, Proposition 2], we get the following fact.

LEMMA 5.3. *If $(x^{(n)})_{n=1}^\infty \subset X$ is an $\overline{fk}$-Cauchy sequence on X, then there is a quasi-orbit $\underline{z} = (z_j)_{j=0}^\infty \in X^\infty$ such that $\overline{fk}(\underline{x}_T^{(n)}, \underline{z}) \to 0$ as $n \to \infty$.*

In fact, the proof of Lemma 5.3 yields that there are $0 = m_0 < m_1 < m_2 < \cdots$ and a subsequence $(x^{(n_j)})_{j=1}^\infty$ of $(x^{(n)})_{n=1}^\infty$ such that $\bar{d}(\{m_j : j \ge 0\}) = 0$ and $\underline{z}$ is given by $z_n = T^n(x^{(n_j)})$ for $m_{j-1} \le n < m_j$. We do not know whether under the assumptions of

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Lemma 5.3 there is a point whose orbit is an $\overline{fk}$-limit of the $\overline{fk}$-Cauchy sequence. We can prove this only by imposing an extra assumption on $T\colon X \to X$. It turns out that the *asymptotic average shadowing property* is a sufficient requirement. Let us recall its definition.

Definition 5.4. A sequence $\underline{z} = (z_j)_{j=0}^{\infty} \in X^{\infty}$ is an *asymptotic average pseudo-orbit* for $T\colon X \to X$ if

$$D_{\mathrm{B}}(T(\underline{z}), \underline{z}) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \rho(T(z_j), z_{j+1}) = 0.$$

We say that $\underline{z} = (z_j)_{j=0}^{\infty} \in X^{\infty}$ is *asymptotically shadowed on average* by $x \in X$ if

$$D_{\mathrm{B}}(\underline{x}_T, \underline{z}) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \rho(T^j(x), z_j) = 0. \tag{7}$$

A system (X, T) has the *asymptotic average shadowing property* if every asymptotic average pseudo-orbit of T is asymptotically shadowed on average by some point.

Note that if $\underline{x}_T$ and $\underline{z}$ satisfy (7), then $D'_{\mathrm{B}}(\underline{x}_T, \underline{z}) = \overline{fk}(\underline{x}_T, \underline{z}) = 0$ as well (see (6) and Lemma 4.16). The asymptotic average shadowing property was introduced by Gu [**41**] and is implied by most versions of the specification property considered in the literature (see [**14**, **33**, **35**, **36**]).

COROLLARY 5.5. *If (X, T) satisfies the asymptotic average shadowing property, then $\overline{fk}$ is a complete pseudometric on X.*

Proof. Fix an $\overline{fk}$-Cauchy sequence $(x^{(n)})_{n=1}^{\infty} \subset X$. Note that the quasi-orbit $\underline{z} \in X^{\infty}$ provided by Lemma 5.3 is an asymptotic average pseudo-orbit and satisfies $\overline{fk}(\underline{x}_T, \underline{z}) = 0$. Pick an $x \in X$ that asymptotically average-shadows $\underline{z}$. Then

$$\overline{fk}(x, x^{(n)}) \le \overline{fk}(\underline{x}_T, \underline{z}) + \overline{fk}(\underline{z}, \underline{x}_T^{(n)}) \to 0 \quad \text{as } n \to \infty.$$

Therefore, x is an $\overline{fk}$-limit of $(x^{(n)})_{n=1}^{\infty}$ in X. □

It turns out that the set of generic quasi-orbits is $\overline{fk}$-closed. More is true: $V(\underline{x})$ depends $\overline{fk}$-continuously on $\underline{x}$ when we consider $V(\underline{x})$ as a point in the space of non-empty closed subsets of $\mathcal{M}(X)$ endowed with the Hausdorff metric D_{H} induced by the Prokhorov metric D_{P} (the latter space is called the hyperspace of $\mathcal{M}(X)$).

Fact 5.6. If $\overline{fk}(\underline{x}, \underline{z}) < \varepsilon$ for some $\varepsilon > 0$, then $D_{\mathrm{H}}(V(\underline{x}), V(\underline{z})) < \varepsilon$, where D_{H} is the Hausdorff metric on the hyperspace of $\mathcal{M}(X)$ endowed with D_{P}.

Proof. If $\overline{fk}(\underline{x}, \underline{z}) < \varepsilon$, then for some $\overline{fk}(\underline{x}, \underline{z}) \le \delta < \varepsilon$ we have $\bar{f}_{\delta}(\underline{x}, \underline{z}) < \delta < \varepsilon$. Therefore, $\bar{f}_{n,\delta}(\underline{x}, \underline{z}) < \delta$ for all n large enough and $D_{\mathrm{P}}(\mathfrak{m}(\underline{x}, n), \mathfrak{m}(\underline{z}, n)) < \varepsilon$ by Fact 4.5. This implies that $D_{\mathrm{H}}(V(\underline{x}), V(\underline{z})) < \varepsilon$. □

Although the proof of the following fact is short, its importance justifies calling it a theorem.

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THEOREM 5.7. *Let $(\underline{x}^{(n)})_{n=1}^{\infty} \subset X^{\infty}$ be such that for each $n \in \mathbb{N}$ there is $\mu_n \in \mathcal{M}(X)$ with $\underline{x}^{(n)} \in \mathrm{Gen}(\mu_n)$. If $\underline{x} \in X^{\infty}$ and $\overline{fk}(\underline{x}, \underline{x}^{(n)}) \to 0$ as $n \to \infty$, then there exists $\mu \in \mathcal{M}(X)$ such that $\mu_n \to \mu$ as $n \to \infty$ in $\mathcal{M}(X)$ and $\underline{x} \in \mathrm{Gen}(\mu)$.*

Proof. By Fact 5.6, $\overline{fk}(\underline{x}, \underline{x}^{(n)}) \to 0$ as $n \to \infty$ implies that $V(\underline{x}^{(n)}) \to V(\underline{x})$ as $n \to \infty$ in the hyperspace of $\mathcal{M}(X)$. Since $\underline{x}^{(n)}$ is generic for μ_n, we have $V(\underline{x}^{(n)}) = \{\mu_n\}$ for $n \in \mathbb{N}$. The family of all singletons is closed in the hyperspace and homeomorphic with $\mathcal{M}(X)$. Thus, $V(\underline{x})$ must also be a singleton, that is, $V(\underline{x}) = \{\mu\}$ for some $\mu \in \mathcal{M}(X)$ and $\mu_n \to \mu$ as $n \to \infty$ in $\mathcal{M}(X)$. □

An immediate consequence of Lemma 5.3 and Theorem 5.7 is the following corollary.

COROLLARY 5.8. *An $\overline{fk}$-Cauchy sequence of generic points uniquely determines an invariant measure.*

Corollary 5.8 allows us to justify the correctness of the next definition.

Definition 5.9. An invariant measure determined by an $\overline{fk}$-Cauchy sequence of generic points is called the *$\overline{fk}$-limit* of the corresponding sequence of measures.

6. *Every GIKN sequence is $\overline{fk}$-Cauchy*

All examples of (γ, κ)-good approximations found in the literature support the conjecture that each pair of periodic orbits satisfying Definition 3.1 is always $(\gamma + (1 - \kappa))$-close with respect to $\overline{fk}$. However, the definition of a (γ, κ)-good approximation does not ensure that the (γ, κ)-projection ψ appearing in Definition 3.1 is a match as in Definition 4.1. This is because a match must be order-preserving, and Definition 3.1 does not stipulate this requirement from a projection. Therefore, the following technical lemma is necessary.

LEMMA 6.1. *If a T-periodic orbit Γ is a (γ, κ)-good approximation of a T-periodic orbit Λ, then for any choice of $x \in \Gamma$ and $z \in \Lambda$, one has $\overline{fk}(x, z) < \gamma + (1 - \kappa)$.*

Proof. It follows from Fact 4.15 that it is enough to find $x_0 \in \Gamma$ and $z_0 \in \Lambda$ such that $\overline{fk}(x_0, z_0) < \gamma + (1 - \kappa)$.

Let $\psi\colon \Delta \to \Lambda$ be the (γ, κ)-projection from Γ to Λ. Pick any $x_0 \in \Delta$ and let $z_0 = \psi(x_0) \in \Lambda$.

By Fact 4.11 it is enough to show that $\bar{f}_{\gamma}(x_0, z_0) < 1 - \kappa$. Define $q = |\Lambda|$. Furthermore, since x_0 and z_0 are periodic, we conclude from Fact 4.9 and Fact 4.4 that it is sufficient to find for any multiple p of $|\Gamma|$ and $|\Lambda|$ a $(p + q, \gamma)$-match π of x_0 with z_0 such that $|\pi| \geq \kappa p$. Let p be a multiple of $|\Gamma|$ and $|\Lambda|$. For $j \geq 1$ define $x_j = T^j(x_0)$ and $z_j = T^j(z_0)$. Note that $(z_j)_{j=0}^{\infty}$ is a q-periodic sequence. We will abuse the notation and treat $\{x_0, \ldots, x_{p-1}\}$, $\{z_0, \ldots, z_{p-1}\}$ as sets with p elements, still denoted by Γ and Λ. Furthermore, we extend ψ to a function from $\Delta \subset \{x_0, \ldots, x_{p-1}\}$ to $\{z_0, \ldots, z_{p-1}\}$ where $|\Delta| = r$ and $r/p \geq \kappa$.

We are going to define the match π by performing an inductive construction with at most $|\Delta|$ steps. At each step we will extend the domain of π by at least one element. Enumerate elements of Δ as $y_0, \ldots, y_{r-1}$ in such a way that the order induced by this indexing of

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Δ coincides with the one induced by enumerating the elements of Γ as $x_0, \dots, x_{p-1}$. By definition, $x_0 = y_0$. Let $\theta\colon \{0, \dots, r-1\} \to \{0, 1, \dots, p-1\}$ be a function such that $\theta(s)$ for $0 \le s < r$ is the position of y_s in the sequence $x_0, \dots, x_{p-1}$, that is, $\theta(s) = j$ if and only if $y_s = x_j$.

We begin with $\mathcal{D}(\pi) = \{0\}$ and $\pi(0) = 0$. By definition, $z_{\pi(0)} = \psi(x_0) = z_0$, hence $\rho(x_0, z_{\pi(0)}) < \gamma$, and π is a $(p+q, \gamma)$-match.

Now assume that we have already performed some number of steps of our construction and we have obtained a $(p+q, \gamma)$-match π such that $0 \le s < r$ is the largest integer satisfying $|\pi| \ge s+1$, $\theta(s) \in \mathcal{D}(\pi)$, $\pi(\theta(s)) \le \theta(s)$ and $z_{\pi(\theta(s))} = \psi(y_s)$. It follows from our construction that $s+1$ is always greater than or equal to the number of steps performed. We can extend π by setting $\pi(\theta(s)+i) = \theta(s)+i$ for $0 < i < q$. Now there are two cases to consider: either the domain of π contains $\{\theta(0), \dots, \theta(r-1)\}$ and we are done, or there is $s < t < r$ which is the smallest integer such that π is not defined at $\theta(t)$. Clearly, in the latter case $\theta(t) - \theta(s) \ge q$. Since $(z_j)_{j=0}^{\infty}$ is a q-periodic sequence there exists $1 \le \ell \le q$ such that $z_{\theta(s)+\ell} = \psi(y_t)$, and $\pi(\theta(s)) < \theta(s) + \ell \le \theta(t)$. We want to set $\pi(\theta(t)) = \theta(s) + \ell$, but to keep π increasing we need to remove first at most $q-1$ elements from $\mathcal{D}(\pi)$ as defined so far, namely those in $\mathcal{D}(\pi) \cap \{\theta(s)+\ell, \dots, \theta(s)+q-1\}$. But then we can set $\pi(\theta(t)+i) = \theta(t)+i = \theta(s)+\ell+i$ for $1 \le i < q$, because

$$z_{\pi(\theta(t)+i)} = T^i(z_{\theta(t)}) \quad \text{and} \quad \rho(T^i(z_{\theta(t)}), T^i(\psi(y_t))) < \gamma.$$

For our new π we see that the largest integer s in $\{0, 1, \dots, r-1\}$ satisfying $|\pi| \ge s+1$, $\theta(s) \in \mathcal{D}(\pi)$, $\pi(\theta(s)) \le \theta(s)$ and $z_{\pi(\theta(s))} = \psi(y_s)$ is larger than or equal to t. Thus, in a finite number of steps our procedure will produce a $(p+q, \gamma)$-match π with $|\mathcal{D}(\pi)| \ge \kappa p$. □

As a consequence, we see that a GIKN measure is the $\overline{fk}$-limit of periodic measures, which immediately implies [8, Lemma 2.5], except for conclusions about ergodicity and support (see also [29, Lemma 2]).

THEOREM 6.2. *If $(\Gamma_n)_{n\in\mathbb{N}}$ is a GIKN sequence of T-periodic orbits and μ_n denotes the ergodic measure supported on Γ_n for $n \in \mathbb{N}$, then for any choice $x_n \in \Gamma_n$ the sequence $(x_n)_{n=1}^{\infty}$ is $\overline{fk}$-Cauchy and μ_n $\overline{fk}$-converges to some μ. In particular, a GIKN measure is an $\overline{fk}$-limit of periodic measures.*

7. $\overline{fk}$-limit of ergodic measures is ergodic

We are going to show that an $\overline{fk}$-limit of a sequence of ergodic measures must be ergodic. To this end, we first present a criterion for the ergodicity of a measure generated by a quasi-orbit. We obtain this by an easy adaptation of an analogous criterion for orbits given by Oxtoby [40].

7.1. *Auxiliary terminology and results.* For $k \in \mathbb{N}$, $\underline{x} = (x_j)_{j=0}^{\infty} \in X^{\infty}$, and $\varphi \in C(X)$ let $A_k(\varphi, \underline{x})$ denote the Birkhoff average along $x_0, x_1, \dots, x_{k-1}$, that is,

$$A_k(\varphi, \underline{x}) = \frac{1}{k} \sum_{j=0}^{k-1} \varphi(x_j) = \int_X \varphi \, d\mathfrak{m}(\underline{x}, k).$$

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For $x \in X$ we write $A_k(\varphi, x)$ for the Birkhoff average along an orbit segment of length k, that is, $A_k(\varphi, x) = A_k(\varphi, \underline{x}_T)$.

Recall that a sequence $\underline{x} \in X^\infty$ (respectively, a point $x \in X$) is generic for some measure μ if and only if for every $\varphi \in C(X)$ the sequence $A_k(\varphi, \underline{x})$ (respectively, $A_k(\varphi, x)$) converges as $k \to \infty$. We denote the corresponding limit by $\varphi^*(\underline{x})$ (respectively, $\varphi^*(x)$). It is easy to see that for any $\ell \in \mathbb{N}$, we have $\varphi^*(\underline{x}) = \varphi^*(\sigma^\ell(\underline{x}))$ (respectively, $\varphi^*(x) = \varphi^*(T^\ell(x))$). For a generic sequence $\underline{x}$ we put

$$A_k^*(\varphi, x_\ell) = |A_k(\varphi, (x_j)_{j=\ell}^\infty) - \varphi^*((x_j)_{j=\ell}^\infty)| = |A_k(\varphi, \sigma^\ell(\underline{x})) - \varphi^*(\underline{x})|.$$

Furthermore, for a generic point x and $\ell \in \mathbb{N}$, we define

$$A_k^*(\varphi, T^\ell(x)) = |A_k(\varphi, T^\ell(x)) - \varphi^*(x)|.$$

Fix $\alpha > 0$, $k \in \mathbb{N}$, and $\varphi \in C(X)$. We say that $\ell \geq 0$ is a *starting point of an* (α, φ)*-bad* k*-segment* in a generic sequence $\underline{x} = (x_j)_{j=0}^\infty$ if $A_k^*(\varphi, \sigma^\ell(\underline{x})) > \alpha$.

The following characterization of ergodic sequences slightly generalizes the one presented by Oxtoby in [40, §4]. It states that a generic sequence $\underline{x} = (x_j)_{j=0}^\infty$ generates an ergodic measure if for every $\varphi \in C(X)$ and $\alpha > 0$ the upper density of the set of integers ℓ being starting points for an (α, φ)-bad k-segments converges to zero as k goes to ∞. Replacing $\underline{x}$ by $\underline{x}_T$, we obtain a criterion for ergodicity of a measure given by a generic point due to Oxtoby. We omit the proof as it follows the same lines as in [40].

THEOREM 7.1. (Oxtoby's criterion) *Let a quasi-orbit $\underline{z} = (z_j)_{j=0}^\infty \in X^\infty$ be generic for some $\mu \in \mathcal{M}_T(X)$. Then μ is ergodic if and only if for every $\varphi \in C(X)$ and $\alpha > 0$ we have*

$$\bar{d}(\{\ell \geq 0 \ : \ |A_k(\varphi, \sigma^\ell(\underline{z})) - \varphi^*(\underline{z})| > \alpha\}) \to 0 \quad \text{as } k \to \infty.$$

Let $n \in \mathbb{N}$, $x \in X$ and $\underline{z} \in X^\infty$. For an (n, δ)-match π of $\underline{x}_T$ with $\underline{z}$ and $k \leq n$ we define for every $\ell \in \mathcal{D}(\pi) \cap \{0, 1, \ldots, n-k\}$ a set

$$\mathcal{D}'_\ell = \{i \in \mathcal{D}(\pi) : \ell \leq i < \ell + k \text{ and } \pi(\ell) \leq \pi(i) < \pi(\ell) + k\}$$

and the (k, δ)*-match* $\pi'_\ell \colon \mathcal{D}'_\ell \to \pi(\mathcal{D}'_\ell)$ *induced by* π at ℓ setting $\pi'_\ell(i) = \pi(i)$ for $i \in \mathcal{D}(\pi'_\ell)$. It is easy to see that π'_ℓ is indeed an (k, δ)-match of $T^\ell(x)$ with $z_{\pi(\ell)}$ satisfying $\mathcal{D}(\pi'_\ell) = \mathcal{D}'_\ell$ and $\mathcal{R}(\pi'_\ell) = \pi(\mathcal{D}'_\ell)$.

We also need the following technical lemma, which asserts that if an orbit $\underline{x}_T$ and a quasi-orbit $\underline{z}$ are sufficiently $\overline{fk}$-close, then for any $\varphi \in C(X)$ and $k \in \mathbb{N}$ one can find a match π which allows one to find a match π of $\underline{x}_T$ and $\underline{z}$ so that the averages of φ over k segments in $\underline{x}_T$ and $\underline{z}$ are also small for most of k-segments.

LEMMA 7.2. *Fix $\varphi \in C(X)$ and $\varepsilon > 0$. Let $\delta > 0$ be such that $y, y' \in X$ and $\rho(y, y') < \delta$ imply $|\varphi(y) - \varphi(y')| < \varepsilon$. If $\underline{z} = (z_j)_{j=0}^\infty \in X^\infty$ is a quasi-orbit and $x \in X$ satisfies $\overline{fk}(\underline{x}_T, \underline{z}) < \delta$, then for every $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ there are an (n, δ)-match π of $\underline{x}_T$ with $\underline{z}$ and a set $A \subset \mathcal{D}(\pi)$ with $|A| > n(1 - 2\sqrt{\delta} - 2\delta) - k$ satisfying*

$$|A_k(\varphi, T^\ell(x)) - A_k(\varphi, \sigma^{\pi(\ell)}(\underline{z}))| \leq \varepsilon + 4\sqrt{\delta}||\varphi||_\infty \quad \text{for every } \ell \in A.$$

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Proof. Fix $k \in \mathbb{N}$, $\varphi \in C(X)$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that for every $n \geq N$ there exists an (n, δ)-match π of $\underline{x}_T$ with $\underline{z}$ satisfying $|\pi| > n(1-\delta)$ and

$$|\{0 \leq j < n \ : T(z_{j+i}) \neq z_{j+i+1} \text{ for some } 0 \leq i < k\}| < n\delta.$$

Fix $n \geq N$. Define

$$A_Z = \{0 \leq j < n \ : T(z_{\pi(j)+i}) \neq z_{\pi(j)+i+1} \text{ for some } 0 \leq i < k\},$$
$$A_R = \{0 \leq j < n-k \ : \ j \in \mathcal{D}(\pi) \text{ and } |\{0 \leq i < k \ : \ \pi(j)+i \notin \mathcal{R}(\pi)\}| \geq \sqrt{\delta}k\},$$
$$A_D = \{0 \leq j < n-k \ : \ j \in \mathcal{D}(\pi) \text{ and } |\{0 \leq i < k \ : \ j+i \notin \mathcal{D}(\pi)\}| \geq \sqrt{\delta}k\}.$$

Note that $|A_Z| \leq n\delta$. To estimate $|A_R|$, define $\mathcal{R}^c(\pi) = \{0, 1, \ldots, n-1\} \setminus \mathcal{R}(\pi)$, and note that

$$|\mathcal{R}^c(\pi)| \geq |\{m \in \mathcal{R}^c(\pi) \ : \ \pi(j) \leq m < \pi(j)+k \text{ for some } j \in A_R\}|.$$

On the other hand, for each $m \in \mathcal{R}^c(\pi)$ the set $\{j \in A_R \ : \ \pi(j) \leq m < \pi(j)+k\}$ has at most k elements; hence, each $j \in A_R$ implies that there are at least $\sqrt{\delta}$ members of $\mathcal{R}^c(\pi)$. Thus, $|\mathcal{R}^c(\pi)| \geq |A_R|\sqrt{\delta}$. This, together with $|\mathcal{R}^c(\pi)| < n\delta$, gives us $|A_R| < n\sqrt{\delta}$. An analogous reasoning leads to $|A_D| < n\sqrt{\delta}$. Define

$$A = (\{0, \ldots, n-k-1\} \cap \mathcal{D}(\pi)) \setminus (A_Z \cup A_D \cup A_R).$$

Then $|A| > n(1-2\sqrt{\delta}-2\delta) - k$ and for every $j \in A$ we have:

(i) the (k, δ)-match π'_j induced by π at j satisfies $|\pi'_j| > (1-2\sqrt{\delta})k$;

(ii) $A_k(\varphi, z_{\pi(j)}) = A_k(\varphi, \sigma^{\pi(j)}(\underline{z}))$.

Therefore,

$$\begin{aligned}
|A_k(\varphi, T^j(x)) - A_k(\varphi, \sigma^{\pi(j)}(\underline{z}))| &= |A_k(\varphi, T^j(x)) - A_k(\varphi, z_{\pi(j)})| \\
&\leq \frac{1}{k} \sum_{i \in \mathcal{D}(\pi'_j)} |\varphi(T^i(x)) - \varphi(z_{\pi(i)})| + \frac{1}{k} \sum_{i \notin \mathcal{D}(\pi'_j)} |\varphi(T^i(x))| + \frac{1}{k} \sum_{i \notin \mathcal{R}(\pi'_j)} |\varphi(z_i)| \\
&\leq \varepsilon + 2\sqrt{\delta}||\varphi||_\infty + 2\sqrt{\delta}||\varphi||_\infty = \varepsilon + 4\sqrt{\delta}||\varphi||_\infty
\end{aligned}$$

and the lemma follows. □

7.2. *$\overline{fk}$-limits of ergodic measures are ergodic.* We prove the main theorem of this section. As the GIKN sequence of periodic orbits is $\overline{fk}$-Cauchy by Theorem 6.2, we see that our Theorem 7.3 generalizes [**29**, Lemma 2]. Furthermore, since $\overline{fk} \leq D_{\mathrm{B}}$, this result also extends [**36**, Theorem 15].

THEOREM 7.3. *If $(x^{(p)})_{p=1}^{\infty} \subset X$ is an $\overline{fk}$-Cauchy sequence of ergodic points then it determines an ergodic measure.*

Proof. By Fact 5.6 there exists a measure $\mu \in \mathcal{M}_T(X)$ such that $\mu_p \to \mu$ as $p \to \infty$ in the weak* topology on $\mathcal{M}_T(X)$. We will apply Oxtoby's criterion (Theorem 7.1) to show that μ is ergodic. Let $\underline{z} = (z_j)_{j=0}^{\infty}$ be a quasi-orbit such that $\bar{f}(\underline{x}_T^{(p)}, \underline{z}) \to 0$ as $p \to \infty$ provided by Lemma 5.3. Clearly, $\underline{z}$ is generic for μ. Fix $\varphi \in C(X)$ and $\alpha > 0$. We need to

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show that for every $\eta > 0$ and all sufficiently large k the set of js initiating an (α, φ)-bad k-segment in $\underline{z}$ has upper density smaller than η.

Note that for every $i, j, k, p \in \mathbb{N}$ we have

$$\begin{aligned} |A_k(\varphi, \sigma^j(\underline{z}) - \varphi^*(\underline{z}))| &\le |A_k(\varphi, \sigma^j(\underline{z})) - A_k(\varphi, T^i(x^{(p)}))| \\ &+ |A_k(\varphi, T^i(x^{(p)})) - \varphi^*(x^{(p)})| + |\varphi^*(x^{(p)}) - \varphi^*(\underline{z})|. \end{aligned} \tag{8}$$

By Theorem 5.7 and Fact 5.6 we can choose $P(\alpha) \in \mathbb{N}$ such that for every $p \ge P(\alpha)$ one has $|\varphi^*(x^{(p)}) - \varphi^*(\underline{z})| \le \alpha/3$.

Let $\delta_0 > 0$ be such that $2\sqrt{\delta_0} + 2\delta_0 < \eta/4$ and $4\sqrt{\delta_0}||\varphi||_\infty < \alpha/6$. Pick $\delta < \delta_0$ such that for all $y, y' \in X$ with $\rho(y, y') < \delta$ one has $|\varphi(y) - \varphi(y')| < \alpha/6$. Note that our choice of constants is motivated by Lemma 7.2. Choose $p \ge P(\alpha)$ such that $\bar{f}(\underline{z}, x_T^{(p)}) < \delta$. Pick $K > 0$ such that for all $k \ge K$ the upper density of the set of is initiating an $(\alpha/3, \varphi)$-bad k-segment in $x_T^{(p)}$ is smaller than $\eta/2$. We are going to prove that each $k \ge K$ is 'sufficiently large' to imply that the upper density of the set of js initiating an (α, φ)-bad k-segment in $\underline{z}$ is smaller than η. To this end fix any $k \ge K$. Let N be sufficiently large to guarantee that $k/N < \eta/4$ and for every $n \ge N$ we have

(i) there exists an (n, δ)-match π of $\underline{z}$ with $x^{(p)}$ such that $|\pi| > (1 - \delta)n$,
(ii) $|\{0 \le j < n : T(z_{j+i}) \ne z_{j+i+1} \text{ for some } 0 \le i < k\}| < n\delta$,
(iii) $|\{0 \le i < n : i \text{ initiates an } (\alpha/3, \varphi)\text{-bad } k\text{-segment in } x_T^{(p)}\}| < n\eta/2$.

We are going to show that for each $n \ge N$ the number of $0 \le j < n$ such that

$$A_k^*(\varphi, \sigma^j(\underline{z})) = |A_k(\varphi, (z_s)_{s=j}^\infty) - \varphi^*(\underline{z})| \le \alpha$$

is larger than $(1 - \eta)n$. To this end let $\tilde{\pi}$ be any extension of π to a bijection from $\{0, \dots, n-1\}$ onto $\{0, \dots, n-1\}$. It follows from the proof of Lemma 7.2 that conditions (i) and (ii) guarantee that the number of integers $0 \le j < n$ for which

$$|A_k(\varphi, \sigma^j(\underline{z})) - A_k(\varphi, T^{\tilde{\pi}(j)}(x^{(p)}))| < \alpha/6 + 4\sqrt{\delta_0}||\varphi||_\infty < \alpha/3$$

is larger than $n(1 - 2\sqrt{\delta} - 2\delta) - k$ and hence larger than $n(1 - \eta/2)$. Some of these js may initiate an $(\alpha/3, \varphi)$-bad k-segment in $x_T^{(p)}$, but condition (iii) bounds from the above the number of such js by $n\eta/2$. Therefore, setting $i = \tilde{\pi}(j)$ in (8), we see that the number of integers $0 \le j < n$ for which all summands on the right-hand side of (8) are bounded above by $\alpha/3$ is larger than $n(1 - \eta)$. Since this holds true for any $n \ge N$ we see that upper density of the set of all js initiating an (α, φ)-bad k-segment in $\underline{z}$ is smaller than η, hence the Oxtoby criterion yields that $\underline{z}$ is an ergodic sequence. □

The ergodicity of GIKN measures is now a straightforward corollary; hence [8, Lemma 2.5] is recovered, except for the statement about the support (which can now be proved in the same way as in [8], since the proof of that part of [8, Lemma 2.5] is independent of the rest).

COROLLARY 7.4. *Every GIKN measure is ergodic.*

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8. *Lower semicontinuity of entropy in* $\overline{fk}$

We are going to show that the function assigning the entropy of the associated measure to a generic point is lower semicontinuous with respect to $\overline{fk}$.

THEOREM 8.1. *If a sequence of measures* $(\mu_n)_{n=1}^{\infty}$ *converges in* $\overline{fk}$ *to* $\mu_0 \in \mathcal{M}_T(X)$*, then*

$$h(\mu_0) \le \liminf_{n\to\infty} h(\mu_n).$$

Before proceeding with the proof, we first note the following technical result allowing us, given an $\overline{fk}$-converging sequence of invariant measures, to replace any partition with a faithful partition without much change in the $\bar{f}$-distance between codings of generic points of the measures in the sequence.

LEMMA 8.2. *If* $\delta > 0$*,* $\mathcal{P} \in \mathbf{P}^k(X)$ *and* $(\mu_j)_{j=0}^{\infty} \subset \mathcal{M}_T^e(X)$*, then there are* $\gamma > 0$ *and* $\mathcal{R} \in \mathbf{P}^{k+1}(X)$ *with* $d_1^{\mu_0}(\mathcal{P}, \mathcal{R}) < \delta$ *satisfying that if* $\underline{x} \in X^{\infty}$ *is generic for* μ_0*,* $j \in \mathbb{N}$*, and* $\underline{z} \in X^{\infty}$ *is generic for* μ_j *with* $\overline{fk}(\underline{x}, \underline{z}) < \gamma$*, then* $\bar{f}(\mathcal{R}(\underline{x}), \mathcal{R}(\underline{z})) < \delta + \overline{fk}(\underline{x}, \underline{z})$ *and* $\mu_j(\partial\mathcal{R}) = \mu_0(\partial\mathcal{R}) = 0$*. The same holds if we replace* $\bar{f}$ *by* $\bar{d}$ *and* $\overline{fk}$ *by* D_{B}*.*

Proof. Fix $\delta > 0$. Using regularity of μ_0 for each $1 \le j \le k$, we can find a compact set $R_j \subset P_j$ such that $\mu_0(P_j \setminus R_j) < \delta/(8k^2)$. For $1 \le i < j \le k$ the sets R_i and R_j are compact and disjoint, hence

$$\Delta = \min\{\rho(x, y) : x \in R_i,\ y \in R_j,\ 1 \le i < j \le k\} > 0.$$

Set $\tilde{\gamma} = \Delta/2$. For each $1 \le j \le k$ and $0 \le c < \tilde{\gamma}$ let R_j^c be the closed c-hull around Q_j, that is, $R_j^c = \{x \in X : \mathrm{dist}(x, Q_j) \le c\}$, and set $R_0^c = X \setminus (R_1^c \cup \cdots \cup R_k^c)$. For each $0 \le c < \tilde{\gamma}$ define the partition $\mathcal{R}^c = \{R_0^c, R_1^c, \ldots, R_k^c\}$. It is easy to see that $d_1^{\mu_0}(\mathcal{P}, \mathcal{R}^{\alpha}) < \delta/2$ and $d_1^{\mu_0}(\mathcal{R}^{\alpha}, \mathcal{R}^{\beta}) < \delta/2$ for any $0 \le \alpha, \beta < \tilde{\gamma}$.

For $1 \le j \le k$ and $0 < c < \tilde{\gamma}$ define a set $\partial_c Q_j = \{x \in X : \mathrm{dist}(x, Q_j) = c\}$. Note that $\partial_c Q_j$ contains (but need not be equal to) the topological boundary of the set $R_j^c = \{x \in X : \mathrm{dist}(x, Q_j) \le c\}$. Consider a family of closed sets $C = \{\partial_c Q_1 \cup \cdots \cup \partial_c Q_k : 0 < c < \tilde{\gamma}\}$. Since elements of C are pairwise disjoint, only countably many of them can have positive μ_j measure for some $j \in \mathbb{N} \cup \{0\}$. Therefore, the set E of all parameters $0 < c < \tilde{\gamma}$ such that for each $c \in E$ the set $\partial_c Q_1 \cup \cdots \cup \partial_c Q_k \in C$ is a μ_j-null set for $j \in \mathbb{N} \cup \{0\}$ has at most countable complement in $(0, \tilde{\gamma})$. Thus, we can pick $\alpha, \beta \in E$ with $\alpha < \beta$ and $\beta - \alpha > \tilde{\gamma}/2$. Set $\gamma = \tilde{\gamma}/2$. Let $\underline{z} \in X^{\infty}$ be a generic sequence for μ_j for some $j \in \mathbb{N}$ and $\bar{f}(\underline{x}, \underline{z}) < \gamma$. Note that $\beta - \alpha > \bar{f}(\underline{x}, \underline{z})$. Define $x'' = \mathcal{R}^{\alpha}(\underline{x})$, $z' = \mathcal{R}^{\beta}(\underline{z})$ and $x' = \mathcal{R}^{\beta}(\underline{x})$ (all three points are considered as elements of the shift space over the alphabet $\{0, 1, \ldots, k\}$).

We claim that $\bar{f}(x'', z') < \bar{f}(\underline{x}, \underline{z}) + \delta/2$. Indeed, if for some $m, n \ge 0$ and $1 \le j \le k$ we have $x_n \in R_j^{\alpha}$ and $\rho(x_n, z_m) < \bar{f}(\underline{x}, \underline{z}) < \beta - \alpha$, then $\mathcal{R}^{\beta}(z_m) = j$. Furthermore, genericity of $\underline{x}$ for μ_0 and $\mu_0(\partial R_0^{\alpha}) = 0$ implies that $d(\{n \ge 0 : x_n \in R_0^{\alpha}\}) = \mu_0(R_0^{\alpha}) < \delta/2$. This proves the claim.

Using again that $\underline{x}$ is generic for μ_0, and $\partial\mathcal{R}^{\alpha}$, $\partial\mathcal{R}^{\beta}$ are μ_0-null, we easily obtain that $\bar{d}(x', x'') \le d_1^{\mu_0}(\mathcal{R}^{\alpha}, \mathcal{R}^{\beta})$. Together with the claim above and $\bar{f}(x', x'') \le \bar{d}(x', x'')$, this completes the proof for the $\bar{f}$ case. The $\bar{d}$-part follows in the same way. □

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Proof of Theorem 8.1. Fix $\varepsilon > 0$. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a measurable partition of X with $h(\mu_0, \mathcal{P}) \geq h(\mu_0) - \varepsilon/3$. Let $\zeta > 0$ be so small that $\bar{f}(y, y') < \zeta$ for two shift generic points $y, y' \in \{0, 1, \dots, k\}^\infty$ implies that the entropies of the corresponding measures differ by at most $\varepsilon/3$ (the existence of such a ζ is guaranteed by [39, Proposition 3.4] and [20]).

Since the function $\mathcal{P} \mapsto h(\mu, \mathcal{P})$ is uniformly continuous on $\mathbf{P}^{k+1}$ [27, Lemma 15.9 (5)], we may pick $0 < \delta < \zeta/2$ such that for any partition $\mathcal{S} = \{S_0, S_1, \dots, S_k\}$ with $d_1^{\mu_0}(\mathcal{P}, \mathcal{S}) < \delta$ we have $|h(\mu_0, \mathcal{P}) - h(\mu_0, \mathcal{S})| < \varepsilon/3$.

Use Lemma 8.2 to find $\gamma > 0$ for μ_0, $\delta/2$ and $\mathcal{P}$. Let $N \in \mathbb{N}$ be such that $\overline{fk}(\underline{x}, \underline{x}_T^{(N)}) < \min\{\gamma, \delta/2\}$. Take the partition $\mathcal{R}$ provided by Lemma 8.2 for $\underline{x}$, $\underline{x}_T^{(N)}$ and μ_N. Let $x' = \mathcal{R}(\underline{x})$ and $x'_N = \mathcal{R}(x_T^{(N)})$. It follows from Lemma 8.2 that x'_N is a generic point for some measure μ'_N with $h(\mu'_N) = h(\mu_N, \mathcal{R}) \leq h(\mu_N)$. By Lemma 8.2 we see that $\bar{f}(x', x'_N) < \delta$. Therefore, $|h(\mu'_0) - h(\mu'_N)| < \varepsilon/3$, hence

$$h(\mu_N) \geq h(\mu_N, \mathcal{S}) = h(\mu'_N) \geq h(\mu_0) - \varepsilon.$$

This completes the proof. □

Note that by [20] entropy is uniformly continuous if $X = \mathscr{A}^\infty$ and we equip $\mathcal{M}_\sigma(\mathscr{A}^\infty)$ with the $\bar{f}$-metric (which, as we noted above, is uniformly equivalent with $\overline{fk}$ induced by a standard metric on $\mathscr{A}^\infty$). This is also the case when the entropy function $\mathcal{M}_T(X) \ni \mu \to h(\mu) \in \mathbb{R}$ is upper semicontinuous with respect to the weak* topology on $\mathcal{M}_T(X)$. For example, this happens if X is a manifold and T is of class C^∞ or, more generally, if T is asymptotically h-expansive (see [19]).

COROLLARY 8.3. *If (X, T) is such that the function $\mathcal{M}_T(X) \ni \mu \to h(\mu) \in \mathbb{R}$ is upper semicontinuous with respect to the weak* topology on $\mathcal{M}_T(X)$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that for $\mu, \mu' \in \mathcal{M}_T(X)$ whenever there are $\underline{x} \in \mathrm{Gen}(\mu)$ and $\underline{x}' \in \mathrm{Gen}(\mu')$ with $\overline{fk}(\underline{x}, \underline{x}') < \delta$, then $|h(\mu) - h(\mu')| < \varepsilon$.*

Without extra assumptions, the entropy function need not be continuous with respect to $\overline{fk}$-convergence.

Example 8.4. Let $\mathscr{A} = \{0\} \cup \{1/k : k \in \mathbb{N}\}$ with the topology inherited from $[0, 1]$. Consider $X = \mathscr{A}^\infty$ with any metric compatible with the product topology. Let $\xi^{(n)}$ be a measure on $\mathscr{A}$ uniformly distributed on $\{1/\ell : 2^n \leq \ell < 2^{n+1}\}$ and $\mu^{(n)}$ denote the product measure on X. It is easy to see that $\mu^{(n)}$ is a shift-invariant measure on $\mathscr{A}^\infty$ and the sequence $(\mu^{(n)})_{n\in\mathbb{N}}$ converges in $\overline{fk}$ to the measure concentrated on a σ-fixed point $(0, 0, \dots) \in \mathscr{A}^\infty$. Furthermore, $h(\mu^{(n)}) = n \log 2$, which means that the entropy function cannot be upper semicontinuous.

9. *GIKN construction leads to loosely Kronecker measure*

We are going to show that the $\overline{fk}$-limit of loosely Kronecker measure is either a periodic measure or a loosely Kronecker measure. Since the GIKN construction yields a measure with uncountable support, every GIKN measure must be loosely Kronecker.

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THEOREM 9.1. *An aperiodic $\overline{fk}$-limit of periodic measures (loosely Kronecker measures) is loosely Kronecker.*

We present the proof at the end of this section. But first, we recall Katok's criterion for standardness (loosely Kronecker) and formulate two technical lemmas we will need for the proof.

9.1. *Katok's criterion.* Let $\mathbf{X} = (X, \mathscr{X}, \mu, T)$ be a measure-preserving system and let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\} \in \mathbf{P}^k(X)$. Following Katok [30, Definition 9.1], we say that the process $(\mathbf{X}, \mathcal{P})$ is (n, ε)-*trivial* if there exists a word $\omega \in \{0, 1, \ldots, k-1\}^n$ such that $\mu_{\mathcal{P}}(B_\varepsilon[\omega]) \geq 1 - \varepsilon$, where $B_\varepsilon[\omega] = \{\omega' \in \{0, 1, \ldots, k-1\}^n : \bar{f}_n(\omega, \omega') < \varepsilon\}$. For every $n \geq 1$, $\varepsilon > 0$ and $\omega \in \{0, 1 \ldots, k-1\}^n$ we clearly have

$$(1 - \mu_{\mathcal{P}}(B_\varepsilon[\omega]))\varepsilon \leq \int_X \bar{f}_n(\mathcal{P}^n(x), \omega)\, d\mu(x) \leq (1 - \mu_{\mathcal{P}}(B_\varepsilon[\omega])) + \varepsilon.$$

Therefore, (cf. [30, Lemma 9.1]) if $\omega \in \{0, 1, \ldots, k-1\}^n$ and $\beta > 0$ are such that

$$\int_X \bar{f}_n(\mathcal{P}^n(x), \omega)\, d\mu(x) = \int_{\Omega_k} \bar{f}_n(u_0u_1 \cdots u_{n-1}, \omega)\, d\mu_{\mathcal{P}}(u) < \beta,$$

then the process $(\mathbf{X}, \mathcal{P})$ is $(n, \sqrt{\beta})$-trivial. Conversely, if the process $(\mathbf{X}, \mathcal{P})$ is $(n, \varepsilon/2)$-trivial, then there is $\omega^{(n)} \in \mathcal{P}^n$ such that

$$\int_X \bar{f}_n(\mathcal{P}(x), \omega^{(n)})\, d\mu(x) = \int_{\Omega_k} \bar{f}_n(u_0u_1 \cdots u_{n-1}, \omega^{(n)})\, \mathrm{d}\mu_{\mathcal{P}}(u) < \varepsilon.$$

A process $(\mathbf{X}, \mathcal{P})$ is *M-trivial* [30, Definition 9.2] if for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for every $n \geq N$ the process $(\mathbf{X}, \mathcal{P})$ is (n, ε)-trivial.

THEOREM 9.2. [30, Theorem 4, (1)⇔(2)] *An aperiodic measure-preserving system $\mathbf{X}$ is loosely Kronecker if and only if for every finite partition $\mathcal{P}$ of X the process $(\mathbf{X}, \mathcal{P})$ is M-trivial.*

9.2. *Proof of Theorem 9.1.* In the proof of our main theorem, we will need the following fact.

LEMMA 9.3. *Let $\mu \in \mathcal{M}_T^e(X)$ and $\varepsilon > 0$. If $\mathcal{P}, \mathcal{R} \in \mathbf{P}^k(X)$ are such that $d_1^\mu(\mathcal{P}, \mathcal{R}) < \varepsilon/3$, then there is $N \in \mathbb{N}$ such that for every $n \geq N$ we have*

$$\int_X \bar{f}_n(\mathcal{P}^n(x), \mathcal{R}^n(x))\, d\mu(x) < \varepsilon.$$

Proof. Let $\mathcal{P} = \{P_0, P_1, \ldots, P_{k-1}\}$, $\mathcal{R} = \{R_0, R_1, \ldots, R_{k-1}\}$ and $\Delta = (P_0 \div R_0) \cup (P_1 \div R_1) \cup \cdots \cup (P_{k-1} \div R_{k-1})$. Observe that the ergodicity of μ implies that for μ-almost every $x \in X$ we have

$$\begin{aligned}\bar{d}_n(\mathcal{P}^n(x), \mathcal{R}^n(x)) &= \frac{1}{n}\sum_{j=0}^{n-1} \chi_\Delta(T^j(x)) \overset{(n\to\infty)}{\longrightarrow} \mu(\Delta) \\ &= \mu(\{x \in X : \mathcal{P}(x) \neq \mathcal{Q}(x)\}) = d_1^\mu(\mathcal{P}, \mathcal{Q}).\end{aligned}$$

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It follows from the Egorov theorem that there are a measurable set $G \subset X$ and $N \in \mathbb{N}$ such that $\mu(G) > 1 - \varepsilon/3$ and for all $n \geq N$ and $x \in G$ one has $\bar{d}_n(\mathcal{P}^n(x), \mathcal{R}^n(x)) < d_1^\mu(\mathcal{P}, \mathcal{R}) + \varepsilon/3$. Notice that $\bar{f}_n(\mathcal{P}^n(x), \mathcal{R}^n(x)) \leq \bar{d}_n(\mathcal{P}^n(x), \mathcal{R}^n(x))$ for all $n \in \mathbb{N}$. Therefore, if $d_1^\mu(\mathcal{P}, \mathcal{R}) < \varepsilon/3$ and $n \geq N$, then

$$\begin{aligned}\int_X \bar{f}_n(\mathcal{P}^n(x), \mathcal{R}^n(x))\, d\mu(x) &\leq \int_X \bar{d}_n(\mathcal{P}^n(x), \mathcal{R}^n(x))\, d\mu(x)\\ &\leq \int_G \bar{d}_n(\mathcal{P}^n(x), \mathcal{R}^n(x))\, d\mu(x) + \mu(X \setminus G)\\ &\leq d_1^\mu(\mathcal{P}, \mathcal{R}) + \varepsilon/3 + \varepsilon/3 \leq \varepsilon. \qquad \square\end{aligned}$$

The rest of this section is devoted to the proof of Theorem 9.1.

Proof of Theorem 9.1. Let $\mu = \mu^{(0)}$ be an $\overline{fk}$-limit of a sequence of loosely Kronecker measures $\mu^{(1)}, \mu^{(2)}, \ldots$. Then there are a sequence $(x^{(n)})_{n=1}^\infty \subseteq X$ and a generic quasi-orbit $\underline{x}$ for $\mu^{(0)}$ such that $\overline{fk}(\underline{x}, \underline{x}_T^{(n)}) \to 0$ as $n \to \infty$ and $x^{(n)}$ is a generic point for $\mu^{(n)}$ for $n \in \mathbb{N}$. Then $\mu^{(0)}$ is ergodic. If $\mu^{(0)}$ is a periodic measure, then there is nothing to show. Assume that $\mu^{(0)}$ is aperiodic.

To apply Katok's criterion (Theorem 9.2) we need to show that for every finite partition $\mathcal{P}$ of X the process $(\mathbf{X}, \mathcal{P})$, where $\mathbf{X} = (X, X_{\mathscr{B}}, \mu^{(0)}, T)$, is M-trivial. To this end we choose a partition $\mathcal{P} \in \mathbf{P}^k(X)$ and fix $\varepsilon > 0$.

We use Lemma 2.3 to find $0 < \alpha < \varepsilon/6$ such that $\bar{f}(\underline{\omega}, \underline{\omega}') < \alpha$ for some $\underline{\omega} \in \mathrm{Gen}(\xi)$, $\underline{\omega}' \in \mathrm{Gen}(\xi')$, where ξ and ξ' being shift-invariant ergodic measures on Ω_{k+1} implies that $\bar{f}(\xi, \xi') < \varepsilon/3$.

We take $\delta < \min\{\alpha/2, \varepsilon/9\}$. We apply Lemma 8.2 to $(\mu^{(n)})_{n=0}^\infty$ to find $\gamma = \gamma(\mu = \mu^{(0)}, \delta, \mathcal{P})$ and a partition $\mathcal{R} \in \mathbf{P}^{k+1}(X)$ with $d_1^{\mu^{(0)}}(\mathcal{P}, \mathcal{R}) < \delta$ and $\mu^{(n)}(\partial\mathcal{R}) = 0$ for $n = 0, 1, \ldots$. Let N be such that $\overline{fk}(\underline{x}, x_T^{(N)}) < \min\{\gamma, \alpha/2\}$. It follows from Lemma 8.2 that

$$\bar{f}(\mathcal{R}(\underline{x}), \mathcal{R}(\underline{x}_T^{(N)})) < \alpha < \varepsilon/6. \tag{9}$$

By Lemma 2.2, the point $\mathcal{R}(\underline{x}_T^{(N)})$ is generic for $\mathcal{R}$-representation measure of $\mu^{(N)}$ denoted by $\mathcal{R}_*\mu^{(N)} = \mu_{\mathcal{R}}^{(N)}$ and $\mathcal{R}(\underline{x})$ is generic for the $\mathcal{R}$-representation $\mu_{\mathcal{R}}^{(0)} = \mathcal{R}_*\mu^{(0)}$ of $\mu^{(0)}$. Inequality (9) and our choice of α imply that $\bar{f}(\mu_{\mathcal{R}}^{(0)}, \mu_{\mathcal{R}}^{(N)}) < \varepsilon/3$. Hence, by the definition of the $\bar{f}$-metric, we find $M' = M'(\varepsilon) > 0$ satisfying that for every $m \geq M'$ there is a coupling λ_m of $\mathcal{R}_*\mu|_{\mathcal{R}^m}$ and $\mathcal{R}_*\mu^{(N)}|_{\mathcal{R}^m}$ for which we have

$$\int_{\mathcal{R}^m \times \mathcal{R}^m} \bar{f}_m(\zeta, \zeta')\, d\lambda_m(\zeta, \zeta') < \varepsilon/3. \tag{10}$$

Using that $\mu^{(N)}$ is loosely Kronecker or periodic, we find $M'' = M''(\varepsilon)$ such that for every $m \geq M''$ there is $\omega^{(m)} \in \mathcal{R}^m$ satisfying

$$\int_X \bar{f}_m(\mathcal{R}^m(x), \omega^{(m)})\, d\mu^{(N)}(x) < \varepsilon/3. \tag{11}$$

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With the above notation, let $m \geq \max\{M', M''\}$. Using (10) and (11), we get (below, $\mathcal{R}^{2m} = \mathcal{R}^m \times \mathcal{R}^m$)

$$\begin{aligned}
&\int_X \bar{f}_m(\mathcal{R}^m(x), \omega^{(m)})\, d\mu^{(0)}(x) \qquad (12)\\
&= \int_{\mathcal{R}^{2m}} \bar{f}_m(\zeta, \omega^{(m)})\, d\lambda_m(\zeta, \zeta')\\
&\leq \int_{\mathcal{R}^{2m}} (\bar{f}_m(\zeta, \zeta') + \bar{f}_m(\zeta', \omega^{(m)}))\, d\lambda_m(\zeta, \zeta')\\
&= \int_{\mathcal{R}^{2m}} \bar{f}_m(\zeta, \zeta')\, d\lambda_m(\zeta, \zeta') + \int_X \bar{f}_m(\mathcal{R}^m(y), \omega^{(m)})\, d\mu^{(N)}(y)\\
&\leq 2\varepsilon/3.
\end{aligned}$$

By Lemma 9.3, since $\mathcal{R}$ is a partition satisfying $d_1^{\mu^{(0)}}(\mathcal{P}, \mathcal{R}) < \delta < \varepsilon/9$, there exists $M \in \mathbb{N}$ such that for every $m \geq M$ we have

$$\int_X \bar{f}_m(\mathcal{P}^m(x), \mathcal{R}^m(x))\, d\mu^{(0)}(x) < \varepsilon/3. \qquad (13)$$

Combining (11) with (12), we see that if $m \geq N = \max\{M, M', M''\}$, then

$$\int_X \bar{f}_m(\mathcal{P}^m(x), \omega^{(m)})\, d\mu^{(0)}(x) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have proved that $(\mathbf{X}, \mathcal{P})$ is M-trivial. Since $\mathcal{P}$ was also arbitrary and $\mu^{(0)}$ is aperiodic we conclude, using Theorem 9.2, that $\mu^{(0)}$ is loosely Kronecker (standard). □

As a consequence of Theorems 6.2, 7.3, and 8.1 (combined with Theorem 3.5 for the second part), we get the following result, which we also call a theorem due to its importance.

THEOREM 9.4. *If μ is an aperiodic GIKN measure resulting from a GIKN sequence $(\Gamma_n)_{n\in\mathbb{N}}$, then μ is a loosely Kronecker measure (hence, ergodic and with zero entropy). In particular, under the assumptions of Theorem 3.5 the resulting measure μ is either a non-hyperbolic periodic measure or a non-hyperbolic loosely Kronecker measure (hence, μ is ergodic and has zero entropy). In all cases above, μ is supported on*

$$\operatorname{supp}\mu = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} \Gamma_n}.$$

Theorem 9.4 may replace [8, Lemma 2.5] and it reveals more information about the resulting measure. This applies, for example, to [15, Theorem B] or [15, Proposition 1.1]. Similar strengthenings are possible for results from [3, 5, 8, 10, 17, 18, 29, 32]. We leave the details to the reader, as presenting them here would require repeating a lot of material from these papers without introducing anything new.

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