Math. Proc. Camb. Phil. Soc. (1992), **112**, 271
Printed in Great Britain

The preimages of submanifolds

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(*Received* 12 *August* 1991; *revised* 30 *January* 1992)

1. *Introduction*

All manifolds in this paper are piecewise linear (or smooth if one wishes).

Let $f: M \to N$ be a proper map between two compact oriented connected n-manifolds and Y be a proper connected k-submanifold of N, where $0 \leqslant k \leqslant n$. Let $\deg(f)$ be the degree of f, $\mathrm{ind}(f)$ be the index of $f_*(\pi_1(M))$ in $\pi_1(N)$, and $R(Y)$ be a regular neighbourhood of Y. Suppose that f is transverse to Y and the components of $f^{-1}(Y)$ are $\tilde{Y}_1, \ldots, \tilde{Y}_l$. Then for some $R(Y)$, we have $\bigcup_i R(\tilde{Y}_i) = f^{-1}(R(Y))$. The n-submanifolds $R(Y)$ and $R(\tilde{Y}_i)$ are oriented by hereditary orientations and

$$\deg(f) = \sum_i \deg\{f|: R(\tilde{Y}_i) \to R(Y)\}.$$

If $\deg(f) \neq 0$, then $1 \leqslant \mathrm{ind}(f) \leqslant |\deg(f)|$.

We say that f is efficient at Y if f is transverse to Y and all the terms in the above summation are non-zero and of the same sign. If Y is not connected, we say that f is efficient at Y if f is efficient at all the components of Y.

Our main result is Theorem 1 below which relates several results on geometric topology, degree of maps, roots of maps and has applications to 3-manifolds.

THEOREM 1. *Let $f: M \to N$ be a proper non-zero degree map between compact oriented connected n-manifolds and Y be a proper connected k-submanifold of N, where $0 \leqslant k \leqslant n$. If Y is not an arc, then f is properly homotopic to f_1 such that f_1 is efficient at Y.*

Two direct corollaries of Theorem 1 and its proof are the following results:

COROLLARY 1. *Suppose that $f: M \to N$ and $Y \subset N$ are as in Theorem 1. Then f is properly homotopic to f_1 such that $f_1^{-1}(Y)$ contains at most $|\deg(f)|$ components.*

COROLLARY 2. *In Corollary 1, if $n \neq 2$ and $k \neq 0$, then 'at most $|\deg(f)|$ components' can be replaced by 'at most $\mathrm{ind}(f)$ components'.*

Before we go further, it is worth making several remarks which imply that the statements above are the best possible results in a reasonable sense.

Remark 1. The condition that Y is not an arc in Theorem 1 cannot be removed. There is a proper map f of degree one from the three-punctured sphere P of dimension n onto the n-ball D such that $f|$ is a non-zero degree on each component of ∂P. Let Y

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be any proper arc of D. If f_1 is properly homotopic to f and is transverse to Y, then $f_1^{-1}(\partial Y)$ contains more than two points. Therefore $f_1^{-1}(Y)$ as a union of arcs contains more than one component.

However, if we assume that $f|_\partial$ is a covering of degree $\deg(f)$, then the condition that Y is not an arc in Theorem 1 can be removed.

Remark 2. Let $f: M \to N$ be a proper zero degree map between compact oriented connected n-manifolds and Y be a point of N. Then f can be deformed so that $f^{-1}(y)$ is an empty set by Kneser [**11**] and Hopf [**6**]. However the condition $\deg(f) \neq 0$ is necessary in Theorem 1. For example, fix $(x_0, y_0) \in S^1 \times S^n$ and define $f: S^1 \times S^n \to S^1 \times S^n$ by $f(x, y) = (x_0, y)$. Then $f(S^1 \times S^n) = x_0 \times S^n$ which intersects $S^1 \times y_0$ at exactly one point transversely. By algebraic topology, it is f_1 homotopic to f, then $f_1(S^1 \times S^n)$ cannot miss $S^1 \times y_0$. Therefore $f_1(S^1 \times S^n)$ cannot miss any proper submanifold containing $S^1 \times y_0$.

Remark 3. If f is transverse to a point y in $\operatorname{int} N$, then $f^{-1}(y)$ must contain at least $|\deg(f)|$ points by the definition of transversality. Since there is a map f with $|\deg(f)| > \operatorname{ind}(f)$, the condition $k \neq 0$ cannot be removed in Corollary 2. In Corollary 4 below, the condition $k \neq 0$ is removed at the expense of transversality. The condition $n \neq 2$ cannot be removed even if we do not require the transversality condition; see [**9**] or Remark 4 following Proposition 1 in Section 2.

The proof of the main part of Theorem 1 can be found in [**3**, **6**, **9**, **15**] for $k = 0$ and $n \neq 2$, in [**11**, **16**] for $k = 0$ and $n = 2$, in [**4**, **18**] for $k = 1$, $n = 3$ and $\deg(f) = 1$, in [**2**] for $k = n-1$, $\operatorname{ind}(f) = 1$ and Y two-sided. In our proof of Theorem 1, for $n \neq 2$ we use only a basic tool in topology of manifolds (Theorem A below), and the proof for the case $k < n$ follows easily from that of the case $k = n$; for $n = 2$, the proof follows directly from a work of Edmonds (Theorem B below).

Corollaries 1 and 2 easily imply the following two known facts.

COROLLARY 3 (Hopf, and also Epstein, for $n \neq 2$; Kneser, and also Skora for $n = 2$). *Let $f: M \to N$ be as in Theorem* 1. *Then f is properly homotopic to f_1 such that there is an n-ball D with the property that $f_1^{-1}(D)$ contains exactly $|\deg(f)|$ n-balls, each of which maps homeomorphically onto D by f_1.*

Let $f: X \to Y$ be a map between topological spaces and $y \in Y$. We say that x_0 is a root of $f(x) = y$ if $f(x_0) = y$.

COROLLARY 4 (Schirmer, and also Lin). *Let $f: M \to N$ be as in Theorem* 1 *and $n \neq 2$. If $y \in \operatorname{int} N$, then f is properly homotopic to f_1 such that $f_1^{-1}(y)$ contains* $\operatorname{ind}(f)$ *points. Furthermore,* $\operatorname{ind}(f)$ *is the minimum number of roots among all maps which are properly homotopic to f.*

Some applications to 3-manifolds will be given in Section 3.

Now we would like to mention some variations and generalizations of Theorem 1.

Any subcomplex Y of N admits a (non-unique) CW-structure $Y_0 \subset Y_1 \subset \ldots \subset Y_n = Y$, where Y_i is contained in the i-skeleton of N, and $Y_i \backslash Y_{i-1}$ is an union of open i-cells.

THEOREM 2. *Let $f: M \to N$ be a proper non-zero degree map between compact oriented connected n-manifolds and Y be a k-dimensional subcomplex of* $\operatorname{int} N$. *Then given any CW-structure $Y_0 \subset Y_1 \subset \ldots \subset Y_n = Y$ of Y, f is properly homotopic to f_1 such that $f_1|: M \backslash f^{-1}(\operatorname{int} R(Y_{i-1})) \to N \backslash \operatorname{int} R(Y_{i-1})$ is efficient at $Y_i \backslash \operatorname{int} R(Y_{i-1})$.*

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Using Theorem 1 and the fact mentioned at the end of Remark 1, Theorem 2 can be proved by inductively homotoping $f_1|: M\backslash f^{-1}(R(A_{ik})) \to N\backslash R(A_{ik})$ rel ∂ so that f is efficient at A_{ik}, where A_{ik} is the kth component of $Y_i\backslash \operatorname{int} R(Y_{i-1})$.

An analogous theorem can be stated for a map between possibly non-orientable manifolds with a non-zero geometric degree. Here, a proper map $f: M \to N$ is called 'efficient' at a connected submanifold Y of N, if f is transverse to Y, and

$$G(f) = \sum_i G(f|: R(Y_i) \to R(Y)),$$

where $G(f)$ is the geometric degree and all terms in the summation are non-zero.

There are three sections in this paper. The remaining part of this section gives some definitions which are involved in the statement of Theorem 1 and some facts which are fundamental in the proof of Theorem 1. In Section 2 we give the proof of Theorem 1. In Section 3 we give some applications to 3-manifolds.

For undefined terminology in this paper, see [**1**] for surfaces, [**5**] for 3-manifolds, [**7**] and [**14**] for piecewise linear topology.

Let Y be a compact k-submanifold of N. We say that Y is proper if $\partial N \cap Y = \partial Y$ for $k < n$, or if the closure of $\partial Y\backslash \partial N$ is a proper $(n-1)$-submanifold of N for $k = n$. This definition of a proper k-submanifold is standard for $k < n$.

For a proper k-submanifold Y of N, where $k < n$, a regular neighbourhood $R(Y)$ of Y has a block normal bundle structure of Y in N (see [**14**]).

Let $f: M \to N$ be a map between two compact connected n-manifolds. We say that f is proper if $f^{-1}(\partial N) = \partial M$ and f induces an (I-)bundle map from $R(\partial M) = f^{-1}(R(\partial N))$ to $R(\partial N)$ for some $R(\partial N)$. We say that a proper map $f: M \to N$ is transverse to a proper k-submanifold Y of N, if $f^{-1}(Y)$ is a proper k-submanifold of M and f induces a bundle map from $f^{-1}(R(Y))$ to $R(Y)$ for some $R(Y)$ for $k < n$, or if f is transverse to the closure of $\partial Y\backslash \partial N$ for $k = n$. The fundamental fact is that if a proper map f restricted to some neighbourhood of a compact subset C of M is transverse to Y, then f is homotopic rel C to a proper map f_1 which is transverse to Y (see [**14**]). We say that f is π_1-surjective if $f_*: \pi_1(M) \to \pi_1(N)$ is surjective. The degree of f is defined by $f_{\#n}[M] = \deg(f)[N]$, where $[M] \in H_n(M, \partial M; \mathbb{Z})$ and $[N] \in H_n(N, \partial N; \mathbb{Z})$ are chosen generators.

The notation I denotes the closed interval and D^l denotes the l-ball. For a subset X of M, $\bar{X}$ denotes the closure of X in M and $\#X$ denotes the number of components of X. For a compact sub-polyhedron X of M, an embedding $\alpha: (I, \operatorname{int} I, \partial I) \to (M, M\backslash X, X)$ is said to be proper if there is an embedding of the l-dimensional 1-handle

$$\alpha^{(l)} = \alpha \times \mathrm{id}_{D^{l-1}}: (I, \operatorname{int} I, \partial I) \times D^{l-1} \to (M, M\backslash X, X).$$

In this case, we call $\alpha(I)$ a properly embedded arc in (M, X).

Let $\beta: (I, \operatorname{int} I, \partial I) \to (M, M\backslash X, X)$ be a map. We say that $\beta(I)$ is strongly homotopic to zero in (M, X) if there is a homotopy $\beta_t: (I, \partial I) \to (M, X), t \in [0, 1]$, such that (1) $\beta_0 = \beta$, (2) $\beta_t(\partial I) = \beta(\partial I)$ for $t \in [0, 1]$, (3) $\beta_t(\operatorname{int} I) \subset M\backslash X$ for $t \in [0, 1)$ and (4) $\beta_1(I) \subset X$.

The following two results are fundamental in the proof of Theorem 1.

THEOREM A. *Suppose that a proper map $f: M \to N$ between compact connected n-manifolds is transverse to a proper k-submanifold $Y \subset N$, where $n > 2$ and $k > 0$. If there is a proper embedding $\alpha: (I, \operatorname{int} I, \partial I) \to (M, M\backslash f^{-1}(Y), f^{-1}(Y))$ such that $f(\alpha(I))$ is strongly homotopic to zero in (N, Y), then f is properly homotopic to a proper map f_1 which is*

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transverse to Y such that: (1) $f_1^{-1}(Y) = f^{-1}(Y) \cup \alpha^{(n-1)}(I \times D^{n-1})$ *if* $k = n$ (*handle adding*); (2) $f_1^{-1}(Y) = (f^{-1}(Y) \backslash \alpha^{(k)}(\partial I \times D^k)) \cup \alpha^{(k)}(I \times \partial D^k)$ *if* $0 < k < n$ (*handle exchanging*).

Various versions or special cases of Theorem A are scattered in the literature (e.g. [**2**, **8**]); the proofs are essentially the same.

THEOREM B (Edmonds; see [**1**] or [**16**]). *If $f: M \to N$ is a proper map of non-zero degree between connected compact oriented surfaces, then f is properly homotopic to a composition of a pinch map followed by a branched cover.*

2. *Proof of Theorem* 1

The proof of Theorem 1 for the case when $n = 1$ is trivial.

Suppose now that $n = 2$. If $Y = N$, then $f^{-1}(Y) = M$, and Theorem 1 is obviously true. If $Y \neq N$, then $N \backslash Y$ contains a 2-dimensional disc D. By Theorem B, we may assume that f is a composition of a pinch map followed by a branched cover. After an isotopy on N, we may assume the image of the pinch point and the branch points (a finite set) are in $\operatorname{int} D$. Then $f|: f^{-1}(R(Y)) \to R(Y)$ is a covering; therefore $\deg(f|: R(\tilde{Y}_i) \to R(Y))$ and $\deg(f)$ have the same sign and thus f is efficient at Y.

From now on, we assume that $n > 2$. First we prove the following proposition.

PROPOSITION 1. *Suppose $f: M \to N$ is a proper map of non-zero degree between manifolds of dimension $n > 2$ and $f_*: \pi_1(M) \to \pi_1(N)$ is surjective. Let Y be a proper k-submanifold of N with components $Y_1, \dots, Y_m$ where each Y_i is not an arc. Then f is properly homotopic to a proper map f_1 such that $f_1^{-1}(Y_i)$ is a connected k-submanifold for any $i \in \{1, \dots, m\}$. Furthermore we may require that f_1 is transverse to Y if $k > 0$.*

Remark 4. Proposition 1 is not true for $n = 2$ even if we do not require the transversality condition. There is a π_1-surjective branched map $f: M \to N$ of degree $d > 1$ for some surfaces M and N. For any $y \in N$, the deficiency $d_f(y)$ of f at y is defined by

$$d_f(y) = \max\{0, |\deg(f)| - \#f^{-1}(y)\}.$$

Then we have $\sum_{y \in N} \operatorname{def}_f(y) \leqslant |\deg(f)|\chi(N) - \chi(M)$ (see [**6**], 1930). Obviously, if we choose $m > |\deg(f)|\chi(N) - \chi(M)$, then we cannot deform f so that $f^{-1}(y_i)$ is a point for all $i \in \{1, \dots, m\}$.

Proof of Proposition 1. The proposition will be proved by considering two cases.

Case 1. $k = n$. Let N_{iu} be the components of $\overline{N \backslash Y}$ neighbouring Y_i, where $u = 1, \dots, u_i, i = 1, \dots, m$. Possibly $N_{iu} = N_{jv}$ for $i \neq j$. All N_{iu}s are manifolds of dimension n. Choose a connected 1-complex Γ_i in $\operatorname{int} Y_i$ with base point y_i which carries $\pi_1(Y_i)$ and a connected 1-complex Γ_{iu} in $\operatorname{int} N_{iu}$ with base point y_{iu} which carries $\pi_1(N_{iu})$. Then for each component S of $\overline{\partial Y \backslash \partial N}$ between Y_i and N_{iu}, pick an arc connecting y_i and y_{iu} which meets S transversely exactly once, where $u = 1, \dots, u_i$ and $i = 1, \dots, m$. By taking the union we get a connected 1-complex Γ which is transverse to ∂Y.

Since $f_*: \pi_1(M) \to \pi_1(N)$ is surjective and $\dim M \geqslant 3$, there exist an embedding $h: \Gamma \to M$ and a homotopy of $f|_{h(\Gamma)}$ such that $f|: h(\Gamma) \to \Gamma$ is a homeomorphism. Extend this homotopy of $f|_{h(\Gamma)}$ to $f: M \to N$ by the homotopy extension property. Since $f|: h(\Gamma) \to N$ is transverse to ∂Y, we can deform f further rel $h(\Gamma)$ such that f is transverse to ∂Y. Let h and f be maps subject to the above conditions such that $f^{-1}(Y)$ has the minimum number of components.

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By the construction, the following conditions are satisfied:

(1) there is one and only one component of $f^{-1}(Y_i)$ (resp. $f^{-1}(N_{iu})$), denoted by $\tilde{Y}_i$ (resp. $\tilde{N}_{iu}$), which contains $h(\Gamma_i)$ (resp. $h(\Gamma_{iu})$);

(2) A component of $f^{-1}(\overline{N\backslash Y})$ neighbouring a component of $f^{-1}(Y_i)$ is a component of $f^{-1}(N_{iu})$ for some N_{iu}, and $\tilde{N}_{iu}$ neighbours $\tilde{Y}_i$;

(3) for each component S of $\overline{\partial Y\backslash\partial N}$ between Y_i and N_{iu}, there is one and only one component of $f^{-1}(S)$, denoted by $\tilde{S}$, which meets $h(\Gamma)$ and lies between $\tilde{Y}_i$ and $\tilde{N}_{iu}$;

(4) all maps $f|:\tilde{Y}_i\to Y_i$ and $f|:\tilde{N}_{iu}\to N_{iu}$ are π_1-surjective.

The following two claims will complete the proof for Case 1.

Claim 1. After a homotopy, we may assume that the only component of $f^{-1}(N_{iu})$ neighbouring $\tilde{Y}_i$ is $\tilde{N}_{iu}$.

Proof. Suppose that for some i there is a component of $f^{-1}(N_{iu})\backslash\tilde{N}_{iu}$, denoted by $\tilde{N}'_{iu}$, which neighbours $\tilde{Y}_i$. Pick any component $\tilde{S}'$ of $f^{-1}(\overline{\partial Y\backslash\partial N})$ between $\tilde{Y}_i$ and $\tilde{N}'_{iu}$ which is a component of $f^{-1}(S)$ for a component S of $\overline{\partial Y\backslash\partial N}$ between Y_i and N_{iu}. By (3), there is a component $\tilde{S}$ of $f^{-1}(S)$ between $\tilde{Y}_i$ and $\tilde{N}_{iu}$.

We can pick a properly embedded arc $\beta: I\to\tilde{Y}_i$ which connects $\tilde{S}$ and $\tilde{S}'$. Now $f\beta(I)$ is an element in $\pi_1(Y_i,S)$. Since S is connected, the inclusion induces an isomorphism $\pi_1(Y_i)\to\pi_1(Y_i,S)$. Since $f|_*:\pi_1(\tilde{Y}_i)\to\pi_1(Y_i)$ is surjective, there is a loop $\gamma(I)$ based at an end-point of $\beta(I)$ such that $f\beta(I)=f\gamma(I)$ in $\pi_1(Y_i,S)$. Let $\alpha=\beta\gamma^{-1}$. Now deform α rel $\alpha(\partial I)$ to be a properly embedded arc in $(\tilde{Y}_i,\tilde{S}\cup\tilde{S}')$. Then $f\alpha(I)$ is homotopic to zero in $\pi_1(Y_i,S)$. Since S is a component of $\overline{\partial Y\backslash\partial N}$, the map $f\alpha(I)$ is strongly homotopic to zero in (Y_i,S); therefore $f\alpha(I)$ is strongly homotopic to zero in $(N,\overline{N\backslash Y})$. Furthermore, we may assume that $\alpha(I)$ is disjoint from $h(\Gamma)$. By Theorem A, f is homotopic rel $h(\Gamma)$ to f_1 such that

$$f_1^{-1}(N_{iu})=f^{-1}(N_{iu})\cup\alpha^{(n-1)}(I\times D^{n-1}).$$

Being connected by the 1-handle $\alpha^{(n-1)}(I\times D^{n-1})$, $\tilde{N}_{iu}$ and $\tilde{N}'_{iu}$ belong to the same component of $f_1^{-1}(N_{iu})$, still denoted by $\tilde{N}_{iu}$, which meets $h(\Gamma)$ and neighbours $\tilde{Y}_i$. For convenience, still denote f_1 by f. Note that f still satisfies the assumption that $\#f^{-1}(Y)$ is minimum. Since $\#f^{-1}(N_{iu})$ is now reduced by 1, Claim 1 can be proved by repeating the above steps finitely many times.

Claim 2. $f^{-1}(Y_i)$ is connected for all i.

Proof. If not, then there is a component $\tilde{Y}'_j$ of $f^{-1}(Y_j)\backslash\tilde{Y}_j$ which neighbours $\tilde{N}_{jv}$, for some $j\in\{1,\dots,m\}$. Pick any component of $f^{-1}(\overline{\partial Y\backslash\partial N})$ between $\tilde{N}_{jv}$ and $\tilde{Y}'_j$, denoted by $\tilde{T}'$, which is a component of $f^{-1}(T)$ for a component T of $\overline{\partial Y\backslash\partial N}$ between Y_j and N_{jv}. By (2) and (3), there is a component $\tilde{T}$ of $f^{-1}(T)$ between $\tilde{Y}_j$ and $\tilde{N}_{jv}$.

There is a properly embedded arc $\alpha: I\to\tilde{N}_{jv}$ which connects $\tilde{T}$ and $\tilde{T}'$ and is disjoint from $h(\Gamma)$. Since $f|_*:\pi_1(\tilde{N}_{jv})\to\pi_1(N_{jv})$ is surjective and T is connected, as before, α can be chosen so that $f\alpha(I)$ is strongly homotopic to zero in (N_{jv},T). Then $f\alpha(I)$ is strongly homotopic to zero in (N,Y). By Theorem A, f is properly homotopic to f_1 such that $f_1^{-1}(Y_j)=f^{-1}(Y_j)\cup\alpha^{(n-1)}(I\times D^{n-1})$ which reduces the number of components of $f^{-1}(Y_j)$. This contradicts the assumption that $\#f^{-1}(Y)$ is minimum. Hence $f^{-1}(Y_i)$ is connected for all $i\in\{1,\dots,m\}$, and the proof for case 1 is completed.

For later use note that $f|:f^{-1}(R(Y_i))\to R(Y_i)$ is π_1-surjective for each i.

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Case 2. $k \leqslant n-1$. Now each component $R(Y_i)$ of $R(Y)$ is a submanifold of dimension n. Since we have proved Theorem 1 for the case $k = n$, we can deform f so that $f^{-1}(R(Y_i))$ is connected and $f|: f^{-1}(R(Y_i)) \to R(Y_i)$ is π_1-surjective. We then make further deformations supported in $R(Y)$ as below so that $f^{-1}(Y_i)$ is connected for each i.

(a) Suppose that $0 < k < n$. Suppose $f^{-1}(Y_j)$ is not connected. Then two components $\tilde{Y}_j$ and $\tilde{Y}'_j$ of $f^{-1}(Y_j)$ are connected by a properly embedded arc $\alpha(I)$ in $(f^{-1}(R(Y_j)), f^{-1}(Y_j))$. Since $\pi_1(R(Y_j), Y_j) = 0$ and $R(Y_j)$ is the regular neighbourhood of Y_j, the map $f\alpha(I)$ is strongly homotopic to zero in $(R(Y_j), Y_j)$. Since Y_j is not an arc by the condition imposed in Theorem 1, each component of $f^{-1}(Y_j)$ is not an arc. Now we can reduce $\#f^{-1}(Y_j)$ by applying Theorem A. After finitely many such steps $f^{-1}(Y_j)$ will be connected.

(b) Suppose that $k = 0$. By the proof of Proposition 1 for the case $k = n$, each $f^{-1}(R(Y_i))$ is an n-ball (obtained by attaching 1-handles to different n-balls). By Hopf's theorem that homotopy classes of proper maps between spheres (balls) are determined by their degrees, we may assume that $f|: f^{-1}(R(Y_i)) \to R(Y_i)$ is a cyclic branched cover with branched set of index $\deg(f)$. Deform f supported in $f^{-1}(R(Y_i))$ so that Y_i belongs to the branched set. Then $f^{-1}(Y_i)$ is connected. ∎

Proof of Theorem 1 *for* $n > 2$, *and proofs of Corollaries* 1, 2, 3, 4.

Suppose that $f: M \to N$ and $Y \subset N$ are as in Theorem 1 and $n > 2$. Let $P: \tilde{N} \to N$ be the covering corresponding to $f_*(\pi_1(M))$. Then there is a π_1-surjective lift $\tilde{f}: M \to \tilde{N}$ of f such that $p\tilde{f} = f$. Let $Y'_1, \ldots, Y'_m$ be the components of $p^{-1}(Y)$. By Proposition 1, we can deform $\tilde{f}$ so that $f^{-1}(Y'_i) = \tilde{Y}_i$ is connected for all i. Hence

$$\deg\{f|: R(\tilde{Y}_i) \to R(Y)\} = \deg\tilde{f}\deg\{p|: R(Y'_i) \to R(Y)\}$$

are all non-zero and have the same sign. Furthermore

$$\#f^{-1}(Y) = \#p^{-1}(Y) \leqslant |\deg(p)| = \operatorname{ind}(f).$$

If $k > 0$, by Proposition 1, we can require that f_1 is transverse to Y; therefore f is efficient at Y.

If $k = 0$, just as in the last step of (b) in the proof of Proposition 1, deform f supported in $f^{-1}(R(Y))$ so that Y misses the branched set. Then we can rechoose a regular neighbourhood $R(Y)$ which misses the branched set. Now f is efficient at Y and $f^{-1}(R(Y))$ contains exactly $|\deg(f)|$ n-balls, and each maps homeomorphically to $R(Y)$ by f.

We have finished the proofs of Theorem 1 and Corollaries 1, 2, and 3.

To prove Corollary 4, note that if y is a point, then the formula above becomes

$$\#f^{-1}(Y) = \#p^{-1}(Y) = |\deg(p)| = \operatorname{ind}(f)$$

and $\operatorname{ind}(f)$ is obviously the minimum number of roots of maps among the proper homotopy class of f. ∎

3. *Degree one maps between* 3-*manifolds*

We are interested in degree one maps between 3-manifolds. It is proved in [**12**] that the existence of degree one maps defines a partial order in the set of geometric 3-manifolds, modulo homotopy equivalences.

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First let us apply Theorem 1 to the case when $\deg f = 1$ and $n = 3$, $N = L_{p,q}$, the 3-dimensional lens space. Let $N = V_1 \cup_\partial V_2$, where V_i is a solid torus. Let Y be a core of V_1. By Theorem 1, f can be homotoped so that f is transverse to Y and $f^{-1}(Y)$ is a connected 1-dimensional submanifold, i.e. a knot K in M. Since the map $f|: K \to Y$ is of degree one, it is homotopic to a homeomorphism. Hence f can be homotoped so that it sends $N(K)$ homeomorphically onto V_1, sends $E(K)(= M \backslash \operatorname{int} N(K))$ onto V_2. Such a map is called a 1-pinch in [**13**], for it can be homotoped so that it 'pinches' $E(K)$ onto the core of V_2. Thus we have obtained

COROLLARY 5. *Any degree* 1 *map from a closed* 3*-manifold* M *onto a lens space* N *is homotopic to a* 1*-pinch. More precisely, given* $N = V_1 \cup V_2$, *there is a knot* K *in* M *such that* f *sends* $N(K)$ *homeomorphically onto* V_1, *and maps* $E(K)$ *onto* V_2.

A fact similar to Corollary 5 is also observed and used by Luft and Sjerve in [**10**].

Corollary 5 implies the classification of the degree one map partial order among lens spaces:

COROLLARY 6. *There exists a degree one map* $f: L_{p,q} \to L_{p',q'}$ *if and only if* (1) $p = kp'$, *where* k *is an integer, and*; (2) $q' = kqc^2$ *in* $\mathbb{Z}_{p'}$, *where* c *is an integer.*

The special case when $p = p'$, together with a theorem of Swarup [**17**] which says that a degree one map inducing an isomorphism on π_1 is a homotopy equivalence, implies the following well-known fact ([**5**], 3·23): there is an orientation-preserving homotopy equivalence $h: L_{p,q} \to L_{p,q'}$ if and only if qq' is a quadratic residue modulo p.

Proof of Corollary 6. Let $L_{p,q} = V_1 \cup_\partial V_2$, $L_{p',q'} = V_1' \cup_\partial V_2'$, where V_i, V_i' are solid tori. If there is such a degree one map f, then $f_*: \pi_1(L_{p,q}) \to \pi_1(L_{p',q'})$ is onto. Thus $p = kp'$ for an integer k. By Corollary 5, there exists a knot K in $L_{p,q}$ such that f maps $N(K)$ onto V_1' homeomorphically, and maps $E(K)$ onto V_2' via a 1-pinch. After a deformation, we may assume that $N(K) \subset V_1$. Let a be the algebraic winding number of K in V_1. Let (λ, μ) (resp. (λ', μ')) be a longitude meridian pair for V_1 (resp. V_1'), and (α, β) be a longitude meridian pair for $N(K)$ such that $\alpha = a\lambda$, $\mu = a\beta$ in $H_1(E(K))$. Then

$$H_1(E(K)) = \{\lambda, \beta \mid p\lambda + qa\beta = 0\}.$$

Consider the map $f_*: H_1(\partial N(K)) \to H_1(\partial V_1)$. We have $f_*(\beta) = \mu'$, $f_*(\alpha + b\beta) = \lambda'$ for some $b \in \mathbb{Z}$. Hence $f_*(p'(\alpha + b\beta) + q'\beta) = p'\lambda' + q'\mu'$, which bounds a meridian disc D in V_2'. Let f be transverse to D. Then $p'(\alpha + b\beta) + q'\beta = p'a\lambda + (p'b + q')\beta$ bounds the surface $f^{-1}(D)$. Hence it must be homologically trivial. It follows that

$$p'a = cp, \quad p'b + q' = cqa, \quad \text{where } c \text{ is an integer.}$$

Hence $k = p/p' = a/c$ and $p'b + q' = cqa = qkc^2$, which implies $q' = kqc^2$ in $\mathbb{Z}_{p'}$. The other implication can be proved by constructing a suitable 1-pinch. ∎

Next we consider degree one maps between any closed oriented 3-manifolds.

COROLLARY 7. *Let* $f: M \to N$ *be a degree one map between closed oriented* 3*-manifolds. Then for any handlebody* H *of genus* g *in* N, f *is homotopic to* f_1 *such that* f_1 *maps* $f_1^{-1}(H)$ *onto* H *homeomorphically.*

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Proof. Since we do not offer a detailed proof of Theorem 2, we will prove Corollary 7 by using Theorem 1 rather than take it as a corollary of Theorem 2. The proof below actually gives an idea of how to prove Theorem 2 using Theorem 1.

A spine Γ of H can be presented as an union of a disjoint union of circles $C_1, \dots, C_g$ and a disjoint union of arcs $A_1, \dots, A_{g-1}$; here A_i connects C_i and C_{i+1} and misses the other C_j's.

By Theorem 1 and the classification of 1-manifolds, we may assume that $f^{-1}(R(C_i))$ is a solid torus and $f|: f^{-1}(R(C_i)) \to R(C_i)$ is a proper map of degree 1. By theorem 13·6 of [5], we may further assume that $f|: f^{-1}(R(C_i)) \to R(C_i)$ is a homeomorphism.

Since the restriction of f on $f^{-1}(R(C_i))$ is transverse to C_i, f can be deformed rel $f^{-1}(R(C_i))$ so that it is transverse to the spine Γ. Since $f|: f^{-1}(\cup R(C_i)) \to \cup R(C_i)$ is a homeomorphism, $f^{-1}(A_i)$ is a union of an arc (connecting $f^{-1}(C_i)$ and $f^{-1}(C_{i+1})$) and some circles. After a further deformation of f rel $f^{-1}(\cup R(C_i))$ we can actually eliminate all of the circle components by the same method as in Proposition 1 (see Remark 1 in Section 1). Thus each component of $f^{-1}(A_i)$ is an arc connecting $f^{-1}(C_i)$ and $f^{-1}(C_{i+1})$. It follows that $f^{-1}(R(\Gamma))$ is a handlebody of genus g and that $f|: f^{-1}(R(\Gamma)) \to R(\Gamma)$ is a proper map of degree 1. Hence f_* is surjective. By the Hopfian property of the free group $\pi_1(f^{-1}(R(\Gamma))) = \pi_1(R(\Gamma))$, f_* is an isomorphism. By 13·6 of [5], we may assume that $f|: f^{-1}(R(\Gamma)) \to R(\Gamma)$ is a homeomorphism. Deform f by composing an ambient isotopy which carries $R(\Gamma)$ to H; then $f|: f^{-1}(H) \to H$ is a homeomorphism. ∎

Let $N = W_1 \cup W_2$ be a Heegaard splitting of N, and Y be the core of W_1. Then Corollary 7 implies that the degree one map $f: M \to N$ can be homotoped so that it maps $f^{-1}(Y)$ onto Y homeomorphically, and 'pinches' the exterior of $f^{-1}(Y)$ onto a core of W_2. Hence any degree one map between closed oriented 3-manifolds is homotopic to a 1-pinch.

Let N be a homotopy 3-sphere. Since there is a degree one map from S^3 onto N, Corollary 7 yields the following theorem, which was initially proved in [4] in an attempt to prove the Poincaré conjecture. Later it was reported in [18].

COROLLARY 8 (Haken, and also Waldhausen). *Given a Heegaard splitting of a homotopy 3-sphere $N = W_1 \cup W_2$, there exists a degree one map $f: S^3 \to N$ so that $f|f^{-1}(W_1)$ is a homeomorphism.*

The authors wish to thank the referee for his helpful suggestions. The second author is partially funded by NSFC.

REFERENCES

[1] A. EDMONDS. Deformation of maps to branched covering in dimension 2. *Ann. of Math.* **110** (1979), 113–125.
[2] A. EDMONDS. Deformation of maps to branched covering in dimension 3. *Math. Ann.* **245** (1979), 273–279.
[3] D. EPSTEIN. The degree of maps. *Proc. London Math. Soc.* **16** (1969), 369–383.
[4] W. HAKEN. On homotopy 3-sphere. *Illinois J. Math.* **10** (1966), 159–180.
[5] J. HEMPEL. *3-manifolds.* Ann of Math. Studies no. 86 (Princeton University Press, 1976).
[6] HOPF. Zur Topologie der Abbidungen von Mannigfaltigkeiten, I, II. *Math. Ann.* **100** (1928), 579–608; **102** (1930), 562–623.
[7] J. HUDSON. *Piecewise Linear Topology* (Benjamin, 1969).
[8] F. LAUDENBACH. *Topologie de la Dimension Trois: Homotopie et Isotopie.* Astérisque no. 12 (Société Mathématique de France, 1974).
[9] X. LIN. On the root classes of mapping. *Acta Math. Sinica (N.S.)* **2** (1986), 199–206.

https://doi.org/10.1017/S030500410007095X Published online by Cambridge University Press

[10] E. Luft and D. Sjerve. Degree 1 maps into lens spaces and free actions on homology spheres. *Topology Appl.* **37** (1990), 131–136.
[11] H. Kneser. Glattung von Flächenabbildungen. *Math. Ann.* **100** (1928), 609–658.
[12] Y. Rong. Degree one maps between geometric 3-manifolds. *Trans. Amer. Math. Soc.*, to appear.
[13] Y. Rong. Maps between Seifert fibered spaces of infinity π_1. Preprint (1990).
[14] C. Rourke and B. J. Sanderson. Block bundle II, transversality. *Ann. of Math.* **87** (1968), 255–277.
[15] H. Schirmer. Mindestzahlen von Koinzidenzpunkt. *J. Reine Angew. Math.* **194** (1955), 21–39.
[16] R. Skora. The degree of maps between surfaces. *Math. Ann.* **276** (1987), 415–423.
[17] G. A. Swarup. On embedded spheres in 3-manifolds. *Proc. Amer. Math. Soc.* **19** (1969), 31–44.
[18] F. Waldhausen. On mappings of handlebodies and Heegaard splittings. In *Topology of Manifolds* (Markham Publishing Company, 1970), pp. 205–211.

https://doi.org/10.1017/S030500410007095X Published online by Cambridge University Press