Bisecting and trisecting the arc of the lemniscate

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1. *Introduction*

In this paper we will discuss the lemniscate curve and show that its arc length can be bisected and trisected using classical ruler and compasses construction. The method dates back to 1718 when Count Giulio Fagnano (1682-1766) first published these constructions [1]. Fagnano was self-educated in mathematics and treated the subject as a hobby. Euler was impressed by his work on this topic and recommended his admission to the Berlin Academy of Science. Euler then generalised Fagnano's work on integrals. In addition, Fagnano was employed to assist in reinforcing the dome of Saint Peter's which was in danger of collapse. Pope Benedict IV rewarded him for his work by publishing his mathematical papers. Fagnano achieved considerable international fame as a mathematician, and rightly so given the outstanding contributions which he made on a number of different topics.

While this topic is old, it is not well advertised, and we believe that our presentation is more accessible than the few we have found in the literature.

2. *The lemniscate*

We begin by deriving a few facts regarding the lemniscate curve shown in Figure 1. Consider two fixed points $F_1(-b,\ 0)$ and $F_2(b,\ 0)$. We can define the lemniscate curve as the locus of points $P(x, y)$ such that the product of the distances from these two fixed points is a constant b^2. Thus $PF_1 \cdot PF_2 = b^2$ or

$$\sqrt{(x+b)^2+y^2}\sqrt{(x-b)^2+y^2} = b^2.$$

Simplifying this we get

$$\left(x^2+y^2\right)^2 = 2b^2\left(x^2-y^2\right).$$

Changing to polar coordinates we have $r^4 = 2b^2r^2\left(\cos^2\theta - \sin^2\theta\right)$ or

$$r^2 = 2b^2\cos 2\theta.$$

Setting $b = \dfrac{1}{\sqrt{2}}$ we get the unit lemniscate with equations

$$\left(x^2+y^2\right)^2 = x^2-y^2 \tag{1}$$

$$\text{and} \qquad r^2 = \cos 2\theta. \tag{2}$$

The remainder of this paper is concerned with the arc length of this unit lemniscate. We will begin by showing that you can bisect an arc starting at the origin to any other point of the lemniscate with straight edge and compasses.

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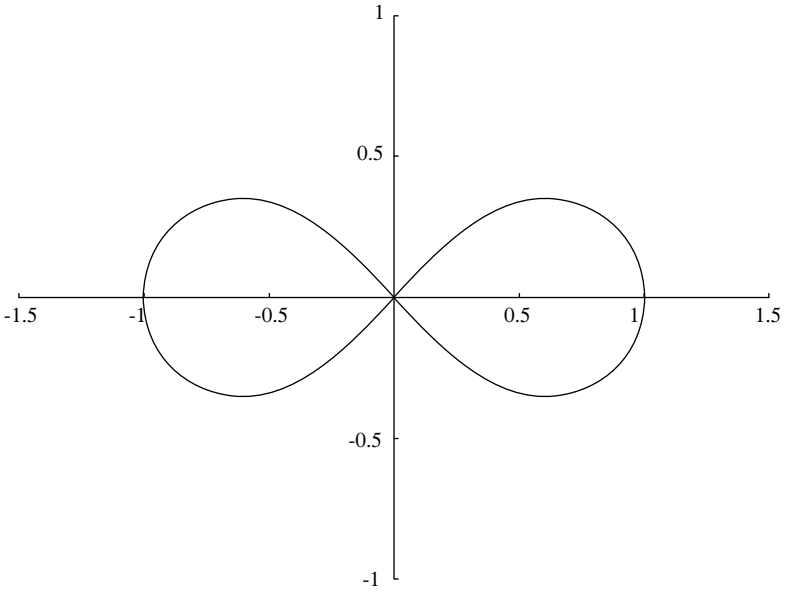


FIGURE 1: The unit lemniscate curve.

For a discussion of the lemniscate curve and its use in deriving a set of advanced trigonometric-like functions see the book by Markushevich [2].

3. *Arc length of the lemniscate*

The standard form for determining the arc length s of a curve given in polar coordinates $r = r(\theta)$ is

$$s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

From (2), $r\dfrac{dr}{d\theta} = -\sin 2\theta$ and $\dfrac{dr}{d\theta} = -\dfrac{\sin 2\theta}{\sqrt{\cos 2\theta}}$. Thus we have

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{\cos 2\theta} = \frac{1 - \cos^2 2\theta}{\cos 2\theta} = \frac{1 - r^4}{r^2}. \tag{3}$$

So $\sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2} = \sqrt{r^2 + \dfrac{1 - r^4}{r^2}} = \dfrac{1}{r}$. From (3) we see that $d\theta = \dfrac{r}{\sqrt{1 - r^4}}\, dr$ and so $s = \displaystyle\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int \frac{dr}{\sqrt{1 - r^4}}$. To determine the length of the lemniscate arc from the origin to the point a distance r from the origin we have

$$s = \int_0^r \frac{dt}{\sqrt{1 - t^4}}.$$

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Half the length of the complete unit lemniscate curve is given by the lemniscate constant

$$L = 2\int_0^1 \frac{dr}{\sqrt{1-r^4}} \approx 2.62205755.$$

4. *Geometric construction*

The reader is familiar with classical geometric construction with straight edge and compasses for bisecting angles, erecting perpendiculars, etc. This paper is about particular constructions that bisect and trisect the lemniscate arcs, however, you will not find any discussion of the actual use of straight edge and compasses. Rather, we find algebraic expressions that indicate how such constructions are made. Gauss is famous for constructing a regular 17-gon at age seventeen. It appears in the last chapter of his seminal book [3], but you will not find the use of straight edge and compasses there either. To do so would be incredibly clumsy.

So how do we indirectly indicate how geometric constructions are performed through the use of algebraic expressions? First let a finite collection of line lengths be given and denoted them by letters $a, b, c, \ldots, g$. Also the unit length is indicated. We then show how, with any two of these line segments, we can construct new line segments that are their sum, difference, product and quotient. In addition we construct the square root of a given line segment. So we now can construct $a + b$, $a - b$, ab, a/b and $\sqrt{a}$ for any given line segments. These five constructions are remarkably simple and are discussed in detail in chapter 3 of the very readable book by Courant and Robbins [4]. We have also shown in an Appendix at the end of this paperhow these five fundamental constructions are made.

Now it is easy to see how any complex algebraic expression composed of only these five operations like

$$\sqrt{\frac{a+\sqrt{c^2/d}}{a-cd}},$$

can be constructed. First construct the square of c then divide this by d. Next take the square root of this construction, and so on.

In fact, expressions of this type are the only ones that are possible to construct. Only finite expressions composed of letters and the symbols for addition, subtraction, multiplication, division and square roots can be constructed. For example, we cannot construct the cube root of an arbitrary line segment. But this is a much more complicated idea.

5. *A strategy for bisecting the lemniscate through geometric construction*

In 1718, [1], a neat strategy was introduced by Count Giulio Fagnano to bisect an arc of the lemniscate curve from the origin to any other arbitrary point on the curve. We know that the length of the lemniscate from $r = 0$

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to $r = r_1$ is given by $s_1 = \int_0^{r_1} \frac{dr}{\sqrt{1 - r^4}}$. Now suppose we can find a transformation, $r = g(u)$, which has the following properties

$$g(0) = 0, \qquad g(u_1) = r_1$$

$$\text{and} \qquad \frac{dr}{\sqrt{1 - r^4}} = 2\frac{du}{\sqrt{1 - u^4}}. \tag{4}$$

Then

$$s_1 = \int_0^{r_1} \frac{dr}{\sqrt{1 - r^4}} = 2\int_0^{u_1} \frac{du}{\sqrt{1 - u^4}}.$$

This last nifty relation shows that the length of the lemniscate arc from radius $u = 0$ to radius $u_1 = g^{-1}(r_1)$ is one half the length of the lemniscate arc from radius $r = 0$ to radius $r = r_1$. Thus we have bisected an arc of the lemniscate by means of the transformation $r = g(u)$.

Suppose the transformations $r = g(u)$ and $u = g^{-1}(r)$ can be made using only a finite combination of addition, subtraction, multiplication, division and the taking of square roots. Then the transformation can be made by geometric construction! Indeed we will demonstrate that the transformation

$$r = g(u) = \frac{2u}{1 + u^4}\sqrt{1 - u^4} \tag{5}$$

has all the necessary properties to show us how to bisect the lemniscate through geometric construction.

6. *Finding the bisection transformation*

Our aim in this section is to show that (5) is an integral of the differential equation

$$\frac{dr}{\sqrt{1 - r^4}} = 2\frac{du}{\sqrt{1 - u^4}}. \tag{6}$$

We could do this by differentiating (5) and showing that it satisfies (6), but the resulting manipulation is clumsy. Instead, we will generate some motivation for the transformation (5).

We first notice that the substitution

$$r = \frac{2t}{1 + t^2} \tag{7}$$

yields $dr = \frac{2(1 - t^2)}{(1 + t^2)^2}dt$ and $\frac{1}{\sqrt{1 - r^2}} = \frac{1 + t^2}{1 - t^2}$ so that

$$\frac{dr}{\sqrt{1 - r^2}} = \frac{2\,dt}{1 + t^2}. \tag{8}$$

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Now if (7) yields the 'rationalisation' (8), maybe the substitution

$$r^2 = \frac{2t^2}{1 + t^4} \tag{9}$$

will transform $\frac{dr}{\sqrt{1 - r^4}}$ in a useful manner. (Notice that if you replace r by r^2 and t by t^2 in (7) you get (9).) Indeed, direct differentiation of (9) yields $dr = \frac{\sqrt{2}(1 - t^4)}{(1 + t^4)^{3/2}}\,dt$ and simple calculation gives us $\frac{1}{\sqrt{1 - r^4}} = \frac{1 + t^4}{1 - t^4}$ so we have

$$\frac{dr}{\sqrt{1 - r^4}} = \frac{\sqrt{2}\,dt}{\sqrt{1 + t^4}}. \tag{10}$$

While (10) is not our differential equation (6), it might be a useful start. The major problem with (10) is the plus sign on the right in the denominator. If we replace t by $e^{i\pi/4}t$, then (9) becomes

$$r^2 = \frac{2e^{i\pi/2}t^2}{1 - t^4} \tag{11}$$

and (10) becomes

$$\frac{dr}{\sqrt{1 - r^4}} = \frac{\sqrt{2}e^{i\pi/4}\,dt}{\sqrt{1 - t^4}} \tag{12}$$

which is more promising. Now do almost the same thing again. Replace t in (9) by $e^{-i\pi/4}u$ and r by t and get

$$t^2 = \frac{2e^{-i\pi/2}u^2}{1 - u^4} \tag{13}$$

and

$$\frac{dt}{\sqrt{1 - t^4}} = \frac{\sqrt{2}e^{-i\pi/4}du}{\sqrt{1 - u^4}}. \tag{14}$$

Now (12) followed by (14) gives us our desired differential equation (6).

The simple algebraic combination of (11) and (13) yields at once (5), the desired solution. We rewrite (5) as

$$r^2 = \frac{4u^2(1 - u^4)}{(1 + u^4)^2}. \tag{15}$$

7. *Geometric construction of the bisection*

We now show that given r in (15) we can find u by geometric construction. We can rewrite (15) as

$$u^8 + \frac{4}{r^2}u^6 + 2u^4 - \frac{4}{r^2}u^2 + 1 = 0.$$

Let us try to factorise this as $(u^4 + Au^2 - 1)(u^4 + Bu^2 - 1) = 0$.

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Expanding the left-hand side we get

$$u^8 + (A + B)u^6 + (AB - 2)u^4 - (A + B)u^2 + 1 = 0.$$

Comparing coefficients we have $A + B = \dfrac{4}{r^2}$ and $AB = 4$. Multiplying the first equation by A we get $A^2 - \dfrac{4}{r^2}A + 4 = 0$. So using the quadratic formula

$$A = \frac{2\left(1 \pm \sqrt{1 - r^4}\right)}{r^2} \tag{16}$$

$$\text{and} \qquad B = \frac{2r^2}{1 \pm \sqrt{1 - r^4}}. \tag{17}$$

Thus given r, we have only to solve one of the equations

$$u^4 + Au^2 - 1 = 0 \qquad \text{and} \qquad u^4 + Bu^2 - 1 = 0$$

to find our solution for the bisection u. Since all this can be done using only a finite combination of addition, subtraction, multiplication, division and the taking of square roots, the geometric construction has been obtained. Finally from $x^2 + y^2 = u^2$ and (1) $x^2 - y^2 = u^4$ we get the point of bisection on the lemniscate

$$x = \sqrt{\frac{u^2 + u^4}{2}} \qquad \text{and} \qquad y = \sqrt{\frac{u^2 - u^4}{2}}. \tag{18}$$

8. *An example of bisection*

We will bisect the full arc in the first quadrant. So we take $r = 1$ in (16) to get $A = 2$. Thus we must find u from $u^4 + 2u^2 - 1 = 0$. The quadratic formula gives us $u = \sqrt{\sqrt{2} - 1}$. So from (18) for the point of bisection on the lemniscate we have

$$x = \sqrt{\frac{2 - \sqrt{2}}{2}} \approx 0.541196100 \qquad \text{and} \qquad y = \sqrt{\frac{3\sqrt{2} - 4}{2}} \approx 0.348310699.$$

9. *Trisection of the full arc*

Fagnano noticed that the transformation (5), with a trivial modification, can also be used to trisect the full arc of the lemniscate in the first quadrant. For this purpose we consider

$$\frac{dr}{\sqrt{1 - r^4}} = -2\frac{du}{\sqrt{1 - u^4}} \tag{19}$$

which is the same as (4) except for the minus sign. We can get (19) from (4) by simply replacing u by $-u$. Now (5) becomes

$$r = g_T(u) = \frac{-2u}{1 + u^4}\sqrt{1 - u^4}, \tag{20}$$

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but (15) is left unchanged.

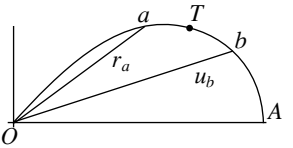


FIGURE 2: Trisecting the arc OA. Arc $Oa = 2$ Arc bA. Thus if points a and b coincide at T, then point T is a point of trisection of arc OA.

Comments on the integrals for arc length

By convention we consider arc length to be a positive quantity. So we expect when $x < 0$ that our integral $\int_{x=0}^{x_0} \frac{dt}{\sqrt{1-t^4}}$ gives us a positive value. However, if the upper limit is negative as in $\int_{x=0}^{-x_1} \frac{dt}{\sqrt{1-t^4}} = -\int_{x=0}^{x_1} \frac{dt}{\sqrt{1-t^4}}$ then we get negative values for the arc length. When this occurs, we will replace the negative value $\int_{x=0}^{-x_1} \frac{dt}{\sqrt{1-t^4}}$ by the positive value $\int_{x=0}^{x_1} \frac{dt}{\sqrt{1-t^4}}$.

Now consider Figure 2 where O, a, T, b and A are points on the lemniscate curve and the values r_a and u_b are the radii from the origin to the points a and b respectively. There are two values of u in (20) that make r vanish, $u = 0$ and $u = 1$. This time take $u = 1$ so that $g_T(1) = r_0 = 0$ and $g_T(u_b) = -r_a$, so we can integrate (19) as

$$\int_{r=0}^{-r_a} \frac{dr}{\sqrt{1-r^4}} = -2\int_{u_a}^{u_b} \frac{du}{\sqrt{1-u^4}}.$$

From the comment above on integrals we should change the left integral to

$$\int_{r=0}^{r_a} \frac{dr}{\sqrt{1-r^4}} = 2\int_{u_b}^{1} \frac{du}{\sqrt{1-u^4}}.$$

Now let point a coincide with point b and call it point T. We write $r_a = r_T = u_b = u_T$ and get

$$\int_0^{r_T} \frac{dr}{\sqrt{1-r^4}} = 2\int_{u_T}^{1} \frac{du}{\sqrt{1-u^4}} = \frac{2}{3} \text{ arc } OA.$$

To find the radius from O to T set $r = u$ in (15) to get $u^2 = \frac{4u^2(1-u^4)}{(1+u^4)^2}$ which simplifies to $u^8 + 6u^4 - 3 = 0$. Thus the radius $u_r = \sqrt[4]{2\sqrt{3}-3} \approx 0.825379$ locates a point of trisection. Using (18) this point has coordinates $x = \sqrt{\frac{1}{2}\left(\sqrt{2\sqrt{3}-3} + (2\sqrt{3}-3)\right)} \approx 0.75675$ and $y = \sqrt{\frac{1}{2}\left(\sqrt{2\sqrt{3}-3} - (2\sqrt{3}-3)\right)} \approx 0.32951$.

The other point of trisection can be obtained by bisecting arc OT. Using

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the method of section 6 we get from (16) $A = \dfrac{2\left(1 + \sqrt{4 - 2\sqrt{3}}\right)}{\sqrt{2\sqrt{3} - 3}}$. Then the root of $u^4 + Au^2 - 1 = 0$ that is of value to us is the radius $u = \sqrt{\dfrac{\sqrt{2\left(1 + \sqrt{4 - 2\sqrt{3}}\right)} - 1 - \sqrt{4 - 2\sqrt{3}}}{\sqrt{2\sqrt{3} - 3}}} \approx 0.435421$. The corresponding point on the lemniscate is numerically (0.33581, 0.27717) as found from (18). Thus the geometric construction of the trisection of the full arc of the lemniscate in the first quadrant has been shown.

10. *Final remarks*

Fagnano did more. He also showed that you could quinquisect the full arc of the lemniscate in the first quadrant by geometric construction! This is briefly discussed in [5, 6, 7].

The later story is fascinating. In 1801 Gauss published his seminal book [3]. In chapter 7 he proved that we can geometrically construct a regular N-gon if $N = 2^k P$, where P is a product of distinct Fermat primes $p_n = 2^{2^n} + 1$. The five known Fermat primes are 3, 5, 17, 257 and 65537. This means that you can geometrically divide the arc of the full circle into N equal segments. Gauss asserted that he had also proved the converse, and was preparing a treatise on this. He warned readers not to waste time trying to construct items like a regular 7 or 9-gon. This treatise never appeared. He also stated, without elaboration, that the same division was possible on the arc of some other curves, including the lemniscate. His diary [8], found 43 years after his death, shows that he was examining this topic as early as 1797 when he was 19 years old. Gauss never published anything on this topic after his 1801 book [3], possibly because of the bad review it received from the Paris Academy.

Some 25 years later, Abel, [9, 10, 11] inspired by Gauss's 1801 book, proved that the full lemniscate arc could be equally divided, by geometric construction, into the same N segments described above for the circle. The converse, that no other subdivisions are possible, was shown by Rosen [11] in 1981.

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Appendix on fundamental constructions

The sum and difference

Finding the sum and difference of two line lengths is immediate from an examination of Figure A1.

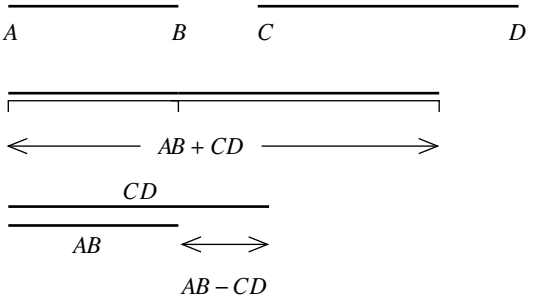


FIGURE A1: Constructing the sum and difference

The product

Finding the product of two line lengths is described in Figure A2.

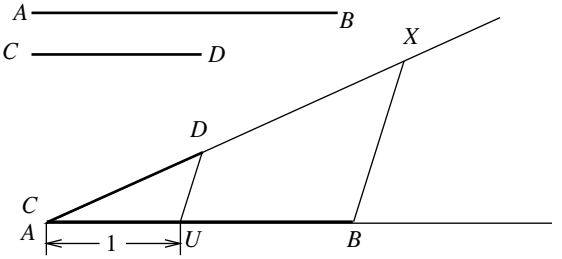


FIGURE A2: Constructing the product

First lay out the unit length AU and AB as shown on a horizontal line. With C coinciding with A lay out CD on a skew line making a convenient angle with the horizontal. Connect U and D. From B, construct a line parallel to UD meeting the skew line at X. Triangles AUD and ABX are similar. Thus $\frac{CD}{1} = \frac{AX}{AB}$ and thus AX is the product of AB and CD.

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The quotient

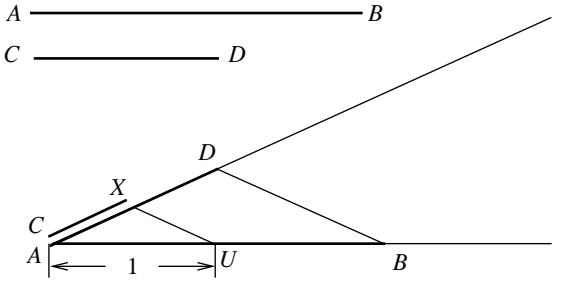


FIGURE A3: Constructing the quotient

As with the product, lay out AB and the unit AU on the horizontal and CD on a convenient skew line. Now draw BD. From U, construct a line parallel to BD meeting the skew line at X. Now triangles AUX and ABD are similar so $\frac{CX}{1} = \frac{CD}{AB}$ and the quotient is found.

The square root

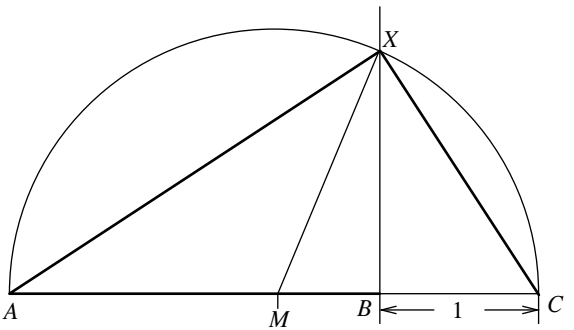


FIGURE A4: Constructing the square root

To find the square root of AB, first lay out AB immediately followed by the unit length forming AC. Bisect AC at M. With M as centre and radius MC, draw the semicircle AXC. Erect a perpendicular line to AC at B intersecting the semicircle at X.

Triangles ABX and BCX are similar. From this we have $\frac{BX}{1} = \frac{AB}{BX}$. Thus BX is the square root of AB.

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