Relations between integrals arising from Baillie's identity on trigonometric series

J. R. NURCOMBE

1. *Introduction*

Baillie's identity is $\sum_{n=1}^{\infty} \frac{\sin n}{n} = \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2} = \frac{\pi - 1}{2}$ (see [1] or [2], p. 240). Its integral analogue, $\int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$, is not difficult to prove (see Lemma 1, below). In this Article, we prove a generalisation of the latter result (see Theorem 1). Theorems 2 and 3 are extensions, involving in addition, powers of cosines in the integrand. Theorem 4 answers a question raised after the proof of Theorem 1, and Theorem 5 collects together the preceding results in the form of three identities between trigonometric integrals. Theorem 6 gives a further generalisation.

2. *Theorem* 1

$$\int_0^{\infty} \frac{\sin^{2n-1} x}{x} dx = \int_0^{\infty} \frac{\sin^{2n} x}{x^2} dx = \frac{\pi}{2}, \quad n = 1, 2, 3, \ldots .$$

For the proof, we require some lemmas.

Lemma 1: $\int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$.

Proof: Writing $x = 2u$ and integrating by parts gives

$$\begin{aligned}
\int_0^{\infty} \frac{\sin x}{x} dx &= \int_0^{\infty} \frac{\sin 2u}{u} du \\
&= \int_0^{\infty} \frac{2 \sin u \cos u}{u} du \\
&= 2 \int_0^{\infty} \frac{\sin u}{u} d(\sin u) \\
&= 2 \left[\frac{\sin^2 u}{u} \right]_0^{\infty} - 2 \int_0^{\infty} \sin u \, d\left(\frac{\sin u}{u} \right) \\
&= 0 - 2 \int_0^{\infty} \sin u \left(\frac{\cos u}{u} - \frac{\sin u}{u^2} \right) du \\
&= - \int_0^{\infty} \frac{\sin 2u}{u} du + 2 \int_0^{\infty} \frac{\sin^2 u}{u^2} du.
\end{aligned}$$

(That $\left[\frac{\sin^2 u}{u} \right]_0^{\infty} = 0$ follows from

$$\lim_{u \to \infty} \frac{\sin^2 u}{u} - \lim_{u \to 0} \frac{\sin^2 u}{u} = 0 - \lim_{u \to 0} \sin u \left(\frac{\sin u}{u} \right) = 0 \times 1 = 0).$$

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It follows that $\int_0^\infty \frac{\sin 2u}{u}du = \int_0^\infty \frac{\sin^2 u}{u^2}du$, and as $\int_0^\infty \frac{\sin u}{u}du = \int_0^\infty \frac{\sin 2u}{u}du$, the left-hand half of Lemma 1 is proved.

That $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$, is a well-known result (see e.g. [2], p. 238).

Lemma 2: $\int_0^\infty \frac{\sin^2 nt}{t \sin t}dt = \int_0^\infty \frac{\sin^2 nt}{t^2}dt = \frac{n\pi}{2}, n = 1, 2, 3, \ldots.$

Proof: Putting $x = nt$ in the right-hand integral of Lemma 1, gives $\int_0^\infty \frac{\sin^2 nt}{nt^2}dt = \frac{\pi}{2}$, so that

$$\int_0^\infty \frac{\sin^2 nt}{t^2}dt = \frac{n\pi}{2}, \quad n = 1, 2, 3, \ldots . \tag{1}$$

Next, using the trigonometric identities,

$$\sin(n + m)x - \sin(n - m)x = 2 \cos nx \sin mx,$$

$$\sin(n + m)x + \sin(n - m)x = 2 \sin nx \cos mx,$$

$$n, m = 1, 2, 3, \ldots,$$

we have

$$\begin{aligned}\sin^2 nx - \sin^2(n-1)x &= \big(\sin nx + \sin(n-1)x\big)\big(\sin nx - \sin(n-1)x\big)\\ &= \big(2 \sin \tfrac{1}{2}(2n-1)x \cos \tfrac{1}{2}x\big)\big(2 \cos \tfrac{1}{2}(2n-1)x \sin \tfrac{1}{2}x\big)\\ &= \sin(2n-1)x \sin x.\end{aligned}$$

Similarly

$$\sin^2(n-1)x - \sin^2(n-2)x = \sin(2n-3)x \sin x,$$

$$\ldots,$$

$$\sin^2 2x - \sin^2 x = \sin 3x \sin x,$$

and

$$\sin^2 x - 0 = \sin^2 x.$$

Adding these equations gives $\sin^2 nx = \sum_{r=1}^{n} \sin\big(2n - (2r-1)\big)x \sin x$, so that

$$\frac{\sin^2 nx}{x \sin x} = \sum_{r=1}^{n} \frac{\sin(2n - (2r-1))x}{x}. \tag{2}$$

Integrating both sides of (2) gives

$$\int_0^\infty \frac{\sin^2 nx}{x \sin x}dx = \int_0^\infty \sum_{r=1}^{n} \frac{\sin(2n - (2r-1))x}{x}dx$$

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$$= \sum_{r=1}^{n} \int_0^\infty \frac{\sin(2n - (2r-1))x}{x} dx.$$

By Lemma 1, each integral in the sum is $\frac{\pi}{2}$, so $\int_0^\infty \frac{\sin^2 nx}{x \sin x} dx = \frac{1}{2} n\pi$. Hence, by (1), the result follows.

Lemma 3: $\sin^2 nx$ can be expressed as a sum of even powers of $\sin x$, $n = 1, 2, 3, \ldots$.

Proof: By induction, for $n = 1$, $\sin^2 x = \sin^2 x$, and for $n = 2$, $\sin^2 2x = 4 \sin^2 x \cos^2 x = 4 \sin^2 x(1 - \sin^2 x) = 4 \sin^2 x - 4 \sin^4 x$. Now suppose $\sin^2 nx$ can be so expressed. We prove the same for $\sin^2 (n+1)x$. Thus

$$\sin^2 (n+1)x = (\sin nx \cos x + \cos nx \sin x)^2$$

$$= \sin^2 nx \cos^2 x + 2 \sin nx \cos x \cos nx \sin x + \cos^2 nx \sin^2 x.$$

Now $\sin^2 nx \cos^2 x = \sin^2 nx(1 - \sin^2 x)$, which is a series of even powers of $\sin x$ by assumption. Similarly, $\cos^2 nx \sin^2 x = (1 - \sin^2 nx) \sin^2 x$ is a series of even powers of $\sin x$. Finally, to prove $\frac{1}{2} \sin 2x \sin 2nx$ can be expressed as a series of even powers of $\sin x$, we again use induction. If $n = 1$, $\frac{1}{2} \sin^2 2x$ satisfies the condition as we have seen. Suppose it holds for $\sin 2x \sin 2nx$. Then

$$\sin 2(n+1)x \sin 2x = (\sin 2nx \cos 2x + \cos 2nx \sin 2x) \sin 2x.$$

Now

$$\sin 2nx \cos 2x \sin 2x = \sin 2nx \sin 2x (1 - 2 \sin^2 x)$$

satisfies the condition be assumption. Finally,

$$\cos 2nx \sin^2 2x = (1 - 2 \sin^2 nx) \sin^2 2x,$$

satisfies the condition by our initial inductive hypothesis.

Remark: It is not difficult to show that the highest occurring power of $\sin x$ in the expansion of $\sin^2 nx$ is $\sin^{2n} x$.

Proof of Theorem 1: By Lemma 3, write

$$\sin^2 nx = a_1 \sin^2 x + a_2 \sin^4 x + \ldots + a_n \sin^{2n} x,$$

where $a_1, a_2, \ldots, a_n$ are integers. Integrating both sides of this equation divided by $x \sin x$, and using Lemma 2, gives

$$\int_0^\infty \frac{\sin^2 nx}{x \sin x} dx = \int_0^\infty \frac{a_1 \sin^2 x + a_2 \sin^4 x + \ldots + a_n \sin^{2n} x}{x \sin x} dx$$

$$\int_0^\infty \frac{\sin^2 nx}{x^2} dx = \int_0^\infty \frac{a_1 \sin^2 x + a_2 \sin^4 x + \ldots + a_n \sin^{2n} x}{x^2} dx = \frac{n\pi}{2}.$$

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Term by term integration then gives

$$\int_0^\infty \frac{a_1 \sin^2 x}{x \sin x}dx + \int_0^\infty \frac{a_2 \sin^4 x}{x \sin x}dx + \ \dots + \int_0^\infty \frac{a_n \sin^{2n} x}{x \sin x}dx$$

$$= \int_0^\infty \frac{a_1 \sin^2 x}{x^2}dx + \int_0^\infty \frac{a_2 \sin^4 x}{x^2}dx + \ \dots + \int_0^\infty \frac{a_n \sin^{2n} x}{x^2}dx.$$

The proof is now completed by induction. If $n = 1$, the first terms on each side of the last equation are equal by Lemma 1 and equal $\frac{1}{2}\pi a_1$. It the sums are equal for the first $(n - 1)$ terms on each side of the equation, then they will also be equal for the first n terms of that equation. Hence the nth terms will be equal, that is

$$\int_0^\infty \frac{a_n \sin^{2n} x}{x \sin x}dx = \int_0^\infty \frac{a_n \sin^{2n} x}{x^2}dx \text{ or } \int_0^\infty \frac{\sin^{2n-1} x}{x}dx = \int_0^\infty \frac{\sin^{2n} x}{x^2}dx.$$

This completes the proof of Theorem 1.

Examples

Substituting $n = 2$ in Theorem 1 gives $\int_0^\infty \frac{\sin^3 x}{x}dx = \int_0^\infty \frac{\sin^4 x}{x^2}dx$. One way of evaluating these integrals is to use the trigonometric identity

$$\sin 3x = 3 \sin x - 4 \sin^3 x,$$

so that

$$\int_0^\infty \frac{\sin^3 x}{x}dx = \int_0^\infty \frac{3}{4}\frac{\sin x}{x}dx - \int_0^\infty \frac{1}{4}\frac{\sin 3x}{x}dx = \left(\frac{3}{4} - \frac{1}{4}\right)\frac{\pi}{2} = \frac{\pi}{4}.$$

Again, $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$, so that

$$\int_0^\infty \frac{\sin^5 x}{x}dx = \int_0^\infty \frac{\sin^6 x}{x^2}dx$$

$$= \int_0^\infty \frac{\sin 5x}{16x}dx - \int_0^\infty \frac{5 \sin x}{16x}dx + \int_0^\infty \frac{20 \sin^3 x}{16x}dx$$

$$= \frac{1}{16}\left(\frac{\pi}{2} - \frac{5\pi}{2} + \frac{20\pi}{4}\right) = \frac{3\pi}{16}.$$

Using the same method, the reader may verify that $\int_0^\infty \frac{\sin^7 x}{x}dx = \frac{5\pi}{32}$.

However, these numerical values do not suggest any general formula, a defect which will be rectified in Theorem 4 when we have established some further results.

Theorem 1 has the following corollary:

Corollary 1: $\int_0^\infty \frac{\cos^2 x \sin^{2n-1} x}{x}dx = \int_0^\infty \frac{\cos^2 x \sin^{2n} x}{x^2}dx, \ n = 1, 2, 3, \dots .$

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Proof:

$$\int_0^\infty \frac{\cos^2 x \sin^{2n-1} x}{x} dx = \int_0^\infty \frac{(1-\sin^2 x)\sin^{2n-1} x}{x} dx$$

$$= \int_0^\infty \frac{\sin^{2n-1} x}{x} dx - \int_0^\infty \frac{\sin^{2n+1} x}{x} dx. \tag{3}$$

$$\text{and } \int_0^\infty \frac{\cos^2 x \sin^{2n} x}{x^2} dx = \int_0^\infty \frac{(1-\sin^2 x)\sin^{2n} x}{x^2} dx$$

$$= \int_0^\infty \frac{\sin^{2n} x}{x^2} dx - \int_0^\infty \frac{\sin^{2n+2} x}{x^2} dx. \tag{4}$$

Now the corresponding terms on the right-hand sides of (3) and (4) are equal by Theorem 1. Hence the result,

3. *Theorem* 2

Corollary 1 introduced $\cos^2 x$ into the integrand of Theorem 1. We now consider the more difficult question of introducing $\cos x$.

Theorem 2: $\int_0^\infty \frac{\cos x \sin^{2n-1} x}{x} dx = \int_0^\infty \frac{\cos x \sin^{2n} x}{x^2} dx, n = 1, 2, 3, \ldots .$

For the proof we need some Lemmas.

Lemma 4: $\int_0^\infty \frac{\cos x \sin nx}{x} dx = \frac{\pi}{2}$, $n = 2, 3, 4, \ldots$ and equals $\frac{\pi}{4}$ when $n = 1$.

Proof: We have

$$\cos x \sin nx = \cos x \sin\big((n-1)+1\big)x$$

$$= \cos x \sin(n-1)x \cos x + \cos x \cos(n-1)x \sin x$$

$$= \cos^2 x \sin(n-1)x + \cos x \sin x \cos(n-1)x$$

$$= \frac{1}{2}(1+\cos 2x)\sin(n-1)x + \frac{1}{2}\sin 2x \cos(n-1)x$$

$$= \frac{1}{2}\sin(n-1)x + \frac{1}{2}\sin(n+1)x.$$

Then $\int_0^\infty \frac{\cos x \sin nx}{x} dx = \int_0^\infty \frac{\sin(n-1)x}{2x} dx + \int_0^\infty \frac{\sin(n+1)x}{2x} dx = \frac{\pi}{2}$ when $n \neq 1$, and $\frac{\pi}{4}$ when $n = 1$, by Lemma 1.

Lemma 5: $\int_0^\infty \frac{\cos x \sin^2 nx}{x \sin x} dx = \frac{n\pi}{2} - \frac{\pi}{4}, \ n = 1, 2, 3, \ldots .$

Proof: Multiplying both sides of (2) by $\cos x$, we obtain

$$\frac{\cos x \sin^2 nx}{x \sin x} = \sum_{r=1}^{n} \frac{\cos x \sin(2n-(2r-1))x}{x}.$$

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Integrating both sides of this equation, we have

$$\int_0^\infty \frac{\cos x \sin^2 nx}{x \sin x} dx = \int_0^\infty \sum_{r=1}^{n} \frac{\cos x \sin (2n-(2r-1))x}{x} dx = (n-1)\frac{\pi}{2} + \frac{\pi}{4},$$

by Lemma 4.

Lemma 6: $\int_0^\infty \frac{\sin^2 x \sin^2 nx}{x^2} dx = \frac{\pi}{4}, n = 1, 2, 3, \ldots.$

Proof: Integrating the left-hand side of Lemma 4 by parts, gives

$$\begin{aligned}\int_0^\infty \frac{\cos x \sin nx}{x} dx &= \int_0^\infty \frac{\sin nx}{x} d(\sin x) \\ &= \left[\frac{\sin nx \sin x}{x}\right]_0^\infty - \int_0^\infty \sin x \, d\left(\frac{\sin nx}{x}\right) \\ &= -\int_0^\infty \sin x \left[\frac{n \cos nx}{x} - \frac{\sin nx}{x^2}\right] dx \\ &= -n\int_0^\infty \frac{\cos nx \sin x}{x} dx + \int_0^\infty \frac{\sin nx \sin x}{x^2} dx = \frac{\pi}{2}, \\ & \qquad n = 2, 3, 4, \ldots .\end{aligned}$$

Now $\int_0^\infty \frac{\cos nx \sin x}{x} dx = 0, n = 2, 3, 4, \ldots$, since

$$\int_0^\infty \frac{\sin nx \cos x + \cos nx \sin x}{x} dx = \int_0^\infty \frac{\sin (n+1)x}{x} dx = \frac{\pi}{2},$$

by Lemma 1. Hence

$$\int_0^\infty \frac{\sin nx \sin x}{x^2} dx = \frac{\pi}{2}, \tag{5}$$

for $n = 2, 3, 4, \ldots$, and this clearly holds for $n = 1$, again by Lemma 1.

Next put $x = 2y$ in (5), so that

$$\int_0^\infty \frac{\sin 2y \sin 2ny}{y^2} dy = \pi, \qquad n = 1, 2, 3, \ldots .$$

Now $\int_0^\infty \frac{\sin^2 (n+1)x}{x^2} dx = \int_0^\infty \left(\frac{\sin nx \cos x + \cos nx \sin x}{x}\right)^2 dx = (n+1)\frac{\pi}{2}$, by Lemma 2.

Squaring out the bracket and expressing the cosine functions in terms of the equivalent sine functions, after some algebra, we find that the last integral equals

$$\int_0^\infty \left(\frac{\sin nx}{x}\right)^2 dx - 2\int_0^\infty \left(\frac{\sin nx \sin x}{x}\right)^2 dx + \frac{1}{2}\int_0^\infty \frac{\sin 2nx \sin 2x}{x^2} dx + \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$$

$$= n\frac{\pi}{2} - 2\int_0^\infty \left(\frac{\sin nx \sin x}{x}\right)^2 dx + \frac{\pi}{2} + \frac{\pi}{2}, \text{ by (5) and Lemma 1.}$$

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As the last expression is equal to $(n + 1)\frac{\pi}{2}$, it follows that

$$\int_0^\infty \frac{\sin^2 x \ \sin^2 nx}{x^2} dx = \frac{\pi}{4}, \text{ as required.}$$

Lemma 7: $\int_0^\infty \frac{\cos x \ \sin^2 nx}{x^2} dx = n\frac{\pi}{2} - \frac{\pi}{4}, \ n = 1, 2, 3, \ldots .$

Proof: Put $x = 2y$, so that

$$\begin{aligned}\int_0^\infty \frac{\cos x \ \sin^2 nx}{x^2} dx &= \int_0^\infty \frac{\cos 2y \ \sin^2 2ny}{2y^2} dy \\ &= \int_0^\infty \frac{(1 - 2\sin^2 y)\sin^2 2ny}{2y^2} dy \\ &= \int_0^\infty \frac{\sin^2 2ny}{2y^2} dy - \int_0^\infty \left(\frac{\sin y \sin 2ny}{y}\right)^2 dy \\ &= n\frac{\pi}{2} - \frac{\pi}{4},\end{aligned}$$

by (1) and Lemma 6.

Proof of Theorem 2: From Lemmas 5 and 7,

$$\int_0^\infty \frac{\cos x \ \sin^2 nx}{x \sin x} dx = \int_0^\infty \frac{\cos x \ \sin^2 nx}{x^2} dx = n\frac{\pi}{2} - \frac{\pi}{4}, \ n = 1, 2, 3, \ldots .$$

Now using Lemma 3, Theorem 2 is proved in an analogous manner to the proof of Theorem 1.

4. *Theorem* 3:

$$\int_0^\infty \frac{\cos x \sin^{2n-1} x}{x} dx = \int_0^\infty \frac{\cos^2 x \sin^{2n} x}{x^2} dx, \ n = 1, 2, 3, \ldots .$$

We need a Lemma.

Lemma 8: $\int_0^\infty \frac{\cos x \sin^2 nx}{x \sin x} dx = n\frac{\pi}{2} - \frac{\pi}{4} = \int_0^\infty \frac{\cos^2 x \sin^2 nx}{x^2} dx, \ n = 1, 2, 3, \ldots .$

Proof: The left-hand equality is Lemma 5. Also

$$\begin{aligned}\int_0^\infty \left(\frac{\cos x \ \sin nx}{x}\right)^2 dx &= \int_0^\infty \frac{(1 - \sin^2 x)\sin^2 nx}{x^2} dx \\ &= \int_0^\infty \left(\frac{\sin nx}{x}\right)^2 dx - \int_0^\infty \left(\frac{\sin x \ \sin nx}{x}\right)^2 dx \\ &= n\frac{\pi}{2} - \frac{\pi}{4}, \text{ by Lemmas 2 and 6, respectively.}\end{aligned}$$

Proof of Theorem 3: Using Lemmas 8 and 3, the conclusion follows in an analogous manner to the proof of Theorem 1.

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5. *Theorem* 4

Earlier, we noted that the numerical examples illustrating Theorem 1 seemed to follow no obvious pattern leading to a general formula. Theorem 4 provides a solution to this problem.

Theorem 4: $\int_0^\infty \frac{\sin^{2n+2}x}{x^2}dx = \frac{\pi}{2}\frac{(2n)!}{2^{2n}(n!)^2} = \int_0^{\pi/2} \sin^{2n}x\,dx,\ n = 0,\ 1,\ 2, \ldots\ .$

For the proof, we need a Lemma.

Lemma 9: $\int_0^\infty \frac{\cos x \sin^{2n-1}x}{x}dx = \frac{1}{2n}\int_0^\infty \frac{\sin^{2n}x}{x^2}dx,\ n = 1,\ 2,\ 3, \ldots\ .$

Proof: Integrating by parts,

$$\begin{aligned}\int_0^\infty \frac{\cos x \sin^{2n-1}x}{x}dx &= \int_0^\infty \frac{\sin^{2n-1}x}{x}d(\sin x)\\ &= \left[\frac{\sin^{2n}x}{x}\right]_0^\infty - \int_0^\infty \sin x\, d\left(\frac{\sin^{2n-1}x}{x}\right)\\ &= -\int_0^\infty \sin x\left[\frac{(2n-1)\sin^{2n-2}x\cos x}{x} - \frac{\sin^{2n-1}x}{x^2}\right]dx\\ &= -(2n-1)\int_0^\infty \frac{\cos x\,\sin^{2n-1}x}{x}\,dx + \int_0^\infty \frac{\sin^{2n}x}{x^2}\,dx.\end{aligned}$$

so that

$$2n\int_0^\infty \frac{\cos x\,\sin^{2n-1}x}{x}\,dx = \int_0^\infty \frac{\sin^{2n}x}{x^2}\,dx,$$

giving the result.

Proof of Theorem 4: By Lemma 9 and Theorem 3,

$$\frac{1}{2n}\int_0^\infty \frac{\sin^{2n}x}{x^2}\,dx = \int_0^\infty \frac{\cos x\,\sin^{2n-1}x}{x}dx = \int_0^\infty \frac{\cos^2 x\,\sin^{2n}x}{x^2}dx.$$

Hence

$$\begin{aligned}\frac{1}{2n}\int_0^\infty \frac{\sin^{2n}x}{x^2}dx &= \int_0^\infty \frac{(1-\sin^2 x)\sin^{2n}x}{x^2}dx\\ &= \int_0^\infty \frac{\sin^{2n}x}{x^2}dx - \int_0^\infty \frac{\sin^{2n+2}x}{x^2}dx,\end{aligned}$$

so that $\int_0^\infty \frac{\sin^{2n+2}x}{x^2}dx = \frac{2n-1}{2n}\int_0^\infty \frac{\sin^{2n}x}{x^2}dx$. It follows that

$$\int_0^\infty \frac{\sin^{2n}x}{x^2}dx = \frac{2n-3}{2n-2}\int_0^\infty \frac{\sin^{2n-2}x}{x^2}dx,$$

and continuing the iteration gives

$$\int_0^\infty \frac{\sin^{2n+2}x}{x^2}dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\ldots\cdot\frac{1}{2}\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{(2n)!}{2^{2n}(n!)^2}\frac{\pi}{2},$$

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using Lemma 1. That these last expressions equal $\int_0^{\pi/2} \sin^{2n} x\, dx$, follow by iteration of the reduction formula

$$I_n = \frac{n-1}{n} I_{n-2}, \quad n = 2, 3, \ldots,$$

where $I_n = \int_0^{\pi/2} \sin^n x\, dx$, $n = 0, 1, 2, \ldots$, (see for example [2], p. 62 for details). This completes the proof of Theorem 4.

Remark: An alternative proof of Theorems 1 and 4 can be obtained from two exercises in [3] (p. 315, nos. 10 and 11). Bromwich gives hints for solution and attributes the results to Joseph Wolstenholme (1829 – 1891).

6. *Summary of results*

In this section, we summarise the results of the preceding sections in Theorem 5 below.

Theorem 5

The following four integrals are equal to one another with the value indicated.

$$\left.\begin{array}{l} \displaystyle\int_0^\infty \frac{\cos x \sin^{2n-1} x}{x}\, dx \\ \displaystyle\int_0^\infty \frac{\cos x \sin^{2n} x}{x^2}\, dx \end{array}\right\} = \left\{\begin{array}{l} \displaystyle\int_0^\infty \frac{\cos^2 x \sin^{2n} x}{x^2} dx \\ \displaystyle\int_0^\infty \frac{\cos^2 x \sin^{2n-1} x}{x}\, dx \end{array}\right\} = \frac{\pi}{2^{2n}} \frac{(2n-2)!}{(n-1)!\, n!}, \; n = 1,2,3, \ldots.$$

Proof: The equality of the integrals follows from Corollary 1 and Theorems 2 and 3. As to their value, we have from Lemma 9 and Theorem 4,

$$\int_0^\infty \frac{\cos x \, \sin^{2n-1} x}{x}\, dx = \frac{1}{2n} \int_0^\infty \frac{\sin^{2n} x}{x^2}\, dx$$
$$= \frac{1}{2n} \frac{\pi}{2} \frac{(2n-2)!}{2^{2n-2}((n-1)!)^2} = \frac{\pi}{2^{2n}} \frac{(2n-2)!}{(n-1)!\, n!}.$$

7. *Theorem* 6

Finally, we generalise Theorem 5.

Theorem 6

The following four integrals are equal to each other.

$$\left.\begin{array}{l} \displaystyle\int_0^\infty \frac{\cos^{2m-1} x \sin^{2n-1} x}{x}\, dx \\ \displaystyle\int_0^\infty \frac{\cos^{2m-1} x \sin^{2n} x}{x^2}\, dx \end{array}\right\} = \left\{\begin{array}{ll} \displaystyle\int_0^\infty \frac{\cos^{2m} x \sin^{2n} x}{x^2}\, dx & m, n = 1, 2, 3, \ldots. \\ \displaystyle\int_0^\infty \frac{\cos^{2m} x \sin^{2n-1} x}{x} dx & m = \tfrac{1}{2}, \text{on LHS} \\ & \text{and 0 on RHS.} \end{array}\right.$$

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Proof: The equations follow from Corollary 1 and Theorems 2 and 3 by, in each case, repeatedly increasing the power of $\cos x$ by 2. For example, by starting with Theorem 3 and a given n,

$$\int_0^\infty \frac{\cos x \sin^{2n-1} x}{x}\,dx = \int_0^\infty \frac{\cos^2 x \sin^{2n} x}{x^2}dx,\ n = 1,\ 2,\ 3,\ \ldots,\text{we also have}$$

$$\int_0^\infty \frac{\cos x \sin^{2n+1} x}{x}\,dx = \int_0^\infty \frac{\cos^2 x \sin^{2n+2} x}{x^2}\,dx.$$

Subtracting these equations gives

$$\int_0^\infty \frac{\cos x \sin^{2n-1} x\,(1-\sin^2 x)}{x}dx = \int_0^\infty \frac{\cos^2 x \sin^{2n} x(1-\sin^2 x)}{x^2}\,dx, \text{or}$$

$$\int_0^\infty \frac{\cos^3 x \sin^{2n-1} x}{x}\,dx = \int_0^\infty \frac{\cos^4 x \sin^{2n} x}{x^2}\,dx.$$ Applying this argument $(m-1)$ times gives

$$\int_0^\infty \frac{\cos^{2m-1} x \sin^{2n-1} x}{x}dx = \int_0^\infty \frac{\cos^{2m} x \sin^{2n} x}{x^2}dx, \qquad m = 1,\ 2,\ 3,\ \ldots,$$

and similarly for Corollary 1 and Theorem 2. This completes the proof. The cases $m = \frac{1}{2}$ on the left-hand side and $m = 0$ on the right, are Theorem 1.

Remark

We shall not prove the value of these integrals, but the interested reader may approve the formula

$$\int_0^\infty \frac{\cos^{2m-1} x\ \sin^{2n} x}{x^2}\,dx = \frac{\pi}{2^{2(n+m-1)}}\frac{(2n-2)!\,(2m-1)!}{(n-1)!\,(m-1)!\,(n+m-1)!},$$

$$m,\ n = 1,\ 2,\ 3,\ \ldots.$$

8. *Context and conclusion*

As implied in the introduction, the inspiration for the above work was Baillie's identity on trigonometric series. Attempts to extend it quickly foundered on the realisation that the analogous integrals were easier to handle and gave more satisfactory results than the corresponding series. The purpose of this section is to give some context to the work and references to related results.

One of the main objectives of [4] is to establish equality (save for the difference of a (rational) constant) between trigonometric sums and their analogous integrals, and effect their evaluation (Baillie's identity and Lemma 1 provide simple examples). Pivotal to this task is the sinc function, (sinus cardinalis), defined as $\operatorname{sinc} x = \dfrac{\sin x}{x}$, $x \in \mathbb{R}$, $x \neq 0$, and $\operatorname{sinc}(0) = 1$. As the authors note, their results work best for powers and other relatives of the sinc function. and in the remarks before their Example 5, they note that introducing cosines into the sums seems to produce easily calculable results

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only in simple cases. Accordingly, it appears unlikely that sums analogous to, and as elegant as, the integrals of our Theorems 5 and 6, can be found.

In [5], the authors sketch a brief history of the evaluation of $\int_0^\infty (\operatorname{sinc} x)^n\,dx$, $(n \in \mathbb{N})$, and related integrals (see the extensive references in [5]). Their objective is to extend the method used in a simple proof of Hardy for evaluating $\int_0^\infty \operatorname{sinc} x\,dx$, to the evaluation of $\int_0^\infty (\operatorname{sinc} x)^n\,dx$, and more generally, of the Fourier transform $\int_0^\infty \frac{\sin^n x}{x^{n-m}}\,e^{i\xi x}\,dx$, $(\xi \in \mathbb{R})$, for integer m and n with $0 \leqslant m < n$. With $\xi = 1$ or 2, and suitable choice of m and n, the reader will immediately recognise the integrands as one of the integrands of Theorem 5. However, the formulae obtained involving B-splines (basis splines), while general and elegant, are complicated to the extent that this does not appear to be a practicable method of discovering the relations between, or evaluating, the integrals of Theorem 5, except in the simplest cases. This is the familiar situation of special cases often giving simpler results than the more general cases.

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J. R. NURCOMBE
8 Chestnut Close,
Saltburn-by-the-Sea,
Cleveland TS12 1PE
e-mail: *nurc@hotmail.co.uk*

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