A PUNCH PROBLEM FOR A TRANSVERSELY ISOTROPIC LAYER

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1. *Introduction.* In this note we consider the axially symmetric indentation of a transversely isotropic layer resting on a rigid foundation. The corresponding half-space problems both for isotropic and transversely isotropic materials have been solved previously (see (1), (2), (3), p. 455) and, more recently, the present problem has been solved for isotropic materials in (4) and (5) using methods different from that given in this paper. In § 2 the problem is reduced to the solution of a Fredholm integral equation of the second kind. In § 3 and subsequently the case of incomplete penetration by the punch, when the radius of the circle of contact is less than that of the end face of the punch, is considered. Finally, in § 4 the results are illustrated by considering a hemispherically ended punch.

2. *Basic equations, reduction to a Fredholm integral equation.* We consider a transversely isotropic, elastic layer $-d \leqslant z \leqslant 0$ of infinite extent resting on a frictionless, rigid foundation at $z = -d$. We suppose the layer is deformed infinitesimally by a rigid, frictionless axially symmetric punch acting along its axis of symmetry, the z-axis, and in contact with the layer over the region $0 \leqslant r \leqslant a$, using cylindrical polar co-ordinates (r, θ, z).

Then, using w to denote the component of the infinitesimal displacement in the z-direction and σ_{zz}, σ_{rz} the normal and shearing stresses in that direction, the boundary conditions on the layer are,

$$\left.\begin{aligned} w &= f(r), & z &= 0 & & (0 \leqslant r \leqslant a), \\ \sigma_{zz} &= 0, & z &= 0 & & (r > a), \\ \sigma_{rz} &= 0, & z &= 0 & & (0 \leqslant r < \infty), \\ w &= 0, & \sigma_{rz} &= 0, & z &= -d \quad (0 \leqslant r < \infty), \end{aligned}\right\} \tag{1}$$

where $w = f(r)$ is the equation of the end surface of the punch plus a constant to allow for the depth of penetration.

In Green and Zerna ((1), (5·12·11), p. 182) potential function solutions to the equations of equilibrium for a transversely isotropic body are given, and we quote the relevant terms here:

$$\left.\begin{aligned} w &= k_1 \frac{\partial \chi_1}{\partial z} + k_2 \frac{\partial \chi_2}{\partial z}, \\ \sigma_{rz} &= c_{44}\left[(1+k_1)\frac{\partial^2 \chi_1}{\partial r\, \partial z} + (1+k_2)\frac{\partial^2 \chi_2}{\partial r\, \partial z}\right], \\ \sigma_{zz} &= (c_{33}k_1 - c_{31}\nu_1)\frac{\partial^2 \chi_1}{\partial z^2} + (c_{33}k_2 - c_{31}\nu_2)\frac{\partial^2 \chi_2}{\partial z^2}, \end{aligned}\right\} \tag{2}$$

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where
$$\nabla_1^2\chi_i+\nu_i\frac{\partial^2\chi_i}{\partial z^2}=0 \quad (i=1,2), \quad \nabla_1^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{3}$$

and ν_i, k_i are constants related to the elastic constants c_{ij} of the material.

Now, using (2), the boundary conditions become

$$\left.\begin{aligned}
(1+k_1)\frac{\partial\chi_1}{\partial z}+(1+k_2)\frac{\partial\chi_2}{\partial z}&=0, && z=0 \quad (0\leqslant r<\infty),\\
k_1\frac{\partial\chi_1}{\partial z}+k_2\frac{\partial\chi_2}{\partial z}&=f(r), && z=0 \quad (0\leqslant r\leqslant a),\\
(c_{33}k_1-c_{31}\nu_1)\frac{\partial^2\chi_1}{\partial z^2}+(c_{33}k_2-c_{31}\nu_2)\frac{\partial^2\chi_2}{\partial z^2}&=0, && z=0 \quad (r>a),\\
(1+k_1)\frac{\partial\chi_1}{\partial z}+(1+k_2)\frac{\partial\chi_2}{\partial z}&=0, && z=-d \quad (0\leqslant r<\infty),\\
k_1\frac{\partial\chi_1}{\partial z}+k_2\frac{\partial\chi_2}{\partial z}&=0, && z=-d \quad (0\leqslant r<\infty).
\end{aligned}\right\} \tag{4}$$

To find a solution of the corresponding half-space problem ((1), §5·8, p. 172), i.e. $d=\infty$, we put

$$\chi_1(r,z)=\frac{\nu_1^{\frac{1}{2}}}{1+k_1}\phi\left(r,\frac{z}{\nu_1^{\frac{1}{2}}}\right), \quad \chi_2(r,z)=-\frac{\nu_2^{\frac{1}{2}}}{1+k_2}\phi\left(r,\frac{z}{\nu_2^{\frac{1}{2}}}\right), \tag{5}$$

and
$$\frac{\partial\phi}{\partial z}(r,z)=\frac{1}{2}\int_{-a}^{a}\frac{g(t)\,dt}{\{r^2+(z+it)^2\}^{\frac{1}{2}}}, \tag{6}$$

which satisfies (3) and the required conditions at infinity. Then

$$\left.\begin{aligned}
\frac{\partial\phi}{\partial z}&=\int_0^r\frac{g(t)\,dt}{(r^2-t^2)^{\frac{1}{2}}}, \quad \text{on} \quad z=0 \quad (0\leqslant r\leqslant a),\\
\frac{\partial^2\phi}{\partial z^2}&=0, \quad \text{on} \quad z=0 \quad (r>a),
\end{aligned}\right\} \tag{7}$$

and the boundary conditions reduce to the integral equation

$$\left(\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right)\int_0^r\frac{g(t)\,dt}{(r^2-t^2)^{\frac{1}{2}}}=f(r),$$

which may be inverted to complete the solution.

For a layer of finite thickness, from (3), we may write $\chi_i(r,z)$ as a Hankel transform, namely

$$\chi_i(r,z)=\int_0^\infty\xi\left[A_i(\xi)\cosh\left(\frac{\xi z}{\nu_i^{\frac{1}{2}}}\right)+B_i(\xi)\sinh\left(\frac{\xi z}{\nu_i^{\frac{1}{2}}}\right)\right]J_0(\xi r)\,d\xi, \tag{8}$$

and by combining the two forms (5) and (6), and (8) we can find a solution to the problem as specified by (4). This method of solution was suggested by Collins (6) and is the one we adopt, reducing the boundary conditions directly to a Fredholm integral equation of the second kind. Alternatively, we could start from (8), reducing the boundary conditions to a dual integral equation which can then be transformed into the Fredholm integral equation by the usual methods.

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Writing

$$\left.\begin{aligned}\chi_1(r,z) &= \frac{\nu_1^{\frac{1}{2}}}{1+k_1}\phi\left(r,\frac{z}{\nu_1^{\frac{1}{2}}}\right)+\int_0^\infty \xi\left[A_1(\xi)\cosh\left(\frac{\xi z}{\nu_1^{\frac{1}{2}}}\right)+B_1(\xi)\sinh\left(\frac{\xi z}{\nu_1^{\frac{1}{2}}}\right)\right]\mathrm{J}_0(\xi r)\,d\xi,\\ \chi_2(r,z) &= -\frac{\nu_2^{\frac{1}{2}}}{1+k_2}\phi\left(r,\frac{z}{\nu_2^{\frac{1}{2}}}\right)+\int_0^\infty \xi\left[A_2(\xi)\cosh\left(\frac{\xi z}{\nu_2^{\frac{1}{2}}}\right)+B_2(\xi)\sinh\left(\frac{\xi z}{\nu_2^{\frac{1}{2}}}\right)\right]\mathrm{J}_0(\xi r)\,d\xi,\end{aligned}\right\} \tag{9}$$

where ϕ is given by (6), and substituting in the boundary conditions (4), we find

$$\left.\begin{aligned}B_1 &= \frac{\nu_1^{\frac{1}{2}}B}{1+k_1}, & B_2 &= -\frac{\nu_2^{\frac{1}{2}}B}{1+k_2},\\ A_1 &= \frac{\nu_1 A}{c_{33}k_1-c_{31}\nu_1}, & A_2 &= -\frac{\nu_2 A}{c_{33}k_2-c_{31}\nu_2},\end{aligned}\right\} \tag{10}$$

and

$$\left.\begin{aligned}B &= \frac{\gamma_1(1-\coth_1)-\gamma_2(1-\coth_2)}{\gamma_1\coth_1-\gamma_2\coth_2}\,\frac{1}{\xi^2}\int_0^a g(t)\cos(\xi t)\,dt,\\ A &= \frac{-\gamma_1\gamma_2(\coth_1-\coth_2)}{\gamma_1\coth_1-\gamma_2\coth_2}\,\frac{1}{\xi^2}\int_0^a g(t)\cos(\xi t)\,dt,\end{aligned}\right\} \tag{11}$$

where

$$\left.\begin{aligned}\gamma_1 &= \frac{(c_{33}k_1-c_{31}\nu_1)}{(1+k_1)\,\nu_1^{\frac{1}{2}}}, & \gamma_2 &= \frac{(c_{33}k_2-c_{31}\nu_2)}{(1+k_2)\,\nu_2^{\frac{1}{2}}},\\ \coth_1 &= \coth\left(\frac{\xi d}{\nu_1^{\frac{1}{2}}}\right), & \coth_2 &= \coth\left(\frac{\xi d}{\nu_2^{\frac{1}{2}}}\right).\end{aligned}\right\} \tag{12}$$

Equations (11) have been obtained by substituting equations (9) and (10) in the boundary conditions on $z=-d$ in (4), and then inverting the Hankel transforms and solving the resulting simultaneous equations for A and B.

The remaining boundary condition becomes

$$\left(\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right)\int_0^r \frac{g(t)\,dt}{(r^2-t^2)^{\frac{1}{2}}}+\int_0^\infty \xi^2\left[\frac{k_1}{\nu_1^{\frac{1}{2}}}B_1(\xi)+\frac{k_2}{\nu_2^{\frac{1}{2}}}B_2(\xi)\right]\mathrm{J}_0(\xi r)\,d\xi = f(r) \quad (0\leqslant r\leqslant a), \tag{13}$$

which, on using the integral representation

$$\mathrm{J}_0(\xi r)=\frac{2}{\pi}\int_0^r \frac{\cos(\xi t)\,dt}{(r^2-t^2)^{\frac{1}{2}}} \tag{14}$$

and (10), (11), can be inverted to give

$$g(t)+\int_0^a g(s)\,K(s,t)\,ds=\left[\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right]^{-1}\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{rf(r)\,dr}{(t^2-r^2)^{\frac{1}{2}}}, \tag{15}$$

a Fredholm integral equation of the second kind for $g(t)$, where

$$K(s,t)=\frac{2}{\pi}\int_0^\infty H(\xi)\cos(s\xi)\cos(t\xi)\,d\xi, \tag{16}$$

and

$$H(\xi)=\frac{\gamma_1(1-\coth_1)-\gamma_2(1-\coth_2)}{\gamma_1\coth_1-\gamma_2\coth_2}. \tag{17}$$

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This integral equation may be solved approximately by numerical or other means, and in the next section an approximate solution is found when d is large compared with the radius of contact a.

The normal stress under the punch is, from (9), (10) and (6),

$$\sigma_{zz}(r,0) = -(\gamma_1-\gamma_2)\frac{1}{r}\frac{\partial}{\partial r}\int_r^a \frac{tg(t)\,dt}{(t^2-r^2)^{\frac{1}{2}}}, \quad \text{on} \quad z=0 \quad (0 \leqslant r \leqslant a), \tag{18}$$

$$= -(\gamma_1-\gamma_2)\left(\int_r^a \frac{g'(t)\,dt}{(t^2-r^2)^{\frac{1}{2}}} - \frac{g(a)}{(a^2-r^2)^{\frac{1}{2}}}\right), \tag{19}$$

assuming $g'(t)$ is continuous and recalling that z tends to zero through negative values. Thus the total pressure exerted by the punch on the layer is

$$P = -2\pi\int_0^a r\sigma_{zz}(r,0)\,dr,$$

$$= -2\pi(\gamma_1-\gamma_2)\int_0^a g(t)\,dt. \tag{20}$$

Similarly the normal displacement on the surface $z=0$ is given by

$$w(r,0) = \left(\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right)\left[\phi_z(r,0)+\int_0^\infty \xi^2 B(\xi)\,J_0(\xi r)\,d\xi\right]$$

$$= \left(\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right)\left[\int_0^a \frac{g(t)\,dt}{(r^2-t^2)^{\frac{1}{2}}} + \int_0^r \frac{1}{(r^2-t^2)^{\frac{1}{2}}}\int_0^a g(s)\,K(s,t)\,ds\,dt\right] \quad \text{for} \quad r \geqslant a. \tag{21}$$

In the case of complete penetration by the punch the radius a is the radius of the punch and is independent of the pressure P and the thickness d of the layer, so that the depth of penetration may be found from (20). For incomplete penetration, however, by a punch with a smooth surface in contact with the layer over the region $0 \leqslant r \leqslant a$, the radius a depends on both P and d, but, from (19), the condition of continuity of the normal stresses over the smooth surface implies

$$g(a) = 0 \tag{22}$$

and this, together with (15) and (20), serves to determine the solution.

In the subsequent work we shall consider the case of incomplete penetration and assume $f''(r)$ is continuous. This is sufficient to ensure that $g'(t)$ is continuous and so (22) follows.

3. *Incomplete penetration.* From the Fredholm equation (15), and (22), we find

$$g(t)+\int_0^a g(s)\{K(s,t)-K(s,a)\}\,ds = \left[\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right]^{-1}\left[\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{rf(r)\,dr}{(t^2-r^2)^{\frac{1}{2}}} - \frac{2}{\pi}\frac{d}{da}\int_0^a \frac{rf(r)\,dr}{(a^2-r^2)^{\frac{1}{2}}}\right]$$

and

$$\int_0^a g(s)\,K(s,a)\,ds = \left[\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right]^{-1}\frac{2}{\pi}\frac{d}{da}\int_0^a \frac{rf(r)\,dr}{(a^2-r^2)^{\frac{1}{2}}}. \tag{23}$$

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It is convenient to write these in a non-dimensional form in terms of R, the largest radial dimension of the end face of the punch. If we substitute

$$\left.\begin{aligned} t = Rx, \quad s = Ry, \\ r = R\rho, \quad a = R\alpha, \\ \frac{g(Rx)}{R} = G(x)\Big/\left[\frac{k_1}{1+k_1} - \frac{k_2}{1+k_2}\right], \quad \frac{f(R\rho)}{R} = F(\rho), \end{aligned}\right\} \tag{24}$$

the equations (23) become

$$G(x) + \int_0^\alpha G(y)[K(Ry, Rx) - K(Ry, R\alpha)]\, R\, dy = \frac{2}{\pi}\left[x\int_0^x \frac{F'(\rho)\, d\rho}{(x^2-\rho^2)^{\frac{1}{2}}} - \alpha\int_0^\alpha \frac{F'(\rho)\, d\rho}{(\alpha^2-\rho^2)^{\frac{1}{2}}}\right] \tag{25}$$

and

$$\int_0^\alpha G(y)\, K(Ry, R\alpha)\, R\, dy = \frac{2}{\pi}\left[F(0) + \alpha\int_0^\alpha \frac{F'(\rho)\, d\rho}{(\alpha^2-\rho^2)^{\frac{1}{2}}}\right]. \tag{26}$$

If the thickness d of the layer is large compared with R we can obtain a solution of (25) as a series in powers of R/d, and, provided the non-dimensional thickness $h = d/R$ is sufficiently large, we assume the series is convergent.

By expanding the cosine terms in the integrand of (16) we can express $RK(Ry, Rx)$ as a power series in $1/h$:

$$\begin{aligned} RK(Ry, Rx) &= \frac{2}{\pi} R\int_0^\infty H(\xi)\cos(Rx\xi)\cos(Ry\xi)\, d\xi \\ &= \sum_{n=0}^\infty \frac{K_n(x, y)}{h^{2n+1}}, \end{aligned} \tag{27}$$

where

$$K_n(x, y) = \frac{(-1)^n[(x+y)^{2n} + (x-y)^{2n}]}{\pi(2n)!\, 2^{2n+1}}\int_0^\infty H\left(\frac{\eta}{2d}\right)\eta^{2n}\, d\eta. \tag{28}$$

Hence, writing

$$\alpha = \sum_{n=0}^\infty \frac{\alpha_n}{h^n} \tag{29}$$

we find

$$\begin{aligned} &R[K(Ry, Rx) - K(Ry, R\alpha)] \\ &\quad = \frac{K_1(x^2-\alpha_0^2)}{h^3} - \frac{2\alpha_0\alpha_1 K_1}{h^4} - \frac{K_1(2\alpha_0\alpha_2 + \alpha_1^2) - K_2\{6y^2(x^2-\alpha_0^2) + x^4 - \alpha_0^4\}}{h^5} + O\left(\frac{1}{h^6}\right) \end{aligned}$$

where $K_0(x, y) = K_0$, $K_1(x, y) = K_1(x^2+y^2)$, $K_2(x, y) = K_2(x^4 + 6x^2y^2 + y^4)$, (30)

the constants $K_0, K_1, K_2, \ldots$ being given by

$$K_n = \frac{(-1)^n}{\pi(2n)!\, 2^{2n}}\int_0^\infty H\left(\frac{\eta}{2d}\right)\eta^{2n}\, d\eta \quad (n = 0, 1, 2, \ldots) \tag{31}$$

from (28) and (30).

Now supposing

$$G(x) = \sum_{n=0}^\infty \frac{G_n(x)}{h^n} \tag{32}$$

a solution can be found to (25) by expanding and comparing coefficients, giving

$$G_0(x) = \frac{2}{\pi}\left[x\int_0^x \frac{F'(\rho)\, d\rho}{(x^2-\rho^2)^{\frac{1}{2}}} - q_0\right], \quad q_0 = \alpha_0\int_0^{\alpha_0} \frac{F'(\rho)\, d\rho}{(\alpha_0^2-\rho^2)^{\frac{1}{2}}}, \tag{33}$$

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$$\left.\begin{aligned}
G_1(x) &= -(2/\pi)\,q_1,\\
G_2(x) &= -(2/\pi)\,q_2,\\
G_3(x) &= -K_1(x^2-\alpha_0^2)\int_0^{\alpha_0} G_0(y)\,dy-(2/\pi)\,q_3,\\
G_4(x) &= -G_0(\alpha_0)\,K_1(x^2-\alpha_0^2)\,\alpha_1+\int_0^{\alpha_0}[2\alpha_0\alpha_1 K_1 G_0(y)-K_1(x^2-\alpha_0^2)\,G_1(y)]\,dy-(2/\pi)\,q_4,\\
G_5(x) &= -\tfrac{1}{2}[G_0'(\alpha_0)\,\alpha_1^2+2G_0(\alpha_0)\,\alpha_2]K_1(x^2-\alpha_0^2)+[2\alpha_0\alpha_1 K_1 G_0(\alpha_0)-K_1(x^2-\alpha_0^2)\,G_0(\alpha_0)]\,\alpha_1\\
&\quad+\int_0^{\alpha_0}[G_0(y)\,\{(2\alpha_0\alpha_2+\alpha_1^2)\,K_1-6y^2(x^2-\alpha_0^2)\,K_2-(x^4-\alpha_0^4)\,K_2\}\\
&\qquad+G_1(y)\,2\alpha_0\alpha_1 K_1-G_2(y)\,K_1(x^2-\alpha_0^2)]\,dy-(2/\pi)\,q_5,
\end{aligned}\right\}\tag{34}$$

yielding $G(x)$ to $O(1/h^6)$, where we have used the notation

$$\sum_{n=0}^{\infty}\frac{q_n}{h^n}=\alpha\int_0^{\alpha}\frac{F'(\rho)\,d\rho}{(\alpha^2-\rho^2)^{\frac{1}{2}}}.\tag{35}$$

From (25), $G(x)$ satisfies the condition $G(\alpha)=0$.

Equations (20) and (24) now combine to give

$$\left[\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right]\frac{P}{\gamma_1-\gamma_2}=-2\pi R^2\int_0^{\alpha}G(x)\,dx,\tag{36}$$

and by substituting from (33) and (34) the q_i, and hence from (35), the α_i may be found after some manipulation. Equation (36) gives finally

$$\left.\begin{aligned}
&\left(\frac{k_1}{1+k_1}-\frac{k_2}{1+k_2}\right)\frac{P}{\gamma_1-\gamma_2}=4R^2\int_0^{\alpha_0}\frac{\rho^2F'(\rho)\,d\rho}{(\alpha_0^2-\rho^2)^{\frac{1}{2}}},\\
\text{defining } \alpha_0,\qquad &\alpha_1=0,\quad \alpha_2=0,\\
&\alpha_3\int_0^{\alpha_0}\frac{[F'(\rho)+\rho F''(\rho)]\,d\rho}{(\alpha_0^2-\rho^2)^{\frac{1}{2}}}=-\tfrac{2}{3}\alpha_0^2K_1\int_0^{\alpha_0}\frac{\rho^2F'(\rho)\,d\rho}{(\alpha_0^2-\rho^2)^{\frac{1}{2}}},\\
&\alpha_4=0,\\
&\alpha_5\int_0^{\alpha_0}\frac{[F'(\rho)+\rho F''(\rho)]\,d\rho}{(\alpha_0^2-\rho^2)^{\frac{1}{2}}}=2\pi\alpha_0^2K_2\int_0^{\alpha_0}G_0(y)\,(y^2+\tfrac{1}{5}\alpha_0^2)\,dy,
\end{aligned}\right\}\tag{37}$$

yielding the radius a of the circle of contact to $O(1/h^6)$. Hence, from (34)

$$\left.\begin{aligned}
&G_1(x)=0,\quad G_2(x)=0,\\
&G_3(x)=-K_1(x^2-\tfrac{1}{3}\alpha_0^2)\int_0^{\alpha_0}G_0(y)\,dy,\\
&G_4(x)=0,\\
&G_5(x)=-K_2\int_0^{\alpha_0}G_0(y)\,\{6y^2(x^2-\tfrac{1}{3}\alpha_0^2)+x^4-\tfrac{1}{5}\alpha_0^4\}\,dy.
\end{aligned}\right\}\tag{38}$$

We can now use (26) to determine $F(0)$ from which the depth of penetration is found. Thus,

$$F(0)=\sum_{n=0}^{\infty}\frac{F_n}{h^n},\tag{39}$$

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and
$$\left.\begin{aligned} F_0 &= -\alpha_0\int_0^{\alpha_0}\frac{F'(\rho)\,d\rho}{(\alpha_0^2-\rho^2)^{\frac{1}{2}}}, & F_1 &= \tfrac{1}{2}\pi K_0\int_0^{\alpha_0}G_0(x)\,dx,\\ F_2 &= 0, & F_3 &= \tfrac{1}{2}\pi K_1\int_0^{\alpha_0}G_0(x)\,(x^2+\tfrac{1}{3}\alpha_0^2)\,dx,\\ F_4 &= 0, & F_5 &= \tfrac{1}{2}\pi K_2\int_0^{\alpha_0}G_0(x)\,(x^4+2x^2\alpha_0^2+\tfrac{1}{5}\alpha_0^4)\,dx. \end{aligned}\right\} \tag{40}$$

Finally, if we substitute the values of $g(t)$ obtained from (24), (33) and (38) into (21), the normal displacement of the free surface $z = 0$ is found to be

$$\begin{aligned} w(r,0) = {} & R\int_0^{\alpha_0}\frac{G_0(x)\,dx}{(\rho^2-x^2)^{\frac{1}{2}}}+\frac{R\pi K_0}{2h}\int_0^{\alpha_0}G_0(x)\,dx\\ & +\frac{RK_1}{h^3}\int_0^{\alpha_0}G_0(x)\left[x^2\tfrac{1}{2}\pi+\tfrac{1}{2}\rho^2\left\{\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{\alpha_0}{\rho}\right)\right\}+\tfrac{1}{2}\alpha_0(\rho^2-\alpha_0^2)^{\frac{1}{2}}+\tfrac{1}{3}\alpha_0^2\sin^{-1}\left(\frac{\alpha_0}{\rho}\right)\right]dx\\ & +\frac{RK_2}{h^5}\int_0^{\alpha_0}G_0(x)\left[x^4\tfrac{1}{2}\pi+3x^2\rho^2\left\{\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{\alpha_0}{\rho}\right)\right\}+3x^2\alpha_0(\rho^2-\alpha_0^2)^{\frac{1}{2}}\right.\\ & \quad +2x^2\alpha_0^2\sin^{-1}\left(\frac{\alpha_0}{\rho}\right)+\tfrac{3}{8}\rho^4\left\{\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{\alpha_0}{\rho}\right)\right\}+\tfrac{1}{8}\alpha_0(\rho^2-\alpha_0^2)^{\frac{1}{2}}(\rho^2+2\alpha_0^2)\\ & \quad \left.+\tfrac{1}{5}\alpha_0^4\sin^{-1}\left(\frac{\alpha_0}{\rho}\right)\right]dx+O\left(\frac{1}{h^6}\right), \quad \text{for} \quad \rho\geqslant\alpha, \end{aligned} \tag{41}$$

where $\rho = r/R$, $\alpha = a/R$.

4. *Particular example: hemispherically ended punch.** As a particular example we consider the case of a punch with a hemispherical end, so that

$$f(r) = f(0)+R-(R^2-r^2)^{\frac{1}{2}}, \quad F(\rho) = f(0)/R+1-(1-\rho^2)^{\frac{1}{2}}, \tag{42}$$

where R is the radius of the hemispherical end. Then, from (33)

$$G_0(x) = \frac{2}{\pi}\left[\tfrac{1}{2}x\log\left(\frac{1+x}{1-x}\right)-\tfrac{1}{2}\alpha_0\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right)\right], \tag{43}$$

and $G_3(x)$, $G_5(x)$ may be found from (38) using the integrals

$$\left.\begin{aligned} \int_0^{\alpha_0}G_0(x)\,dx &= \frac{1}{2\pi}\left[2\alpha_0-(1+\alpha_0^2)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right)\right],\\ \int_0^{\alpha_0}x^2G_0(x)\,dx &= \frac{1}{4\pi}\left[2\alpha_0(1+\tfrac{1}{3}\alpha_0^2)-(1+\tfrac{1}{3}\alpha_0^4)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right)\right],\\ \int_0^{\alpha_0}x^4G_0(x)\,dx &= \frac{1}{6\pi}\left[2\alpha_0(1+\tfrac{1}{3}\alpha_0^2+\tfrac{1}{5}\alpha_0^4)-(1+\tfrac{1}{5}\alpha_0^6)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right)\right]. \end{aligned}\right\} \tag{44}$$

* The problem has been considered by Segedin (7) for an isotropic half-space using a different method. See also Sneddon (8).

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The maximum depth of penetration is $-f(0) = -RF(0) = -R\sum_{n=0}^{\infty} F_n/h^n$ and it follows directly from (40), (42) and (43) that

$$\left.\begin{aligned}
F_0 &= -\tfrac{1}{2}\alpha_0 \log\left(\frac{1+\alpha_0}{1-\alpha_0}\right),\\
F_1 &= -\tfrac{1}{4}K_0\left[(1+\alpha_0^2)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right) - 2\alpha_0\right],\\
F_2 &= 0,\\
F_3 &= -\tfrac{1}{8}K_1\left[(1+\tfrac{2}{3}\alpha_0^2+\alpha_0^4)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right) - 2\alpha_0(1+\alpha_0^2)\right],\\
F_4 &= 0,\\
F_5 &= -\tfrac{1}{12}K_2\left[(1+3\alpha_0^2+\tfrac{3}{5}\alpha_0^4+\tfrac{9}{5}\alpha_0^6)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right) - 2\alpha_0(1+\tfrac{10}{3}\alpha_0^2+\tfrac{9}{5}\alpha_0^4)\right].
\end{aligned}\right\} \tag{45}$$

Similarly from (37), (42) and (43) we find the radius of the circle of contact is given by $a = R\alpha = R\sum_{n=0}^{\infty} \alpha_n/h^n$, where

$$\left.\begin{aligned}
\left(\frac{k_1}{1+k_1} - \frac{k_2}{1+k_2}\right)\frac{P}{(\gamma_1-\gamma_2)R^2} &= (1+\alpha_0^2)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right) - 2\alpha_0,\\
\alpha_1 &= 0, \quad \alpha_2 = 0,\\
\alpha_3 &= -\frac{K_1\alpha_0^2(1-\alpha_0^2)}{3}\left[\frac{(1+\alpha_0^2)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right) - 2\alpha_0}{(1-\alpha_0^2)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right) + 2\alpha_0}\right],\\
\alpha_4 &= 0,\\
\alpha_5 &= -\frac{K_2\alpha_0^2(1-\alpha_0^2)}{15}\left[\frac{(15+6\alpha_0^2+11\alpha_0^4)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right) - 2\alpha_0(15+11\alpha_0^2)}{(1-\alpha_0^2)\log\left(\frac{1+\alpha_0}{1-\alpha_0}\right) + 2\alpha_0}\right].
\end{aligned}\right\} \tag{46}$$

The equations defining F_0 and α_0 in (45) and (46) are identical in form to those obtained by Segedin in the isotropic case for the half-space problem.

To evaluate the normal displacement of the surface $z = 0$ due to the punch by equation (41), the only integral required in addition to (44) is

$$R\int_0^{a} \frac{G_0(x)\,dx}{(\rho^2-x^2)^{\frac{1}{2}}} = -\frac{1}{\pi}a_0\log\left(\frac{R+a_0}{R-a_0}\right)\sin^{-1}\left(\frac{a_0}{r}\right) - \frac{1}{\pi}(r^2-a_0^2)^{\frac{1}{2}}\log\left(\frac{R+a_0}{R-a_0}\right) + \frac{2R}{\pi}\sin^{-1}\left(\frac{a_0}{r}\right) + \frac{2R}{\pi}l(r) \quad (a_0 \leqslant r < \infty), \tag{47}$$

and

$$l(r) = -\frac{(R^2-r^2)^{\frac{1}{2}}}{R}\tan^{-1}\left\{\frac{a_0(R^2-r^2)^{\frac{1}{2}}}{R(r^2-a_0^2)^{\frac{1}{2}}}\right\} \quad (a_0 \leqslant r \leqslant R),$$

$$= \frac{(r^2-R^2)^{\frac{1}{2}}}{2R}\log\left\{\frac{R(r^2-a_0^2)^{\frac{1}{2}} + a_0(r^2-R_0^2)^{\frac{1}{2}}}{R(r^2-a_0^2)^{\frac{1}{2}} - a_0(r^2-R^2)^{\frac{1}{2}}}\right\} \quad (r > R),$$

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writing $a_0 = R\alpha_0$, $r = R\rho$. This is the normal displacement of the plane surface of a semi-infinite medium due to indentation by a hemispherically ended punch of radius R, where a_0 is the radius of the circle of contact.

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