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ON A MEMORY GAME AND PREFERENTIAL ATTACHMENT GRAPHS

HÜSEYIN ACAN,* *Monash University*
PAWEŁ HITCZENKO,** *Drexel University*

Abstract

In their recent paper Velleman and Warrington (2013) analyzed the expected values of some of the parameters in a memory game; namely, the length of the game, the waiting time for the first match, and the number of lucky moves. In this paper we continue this direction of investigation and obtain the limiting distributions of those parameters. More specifically, we prove that when suitably normalized, these quantities converge in distribution to a normal, Rayleigh, and Poisson random variable, respectively. We also make a connection between the memory game and one of the models of preferential attachment graphs. In particular, as a by-product of our methods, we obtain the joint asymptotic normality of the degree counts in the preferential attachment graphs. Furthermore, we obtain simpler proofs (although without rate of convergence) of some of the results of Peköz *et al*. (2014) on the joint limiting distributions of the degrees of the first few vertices in preferential attachment graphs. In order to prove that the length of the game is asymptotically normal, our main technical tool is a limit result for the joint distribution of the number of balls in a multitype generalized Pólya urn model.

Keywords: Convergence in distribution; generalized Pólya urn; preferential attachment graph; memory game

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Secondary 05C82; 60B12; 60C05

1. Introduction

Consider the following game of memory played with a deck of cards, which has been studied by Velleman and Warrington [20]. There are $2n$ cards with n different labels, two cards for each label. They are shuffled and put in a row, all facing down. A player proceeds in *moves* which consist of flipping over a card and then a second card. If the cards match, he/she removes the cards from the deck; otherwise he/she flips them over again. The goal is to finish the game using the least number of moves. Each initial configuration of the cards corresponds to a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$. Following Velleman and Warrington, we assume that the player has a perfect memory and the initial configuration is given by a permutation chosen uniformly at random.

The player uses the following optimal strategy. He/she starts from the leftmost card and flips over the cards from left to right. Before starting round t, if he/she knows the places of two identical cards, i.e. if he/she flipped over one of them at round $t-1$ and one of them before round $t-1$ then he/she removes the pair at round t. Otherwise, he/she flips over the first (i.e.

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* Current address: Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA.
Email address: huseyin.acan@rutgers.edu
** Postal address: Department of Mathematics, Drexel University, Philadelphia, PA 19104, USA.
Email address: phitczen@math.drexel.edu

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leftmost) card that has not been flipped over yet. In this case, if the matching card has been flipped over previously, then he/she flips it over once again and removes the pair from the deck; if not, he/she flips over the next unflipped card. Note that each card is flipped over at least once and at most twice. Thus, the player finishes in at least n rounds and at most $2n$ rounds.

We refer the reader to [20] for more details and references about the game. In particular, the authors of [20] studied the expected values of three parameters: the length of the game (i.e. the total number of moves), the number of moves until the first match, and the number of lucky moves, where a lucky move refers to a move at which two (necessarily consecutive) cards of the same label are flipped for the first (and the last) time in the same round.

In this paper we take the analysis one step further and identify the limiting distributions of those three parameters as the size of the deck n increases to ∞; see Theorems 1, 3, and 5 below for precise statements. Furthermore, we describe below a connection between the memory game and a version of a model of the preferential attachment graphs introduced in [4], mathematically formalized in [6], and subsequently studied in a number of papers; see, for example, [5], [7], [18], and the references therein. Thus, our methods may be used to study the preferential attachment models. In particular, our analysis allows us to find the asymptotic joint degree distribution of the first k vertices of a preferential attachment graph, a result that was originally obtained (in a more precise version) in [16]. We also infer the joint asymptotic normality of the degree counts in such graphs.

This connection between the memory game and the preferential attachment graphs is made through *linearized chord diagrams* (LCDs), i.e. pairings of $2n$ points. (LCDs appear in various fields in mathematics; see, for example, [2], [3], [8]–[11], [17], and [19].) First, the connection between the memory game and LCDs is made by noting that each game is equivalent to a unique game in which the second occurrence of card i precedes the second occurrence of card j for $i < j$. Such games (called standard in [20]) correspond to LCDs by pairing two cards with the same label.

On the other hand, Bollobás and Riordan [6] used LCDs to generate a preferential attachment random graph introduced in [4]. We briefly recall that this graph, $\mathcal{G}_n$, is obtained from the following discrete-time random graph process. Start with $\mathcal{G}_1$, the graph with one vertex v_1 and a loop on v_1. For $n > 1$, $\mathcal{G}_n$ is obtained from $\mathcal{G}_{n-1}$ by creating a new vertex v_n and adding a random edge e incident to v_n, where

$$\mathbb{P}(e = v_i v_n) = \begin{cases} \dfrac{D_{\mathcal{G}_{n-1}}(v_i)}{2n-1} & \text{if } 1 \le i \le n-1, \\ \dfrac{1}{2n-1} & \text{if } i = n, \end{cases}$$

and where $D_{\mathcal{G}}(v)$ denotes the degree of the vertex v in the graph $\mathcal{G}$. (A loop contributes 2 to the degree.)

Bollobás and Riordan proposed the following method to generate $\mathcal{G}_n$. Let C_n be a random LCD (i.e. random matching of $2n$ points on a line). Connect the points in each pair by an arc (chord) above the line. For each arc, there is a left and a right endpoint. Proceeding from left to right, identify all endpoints up to and containing the first right endpoint to form the vertex v_1. Then, starting with the next endpoint, identify all the endpoints up to the second right endpoint to form v_2, and so on. Then, for each arc, create an edge (possibly a loop) between the vertices corresponding to the left and right endpoints of the arc; see Figure 1. The graph $\mathcal{G}_{n+1}$ can be obtained from $\mathcal{G}_n$ as follows. Create one endpoint to the right of the last endpoint in C_n and

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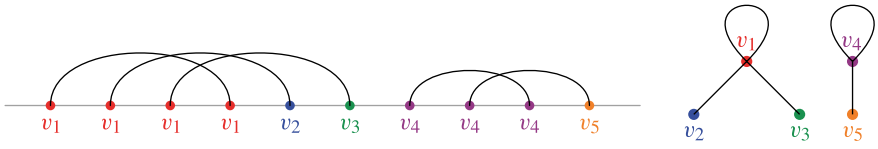


FIGURE 1: A linearized chord diagram and the corresponding graph.

then create another endpoint in one of the $2n+1$ intervals, choosing the interval uniformly at random. Finally, join these two endpoints by an arc and modify the graph to obtain $\mathcal{G}_{n+1}$.

Using this description, Bollobás *et al.* [7] proved that this graph has power-law degree distribution, which had been previously observed in [4] using experimental methods. (Power-law degree distribution means that the number of vertices with degree k is proportional to $nk^{-\gamma}$ as $n \to \infty$ for some $\gamma > 0$. Bollobás *et al.* [7] showed that $\gamma = 3$ in particular.)

Recently, Peköz *et al.* [15] studied the degree of a fixed vertex in $\mathcal{G}_n$. In particular, they found an asymptotic distribution for $D_{\mathcal{G}_n}(v_i)$, along with a rate of convergence, as $n \to \infty$. In a more recent work [16], they extended their result to the joint distribution of the first k vertices. One consequence of our approach is a simpler proof for the asymptotic joint degree distribution of the first k vertices for any fixed k using chord diagrams and the above description of $\mathcal{G}_n$.

In the next section we will introduce our notation and present our main results. In the subsequent sections, we will give the proofs of these results.

2. Main results

The *size* of a memory game is the number of pairs of cards a player starts with. Suppose that we have an infinite number of cards labeled $1, 2, \ldots$ in pairs, and each pair has a blue copy and a red copy. The colors are only on one side, and when a card is face down one cannot see its color. A game of size n is simply a permutation of the first n pairs of cards. Since the colors are not essential for the game, we may assume that a blue card always precedes the red card with the same label. Let $\mathcal{T}_n$ denote the set of such permutations with n pairs. Thus, the cardinality of $\mathcal{T}_n$ is $(2n)!/2^n$. Also let $\mathcal{T}_n^*$ denote the subset of $\mathcal{T}_n$ such that the red card with label i comes before the red card with label $i+1$ for every $i \in [n-1]$. Following [20], we call $\mathcal{T}_n^*$ the set of *standard deals* (denoted by M_n in [20]). It is easy to see that the cardinality of $\mathcal{T}_n^*$ is the product of odd positive integers 1 through $2n-1$.

A permutation $\sigma \in \mathcal{T}_n$ can be standardized by relabeling the pairs of cards so that the red cards appear in increasing order. Such a relabeling does not change the game's characteristics that we are interested in. Note that there is a unique standard deal corresponding to a permutation in $\mathcal{T}_n$ and, conversely, there are $n!$ permutations equivalent to a given standard deal in $\mathcal{T}_n^*$. A (uniformly) random game corresponds to a (uniformly) random element of $\mathcal{T}_n$ or $\mathcal{T}_n^*$. We are free to choose either interpretation for our analyses.

Red cards partition a permutation in $\mathcal{T}_n$ into n disjoint blocks. Each block consists of consecutive cards in the permutation, and for $1 \le k \le n$, the kth block ends with the kth red card in the permutation. The *length of a block* is the number of cards it contains. We denote by $B_{n,i}$ the number of blocks of length i in a random game with n pairs of cards. We have

$$\sum_{i \ge 1} B_{n,i} = n$$

and $B_{n,i} = 0$ for every $i > n+1$.

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We will use the symbol $(n)_k$ for the falling factorial. In particular, $(i+2)_3 = i(i+1)(i+2)$ will appear frequently throughout this paper. The following result gives the joint asymptotic distribution of the block counts and is crucial for our analysis.

Theorem 1. *As $n \to \infty$,*

$$\frac{1}{\sqrt{n}}\left(B_{n,i} - \frac{4n}{(i+2)_3}\right)_{i=1}^{\infty} \xrightarrow{\mathrm{D}} (W_i)_{i=1}^{\infty}, \tag{1}$$

where $\xrightarrow{\mathrm{D}}$ denotes the convergence in distribution and (W_i) are jointly Gaussian with mean 0 and covariance matrix $\Sigma = [\sigma_{ij}]$ given by

$$\sigma_{ij} = \begin{cases} \dfrac{16}{(i+2)_3(j+2)_3} - \dfrac{24}{(i+j+2)_4} & \text{if } i \neq j, \\ \dfrac{4}{(j+2)_3} + \dfrac{16}{(j+2)_3^2} - \dfrac{24}{(2j+2)_4} & \text{if } i = j. \end{cases} \tag{2}$$

Remark 1. In the preferential attachment graphs interpretation, $B_{n,i}$ is the number of vertices of degree i. Thus, the above theorem gives the joint asymptotic normality for the degree counts in the preferential attachment graphs. To the best of the authors' knowledge this result is new.

The following theorem gives us the total number of even length blocks. This is closely related to the *length of the game* defined as the minimum number of moves required to finish the game.

Theorem 2. *Let $Y_n = \sum_{i\geq 1} B_{n,2i}$. Then, as $n \to \infty$,*

$$\frac{1}{\sqrt{n}}(Y_n - (3 - 4\ln 2)n) \xrightarrow{\mathrm{D}} N(0, \sigma^2),$$

where $\sigma^2 = (4\ln 2)^2 + 8\ln 2 - 13 \sim 0.2324$.

Denote by G_n the length of a random game with n pairs of cards. Using the previous theorem, we obtain the distribution of G_n, which is stated next.

Corollary 1. *As $n \to \infty$, we have*

$$\frac{G_n - (3 - 2\ln 2)n}{\sqrt{n}} \xrightarrow{\mathrm{D}} N\left(0, \frac{\sigma^2}{4}\right),$$

where σ^2 is as in Theorem 2.

Proof. The length of the game G_n satisfies

$$G_n = \frac{3}{2}n + \frac{1}{2}\sum_{i\geq 1} B_{n,2i} - L_n$$

as observed by Velleman and Warrington [20, Section 4]. Here L_n denotes the number of lucky moves (defined in [20] and also below). By Theorem 5, L_n is bounded in probability. The result now follows from Theorem 2. □

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Denote by $D_{n,i}$ the length of the ith block in a random game. In the random preferential attachment graph described in Section 1, $D_{n,i}$ corresponds to the degree of the ith vertex. The distribution of $D_{n,i}$ is given by Peköz *et al.* [15]. The distribution of $D_{n,1}$ is given in the following theorem. In Section 3 we provide a short proof for this theorem.

Theorem 3. *Let X be a standard Weibull random variable with parameter 2, i.e. a random variable whose probability density function is $2x\mathrm{e}^{-x^2}$ if $x > 0$ and 0 otherwise. Then, as $n \to \infty$,*

$$\frac{D_{n,1}}{2\sqrt{n}} \xrightarrow{\mathrm{D}} X.$$

The next theorem is a generalization of the previous theorem. This result recovers, albeit without any error bound, the result of Peköz *et al.* [16, Theorem 1.1].

Theorem 4. *As $n \to \infty$, for any fixed positive integer k,*

$$\left(\frac{D_{n,1}}{2\sqrt{n}}, \dots, \frac{D_{n,k}}{2\sqrt{n}}\right) \xrightarrow{\mathrm{D}} (X_1, \dots, X_k),$$

where the joint density function of $(X_1, \dots, X_k)$ is given by

$$f(x_1, \dots, x_k) = \begin{cases} 2^k\, x_1(x_1 + x_2) \cdots (x_1 + \cdots + x_k)\mathrm{e}^{-(x_1+\cdots+x_k)^2} & \text{if } x_1, \dots, x_k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

A *lucky move* is a move at which we pick two matching cards by mere chance. This happens when two matching cards are next to each other and neither of these two cards has been flipped over before this move. We denote by L_n the number of lucky moves in a random game with n pairs of cards.

Theorem 5. *As $n \to \infty$, the number of lucky moves L_n converges in distribution to a Poisson random variable with parameter $\ln 2$, i.e.*

$$L_n \xrightarrow{\mathrm{D}} \mathrm{Po}(\ln 2).$$

3. Length of the first block

Recall that the length of the first block corresponds to the degree of the first vertex in the random preferential attachment graph $\mathcal{G}_n$. When standard deals are used, the first match occurs at the position of the red card with value 1. For $\sigma \in \mathcal{T}_n^*$, Velleman and Warrington [20] denoted this position by $f(\sigma)$ and they defined a sequence

$$a(n, j) := \{\pi \in \mathcal{T}_n^* \colon f(\pi) = j\}.$$

They noted ([20, Section 3]) that the numbers $a(n, j)$ satisfy the recurrence

$$a(n, j) = (2n - 1 - j)a(n - 1, j) + (j - 1)a(n - 1, j - 1)$$

with the boundary conditions $a(1, 2) = 1$ and $a(n, j) = 0$ if $j < 2$ or $j > n + 1$. They used this relation to obtain the expression for the expected position of the first match. Their argument quickly leads to the following statement.

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Proposition 1. *Let*

$$A_n(x) = \sum_{j=2}^{n+1} a(n, j)x^j$$

be the generating polynomial of the sequence $(a(n, j))$. *Then, for* $n \geq 2$,

$$A_n(x) = (2n-1)A_{n-1}(x) + x(x-1)A'_{n-1}(x), \qquad A_1(x) = x^2. \tag{3}$$

Furthermore, (see [20, Theorem 5] for the first statement) as $n \to \infty$,

$$\mathbb{E}[D_{n,1}] = \frac{2^{2n}}{\binom{2n}{n}} \sim \sqrt{\pi n} \tag{4}$$

and

$$\operatorname{var}(D_{n,1}) = (4-\pi)n + O(\sqrt{n}). \tag{5}$$

Proof. Let $h(j)$ be any sequence and set

$$H_n := \sum_{j=2}^{n+1} h(j)a(n, j).$$

Then, by [20, Lemma 4] (where $h(j) = j$ is used but the same argument works for any sequence $(h(j))$), we obtain

$$H_n = (2n-1)H_{n-1} + \sum_{j=2}^{n} j(h(j+1) - h(j))a(n-1, j).$$

If $h(j) = x^j$ then $H_n = A_n(x)$, $h(j+1) - h(j) = x^j(x-1)$, and we obtain (3).

To prove (4) and (5) let $r \geq 1$. The Leibniz formula and (3) imply that

$$\begin{aligned}
A_n^{(r)}(x) &= (2n-1)A_{n-1}^{(r)}(x) + (x(x-1)A'_{n-1}(x))^{(r)} \\
&= (2n-1)A_{n-1}^{(r)}(x) + \sum_{j=0}^{r} \binom{r}{j} (x(x-1))^{(j)} A_{n-1}^{(r-j+1)}(x) \\
&= (2n-1)A_{n-1}^{(r)}(x) + x(x-1)A_{n-1}^{(r+1)}(x) + r(2x-1)A_{n-1}^{(r)}(x) + r(r-1)A_{n-1}^{(r-1)}(x) \\
&= (2n-1+r(2x-1))A_{n-1}^{(r)}(x) + r(r-1)A_{n-1}^{(r-1)}(x) + x(x-1)A_{n-1}^{(r+1)}(x).
\end{aligned}$$

It follows that

$$A_n^{(r)}(1) = (2n-1+r)A_{n-1}^{(r)}(1) + r(r-1)A_{n-1}^{(r-1)}(1).$$

When $r = 0$, we obtain

$$A_n(1) = (2n-1)A_{n-1}(1) = \cdots = (2n-1)!!$$

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as was observed by Velleman and Warrington [20, Lemma 3]. This yields the following recurrence for the factorial moments:

$$\begin{aligned}\mathbb{E}[(D_{n,1})_r] &= \frac{A_n^{(r)}(1)}{A_n(1)} \\ &= (2n-1+r)\frac{A_{n-1}^{(r)}(1)}{A_n(1)} + r(r-1)\frac{A_{n-1}^{(r-1)}(1)}{A_n(1)} \\ &= \frac{2n-1+r}{2n-1}\frac{A_{n-1}^{(r)}(1)}{A_{n-1}(1)} + \frac{r(r-1)}{2n-1}\frac{A_{n-1}^{(r-1)}(1)}{A_{n-1}(1)} \\ &= \frac{2n-1+r}{2n-1}\mathbb{E}[(D_{n-1,1})_r] + \frac{r(r-1)}{2n-1}\mathbb{E}[(D_{n-1,1})_{r-1}]\end{aligned}$$

with the initial condition specified by $D_{1,1} = 2$. Letting $r = 1$, we have

$$\mathbb{E}[D_{n,1}] = \frac{2n}{2n-1}\mathbb{E}[D_{n-1,1}] = \frac{(2n)!!}{(2n-1)!!} = \frac{2^{2n}}{\binom{2n}{n}} \sim \sqrt{\pi n},$$

which proves (4).

When we set $r = 2$, we obtain

$$\begin{aligned}\mathbb{E}[(D_{n,1})_2] &= \frac{2n+1}{2n-1}\mathbb{E}[(D_{n-1,1})_2] + \frac{2}{2n-1}\mathbb{E}[D_{n-1,1}] \\ &= \frac{2n+1}{2n-1}\mathbb{E}[(D_{n-1,1})_2] + \frac{2^{2n-1}}{(2n-1)\binom{2n-2}{n-1}}.\end{aligned}$$

Letting $g_n := \mathbb{E}[(D_{n,1})_2]/(2n+1)$ yields

$$g_n = g_{n-1} + \frac{2^{2n-1}}{(2n+1)(2n-1)\binom{2n-2}{n-1}}$$

with $g_1 = \frac{2}{3}$. This is solved by

$$g_n = 2 - \frac{2^{2n+2}}{(n+1)\binom{2n+2}{n+1}} \sim 2 - \sqrt{\frac{\pi}{n+1}}.$$

Hence,

$$\mathbb{E}[(D_{n,1})_2] = 2(2n+1) - \frac{2n+1}{n+1}\frac{2^{2n+2}}{\binom{2n+2}{n+1}} = 4n + O(\sqrt{n})$$

and, thus,

$$\operatorname{var}(D_{n,1}) = \mathbb{E}[(D_{n,1})_2] + \mathbb{E}[D_{n,1}] - (\mathbb{E}[D_{n,1}])^2 = (4-\pi)n + O(\sqrt{n}),$$

which proves (5). □

Remark 2. Presumably, this process could be continued to determine the asymptotics of the factorial moments, and possibly identify the limiting distribution of $D_{n,1}$. For instance, for $r = 3$, we can obtain

$$\mathbb{E}[(D_{n,1})_3] = 6\left(\frac{\sqrt{\pi}(n+2)n!}{\Gamma(n+1/2)} - 4n - 3\right) \sim 6\sqrt{\pi}n^{3/2}.$$

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However, computations become progressively more complicated and hence we use a different approach to find an asymptotic distribution for $D_{n,1}$.

The following simple lemma will be used throughout this paper.

Lemma 1. *Let $n \to \infty$ and let $m = m(n)$ be a positive integer-valued function such that $m = o(n^{2/3})$. We have*

$$(n)_m = n^m \mathrm{e}^{-\binom{m}{2}/n}\left(1 + O\left(\frac{m^3}{n^2}\right)\right).$$

Proof. We have

$$(n)_m = n^m \prod_{j=1}^{m-1}\left(1 - \frac{j}{n}\right) = n^m \exp\left(\sum_{j=1}^{m-1} \ln\left(1 - \frac{j}{n}\right)\right). \tag{6}$$

Using the expansion $\ln(1-x) = -x + O(x^2)$ valid for $x \to 0$ and the assumption $m = o(n^{2/3})$, we obtain

$$\sum_{j=1}^{m-1} \ln\left(1 - \frac{j}{n}\right) = \sum_{j=1}^{m-1}\left(-\frac{j}{n} + O\left(\frac{j^2}{n^2}\right)\right) = -\frac{\binom{m}{2}}{n} + O\left(\frac{m^3}{n^2}\right).$$

Substituting this into (6) and using $\mathrm{e}^x \sim 1 + x$ for $x \to 0$ proves the lemma. □

Proof of Theorem 3. For the proof, we use the space of all deals $\mathcal{T}_n$. Let r_1 denote the first red card in the random permutation. If the length of the first block is t then the position of r_1 is t, in which case $t-1$ blue cards precede r_1, one of them being the blue counterpart of r_1. Now we want to find the number of games such that the first red card appears at position t. To this end, we first choose r_1, which can be done in n ways. Then we choose and permute the blue cards preceding r_1, which can be done in $\binom{n-1}{t-2}(t-1)!$ ways. Note that we choose only $t-2$ blue cards since one of the $t-1$ is already known. Then we permute the remaining cards that will appear after r_1, and this can be done in $(2n-t)!/2^{n-t+1}$ ways. Finally, to find the probability $\mathbb{P}(D_{n,1} = t)$, we need to divide the product by $(2n)!\,2^{-n}$, the size of $\mathcal{T}_n$. Hence, for $2 \le t \le n+1$,

$$\mathbb{P}(D_{n,1} = t) = n\binom{n-1}{t-2}(t-1)!\,\frac{(2n-t)!}{2^{n-t+1}}\,\frac{2^n}{(2n)!} = \frac{t-1}{2n-t+1}\,\frac{2^{t-1}\,(n)_{t-1}}{(2n)_{t-1}}. \tag{7}$$

Let $t = 2\alpha\sqrt{n}$ for some positive α. Using Lemma 1, we have

$$\frac{2^{t-1}(n)_{t-1}}{(2n)_{t-1}} = \exp\left(-\frac{t^2}{4n} + O\left(\frac{t}{n}\right) + O\left(\frac{t^3}{n^2}\right)\right)$$

and, consequently,

$$\mathbb{P}\left(\frac{D_{n,1}}{2\sqrt{n}} = \alpha\right) = \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)\frac{1}{2\sqrt{n}}\,2\alpha\mathrm{e}^{-\alpha^2}.$$

Thus, $D_{n,1}/2\sqrt{n}$ converges in distribution to a random variable X with density function $f(x)$, where $f(x) = 0$ for $x < 0$, and $f(x) = 2x\mathrm{e}^{-x^2}$ for $x \ge 0$. In particular, we have

$$F_n(z) := \mathbb{P}\left(\frac{D_{n,1}}{2\sqrt{n}} \le z\right) \to 1 - \mathrm{e}^{-z^2} \quad \text{for any } z \ge 0. \qquad \square$$

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With a little more work, we can bound $|F_n(\alpha)-F(\alpha)|$, where F is the cumulative distribution function of X. Define $e_t = (t/(2n))\exp(-t^2/4n)$. Note that the sequence $\{e_t\}_{t\geq 1}$ is log-concave. Now fix $\alpha > 0$ and let $N_\alpha = \lfloor 2\alpha\sqrt{n}\rfloor$. By (7), we have

$$F_n(\alpha) = \sum_{t=2}^{N_\alpha} \mathbb{P}(D_{n,1} = t) = \sum_{t=1}^{N_\alpha - 1} \frac{t}{2n-t}\frac{2^t(n)_t}{(2n)_t} = \sum_{t=1}^{N_\alpha - 1}\left(\frac{t}{2n} + O\left(\frac{t^2}{n^2}\right)\right)\frac{2^t(n)_t}{(2n)_t}.$$

Using Lemma 1,

$$\frac{2^t(n)_t}{(2n)_t} = \exp\left(-\frac{t(t-1)}{4n}\right)\left(1 + O\left(\frac{t^3}{n^2}\right)\right).$$

Combining the previous two equations, we obtain

$$F_n(\alpha) = \sum_{t=1}^{N_\alpha - 1} e_t\left(1 + O\left(\frac{t}{n} + \frac{t^3}{n^2}\right)\right) = \sum_{t=1}^{N_\alpha - 1} e_t + O\left(\frac{1}{\sqrt{n}}\right).$$

In fact, by a more careful analysis of (6), we can write

$$\left|F_n(\alpha) - \sum_{t=1}^{N_\alpha - 1} e_t\right| \leq \frac{C(1+\alpha+\alpha^3)}{\sqrt{n}}$$

for an absolute constant C uniformly for all α. Considering $\sum e_t$ as a Riemann sum, by the log-concavity of $\{e_t\}$, it is easy to see that for some constant C_2,

$$\left|\sum_{t=1}^{N_\alpha - 1} e_t - F(\alpha)\right| \leq C_2 e^*,$$

where e^* is the maximum of $\{e_t\}$, which is at most $1/\sqrt{2en}$. Using the last two equations, for all $\alpha > 0$,

$$|F_n(\alpha) - F(\alpha)| \leq C_3(1+\alpha+\alpha^3)n^{-1/2}$$

for an absolute constant C_3. On the other hand, for $\alpha \geq \sqrt{\log n}$,

$$1 - F(\alpha) = \mathrm{e}^{-\alpha^2} = O\left(\frac{1}{n}\right)$$

and

$$|F_n(\alpha)-F(\alpha)| \leq |1-F(\alpha)|+|F_n(\alpha)-1| \leq O\left(\frac{1}{n}\right)+1-F_n(\sqrt{\log n}) = O((\log n)^{3/2}n^{-1/2}).$$

Proof of Theorem 4. Let $t_1, \ldots, t_k$ be positive integers and let $t = t_1 + \cdots + t_k$. Similarly to (7), we write

$$\mathbb{P}(D_{n,1} = t_1, \ldots, D_{n,k} = t_k) = \binom{n}{k} k!\,(t_1 - 1)(t_1 + t_2 - 3)\cdots(t_1 + \cdots + t_k - 2k + 1)$$
$$\times \binom{n-k}{t-2k}(t-2k)!\,\frac{(2n-t)!}{2^{n-t+k}}\frac{2^n}{(2n)!}.$$

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Here $(t_1 - 1)$ is the number of possible positions for the blue partner of the first red card, $(t_1 + t_2 - 3)$ is the number of positions for the blue partner of the second red card, and so on. Simplifying this, we obtain

$$\mathbb{P}(D_{n,1} = t_1, \dots, D_{n,k} = t_k) = \frac{2^{t-k}\,(n)_{t-k}}{(2n)_{t-k}} \frac{(t_1 - 1)\cdots(t_1 + \cdots t_k - 2k + 1)}{(2n - t + k)_k}.$$

Letting $t_i = 2\alpha_i\sqrt{n}$ for positive α_i, the right-hand side can be expressed as

$$\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)(\mathrm{e}^{-(\alpha_1+\cdots+\alpha_k)^2})\left(\frac{\alpha_1}{\sqrt{n}}\right)\cdots\left(\frac{\alpha_1 + \cdots + \alpha_k}{\sqrt{n}}\right),$$

from which the theorem follows. □

4. Blocks of a given size

In this section we will estimate the expected value and the variance of $B_{n,i}$. We need these estimates in the next sections. We start with the following easy lemmas.

Lemma 2. *For N, i, and j, we have*

$$\sum_k \binom{N-k}{i}\binom{k}{j} = \binom{N+1}{i+j+1}, \tag{8}$$

$$\sum_k \binom{k}{l}\binom{N-k}{i}\binom{N-k-i}{j} = \binom{i+j}{i}\binom{N+1}{i+j+l+1}. \tag{9}$$

Proof. The first assertion is [12, Equation (5.26)] with $q = 0$. For the second equation note that

$$\binom{N-k}{i}\binom{N-k-i}{j} = \frac{(N-k)!}{i!\,(N-k-i)!}\frac{(N-k-i)!}{j!\,(N-k-i-j)!} = \binom{i+j}{i}\binom{N-k}{i+j}.$$

Hence, by (8), the left-hand side of (9) is

$$\binom{i+j}{i}\sum_k \binom{k}{l}\binom{N-k}{i+j} = \binom{i+j}{i}\binom{N+1}{i+j+l+1}.$$

The next lemma follows from simple algebra and the proof is omitted. □

Lemma 3. *For $f_{xy} := 6\binom{x+y-1}{y} + 3\binom{x+y}{y+1} + \binom{x+y+1}{y+2}$, we have*

$$f_{ij} + f_{ji} = \frac{(i+j+4)!}{(i+2)!\,(j+2)!}.$$

4.1. Expected number of blocks of a given size

It has been shown by Velleman and Warrington [20] that the expected value of the number of blocks of size i is asymptotic to $4n/(i+2)_3$ as the number of pairs of cards n goes to ∞. We will need a more quantitative version of their result.

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Proposition 2. *As $n \to \infty$ and for $i = o(n^{1/2})$,*

$$\mathbb{E}[B_{n,i}] = \mathbb{E}[\xi_i] + \frac{4n}{(i+2)_3}\left(1 + O\left(\frac{i^2}{n}\right)\right), \tag{10}$$

where ξ_i is the indicator of the event that the first block has size i. (Note, in particular, that $\xi_1 = 0$.) Furthermore, for every $n \geq 1$ and $1 \leq i \leq n+1$,

$$\mathbb{E}[B_{n,i}] \leq \mathbb{E}[\xi_i] + \frac{4\mathrm{e}^{1/2}n}{(i+2)_3}. \tag{11}$$

Proof. We use the set of permutations $\mathcal{T}_n$ throughout the proof. For $x \in [n]$, let x_{B} and x_{R} denote the blue and the red cards labeled with x, respectively. Recall that the block sizes are determined by the positions of the red cards. For $a \neq b$, let $I^i_{a,b}$ denote the indicator random variable which takes the value 1 if the following hold in the random permutation of cards:

(i) a_{R} comes before b_{R};

(ii) there is no red card between a_{R} and b_{R};

(iii) there are exactly $i-1$ blue cards between a_{R} and b_{R}.

If all three of these events hold then the contribution of the pair $(a_{\mathrm{R}}, b_{\mathrm{R}})$ to $B_{n,i}$ is 1. Thus, we have

$$B_{n,i} = \xi_i + \sum_{a \neq b} I^i_{a,b} \tag{12}$$

and

$$\mathbb{E}[B_{n,i}] = \mathbb{E}[\xi_i] + \sum_{a \neq b} \mathbb{E}[I^i_{a,b}] = \mathbb{E}[\xi_i] + n(n-1)\mathbb{E}[I^i_{1,2}], \tag{13}$$

where the last identity follows from symmetry. For computing $\mathbb{E}[I^i_{1,2}]$, we can first choose the positions of 1_{R} and 2_{R}, then choose cards (all blue) to put between 1_{R} and 2_{R}, and finally permute everything and divide by the total number of configurations. We write

$$\mathbb{E}[I^i_{1,2}] = E_1 + E_2,$$

where E_1 is the contribution of those configurations such that both 1_{B} and 2_{B} appear before 1_{R}, and E_2 is the rest. Of course, E_2 is positive if and only if $2 \leq i \leq n$. Now, as explained below,

$$E_1 = \sum_{k \geq 2} k(k-1)\binom{n-2}{i-1}(i-1)!\binom{2n-k-i-1}{i-1}(i-1)!\,\frac{(2n-2i-2)!/2^{n-i-1}}{(2n)!/2^n}.$$

Explanation. In the sum, $k+1$ represents the position of 1_{R}. This implies that the position of 2_{R} is $k+i+1$. We choose the positions of 1_{B} and 2_{B} in $k(k-1)$ ways. We choose $i-1$ cards out of $n-2$ blue cards to put in between 1_{R} and 2_{R}, and permute these in $(i-1)!$ ways. Next, we choose $(i-1)$ positions from $\{k+i+2, \ldots, 2n\}$ for the red partners of the blue cards sandwiched between 1_{R} and 2_{R}, and permute these red partners in $(i-1)!$ ways. The rest of the numerator is for permuting all the remaining cards.

After simple cancellations, we obtain

$$E_1 = \frac{2^{i+2}(i-1)!\,(n-2)_{(i-1)}}{(2n)_{(2i+2)}} \sum_{k \geq 2} \binom{k}{2}\binom{2n-k-i-1}{i-1}.$$

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Using (8) for the right-hand side above and simplifying the resulting expression, we have

$$E_1 = \frac{2^{i+2}(i-1)!\,(n-2)_{(i-1)}}{(2n)_{(2i+2)}}\binom{2n-i}{i+2} = \frac{2^{i+2}}{(i+2)_3}\frac{(n-2)_{(i-1)}}{(2n)_i}. \tag{14}$$

Similarly, for $i \geq 2$, we have

$$\begin{aligned} E_2 &= \sum_{k\geq 1} k\binom{n-2}{i-2}(i-1)!\binom{2n-k-i-1}{i-2}(i-2)!\,\frac{(2n-2i)!/2^{n-i}}{(2n)!/2^n} \\ &= \frac{2^i(i-1)!\,(n-2)_{(i-2)}}{(2n)_{(2i)}}\sum_{k\geq 1}\binom{k}{1}\binom{2n-k-i-1}{i-2} \\ &= \frac{2^i}{i}\frac{(n-2)_{(i-2)}}{(2n)_i}. \end{aligned} \tag{15}$$

Combining (14) and (15), we obtain

$$n(n-1)(E_1+E_2) = \frac{2^{i+2}(n)_{i+1}}{(i+2)_3(2n)_i} + \frac{2^i(n)_i}{i(2n)_i} = \frac{4n}{(i+2)_3}\frac{2^i(n)_i}{(2n)_i}\left(1+\frac{i^2-i+2}{4n}\right). \tag{16}$$

It follows from Lemma 1 that, for $i = o(\sqrt{n})$,

$$\frac{2^i(n)_i}{(2n)_i} = \mathrm{e}^{-i(i-1)/(4n)}\left(1+O\left(\frac{i^3}{n^2}\right)\right) = 1+O\left(\frac{i^2}{n}\right).$$

This, combined with (13) and (16), implies (10). Furthermore, for $1 \leq i \leq n+1$,

$$\frac{2^i(n)_i}{(2n)_i} = \prod_{j=1}^{i-1}\frac{1-j/n}{1-j/2n} = \prod_{j=1}^{i-1}\left(1-\frac{j}{2n-j}\right) \leq \prod_{j=1}^{i-1}\left(1-\frac{j}{2n}\right) \leq \exp\left\{-\frac{i(i-1)}{4n}\right\}.$$

Hence, applying $1+x \leq \mathrm{e}^x$ to the last factor in (16), we obtain

$$\frac{2^i(n)_i}{(2n)_i}\left(1+\frac{i^2-i+2}{4n}\right) \leq \exp\left\{-\frac{i(i-1)}{4n}+\frac{i^2-i+2}{4n}\right\} \leq \mathrm{e}^{1/2},$$

which proves (11). □

4.2. Bound on the variance of the number of blocks of a given size

In this section we estimate $\mathrm{var}(B_{n,i})$. The following is sufficient for our purposes.

Lemma 4. *There exists an absolute constant M such that, for $i \leq n^{1/3}$,*

$$\mathrm{var}(B_{n,i}) \leq M\frac{n}{i^3}. \tag{17}$$

Proof. By (12),

$$\begin{aligned} \mathbb{E}[B_{n,i}^2] &= \mathbb{E}[\xi_i] + \mathbb{E}\left[\sum_{((a,b),(c,d))} I_{a,b}^i I_{c,d}^i\right] + 2\mathbb{E}\left[\sum_{(a,b,c)} I_{a,b}^i I_{b,c}^i\right] + 2\mathbb{E}\left[\xi_i\sum_{(a,b)} I_{a,b}^i\right] \\ &= \mathbb{E}[\xi_i] + \Sigma_1 + 2\Sigma_2 + 2\Sigma_3, \end{aligned} \tag{18}$$

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where different letters in the sums denote distinct numbers from $[n]$ and Σ_k denotes the kth sum before the last equality sign. We will see that the main contribution to $\mathbb{E}[B_{n,i}^2]$ comes from Σ_1.

By Proposition 2, there are some absolute positive constants K_1 and K_2 such that $\mathbb{E}[B_{n,i}]$ lies in the interval $[K_1 n/i^3, K_2 n/i^3]$ for all $i \le n^{1/3}$ as $n \to \infty$. Also, since there are only n blocks, we have the trivial upper bound $B_{n,i} \le n$. Hence,

$$2\Sigma_3 = 2\mathbb{E}[\xi_i(B_{n,i} - \xi_i)] \le 2n\mathbb{E}[\xi_i] \le 2i, \tag{19}$$

since, using (7), we have

$$\mathbb{E}[\xi_i] = \mathbb{P}(D_{n,1} = i) = \frac{i-1}{2n-i+1}\frac{2^{i-1}(n)_{i-1}}{(2n)_{i-1}} \le \frac{i}{n} \quad \text{for } i \le n+1.$$

Next we estimate Σ_1.

Computation of Σ_1. Let $A = A(n,i)$ be the following event:

(a) $\{1_{\mathrm{R}}, 2_{\mathrm{R}}, 3_{\mathrm{R}}, 4_{\mathrm{R}}\}$ appear in the order $1_{\mathrm{R}}, 2_{\mathrm{R}}, 3_{\mathrm{R}}, 4_{\mathrm{R}}$ in the permutation;

(b) 1_{R} and 2_{R} are separated by $i-1$ blue cards;

(c) 3_{R} and 4_{R} are separated by $i-1$ blue cards.

Let p_i be the probability of A. Note that, for each permutation in A, the contribution of the 4-tuple $(1_{\mathrm{R}}, 2_{\mathrm{R}}, 3_{\mathrm{R}}, 4_{\mathrm{R}})$ to Σ_1 is 2, and we have

$$\Sigma_1 = 2\binom{n}{4}4!\, p_i = 2(n)_4 p_i.$$

We now estimate p_i for $i = O(n^{1/3})$. Before we start, we introduce some notation. The positions of the red cards $1_{\mathrm{R}}, 2_{\mathrm{R}}, 3_{\mathrm{R}}$, and 4_{R} determine five disjoint intervals in a permutation. We denote the jth interval by I_j for $1 \le j \le 5$. Depending on which interval(s) the blue cards $\{1_{\mathrm{B}}, 2_{\mathrm{B}}, 3_{\mathrm{B}}, 4_{\mathrm{B}}\}$ lie, we have different cases. Let t_u be the number of blue cards from $\{1_{\mathrm{B}}, 2_{\mathrm{B}}, 3_{\mathrm{B}}, 4_{\mathrm{B}}\}$ in the interval I_u for $1 \le u \le 4$. Note that I_5 cannot contain any of these four blue cards. Since x_{B} must appear before x_{R} for any $x \in [n]$, we have the following constraints for the nonnegative integers t_1, t_2, t_3, and t_4:

$$\sum_{u=1}^{v} t_u \ge v \quad \text{for } v \in \{1,2,3\}; \tag{20}$$

$t_1 + t_2 + t_3 + t_4 = 4$. Both intervals I_2 and I_4 have lengths $i-1$, i.e. there are $i-1$ positions in each interval. We denote by k and ℓ the lengths of I_1 and I_3, respectively. Thus, the length of I_5 is $(2n-4)-k-\ell-2(i-1) = (2n-2i-2-k-\ell)$. For a given $\boldsymbol{t} = (t_1, t_2, t_3, t_4)$ meeting the conditions in (20), let $K(\boldsymbol{t})$ be the number of allocations of $\{1_{\mathrm{B}}, \dots, 4_{\mathrm{B}}\}$ into intervals I_1, I_2, I_3, and I_4, respecting $\boldsymbol{t}$. By considering the number of blue cards in the intervals and the

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red partners of the blue cards in I_2 and I_4, we obtain

$$\begin{aligned} p_i = \sum_{\boldsymbol{t}} K(\boldsymbol{t}) \sum_{k,\ell} \binom{k}{t_1} t_1! \binom{i-1}{t_2} t_2! \binom{\ell}{t_3} t_3! \binom{i-1}{t_4} t_4! \\ \times \binom{n-4}{2i-2-t_2-t_4} (2i-2-t_2-t_4)! \\ \times \binom{2n-2i-k-\ell-2}{i-1-t_4} (i-1-t_4)! \\ \times \binom{2n-3i-k-1-t_3+t_4}{i-1-t_2} (i-1-t_2)! \\ \times \frac{(2n-4i-4+2t_2+2t_4)!}{2^{n-2i-2+t_2+t_4}} \frac{2^n}{(2n)!}. \end{aligned}$$

After cancellations and taking the factors that do not depend on k or ℓ out of the inner sum, we have

$$\begin{aligned} p_i = \sum_{\boldsymbol{t}} K(\boldsymbol{t}) (i-1)!^2 t_1! t_3! (n-4)_{2i-2-t_2-t_4} \frac{2^{2i+2-t_2-t_4}}{(2n)_{4i+4-2t_2-2t_4}} \\ \times \sum_k \binom{k}{t_1} \binom{2n-3i-k-1-t_3+t_4}{i-1-t_2} \sum_\ell \binom{\ell}{t_3} \binom{2n-2i-k-\ell-2}{i-1-t_4}. \end{aligned}$$

Using Lemma 2,

$$\sum_\ell \binom{\ell}{t_3} \binom{2n-2i-k-\ell-2}{i-1-t_4} = \binom{2n-2i-k-1}{i+t_3-t_4},$$

so we need the sum

$$\sum_k \binom{k}{t_1} \binom{2n-3i-k-1-t_3+t_4}{i-1-t_2} \binom{2n-2i-k-1}{i+t_3-t_4}. \tag{21}$$

For the product of the second and the third terms in the sum, we write

$$\begin{aligned} \binom{2n-3i-k-1-t_3+t_4}{i-1-t_2} \binom{2n-2i-k-1}{i+t_3-t_4} \\ = \binom{2i-1-t_2+t_3-t_4}{i-1-t_2} \binom{2n-2i-k-1}{2i-1-t_2+t_3-t_4}. \end{aligned}$$

Using this identity, the sum in (21) can be written as

$$\binom{2i-1-t_2+t_3-t_4}{i-1-t_2} \sum_k \binom{k}{t_1} \binom{2n-2i-k-1}{2i-1-t_2+t_3-t_4}.$$

Using Lemma 2 once again, this can be expressed as

$$\binom{2i-1-t_2+t_3-t_4}{i-1-t_2} \binom{2n-2i}{2i+t_1-t_2+t_3-t_4}.$$

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Combining the results in the previous lines and using the identity $4-2t_2-2t_4 = t_1-t_2+t_3-t_4$, we obtain

$$\begin{aligned}(n)_4 p_i &= \sum_{\boldsymbol{t}} K(\boldsymbol{t})(i-1)!^2 t_1!\, t_3!\, (n)_{2i+2-t_2-t_4} \\ &\quad \times \frac{2^{2i+2-t_2-t_4}}{(2n)_{4i+4-2t_2-2t_4}} \binom{2i-1-t_2+t_3-t_4}{i-1-t_2} \binom{2n-2i}{2i+t_1-t_2+t_3-t_4} \\ &= \sum_{\boldsymbol{t}} K(\boldsymbol{t}) t_1!\, t_3!\, \frac{(i-1)!^2}{(i-1-t_2)!\,(i+t_3-t_4)!} \frac{(2i-1-t_2+t_3-t_4)!}{(2i+t_1-t_2+t_3-t_4)!} \\ &\quad \times \frac{2^{2i+2-t_2-t_4}(n)_{2i+2-t_2-t_4}}{(2n)_{2i}}. \end{aligned} \tag{22}$$

Defining $a := 2i+2-t_2-t_4$ and using Lemma 1 for the falling factorials in (22), we obtain

$$\begin{aligned}(n)_4 p_i &= \sum_{\boldsymbol{t}} K(\boldsymbol{t}) t_1!\, t_3!\, \frac{(i-1)!^2}{(i-1-t_2)!\,(i+t_3-t_4)!} \frac{(2i-1-t_2+t_3-t_4)!}{(2i+t_1-t_2+t_3-t_4)!} \\ &\quad \times (2n)^{t_1+t_3-2} \exp\left(\frac{-2\binom{a}{2}+\binom{2i}{2}}{2n}\right)\left[1+O\left(\frac{1}{n}\right)\right], \end{aligned}$$

where

$$\frac{-2\binom{a}{2}+\binom{2i}{2}}{2n} = -\frac{i^2}{n} + O\left(\frac{i}{n}\right).$$

Hence, for $i = o(n^{1/3})$, we have

$$(n)_4 p_i = \left(1+O\left(\frac{i^2}{n}\right)\right) \sum_{\boldsymbol{t}} K(\boldsymbol{t}) t_1!\, t_3!\, (2n)^{t_1+t_3-2} F(\boldsymbol{t}), \tag{23}$$

where

$$F(\boldsymbol{t}) = \frac{(i-1)!^2}{(i-1-t_2)!\,(i+t_3-t_4)!} \frac{(2i-1-t_2+t_3-t_4)!}{(2i+t_1-t_2+t_3-t_4)!}. \tag{24}$$

Note that there exists an absolute constant γ such that for any $i \geq 1$ and $\boldsymbol{t}$,

$$F(\boldsymbol{t}) \leq \gamma\, i^{-1-t_3+t_2+t_4}\, i^{-1-t_1} = \gamma\, i^{2-2t_1-2t_3}.$$

Consequently, the sum in (23) over the vectors $\boldsymbol{t} = (t_1, t_2, t_3, t_4)$ with $t_1 + t_3 \leq 3$ is only $O(n/i^4)$. Also, there are three vectors $\boldsymbol{t}$ with the sum of the first and the third components being 4: $\boldsymbol{t}_1 = (4,0,0,0)$, $\boldsymbol{t}_2 = (3,0,1,0)$, and $\boldsymbol{t}_3 = (2,0,2,0)$. For these $\boldsymbol{t}$, we have $K(\boldsymbol{t}_1) = 1$, $K(\boldsymbol{t}_2) = 2$, and $K(\boldsymbol{t}_3) = 1$. Then

$$(n)_4 p_i = O\left(\frac{n}{i^4}\right) + \left(1+O\left(\frac{i^2}{n}\right)\right)(2n)^2(24F(\boldsymbol{t}_1) + 12F(\boldsymbol{t}_2) + 4F(\boldsymbol{t}_3)). \tag{25}$$

Using (24), we have

$$24F(\boldsymbol{t}_1) + 12F(\boldsymbol{t}_2) + 4F(\boldsymbol{t}_3) = 4\frac{(i-1)!^2}{(2i+4)!}\left[6\binom{2i-1}{i} + 3\binom{2i}{i+1} + \binom{2i+1}{i+2}\right].$$

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Using Lemma 3 with $i = j$, we obtain

$$\left[6\binom{2i-1}{i} + 3\binom{2i}{i+1} + \binom{2i+1}{i+2}\right] = \frac{(2i+4)!}{2(i+2)!^2}$$

and, consequently,

$$24F(\boldsymbol{t}_1) + 12F(\boldsymbol{t}_2) + 4F(\boldsymbol{t}_3) = \frac{2}{((i+2)_3)^2}.$$

Finally, using this equation and (25), we have

$$\Sigma_1 = 2(n)_4 p_i = O\left(\frac{n}{i^4}\right) + \left(1 + O\left(\frac{i^2}{n}\right)\right)\frac{16n^2}{((i+2)_3)^2} = \frac{16n^2}{((i+2)_3)^2} + O\left(\frac{n}{i^4}\right). \quad (26)$$

Computation of Σ_2. Let A' denote the event $\{I^i_{1,2} = 1 = I^i_{2,3}\}$ in a game with n pairs of cards and let $q = \mathbb{P}(A')$. Thus, we have $\Sigma_2 = (n)_3 q$ and we need to estimate q. On the event A', 1_{R} appears before 2_{R}, which appears before 3_{R}. These three red cards determine four intervals, I_1 through I_4, numbered in the order they appear. Both I_2 and I_3 contain $i-1$ blue cards and no red cards. For $1 \le j \le 3$, let t_j denote the number of elements from $\{1_{\mathrm{B}}, 2_{\mathrm{B}}, 3_{\mathrm{B}}\}$ contained in I_j. We have the following possibilities for $\boldsymbol{t} = (t_1, t_2, t_3)$:

$$\boldsymbol{t}_1 = (3,0,0), \qquad \boldsymbol{t}_2 = (2,1,0), \qquad \boldsymbol{t}_3 = (2,0,1), \qquad \boldsymbol{t}_4 = (1,2,0), \qquad \boldsymbol{t}_5 = (1,1,1).$$

Let K_u be the number of ways to place $\{1_{\mathrm{B}}, 2_{\mathrm{B}}, 3_{\mathrm{B}}\}$ in I_1, I_2, and I_3 respecting $\boldsymbol{t}_u$. We have $K_1 = 1$, $K_2 = 2$, $K_3 = 1$, $K_4 = 1$, and $K_5 = 1$. Let k denote the length of I_1, so the length of I_4 is $2n - 2i - k - 1$. We need to put $2i - 2 - t_2 - t_3$ blue cards other than $\{1_{\mathrm{B}}, 2_{\mathrm{B}}, 3_{\mathrm{B}}\}$ in $I_2 \cup I_3$ and we need to put the red counterparts of these blue cards in I_4. Hence, using Lemma 2 in the last step below, we have

$$\begin{aligned}
q &= \sum_{u=1}^{5}\sum_{k} K_u \binom{k}{t_1}\binom{i-1}{t_2}\binom{i-1}{t_3} t_1!\,t_2!\,t_3! \binom{n-3}{2i-2-t_2-t_3}(2i-2-t_2-t_3)! \\
&\qquad \times \binom{2n-k-2i-1}{2i-2-t_2-t_3}(2i-2-t_2-t_3)!\frac{(2n-4i-2+2t_2+2t_3)!}{2^{n-2i-1+t_2+t_3}}\frac{2^n}{(2n)!} \\
&= (i-1)!^2 \sum_{u=1}^{5} K_u t_1! \binom{2i-2-t_2-t_3}{i-1-t_2}(n-3)_{2i-2-t_2-t_3} \\
&\qquad \times \frac{2^{2i+1-t_2-t_3}}{(2n)_{4i+2-2t_2-2t_3}} \sum_{k}\binom{k}{t_1}\binom{2n-k-2i-1}{2i-2-t_2-t_3} \\
&= (i-1)!^2 \sum_{u=1}^{5} K_u t_1! \binom{2i-2-t_2-t_3}{i-1-t_2}(n-3)_{2i-2-t_2-t_3} \\
&\qquad \times \frac{2^{2i+1-t_2-t_3}}{(2n)_{4i+2-2t_2-2t_3}} \binom{2n-2i}{2i-1+t_1-t_2-t_3}.
\end{aligned}$$

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Now, using $2i-1+t_1-t_2-t_3 = 2i+2-2t_2-2t_3$, we simplify the last expression and write

$$\begin{aligned} q &= \sum_{u=1}^{5} K_u t_1!\,(i-1)!^2 \binom{2i-2-t_2-t_3}{i-1-t_2} (n-3)_{2i-2-t_2-t_3} \frac{2^{2i+1-t_2-t_3}}{(2n)_{2i}} \\ &\qquad \times \frac{1}{(2i+2-2t_2-2t_3)!} \\ &= \sum_{u=1}^{5} O\left(\frac{1}{i^{4-2t_2-2t_3}} \frac{2^{2i}(n)_{2i}}{(2n)_{2i}} \frac{1}{n^{2+t_2+t_3}} \right) \\ &= \sum_{u=1}^{5} O\left(\frac{1}{i^{4-2t_2-2t_3}} \frac{1}{n^{2+t_2+t_3}} \right) \\ &= \sum_{u=1}^{5} O\left(\frac{1}{i^4 n^2} \left(\frac{i^2}{n} \right)^{t_2+t_3} \right). \end{aligned}$$

Since $i = o(n^{1/3})$, the main contribution comes from $u = 1$, i.e. when $(t_1, t_2, t_3) = (3, 0, 0)$, in which case we have $O(n^{-2}i^{-4})$. In other words, we have $q = O(n^{-2}i^{-4})$. Then,

$$\Sigma_2 = (n)_3 q = O\left(\frac{n}{i^4} \right). \tag{27}$$

Combining (18), (19), (26), and (27), we obtain (17) for some absolute constant M. □

As a corollary of the previous two lemmas, we obtain the following concentration result for the $B_{n,i}$s. Let $\overline{B}_{n,i} = B_{n,i} - \mathbb{E}[B_{n,i}]$ for $i \geq 1$.

Corollary 2. *Let $l = o(n^{1/3})$ and let $\omega \to \infty$. Then*

$$\mathbb{P}(|\overline{B}_{n,1}| \leq \sqrt{n\omega}, \ldots, |\overline{B}_{n,\ell}| \leq \sqrt{n\omega}) = 1 - O\left(\frac{1}{\omega} \right).$$

Proof. Using Lemma 4, and Chebyshev's inequality, for any $i \leq l$, we obtain

$$\mathbb{P}(|\overline{B}_{n,i}| \geq \sqrt{n\omega}\,) \leq \frac{M}{i^3 \omega},$$

where M is a constant from Lemma 4. Hence,

$$\sum_{i=1}^{\ell} \mathbb{P}(|\overline{B}_{n,i}| \geq \sqrt{n\omega}\,) = O\left(\frac{1}{\omega} \right),$$

from which the corollary follows. □

5. Lucky moves

The asymptotics of the expected value of lucky moves in a random game of size n was derived in [20, Corollary 18]. Here we prove Theorem 5, which gives the limiting distribution of the number of lucky moves as $n \to \infty$.

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Proof of Theorem 5. Let us write L instead of L_n for simplicity of notation. We will compute, asymptotically, the factorial moments $\mathbb{E}[(L)_k]$ for $k \geq 1$. Fix k and let $\mathcal{E}$ be the event that the pairs of cards labeled $n-k+1, \ldots, n$ constitute lucky moves in a random deal in $\mathcal{T}_n$. Clearly,

$$\mathbb{E}[(L)_k] = (n)_k \mathbb{P}(\mathcal{E}) \sim n^k \mathbb{P}(\mathcal{E}).$$

Thus, we need to find $\mathbb{P}(\mathcal{E})$ asymptotically. To this end, we use the following algorithm to generate a random deal with n pairs of cards.

Start with an uncolored deck of cards and put the cards in a row one by one in the following way. At the first round put one of the two cards labeled 1 in a row. For $k \geq 1$, at round $k+1$, insert a card with label $\lceil (k+1)/2 \rceil$ into an interval chosen uniformly at random from the $(k+1)$ intervals determined by the first k cards already in the row, where an interval refers to a space either between two cards, or before the first card, or after the last card. After a pair of identical cards are inserted, which happens after each even-numbered step, color the one on the left with blue and the one on the right with red. We denote by $(a_1, \ldots, a_k)$ the permutation after the kth step of this algorithm, so $(a_1, \ldots, a_{2n})$ denotes a random game from $\mathcal{T}_n$.

After m pairs of cards are put down, there are $2m+1$ intervals. Let them be $I_1, \ldots, I_{2m+1}$ from left to right. For $m \geq 1$, let $S_0^{(m)}$ denote the set of intervals following red cards in $(a_1, \ldots, a_{2m})$ together with the first interval, i.e.

$$S_0^{(m)} := \{I_i : a_{i-1} \text{ is red}\} \cup \{I_1\}.$$

Recursively, for $t \geq 1$, define

$$S_t^{(m)} := \{I_i : a_{i-1} \text{ is blue and } i-1 \in S_{t-1}^{(m)}\}.$$

Let $S^{(m)} := \bigcup_{i \geq 0} S_{2i}^{(m)}$ be the set of 'good' intervals. A good interval is an interval such that if two identical cards are inserted into this interval, then those two cards constitute a lucky move. Note that the intervals are alternately good and bad as long as a red card does not interfere. The sets $S_i^{(m)}$ partition the set $[2m+1]$ and

$$|S^{(m)}| \geq |S_0^{(m)}| = m+1.$$

Event $\mathcal{E}$ is equivalent to the following event. Recursively, for $1 \leq j \leq k$, the two cards labeled $n-k+j$, which are inserted at rounds $2n-2k+2j-1$ and $2n-2k+2j$ in the algorithm are inserted next to each other and the first one is inserted into a good interval, i.e. into an interval in $S^{(n-k+j-1)}$. Note that, after such an insertion, both the number of good intervals and the number of bad intervals increase by 1, i.e. $|S^{(n-k+j)}| = |S^{(n-k+j-1)}| + 1$. Let s denote the number of good intervals after $n-k$ pairs of cards are placed, i.e. $s = |S^{(n-k)}|$. Since $s \geq n-k+1$, conditionally on s, the probability of $\mathcal{E}$ is

$$\prod_{j=0}^{k-1} \frac{s+j}{\binom{2n-2k+2j+1}{2} + 2n - 2k + 2j + 1} \sim \left(\frac{s}{2n^2}\right)^k,$$

where the numerator on the left-hand side is the number of good intervals and the denominator is the number of ways to insert the two cards with labels $n-k+j+1$. Since $|S_0^{(n-k)}| = n-k+1$ and $|S_{2i}^{(n-k)}| = \sum_{j \geq 2i+1} B_{n-k,j}$ for $i \geq 1$, we have

$$s = \sum_{i \geq 0} |S_{2i}^{(n-k)}| = 1 + (n-k) + \sum_{i \geq 1} \sum_{j \geq 2i+1} B_{n-k,j}.$$

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Hence, by Proposition 2 and the identity

$$\sum_{j\ge 2i+1} \frac{4}{(j+2)_3} = \sum_{j\ge 2i+1} \left(\frac{2}{j} - \frac{4}{j+1} + \frac{2}{j+2} \right) = \frac{2}{2i+1} - \frac{2}{2i+2}, \tag{28}$$

we obtain

$$\begin{aligned}
\mathbb{E}[s] &= n - k + 1 + \sum_{i\ge 1} \sum_{j\ge 2i+1} \mathbb{E}[B_{n-k,j}] \\
&\sim n + \sum_{i\ge 1} \sum_{j\ge 2i+1} \frac{4n}{(j+2)_3} \\
&= n + n \sum_{i\ge 1} \left(\frac{2}{2i+1} - \frac{2}{2i+2} \right) \\
&= 2n \sum_{i\ge 0} \left[\frac{1}{2i+1} - \frac{1}{2i+2} \right] \\
&= (2\ln 2)n.
\end{aligned}$$

Next we show that $s/\mathbb{E}[S] \to 1$ in probability. For this, it is enough to show that, for every $\varepsilon > 0$,

$$\mathbb{P}(|s - \mathbb{E}[s]| > \varepsilon n) \to 0 \quad \text{as } n \to \infty.$$

This probability is bounded by the sum

$$\mathbb{P}\left(\left| \sum_{i=1}^{l} \sum_{j\ge 2i+1} \overline{B}_{n-k,j} \right| > \frac{\varepsilon n}{2} \right) + \mathbb{P}\left(\left| \sum_{i>l} \sum_{j\ge 2i+1} \overline{B}_{n-k,j} \right| > \frac{\varepsilon n}{2} \right).$$

Since $\sum_j B_{m,j} = m$ for every $m \ge 1$, we have

$$\begin{aligned}
\mathbb{P}\left(\left| \sum_{i=1}^{l} \sum_{j\ge 2i+1} \overline{B}_{n-k,j} \right| > \frac{\varepsilon n}{2} \right) &= \mathbb{P}\left(\left| \sum_{i=1}^{l} \sum_{j=1}^{2i} \overline{B}_{n-k,j} \right| > \frac{\varepsilon n}{2} \right) \\
&\le \mathbb{P}\left(\text{there exists } 1 \le j \le 2l : |\overline{B}_{n-k,j}| > \frac{\varepsilon n}{2l(l+1)} \right).
\end{aligned}$$

By Corollary 2 applied with $\omega = \varepsilon^2 n/(2l(l+1))^2$, the right-hand side above goes to 0 for $l = o(n^{1/4})$.

Now consider the second probability. Since $\mathbb{P}(|\overline{X}| > t) \le 2\mathbb{E}[X]/t$ for a nonnegative random variable X and any $t > 0$, using Proposition 2, we obtain

$$\begin{aligned}
\mathbb{P}\left(\left| \sum_{i>l} \sum_{j\ge 2i+1} \overline{B}_{n-k,j} \right| > \frac{\varepsilon n}{2} \right) &\le \frac{4}{\varepsilon n} \sum_{i>l} \sum_{j\ge 2i+1} \mathbb{E}[B_{n-k,j}] \\
&\le \frac{4}{\varepsilon n} \sum_{i>l} \sum_{j\ge 2i+1} \left(\mathbb{E}[\xi_j] + \frac{4\mathrm{e}^{1/2} n}{(j+2)_3} \right) \\
&\le \frac{4}{\varepsilon n} \sum_{i>l} \left(\mathbb{P}(D_{n-k,1} \ge 2i+1) + \frac{cn}{i^2} \right) \\
&\le \frac{4}{\varepsilon n} \left(\mathbb{E}[D_{n-k,1}] + \frac{cn}{l} \right).
\end{aligned}$$

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By Proposition 1, $\mathbb{E}[D_{n-k,1}] = O(\sqrt{n})$ so that the right-hand side goes to 0 as long as $l \to \infty$ with $n \to \infty$. Thus,

$$\mathbb{E}[(L)_k] \sim n^k \left(\frac{(2\ln 2)n}{2n^2} \right)^k = (\ln 2)^k,$$

and L converges in distribution to Po(ln 2). □

6. Length of the game

In this section we prove Theorem 2 and, thus, also the asymptotic normality of the length of the game (see Corollary 1 in Section 2). The proof is an application of Theorem 1 which will be discussed at the end of this section.

We first note that by using (13), (16), the fact that $\sum_{i\geq 1} \mathbb{E}[\xi_i] = 1$, and by letting i_n be an integer of order $\sqrt{n/\log n}$, we have

$$\begin{aligned}
\sum_{i\geq 1} \mathbb{E}[B_{n,2i}] &= \sum_{i=1}^{i_n} \mathbb{E}[B_{n,2i}] + \sum_{i>i_n} \mathbb{E}[B_{n,2i}] \\
&= \sum_{i=1}^{i_n} \left(\mathbb{E}[\xi_{2i}] + \frac{4n}{(2i+2)_3} + O\left(\frac{1}{i}\right) \right) + O\left(\sum_{i>i_n} \frac{n}{i^3} \right) \\
&= \sum_{i=1}^{\infty} \frac{4n}{(2i+2)_3} + O(\log n) + O\left(\frac{n}{i_n^2} \right) \\
&= (3 - 4\ln 2)n + O(\log n),
\end{aligned}$$

where the last equality follows by the partial fraction decomposition used in (28). Hence,

$$\frac{1}{\sqrt{n}}(Y_n - (3 - 4\ln 2)n) = \frac{1}{\sqrt{n}} \sum_{i\geq 1} \overline{B}_{n,2i} + O\left(\frac{\log n}{\sqrt{n}} \right)$$

and it suffices to show that the first term on the right-hand side is asymptotically normal.

For $j \geq 1$, let $V_j = \sum_{i=1}^{j} W_{2i}$ and $V = \sum_{i=1}^{\infty} W_{2i}$, where (W_i) are defined in Theorem 1. Pick any $\varepsilon > 0$. Pick N_0 such that for every $N \geq N_0$, we have $\mathbb{P}(|V - V_N| > \varepsilon/2) < \varepsilon/3$. This is possible since the variance of $V - V_N$ goes to 0 as $N \to \infty$ and hence $V - V_N$ goes to 0 in probability. Take any $x \in \mathbb{R}$ and any $N \geq N_0$ and pick n_0 such that, for $n \geq n_0$,

$$\left| \mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i\leq N} \overline{B}_{n,2i} \leq x + \frac{\varepsilon}{2} \right) - \mathbb{P}\left(V_N \leq x + \frac{\varepsilon}{2} \right) \right| < \frac{\varepsilon}{3}$$

and

$$\left| \mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i\leq N} \overline{B}_{n,2i} \leq x - \frac{\varepsilon}{2} \right) - \mathbb{P}\left(V_N \leq x - \frac{\varepsilon}{2} \right) \right| < \frac{\varepsilon}{3}.$$

This is possible since by Theorem 1 there is a joint convergence, and thus also convergence of any finite sums.

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Now, for $n \ge n_0$, we have

$$\begin{aligned}\mathbb{P}\Big(\frac{1}{\sqrt{n}}\sum_i \overline{B}_{n,2i} \le x\Big) &= \mathbb{P}\Big(\frac{1}{\sqrt{n}}\sum_{i\le N} \overline{B}_{n,2i} + \frac{1}{\sqrt{n}}\sum_{i>N} \overline{B}_{n,2i} \le x\Big)\\ &\le \mathbb{P}\Big(\frac{1}{\sqrt{n}}\sum_{i\le N} \overline{B}_{n,2i} \le x + \frac{\varepsilon}{2}\Big) + \mathbb{P}\Big(\frac{1}{\sqrt{n}}\Big|\sum_{i>N} \overline{B}_{n,2i}\Big| > \frac{\varepsilon}{2}\Big)\\ &\le \mathbb{P}\Big(V_N \le x + \frac{\varepsilon}{2}\Big) + \frac{\varepsilon}{3} + \mathbb{P}\Big(\frac{1}{\sqrt{n}}\Big|\sum_{i>N} \overline{B}_{n,2i}\Big| > \frac{\varepsilon}{2}\Big)\\ &\le \mathbb{P}(V \le x + \varepsilon) + \frac{2\varepsilon}{3} + \mathbb{P}\Big(\frac{1}{\sqrt{n}}\Big|\sum_{i>N} \overline{B}_{n,2i}\Big| > \frac{\varepsilon}{2}\Big).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{P}\Big(\frac{1}{\sqrt{n}}\sum_i \overline{B}_{n,2i} \le x\Big) &\ge \mathbb{P}\Big(\frac{1}{\sqrt{n}}\sum_i \overline{B}_{n,2i} \le x, \frac{1}{\sqrt{n}}\Big|\sum_{i>N} \overline{B}_{n,2i}\Big| \le \frac{\varepsilon}{2}\Big)\\ &\ge \mathbb{P}(V \le x - \varepsilon) - \frac{2\varepsilon}{3} - \mathbb{P}\Big(\frac{1}{\sqrt{n}}\Big|\sum_{i>N} \overline{B}_{n,2i}\Big| > \frac{\varepsilon}{2}\Big).\end{aligned}$$

As we will show below, n_0 and N_0 can be chosen so that

$$\mathbb{P}\Big(\frac{1}{\sqrt{n}}\Big|\sum_{i>N} \overline{B}_{n,2i}\Big| > \frac{\varepsilon}{2}\Big) \le \frac{\varepsilon}{3} \quad \text{for } n \ge n_0,\ N \ge N_0. \tag{29}$$

We then have

$$\mathbb{P}(V \le x - \varepsilon) - \varepsilon \le \mathbb{P}\Big(\frac{1}{\sqrt{n}}\sum_i \overline{B}_{n,2i} \le x\Big) \le \mathbb{P}(V \le x + \varepsilon) + \varepsilon.$$

Letting $\varepsilon \to 0$ and using the fact that the distribution of V is continuous, we conclude that

$$\mathbb{P}\Big(\frac{1}{\sqrt{n}}\sum_i \overline{B}_{n,2i} \le x\Big) \to \mathbb{P}(V \le x).$$

Finally, note that $V \xrightarrow{\mathrm{D}} N(0, \sigma^2)$, where

$$\begin{aligned}\sigma^2 &= \operatorname{var}\Big(\sum_{i\ge 1} W_{2i}\Big)\\ &= \sum_{i\ge 1}\sigma_{2i,2i} + 2\sum_{1\le i<j}\sigma_{2i,2j}\\ &= 4\sum_{i=1}^{\infty}\frac{1}{(2i+2)_3} + 16\sum_{i,j=1}^{\infty}\frac{1}{(2i+2)_3(2j+2)_3} - 24\sum_{i,j=1}^{\infty}\frac{1}{(2i+2j+4)_4}\\ &= (3-4\ln 2)(4-4\ln 2) - 24\sum_{j=1}^{\infty}\frac{1}{(4j+4)_4} - 48\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}\frac{1}{(2i+2j+4)_4}\\ &= (3-4\ln 2)(4-4\ln 2) - 24\Big(-\frac{1}{24} + \frac{\ln 2}{4} - \frac{\pi}{24}\Big) - 48\Big(\frac{13}{24} + \frac{\pi}{48} - \frac{7\ln 2}{8}\Big)\\ &= (4\ln 2)^2 + 8\ln 2 - 13.\end{aligned}$$

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It remains to prove (29). Pick $i_n > N$ of order $n^{(1/3)-\eta}$, where $0 < \eta < \frac{1}{12}$. The left-hand side of (29) is bounded by

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}\left|\sum_{N<i\le i_n}\overline{B}_{n,2i}\right| > \frac{\varepsilon}{4}\right) + \mathbb{P}\left(\frac{1}{\sqrt{n}}\left|\sum_{i>i_n}\overline{B}_{n,2i}\right| > \frac{\varepsilon}{4}\right). \tag{30}$$

The first probability by Chebyshev's inequality is at most

$$\frac{16}{\varepsilon^2 n}\operatorname{var}\left(\sum_{N<i\le i_n} B_{n,2i}\right) = \frac{16}{\varepsilon^2 n}\left(\sum_{N<i\le i_n}\operatorname{var}(B_{n,2i}) + 2\sum_{N<i<j\le i_n}\operatorname{cov}(B_{n,2i}, B_{n,2j})\right).$$

It follows from Lemma 4 and the choice of i_n that the first sum is $O(n/N^2)$. By the Cauchy–Schwarz inequality,

$$|\operatorname{cov}(B_{n,2i}, B_{n,2j})| \le (\operatorname{var}(B_{n,2i})\operatorname{var}(B_{n,2j}))^{1/2} = O\left(\frac{n}{i^{3/2}j^{3/2}}\right).$$

Thus, the second sum is of order at most

$$n\sum_{i>N}\frac{1}{i^{3/2}}\sum_{j>i}\frac{1}{j^{3/2}} \le cn\sum_{i>N}\frac{1}{i^{3/2}}\frac{1}{i^{1/2}} = O\left(\frac{n}{N}\right)$$

so that the first probability in (30) is $O(1/N)$. The second probability, by Markov and (11), is bounded by

$$\begin{aligned}
\frac{4}{\varepsilon\sqrt{n}}\mathbb{E}\left[\left|\sum_{i>i_n}\overline{B}_{n,2i}\right|\right] &\le \frac{8}{\varepsilon\sqrt{n}}\mathbb{E}\left[\sum_{i>i_n}B_{n,2i}\right] \\
&\le \frac{8}{\varepsilon\sqrt{n}}\sum_{i>i_n}\left(\mathbb{E}[\xi_i] + \frac{4\mathrm{e}^{1/2}n}{i(i+1)(i+2)}\right) \\
&= \frac{8}{\varepsilon\sqrt{n}}\mathbb{P}(D_{n,1} > i_n) + O\left(\frac{\sqrt{n}}{i_n^2}\right) \\
&= O\left(\frac{1}{\sqrt{n}}\right) + O(n^{2\eta-1/6}) \\
&= o(1),
\end{aligned}$$

by our choice of i_n. Thus, for a given $\varepsilon > 0$ each of the probabilities in (30) can be made smaller than $\varepsilon/6$ by choosing N_0 and n_0 sufficiently large. This implies (29).

Proof of Theorem 1. We consider the following dynamic version of the game. For $k \ge 1$, at step k, we insert the pair of cards labeled k into the game so that the position of the blue card is chosen uniformly at random from $2k-1$ positions first and then the red card goes to the end of the row. We can represent the evolution of blocks as a generalized Pólya urn model with infinitely many types of balls: $0, 1, 2, \ldots$. Drawing a ball of type $i \ge 1$ corresponds to inserting the blue card of the new pair in an existing block of length i. Drawing a type 0 ball represents a situation when the blue and red cards of the arriving pair are both put at the end thus creating a block of size 2.

The evolution of the urn is as follows. We start with one ball of type 0. If, at any time, a ball of type i, $i \ge 1$, is drawn, it is replaced by one ball of type 1 and one ball of type $i+1$. If

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a ball of type 0 is drawn, it is returned to the urn along with one ball of type 2 and the process is continued in the same manner. After n draws, there will be one ball of type 0 and n balls of other types. With this interpretation, the number of blocks of length i when $2n$ cards have been played, $B_{n,i}$, is the number of balls of type i in the urn after the nth draw; their asymptotic distributions have been studied widely in the literature.

In fact, if not for the ball of type 0 it would be *exactly* the situation of outdegrees in a random plane recursive tree studied by Janson in [14] (see Theorem 1.3 in particular) and in a much more general setting in [13] except that his $i+1$ (and $i+2$, $i+3$) are our i, $i+1$, and $i+2$, respectively.

To deal with this type 0 ball, we wish to apply [21, Theorem 2.1]. In [21], a type 0 ball is referred to as an *immigration ball*. Let us call the vector $(a_{m,k})_{k=0}^{K}$ representing the number of balls added to the urn after an immigration ball is drawn the *immigration vector*. Our immigration vector $(a_{m,k})_{k=0}^{K}$ (see [21, p. 646, item (a)]) is time independent and is given by

$$[a_{m,0}, \ldots, a_{m,k}] = [0, 0, 1, 0, \ldots, 0].$$

Similarly, the addition (or replacement) matrices are time independent, nonrandom, and the number of balls in the urn increases when a nonimmigration ball is drawn. Thus, our situation falls into the second of the three cases described in [21, Section 2.3] and in this case [21, Theorem 2.1] states that the immigration is asymptotically negligible and the urn has the same asymptotics as without immigration.

The urn without immigration was studied by Janson [13], [14], who reduced infinitely many types to finitely many and used a 'superball trick'; see [13, Remark 4.2].

After these reductions, in Janson's notation (and including the immigration balls), if $\xi_i = (\xi_{0,i}, \xi_{1,i}, \ldots, \xi_{M,i})$ is the replacement (column) vector when a ball of type i is drawn, then with $\delta_{i,j}$ denoting the Kronecker delta, $\xi_{0,j} = 2\delta_{2,j}$, i.e. $\xi_0 = (0, 0, 2, 0 \ldots, 0)$.

For $1 \le i < M$,

$$\xi_{i,j} = \delta_{1,j} - i\delta_{i,j} + (i+1)\delta_{i+1,j} = (0, 1, 0 \ldots, 0, -i, i+1, 0, \ldots, 0),$$

where $-i$ is on the $(i+1)$th position (we start enumeration at 0). Finally,

$$\xi_{M,j} = \delta_{1,j} + \delta_{M,j} = (0, 1, 0, \ldots, 0, 1).$$

The replacement matrix A_0 (which has vectors ξ_j as its columns) is

$$A_0 = \boldsymbol{H}' = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 2 & 2 & -2 & 0 & \ldots & 0 & 0 \\ & & \ldots & & \ldots & & \\ 0 & 0 & \ldots & 0 & M-1 & -M+1 & 0 \\ 0 & 0 & \ldots & 0 & 0 & M & 1 \end{bmatrix},$$

where $\boldsymbol{H}$ was the notation used in [21]. Note that if A is a matrix obtained from A_0 by removing its first column and its first row, then it is exactly the matrix considered by Janson [14] for the random plane recursive tree (with i shifted by 1 since he starts enumeration at $i = 0$ and ours, after discarding balls of type 0, starts at $i = 1$). It follows from his argument (see the proof of [14, Theorem 1.3] and Remark 3(ii) below) that the eigenvalues of A are $2, -1, \ldots, -(M-1)$

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and, consequently, that (1) holds with covariances of (W_i) given by

$$\sigma_{i,j} = 2\sum_{k=0}^{i-1}\sum_{l=0}^{j-1}\frac{(-1)^{k+l}}{k+l+4}\binom{i-1}{k}\binom{j-1}{l}\left(\frac{2(k+l+4)!}{(k+3)!\,(l+3)!} - 1 - \frac{(k+1)(l+1)}{(k+3)(l+3)}\right).$$

As was shown in [1, Proposition 1], this expression simplifies to (2). □

Remark 3. (i) It is tempting to incorporate the ball of type 0 in the general framework of Janson, i.e. to consider A_0 as the replacement matrix. An additional eigenvalue is 0, so we are still in the regime $\mathrm{Re}(\lambda_2) < \frac{1}{2}\lambda_1$ so that Janson's general theory [13] would apply. Accordingly, for the joint distribution of $(B_{n,i})_{i\geq 0}$, we would have

$$\frac{1}{n}(B_{n,i})_{i=0}^{\infty} \to \left(0, \left(\frac{4}{(i+2)_3}\right)_{i=1}^{\infty}\right) \quad \text{almost surely as } n\to\infty$$

and

$$\frac{1}{\sqrt{n}}\left(B_{n,0}-1, \left(B_{n,i} - \frac{4n}{(i+2)_3}\right)_{i=1}^{\infty}\right) \xrightarrow{\mathrm{D}} (W_i)_{i=0}^{\infty} \quad \text{as } n\to\infty,$$

where (W_i) are jointly Gaussian with covariances $\sigma_{i,j}$ given by (2) if $i, j \geq 1$, and $\sigma_{ij} = 0$ if $i = 0$ or $j = 0$. In particular, W_0 is degenerate, as it should be. The formal difficulty is that the condition in [13, Equation (A6)] is not satisfied: there are two equivalence classes $\{0\}$ and $\{1, 2, \dots\}$ and the first one is dominating while the second is not (a ball of type 0 is never put in unless it is drawn). Probably Janson's theory could be extended to include such cases, but we do not know this.

(ii) Janson found the eigenvalues and eigenvectors of A by directly solving the system of linear equations. Alternatively, as he himself suggested, one can proceed by induction. Indeed, after expanding $A_0 - \lambda I$ with respect to the first row and then repeatedly expanding the resulting cofactors with respect to their last rows, we see that the characteristic polynomial of A_0 is

$$p_{A_0}(\lambda) = (-1)^{M-1}\lambda(\lambda-2)\prod_{j=0}^{M-1}(\lambda+j).$$

The eigenvalues then may be computed by solving the system $(A_0-\lambda I)v = 0$. For example, when $\lambda = 2$ this gives $v_0 = 0$ and

$$\lambda v_1 = \sum_{i=2}^{M} v_i, \qquad j v_{j-1} = (\lambda+j)v_j, \quad 2\leq j < M, \qquad M v_{M-1} = (\lambda-1)v_M,$$

which are solved by $v_{M-1} = v_M/M$, and

$$\begin{aligned} v_{M-1} &= \frac{j+3}{j+1}v_{j+1} \\ &\vdots \\ &= \frac{(j+3)(j+4)\cdots(M+1)}{(j+1)(j+2)\cdots M}v_M \\ &= \frac{M+1}{(j+1)(j+2)}v_M, \qquad 1\leq j\leq M-2. \end{aligned}$$

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Choosing $v_M = 2/(M+1)$ normalizes v so that $\sum_j v_j = 1$. This is exactly what Janson gives (see [14, Equation (5.6)]) except that our j is his $i+1$.

The other eigenvectors (left and right) can be computed in a similar fashion. We skip further details.

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