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Complexity and varieties for infinitely generated modules, II

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1. *Introduction*

It has now been almost twenty years since Alperin introduced the idea of the complexity of a finitely generated kG-module, when G is a finite group and k is a field of characteristic $p > 0$. In proving one of the first major results in the area [**1**], Alperin and Evens demonstrated the connection of the study of complexity for modules to the group cohomology. That connection eventually led to the categorization of modules according to their associated varieties in the maximal ideal spectrum of the cohomology ring $H^*(G, k)$. In all of the work that has followed, two principles have proved to be extremely important. The first is that the associated variety of a module is directly related to the structure of the module through the rank variety which is defined by the matrix representation of the module. The second major result is the tensor product theorem which says that the variety associated to a tensor product $M \otimes_k N$ is the intersection of the varieties associated to the modules M and N. In this paper we generalize these results to infinitely generated kG-modules.

Our investigation of homological properties of infinitely generated kG-modules was started with the study of complexity in [**5**]. In that paper we defined the complexity and took our first stab at the problem of defining associated support varieties. As we pointed out, there are numerous problems to be encountered. For one thing, cohomology is not finitely generated, and hence the definition of the associated variety as the variety of the annihilator of the cohomology does not work well. Instead, what we define is an extended notion of the rank variety of a module. The definition is valid when the group is an elementary abelian p-group, and when the module is finitely generated then the new rank variety is essentially the same as the old. The main problem with the definition is that the new rank variety is not really a variety, but only a collection of sets which are closed in the Zariski topology. All indications are that this particular feature is unavoidable. On the other hand the new

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rank variety does satisfy the tensor product theorem and can be extended to groups which are not elementary abelian.

The next three sections of the paper consist of preliminaries aimed at the proof of the new version of 'Dade's Lemma' and the characterization of the rank variety in terms of generic points. As we saw in [**5**], it is possible to construct a module whose rank variety in the old sense is simply affine r-space with all of the points removed. At the same time the module in the example is far from projective. Consequently it is necessary to extend the field in order to distinguish the module from a projective module. This is exactly the point: the new rank variety can be given as the old rank variety taken over a large extension K of k. Then Dade's lemma says that the module is projective if and only if its variety is trivial. On the other hand a point in the variety over K must be generic for some irreducible subvariety over k. Hence, with some preliminaries we are able to identify the new rank variety with a subset of the collection of all irreducible closed subvarieties of affine r-space.

In Section 5 we begin the development of the cohomological variety by showing how the cohomology can be adapted to give another proof of Dade's lemma. As a preliminary to later sections we also discuss a few of the details of the problems that are encountered in the communication between the cohomological and the rank varieties. Section 5 provides a proof of the tensor product theorem for rank varieties. Although it is not directly related to the calculation of the cohomological varieties it is an important tool in the development of some of the properties. In particular in Section 10 it is used in showing that the rank variety is isomorphic (actually only isogenous) to the cohomological variety in the case in which the group is elementary abelian and in showing that the tensor product theorem holds for the cohomological variety in general. The latter is accomplished by restriction to the elementary abelian p-subgroups.

The actual definition of the cohomological variety is a two tiered process. First we must recall, in Section 8, the development of the idempotent modules given in [**18**]. Our summary of the facts about these modules is very brief, but it is here that we really use the cohomology to give the name 'cohomological' to the variety. The fact is that the idempotent modules of interest are defined by the colimits of triangles defined by cohomology and enjoy a universal property with respect to the subcategory of finitely generated modules defined by subvarieties of the spectrum of the cohomology ring. The cohomological variety is defined in Section 10 in terms of a tensor product property involving the idempotent modules. The rank varieties of the idempotent modules, given in Section 9, play an important part in the theory. We end the paper by showing the relationship of this cohomological variety with an earlier idea defined in [**5**].

We should mention that the results of this paper have already seen some significant applications. The third author has used the extended version of Dade's lemma in his characterization of the thick subcategories of the stable category of finitely generated kG-modules in the case where G is a finite p-group. The first author has used some of the results to settle most of the questions raised in [**6**] on modules with vanishing cohomology. In addition, the second author in collaboration with Peng has used relative cohomology to develop idempotent systems of finitely generated modules which represent ideals in the cohomology rings and not just subvarieties of the spectrum [**10**].

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2. *Generic points*

We begin with a discussion of generic points in this and the next section. The information is in some sense classical (see for example Zariski and Samuel[**19**], chapters VI and VII for a discussion considerably more general than ours), but it is hard to find the material in the form in which we wish to use it.

Let k be an algebraically closed field of characteristic p, and let $E = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian p-group of rank r. Let $V_E^r(k)$ denote the affine r-space $J(kE)/J^2(kE)$. Then $V_E^r(k)$ has as coordinate vectors the images x_i of the elements $X_i = g_i - 1 \in J(kE)\,(1 \leqslant i \leqslant r)$. We write y_i for the linear function on $V_E^r(k)$ which takes the value one on x_i and zero on the remaining $x_j\,(j \neq i)$. Then the coordinate ring of $V_E^r(k)$ is the polynomial ring

$$k[V_E^r(k)] = k[y_1, \dots, y_r].$$

In order to define rank varieties for infinitely generated modules, we must first introduce a field extension of k large enough to define appropriate generic points. We denote by K an algebraically closed field of large transcendence degree over k. We shall see that such a field is big enough so that the variety $V_E^r(K)$ contains generic points for every closed irreducible subvariety V of $V_E^r(k)$ of dimension less than or equal to the transcendence degree of K over k.

Let V be a closed irreducible subvariety of $V_E^r(k)$, with corresponding prime ideal $\mathfrak{p} \subseteq k[V_E^r(k)]$. Then the coordinate ring of V is $k[V] = k[V_E^r(k)]/\mathfrak{p}$, and the function field $k(V)$ is the field of fractions of $k[V]$. Writing $\overline{y}_i$ for the image of y_i in $k(V)$, we say that the point defined over the extension field $k(V)$ of k:

$$\overline{y}_1 x_1 + \dots + \overline{y}_r x_r \in V_E^r(k(V))$$

is 'the' generic point for V. It is a canonically defined element of $V_E^r(k(V))$, which is independent of the choice of basis $x_1, \dots, x_r$.

If K is algebraically closed and has transcendence degree over k at least $\dim(V)$, then the function field $k(V)$ may be embedded as a subfield of K containing k (and in such a way that the embedding is an isomorphism on k). Let $t_1, \dots, t_r$ be the images in K of $\overline{y}_1, \dots, \overline{y}_r$ under such an embedding. Then there is a corresponding generic point

$$\alpha_V = t_1 x_1 + \dots + t_r x_r.$$

Conversely, given any point in $V_E^r(K)$, say $\alpha = t_1 x_1 + \dots + t_r x_r$, let $\mathfrak{p}$ be the ideal in $k[y_1, \dots, y_r]$ consisting of the polynomial relations satisfied by $t_1, \dots, t_r$ over k. Since k is a field, the ideal $\mathfrak{p}$ is prime, and so there is a corresponding closed irreducible subvariety $V \subseteq V_E^r(k)$ whose coordinate ring is $k[V] = k[V_E^r(k)]/\mathfrak{p}$. Thus there is a uniquely determined homomorphism from $k(V)$ into K with the property that the corresponding extension of scalars $V_E^r(k(V)) \hookrightarrow V_E^r(K)$ sends the generic point of $V_E^r(k(V))$ to α; namely the homomorphism sending each $\overline{y}_i$ to the corresponding t_i. In this situation, we say that α is 'a' generic point for V.

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PROPOSITION 2·1. *Each point in $V_E^r(K)$ is generic for a uniquely determined closed irreducible subvariety of $V_E^r(k)$. If K is algebraically closed of transcendence degree d over k, then $V_E^r(K)$ contains generic points for all closed irreducible subvarities of dimension at most d in $V_E^r(k)$.*

Proof. This may be found in Section $\mathbf{5}^{\text{bis}}$ of chapter VI of Zariski and Samuel [**19**].

3. *Generic points for projective varieties*

As in the last section, we let

$$k[V_E^r(k)] = k[y_1, \dots, y_r].$$

Regarding this as a graded ring with each y_i in degree one, it is the graded coordinate ring of the projective space $\bar{V}_E^r(k)$ of dimension $r-1$ over k, whose points correspond to the lines through the origin in $V_E^r(k)$. A closed homogeneous irreducible subvariety V corresponds to a closed irreducible subvariety $\bar{V}$ of $\bar{V}_E^r(k)$, and the corresponding prime ideal in $k[V_E^r(k)]$ is homogeneous.

If $\bar{V}$ is a closed irreducible subvariety of $\bar{V}_E^r(k)$, corresponding to a homogeneous prime ideal $\mathfrak{p}$ in $k[V_E^r(k)]$, then $k[V] = k[V_E^r(k)]/\mathfrak{p}$ is a graded integral domain. Localizing at the multiplicative subset consisting of all nonzero positive degree homogeneous elements, we obtain a $\mathbb{Z}$-graded ring whose degree zero elements form a field denoted $k(\bar{V})$. This is called the function field of $\bar{V}$. The homogeneous elements $\bar{y}_i$ are not in $k(\bar{V})$, but rather all the ratios $\bar{y}_i/\bar{y}_j$ are in $k(\bar{V})$ whenever $\bar{y}_j$ is nonzero in $k[V]$. It follows that there is a well-defined line

$$(\bar{y}_1 : \bar{y}_2 : \dots : \bar{y}_r) \in \bar{V}_E^r(k(\bar{V})).$$

For example, if $\bar{y}_j \neq 0$ then the point $(\bar{y}_1/\bar{y}_j, \dots, 1, \dots, \bar{y}_r/\bar{y}_j)$ is in this line. We say that this point in $\bar{V}_E^r(k(\bar{V}))$ is 'the' generic point for $\bar{V}$. It is a canonically defined element, independent of the choice of basis $x_1, \dots, x_r$.

If K is an algebraically closed field, having transcendence degree over k at least $\dim(\bar{V}) = \dim V - 1$, then the function field $k(\bar{V})$ may be embedded as a subfield of K containing k. Moreover, it may be embedded in such a way that the embedding is an isomorphism on k. Let $t_1, \dots, t_r$ be the images of $\bar{y}_1/\bar{y}_j, \dots, 1, \dots, \bar{y}_r/\bar{y}_j$ in K, for some choice of j so that $\bar{y}_j$ is nonzero in $k[V]$. Then there is a corresponding generic point

$$(t_1 : \dots : t_r) \in \bar{V}_E^r(K).$$

It does not depend on the choice of j, because the ratios of the t_is do not depend on this choice.

Conversely, given any point $(t_1 : \dots : t_r) \in \bar{V}_E^r(K)$, let $\mathfrak{p}$ be the ideal in $k[y_1, \dots, y_r]$ generated by the homogeneous polynomial relations between the t_is over k. This ideal only depends on the ratios of the t_is.

PROPOSITION 3·1. *Provided K has transcendence degree at least r over k, each line through the origin in $V_E^r(K)$ contains a generic point for a uniquely determined closed homogeneous irreducible subvariety of $V_E^r(k)$. For any point on such a line, this is the smallest homogeneous subvariety containing the subvariety for which the point is generic.*

Proof. See the discussion in Section 4^{bis} of chapter VII of Zariski and Samuel [**19**].

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If V is not homogeneous, then there is an element $y \in K$ which is transcendental over the subfield of K generated by k and the (defined) ratios of the λ_is. The point $y\alpha = y\lambda_1 x_1 + \ldots + y\lambda_r x_r$ is in the same line as α. Given any polynomial satisfied by $y\lambda_1, \ldots, y\lambda_r$ over k, we may separate out the different powers of y to see that each homogeneous part of the polynomial must also be satisfied. So $y\alpha$ is a generic point for the smallest closed homogeneous irreducible subvariety containing V.

Remark 3·2. For $V_E^r(K)$ to have generic points for all closed irreducible subvarieties of $V_E^r(k)$, it is necessary and sufficient for K to have transcendence degree at least r over k. For $V_E^r(K)$ to have lines through the origin corresponding to all of the closed homogeneous irreducible subvarieties of $V_E^r(k)$, it is only necessary for K to have transcendence degree at least $r-1$ over k. The extra transcendence degree simply provides enough generic points on these lines.

4. *A Kronecker quiver lemma*

The following lemma may be regarded as an infinite dimensional substitute for eigenvalues, see Remark 4·2.

LEMMA 4·1. *Suppose that V and W are vector spaces over an algebraically closed field k, and that K is a non-trivial extension of k. Suppose that f and g are linear maps from V to W with the property that for every pair of scalars λ and μ in K, not both zero, the linear map $\lambda f + \mu g : K \otimes_k V \to K \otimes_k W$ is an isomorphism. Then both V and W are the zero vector space.*

Proof. Use g to identify V and W. Then f is a k-endomorphism of V with the property that for all $\mu \in K$, the endomorphism $1 \otimes f + \mu \otimes \mathrm{Id}$ of $K \otimes_k V$ is invertible. The action of f makes V into a $k[t]$-module, with t acting as f does. Since $f + \mu . \mathrm{Id}$ is invertible for all $\mu \in k$, and k is algebraically closed, V is a $k(t)$-module. Now in the ring $K \otimes_k k(t)$, any element of the form $1 \otimes t + \mu \otimes 1$ with μ an element of K which is not in k, is not invertible. If V is nonzero then it has $k(t)$ as a summand as a $k(t)$-module, so that $1 \otimes f + \mu \otimes \mathrm{Id}$ is not invertible. This contradicts the hypothesis, and so the lemma is proved.

Remark 4·2. Of course, if V and W are finite dimensional, then passing to an extension field K is not necessary. One just uses the theory of eigenvalues to see that some $f + \mu g$ has a nonzero kernel.

5. *Rank varieties and Dade's lemma*

Let $E = \langle g_1, \ldots, g_r \rangle$ be an elementary abelian p-group of rank r, and let $K \supseteq k$ be algebraically closed fields of characteristic p. If M is a kE-module (not necessarily finitely generated), we define $V_E^r(K \otimes_k M)$ as follows. Given a point $0 \neq \alpha = \lambda_1 x_1 + \ldots + \lambda_r x_r$ in $V_E^r(K)$, we set

$$u_\alpha = 1 + \lambda_1 X_1 + \ldots + \lambda_r X_r,$$

where $X_i = g_i - 1 \in kE \subseteq KE$. Then u_α is a unit of order p in the group algebra KE, so we may ask whether $(K \otimes_k M) \downarrow_{\langle u_\alpha \rangle}$ is free. The set $V_E^r(K \otimes_k M)$ is defined to consist of the origin, together with the set of points $\alpha \in V_E^r(K)$ such that $(K \otimes_k M) \downarrow_{\langle u_\alpha \rangle}$ is not free.

We call the above defined subgroup $\langle u_\alpha \rangle$ of the group algebra KE a *cyclic shifted*

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subgroup of E. More generally, if $\alpha_1, \dots, \alpha_s$ are linearly independent elements of $V_E^r(K)$, we define the subgroup $\langle u_{\alpha_1}, \dots, u_{\alpha_s} \rangle$ of KE to be the *shifted subgroup* of E corresponding to $\alpha_1, \dots, \alpha_s$.

In the case of a finitely generated module M, Dade's lemma [13] says that M is projective if and only if $V_E^r(M) = \{0\}$. For an infinitely generated module, this is no longer true. But this is, in some sense, because $V_E^r(k)$ does not have enough points.

LEMMA 5·1. *Suppose that g is an element of order p in the centre of a finite p-group G, and that M is a kG-module. Then M is projective if and only if $M\downarrow_{\langle g \rangle}$ is projective and $M/(g-1)M$ is a projective $k(G/\langle g \rangle)$-module.*

Proof. The 'only if' part is clear. So suppose that $M\downarrow_{\langle g \rangle}$ is projective and that the quotient $M/(g-1)M$ is a projective $k(G/\langle g \rangle)$-module. If $m \in M$ is a G-fixed point, then there is an element $m' \in M$ such that $m = (g-1)^{p-1} m' = \sum_{i=0}^{p-1} g^i m'$. If $h \in G$ then $(g-1)^{p-1}(h-1)m' = (h-1)m = 0$, and so there is an element $m'' \in M$ with $(h-1)m' = (g-1)m''$. Thus $m' + (g-1)M$ is a $G/\langle g \rangle$-fixed point in $M/(g-1)M$. So there is an element $m''' \in M$ such that $\sum_{h \in G/\langle g \rangle} h(m''') - m' \in (g-1)M$. Here, the sum runs over a set of coset representatives of $\langle g \rangle$ in G. Thus we have that $\sum_{h \in G} h(m''') = \sum_{i=0}^{p-1} g^i m' = m$. Hence every fixed point is a trace, and so M is a projective kG-module.

THEOREM 5·2 (Dade's lemma for infinitely generated modules). *Suppose that k is an algebraically closed field, and that K is an algebraically closed field of transcendence degree at least $r-1$ over k. Let E be an elementary abelian p-group of rank r. Then a kE-module M is projective if and only if $V_E^r(K \otimes_k M) = \{0\}$.*

Proof. We offer two proofs of this theorem, both of which rely on Lemma 4·1. The second proof is given in the next section. Our first proof is a modification of the one in Dade[13], in which his eigenvalue argument has been replaced by the use of Lemma 4·1. For the sake of easy reading, we reproduce his argument with appropriate modifications.

If M is projective, then it is clear that $V_E^r(K \otimes_k M) = \{0\}$. So assume that $V_E^r(K \otimes_k M) = \{0\}$. We argue by induction on the rank r of E. In case $r = 1$, the result is obvious, so we begin with the case in which $r = 2$. We may assume that M has no projective summands. Let $M_1 = J(kE)^p M$, $M_2 = J(kE)^{p-1} M$, and M_3 be the submodule of M consisting of the elements m such that $J(kE)m \subseteq M_2$. Then $M \supseteq M_3 \supseteq M_2 \supseteq M_1$, and $J(kE)$ annihilates M_3/M_2 and M_2/M_1. Let $\hat{M} = K \otimes_k M$, $\hat{M}_1 = K \otimes_k M_1 = J(KE)^p \hat{M}$, $\hat{M}_2 = K \otimes_k M_2 = J(KE)^{p-1} \hat{M}$, and $\hat{M}_3 = K \otimes_k M_3$ consists of the elements m in $\hat{M}$ such that $J(KE)m \subseteq \hat{M}_2$. We claim that for any λ and μ in K not both zero, the map ξ induced by $x = \lambda x_1 + \mu x_2$ from $\hat{M}_3/\hat{M}_2$ to $\hat{M}_2/\hat{M}_1$ is an isomorphism.

To see that it is injective, if $m + \hat{M}_2$ is in the kernel of ξ, then $xm \in \hat{M}_1$. Now $xJ(KE)^{p-1} = J(KE)^p$, so $x\hat{M}_2 = \hat{M}_1$, and $xm = xm'$ for some $m' \in \hat{M}_2$., Then $x(m-m') = 0$, and since the restriction of $\hat{M}$ to $\langle 1+x \rangle$ is free, it follows that $m - m' = x^{p-1} m''$ for some $m'' \in \hat{M}$. Thus $m = m' + x^{p-1} m'' \in \hat{M}_2$.

To see that it is surjective, we must prove that $x\hat{M}_3 = \hat{M}_2$. Choose an element $y = \lambda' x_1 + \mu' x_2$ not in the line containing x. Then $J(KE)^{p-1}$ is K-linearly spanned by y^{p-1} and $xJ(KE)^{p-2}$, and we have $J(KE)^{p-2}\hat{M} \subseteq \hat{M}_3$, and so $\hat{M}_2 = y^{p-1}\hat{M} + x\hat{M}_3$. So if $x\hat{M}_3$ is a proper subspace of $\hat{M}_2$, then there is an element $m \in \hat{M}$ such that

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$y^{p-1}m \notin x\hat{M}_3$. Since $\hat{M}$ has no projective summands, we have $x^{p-1}y^{p-1}m = 0$. Because the restriction of $\hat{M}$ to $\langle 1+x\rangle$ is free, it follows that there is an element $m' \in \hat{M}$ such that $y^{p-1}m = xm'$. Then $xym' = y^p m = 0$, and again from the fact that the restriction of $\hat{M}$ to $\langle 1+x\rangle$ is free, it follows that there is an element $m'' \in \hat{M}$ such that $x^{p-1}m'' = ym'$. So both xm' and ym' are in $\hat{M}_2$, which means that $m' \in \hat{M}_3$. But then $y^{p-1}m = xm' \in x\hat{M}_3$, contradicting the choice of m.

Now we may apply Lemma 4·1 to the maps induced by x_1 and x_2 from M_3/M_2 to M_2/M_1 to deduce that $M_1 = M_2 = M_3$. Thus $J(kE)^{p-1}M = J(kE)^p M$, which implies that $J(kE)^{p-1}M = 0$. But then the fact that the restriction to $\langle 1+x_1\rangle$ is projective implies that $M = 0$. This completes the proof in the case $r = 2$.

If $r \geqslant 3$, and the theorem is true for groups of rank at most $r-1$, we argue as follows. Let $\bar{M} = M/x_r M$, regarded as a module over the subgroup E' generated by $g_1, \ldots, g_{r-1}$. For each choice of $\lambda_1, \ldots, \lambda_{r-1} \in K$, not all zero, let K_0 be the algebraic closure in K of the subfield generated by k together with the ratios between $\lambda_1, \ldots, \lambda_{r-1}$. In other words, if $1 \leqslant m \leqslant r-1$ with $\lambda_m \neq 0$, then K_0 is the subfield generated by $\lambda_1/\lambda_m, \ldots, \lambda_{r-1}/\lambda_m$. So K_0 has transcendence degree at most $r-2$ over k, and so K is strictly bigger than K_0. Applying the theorem to the rank two subgroup of $K_0 E$ generated by $1+(\lambda_1/\lambda_m)x_1+\ldots+(\lambda_{r-1}/\lambda_m)x_{r-1}$ and g_r, we see that $V_{E'}^r(K \otimes_k \bar{M}) = \{0\}$. It follows that $\bar{M}$ is projective as a kE'-module. Since the restriction of M to the subgroup generated by g_r is also projective, we may apply lemma 5·1 to deduce that M is a projective kE-module. This completes our first proof of theorem. ∎

At this point we could define the rank variety of a kE-module M to be the subset $V_E^r(K \otimes_k M)$ of $V_E^r(K)$, where K is any algebraically closed extension of k having sufficiently large transcendence degree. Of course, sufficiently large means that $V_E^r(K)$ should have generic points for all closed homogeneous irreducible subvarieties of $V_E^r(k)$ (see Remark 3·2). The obvious defect of this definition is that it would seem to depend on the choice of the extension K. One way of remedying this defect is to replace each line through the origin in $V_E^r(K)$ by the closed homogeneous irreducible subvariety for which it is generic. To make sense of this definition, we need the following.

PROPOSITION 5·3. *Let M be a kE-module and let $V \subseteq V_E^r(k)$ be a closed homogeneous irreducible subvariety. Suppose that K and K' are algebraically closed extensions of transcendence degree at least* $\dim V$, *and that $\alpha \in V_E^r(K)$* and *$\beta \in V_E^r(K')$ are generic points for V. Then $\alpha \in V_E^r(K \otimes_k M)$ if and only if $\beta \in V_E^r(K' \otimes_k M)$.*

Proof. For notation, we refer the reader to the discussion in Section 2. Let $X_i = g_i - 1$, where $E = \langle g_1, \ldots, g_r\rangle$. For each i, let x_i be the image of X_i in $J(kE)/J^2(kE)$, so that the generic point of V in $V_E^r(k(V))$ is

$$\gamma = \bar{y}_1 x_1 + \cdots + \bar{y}_r x_r.$$

To avoid confusion, we identify $V_E^r(k(V))$ with $k(V)^r$ and $V_E^r(K)$ with K^r. In this notation, the generic point of V is $\gamma = (\bar{y}_1, \ldots, \bar{y}_r)$. Write $\alpha = (\alpha_1, \ldots, \alpha_r) \in K^r$, $\beta = (\beta_1, \ldots, \beta_r) \in (K')^r$. There exist uniquely determined field homomorphisms $\sigma: k(V) \to K$ and $\tau: k(V) \to K'$ such that for all i, $\sigma(\bar{y}_i) = \alpha_i$ and $\tau(\bar{y}_i) = \beta_i$, and such that σ and τ extend the embedding of k into K.

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Now a non-zero element $\alpha \in V_E^r(K)$ is in $V_E^r(K \otimes_k M)$ if and only if $K \otimes_k M$ is not free as a $K\langle u_\alpha \rangle$-module, where $u_\alpha = 1 + \sum_i \alpha_i \otimes X_i \in K \otimes_k kE$. Let

$$u_\gamma = 1 + \sum_i \bar{y}_i \otimes X_i \in k(V) \otimes_k kE.$$

We note that there are isomorphisms

$$K \otimes_{k(V)} k(V) \otimes_k kE \to K \otimes_k kE \cong KE$$

and

$$K \otimes_{k(V)} k(V) \otimes_k M \to K \otimes_k M$$

given by $a \otimes b \otimes u \to a.\sigma(b) \otimes u$. The former isomorphism sends $1 \otimes u_\gamma$ to u_α. Clearly $k(V) \otimes_k M$ is free as a $k(V)\langle u_\gamma \rangle$-module if and only if $K \otimes_{k(V)} k(V) \otimes_k M$ is free as a $K\langle 1 \otimes u_\gamma \rangle$-module. This is because the latter is obtained from the former by extension of scalars. So via the above isomorphism, $K \otimes_k M$ is free as a $K\langle u_\alpha \rangle$-module if and only if $k(V) \otimes_k M$ is free as a $k(V)\langle u_\gamma \rangle$-module. Hence we are done, because we can make the same argument with α replaced by β and σ replaced by τ.

Definition 5·4. Let $\mathscr{V}_E^r(k)$ be the set of all non-zero closed homogeneous irreducible subvarieties of $V_E^r(k)$. For any kE-module M, let $\mathscr{V}_E^r(M)$ be the set of all $V \in \mathscr{V}_E^r(k)$ such that for any algebraically closed extension K of transcendence degree at least $\dim V$ and any generic point $\alpha \in V_E^r(K)$ for V, we have $\alpha \in V_E^r(K \otimes_k M)$.

It follows from Proposition 3·1 that whenever K is an algebraically closed extension of transcendence degree at least $r-1$ over k, then $V_E^r(K \otimes_k M)$ can be recovered from $\mathscr{V}_E^r(M)$.

Remarks 5·5. (i) Recall that a closed homogeneous irreducible set in $V_E^r(k)$ corresponds to a homogeneous prime ideal in $k[y_1, \ldots, y_r]$. So $\mathscr{V}_E^r(M)$ is a subset of the projective scheme $\mathbf{Proj}\, k[y_1, \ldots, y_r]$.

(ii) If M is finitely generated then

$$\mathscr{V}_E^r(M) = \{V \in \mathscr{V}_E^r(k) \mid V \subseteq V_E^r(M)\}.$$

COROLLARY 5·6. *A kE-module M is projective if and only if $\mathscr{V}_E^r(M) = \varnothing$.*

Proof. This follows from Proposition 5·3 and Theorem 5·2.

6. *Cohomology, Bocksteins, and a different view of Dade's lemma*

In this section we introduce some cohomological considerations which also play a role in the later sections of the paper. The explanation of the Bockstein and the map on varieties is a bit technical but also unavoidable. The proof of Dade's lemma that we obtain from this construction is in some sense a generalization of the cohomological proof in Carlson [7].

Again let $E = \langle g_1, \ldots, g_r \rangle$ be an elementary abelian p-group of rank r. Recall that

$$H^*(E, k) = \begin{cases} k[\zeta_1, \ldots, \zeta_r] \otimes \Lambda(\eta_1, \ldots, \eta_r) & \text{if} \quad p > 2 \\ k[\eta_1, \ldots, \eta_r] & \text{if} \quad p = 2. \end{cases}$$

Here $\deg(\zeta_i) = 2$, $\deg(\eta_i) = 1$ and Λ is the exterior algebra generated by $\eta_1, \ldots, \eta_r$. The generators of the cohomology may be chosen so that $\mathrm{res}_{E, \langle g_i \rangle}(\eta_j) = 0$ and $\mathrm{res}_{E, \langle g_i \rangle}(\zeta_j) = 0$ whenever $j \neq i$, and also that $\mathrm{res}_{E, \langle g_i \rangle}(\eta_i)$ and $\mathrm{res}_{E, \langle g_i \rangle}(\zeta_i)$ are the 'canonical' generators in degrees one and two respectively.

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In the case that $k = \mathbb{F}_p$, there is a linear cohomology operation $\beta: H^n(E, \mathbb{F}_p) \to H^{n+1}(E, \mathbb{F}_p)$ called the Bockstein map. On degree one elements it is given by $\beta(\eta_i) = \zeta_i$, where ζ_i is taken to mean η_i^2 if $p = 2$. Some problems arise when we attempt to extend β to cohomology with coefficients in a larger field k. First notice that $H^1(E, k)$ is naturally isomorphic to $\mathrm{Hom}(E, k)$ which we may identify as the k-linear dual of $J(kE)/J^2(kE) = V_E^r(k)$. Likewise the k-linear dual of $H^2(E, k) \cap \mathrm{Im}(\beta)$ can be identified with the maximal ideal spectrum of the polynomial ring $k[\zeta_1, \ldots, \zeta_r]$. The latter is denoted $V_E(k)$. Hence the Bockstein β should induce a map of k-linear duals $\beta^*: V_E(k) \to V_E^r(k)$ which is certainly bijective. The problem is to make the map commute with restrictions to cyclic shifted subgroups.

The technicality needed here is the following. Suppose that $\zeta = f(\zeta_1, \ldots, \zeta_r)$ is a polynomial of degree n in the degree two generators $\zeta_1, \ldots, \zeta_r$ (so that ζ has degree $2n$). Let $u_\alpha = 1 + \lambda_1 X_1 + \ldots + \lambda_r X_r$ generate a cyclic shifted subgroup in kE (see Section 5 for notation). Then

$$\mathrm{res}_{E, \langle u_\alpha \rangle}(\zeta) = f(\lambda_1^p, \ldots, \lambda_r^p)\, \mu,$$

where μ is the canonical degree two generator of $H^2(\langle u_\alpha \rangle, k)$ (see for example [7]). It follows that in order to make the Bockstein commute with restriction to cyclic shifted subgroups on kE, it is necessary to extend it semilinearly, using the Frobenius map. That is, we set

$$\beta\left(\sum_i \lambda_i \eta_i\right) = \sum_i \lambda_i^p \beta(\eta_i).$$

The reason for doing this is that the restriction to a cyclic shifted subgroup can be viewed as an element of the dual space of $H^1(E, k)$ or $H^2(E, k) \cap \mathrm{Im}(\beta)$. So the map on the dual space is a bijective semilinear homomorphism

$$\beta^*: V_E(k) \to V_E^r(k)$$

given by

$$\beta^*(\alpha_1, \ldots, \alpha_r) = (\alpha_1^p, \ldots, \alpha_r^p).$$

As a map of varieties it is an isogeny but not an isomorphism since its inverse is not a polynomial map. Summing up, we have the following.

Proposition 6·1 (Avrunin and Scott [3]). *The Bockstein homomorphism induces a semilinear isogeny $\beta^*: V_E(k) \to V_E^r(k)$ which commutes with restrictions to shifted subgroups of E.*

With the information above we can now present our second proof of Dade's lemma. This proof is slightly shorter but makes use of more sophisticated ideas.

Second proof of Theorem 5·2. We proceed by induction on the rank r of E. If $r = 1$, the theorem is obvious. So we shall assume that $r > 1$ and that the theorem is true for elementary abelian p-groups of rank $r-1$. Let K' be an algebraically closed field with $k \subseteq K' \subseteq K$, and such that K' has transcendence degree $r-2$ over k.

Choose two linearly independent elements η and η' in $H^1(E, \mathbb{F}_p)$, so that their Bocksteins $\beta(\eta)$ and $\beta(\eta')$ may be regarded as algebraically independent elements of $H^2(E, \mathbb{F}_p) \subseteq H^2(E, K)$. The idea will be to show that for all λ and μ in K, not both zero, the maps f and g induced by $\beta(\eta)$ and $\beta(\eta')$:

$$f, g: \widehat{\mathrm{Ext}}^0_{K'E}(K', K' \otimes_k M) \to \widehat{\mathrm{Ext}}^2_{K'E}(K', K' \otimes_k M)$$

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($\widehat{\mathrm{Ext}}$ = Tate Ext) have the property that for all λ and μ in K, not both zero, the map

$$\lambda f+\gamma g\colon \widehat{\mathrm{Ext}}^0_{KE}(K,K\otimes_k M)\to \widehat{\mathrm{Ext}}^2_{KE}(K,K\otimes_k M)$$

is an isomorphism. (Note that $\widehat{\mathrm{Ext}}$ behaves well under field extensions.) Then we deduce from Lemma 4·1 that $\widehat{\mathrm{Ext}}^0_{K'E}(K',K'\otimes_k M)=0$, so that $\widehat{\mathrm{Ext}}^0_{kE}(k,M)=0$, and hence M is projective.

To prove that $\lambda f+\gamma g$ is an isomorphism (λ and μ not both zero), we argue as follows. Let E' be a shifted subgroup of E of rank $r-1$ over K, corresponding to the kernel of

$$\lambda^{1/p}\eta+\mu^{1/p}\eta'\in H^1(E,K)\cong \mathrm{Hom}\,(J(KE)/J^2(KE),K).$$

Then the element $\lambda\beta(\eta)+\mu\beta(\eta')=\beta(\lambda^{1/p}\eta+\mu^{1/p}\eta')\in H^2(E,K)\cong \mathrm{Ext}^2_{KE}(K,K)$ is represented by the exact sequence

$$0\to K\to K_{KE'}\uparrow^{KE}\to K_{KE'}\uparrow^{KE}\to K\to 0.$$

We would like to say that tensoring this sequence with M gives us a sequence of the form

$$0\to K\otimes_k M\to (K\otimes_k M)\downarrow_{KE'}\uparrow^{KE}\to (K\otimes_k M)\downarrow_{KE'}\uparrow^{KE}\to K\otimes_k M\to 0.$$

For then, by the inductive hypothesis, $(K'\otimes_k M)\downarrow_{K'E'}$ is projective, and hence so is $(K\otimes_k M)\downarrow_{KE'}\uparrow^{KE}$. This implies that $\lambda\beta(\eta)+\mu\beta(\eta')$ induces a stable isomorphism from $\Omega^2(K\otimes_k M)$ to $K\otimes_k M$, so that $\lambda f+\mu g$ is an isomorphism as required.

It is not quite true that tensoring the above sequence with M gives us a sequence of the desired form, because shifted subgroups do not behave in quite the same way as subgroups in this respect. However, we can argue as follows. The map $\lambda f+\mu g$ can be defined using Yoneda products instead of cup products. This means that we can change the Hopf algebra structure on KE without altering the effect of this map. So we change the group basis so that E' is an actual subgroup of E rather than just a shifted subgroup. With this Hopf algebra structure, tensoring the sequence with M has the desired effect. This completes our second proof of the Theorem 5·2.

7. *The tensor product theorem*

The main problem with proving the tensor product theorem, namely, that the rank variety of a tensor product of two modules is the intersection of their rank varieties, is that tensor product does not commute with restriction to cyclic shifted subgroups. To get around this, we begin with an external form of the tensor product theorem and then restrict to a diagonal subgroup to get an internal tensor product theorem.

For finitely generated modules the tensor product theorem was proved in Carlson [**7**]. Avrunin and Scott verified the theorem for cohomological varieties when they proved Proposition 6·1. The proof that we give here follows the lines of a second proof in [**3**].

Let $k\subseteq K$ be an extension of fields of characteristic p. We begin with the remark that for k-vector spaces M and N, there is a natural isomorphism

$$K\otimes_k (M\otimes_k N)\cong (K\otimes_k M)\otimes_K (K\otimes_k N).$$

Remark 7·1. In contrast to the tensor product, in general $K\otimes_k \mathrm{Hom}_k(M,N)$ can be a proper subspace of $\mathrm{Hom}_K(K\otimes_k M,K\otimes_k N)$.

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THEOREM 7·2. *Suppose that E and E' are elementary abelian p-groups, M is a kE-module, and N' is a kE'-module. Then*

$$V^r_{E\times E'}(K\otimes_k(M\otimes_k N)) = V^r_E(K\otimes_k M)\times V^r_{E'}(K\otimes_k N)$$

as subvarieties of $V^r_{E\times E'}(K) = V^r_E(K)\times V^r_{E'}(K)$.

Proof. For points in either $V^r_E(K)\times\{0\}$ or $\{0\}\times V^r_{E'}(K)$, the statement is clear. If $0\neq\alpha\in V^r_E(K)$ and $0\neq\beta\in V^r_{E'}(K)$, then we have elements $u_\alpha\in KE$ and $u_\beta\in KE'$, so that $\langle u_\alpha\rangle\times\langle u_\beta\rangle$ is a rank two subgroup of $K(E\times E') = KE\otimes_K KE'$. Since restriction to cyclic shifted subgroups does commute with taking external tensor products, we have

$$(K\otimes_k(M\otimes_k N))\downarrow_{\langle u_\alpha\rangle\times\langle u_\beta\rangle}\cong(K\otimes_k M)\downarrow_{\langle u_\alpha\rangle}\otimes_K(K\otimes_k N)\downarrow_{\langle u_\beta\rangle}.$$

We need to restrict further to $u_{(\alpha,\beta)}\in K(\langle u_\alpha\rangle\times\langle u_\beta\rangle) = K\langle u_\alpha\rangle\otimes_K K\langle u_\beta\rangle$. This generates a shifted subgroup of $\langle u_\alpha\rangle\times\langle u_\beta\rangle$ corresponding to a point in $V_{\langle u_\alpha\rangle\times\langle u_\beta\rangle}$ not in the subsets $V_{\langle u_\alpha\rangle}$ or $V_{\langle u_\beta\rangle}$.

For the group algebra of a cyclic group, every module (including infinite dimensional ones) is a direct sum of indecomposables of dimensions between 1 and p (see for example Köthe[**15**] or Auslander[**2**]). The variety of an external tensor product of such indecomposables is:

(i) the origin $\{0\}$ if both have dimension p,

(ii) the line $V_{\langle u_\alpha\rangle}$ *or* $V_{\langle u_\beta\rangle}$ if one has dimension p and the other does not, and

(iii) The whole of $V_{\langle u_\alpha\rangle\times\langle u_\beta\rangle}$ if neither has dimension p (because then the dimension of the tensor product is not divisible by p).

It follows that the restriction $K\otimes_k(M\otimes_k N)\cong(K\otimes_k M)\otimes_K(K\otimes_k N)$ to $\langle u_{(\alpha,\beta)}\rangle$ is not projective if and only if both $(K\otimes_k M)\downarrow_{\langle u_\alpha\rangle}$ and $(K\otimes_k N)\downarrow_{\langle u_\beta\rangle}$ have a summand of dimension less than p; namely if neither of them is projective. Thus $(\alpha,\beta)\in V^r_{E\times E'}(K\otimes_k(M\otimes_k N))$ if and only if both $\alpha\in V^r_E(K\otimes_k M)$ and $\beta\in V^r_{E'}(K\otimes_k N)$.

PROPOSITION 7·3. *If $E_1\leqslant E_2$ are elementary abelian p-groups and M is a kE_2-module, then regarding $V^r_{E_1}(K)$ as a subvariety of $V^r_{E_2}(K)$, we have that*

$$V^r_{E_1}(K\otimes_k M) = V^r_{E_1}(K)\cap V^r_{E_2}(K\otimes_k M).$$

Proof. This is clear from the definitions, since whether a point in $V^r_{E_1}(K)$ is in the rank variety does not depend on whether it is regarded as a point in $V^r_{E_1}(K)$ or as a point in $V^r_{E_2}(K)$.

THEOREM 7·4. *Let E be an elementary abelian p-group, and let M and N be kE-modules. Then*

$$V^r_E(K\otimes_k(M\otimes_k N)) = V^r_E(K\otimes_k M)\cap V^r_E(K\otimes_k N).$$

Proof. Let $\Delta: E\to E\times E$ be the diagonal embedding defined by $\Delta(g) = (g,g)$. Then the internal tensor product $M\otimes_k N$ as a kE-module is equal to the restriction of the external tensor product via the diagonal embedding Δ. So we have

$$\begin{aligned} V^r_E(K\otimes_k(M\otimes_k N)) &= V^r_{\Delta(E)}(K)\cap V^r_{E\times E}(K\otimes_k(M\otimes_k N)) \\ &= V^r_{\Delta(E)}(K)\cap(V^r_E(K\otimes_k M)\times V^r_E(K\otimes_k N)) \\ &= V^r_E(K\otimes_k M)\cap V^r_E(K\otimes_k N). \end{aligned}$$

This completes the proof of the theorem.

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COROLLARY 7·5. (*See Definition* 5·4) *Let E be an elementary abelian p-group and let M and N be kE-modules. Then $\mathscr{V}_E^r(M \otimes_k N) = \mathscr{V}_E^r(M) \cap \mathscr{V}_E^r(N)$.*

In particular, $M \otimes_k N$ is projective if and only if $\mathscr{V}_E^r(M) \cap \mathscr{V}_E^r(N) = \varnothing$.

Proof. The first part follows from Theorem 7·4, and the second part then follows from Corollary 5·6.

PROPOSITION 7·6. *Let M be a kE-module, and let $u, v \in kE$ be units of order p with $u - v \in J^2(kE)$. Then $M\downarrow_{\langle u \rangle}$ is free if and only if $M\downarrow_{\langle v \rangle}$ is free.*

Proof. The proof of this is exactly parallel to the proof of lemma 6·4 of Carlson [**7**], but with the reference to theorem 5·6 of [**7**] replaced by a reference to the above Theorem 7·4.

Question 7·7. What is the rank variety of $M^* = \mathrm{Hom}_k(M, k)$, or more generally of $\mathrm{Hom}_k(M, N)$? The difficulty with this question is related to Remark 7·1.

8. *Idempotent modules*

In preparation for a discussion of the cohomological variety of a module in the next section, we summarize the necessary material on idempotent modules in the stable category from Rickard [**18**].

If $\mathscr{C}$ is a thick subcategory of the stable category of finitely generated kG-modules $\mathbf{stmod}(kG)$, we denoted by $\mathscr{C}^{\oplus}$ the smallest full subcategory of the stable module category $\mathbf{StMod}(kG)$ that contains $\mathscr{C}$ and is closed under arbitrary direct sums. Then for each object X in $\mathbf{StMod}(kG)$, there is a distinguished triangle

$$\mathscr{E}_{\mathscr{C}}(X) \to X \to \mathscr{F}_{\mathscr{C}}(X) \to \Omega^{-1}\mathscr{E}_{\mathscr{C}}(X)$$

characterized by the following properties. The object $\mathscr{E}_{\mathscr{C}}(X)$ lies in $\mathscr{C}^{\oplus}$, and

$$\mathrm{Hom}_{\mathbf{StMod}(kG)}(Y, \mathscr{F}_{\mathscr{C}}(X)) = 0$$

for every Y in $\mathscr{C}$, and hence also for every Y in $\mathscr{C}^{\oplus}$. We express this by saying that $\mathscr{F}_{\mathscr{C}}(X)$ is **local** with respect to $\mathscr{C}$ or with respect to $\mathscr{C}^{\oplus}$.

Corresponding to each collection $\mathscr{W}$ of closed homogeneous subvarieties $V \subseteq V_G(k)$ closed under specialization (i.e. $V \in \mathscr{W}$, $W \subseteq V$ implies $W \in \mathscr{W}$) and finite unions, there is a thick subcategory $\mathscr{C}_{\mathscr{W}}$ of $\mathbf{stmod}(kG)$ consisting of the modules whose variety is an element of $\mathscr{W}$. We write

$$E(\mathscr{W}) \to k \to F(\mathscr{W}) \to \Omega^{-1}E(\mathscr{W})$$

for the corresponding triangle of the trivial module. Then the corresponding triangle of any module X is given by

$$E(\mathscr{W}) \otimes X \to X \to F(\mathscr{W}) \otimes X \to \Omega^{-1}E(\mathscr{W}) \otimes X.$$

The modules $E(\mathscr{W})$ and $F(\mathscr{W})$ are idempotent in the stable category, in the sense that there are isomorphisms $E(\mathscr{W}) \otimes E(\mathscr{W}) \cong E(\mathscr{W}) \oplus$ (projective) and $F(\mathscr{W}) \otimes F(\mathscr{W}) \cong F(\mathscr{W}) \oplus$ (projective).

A couple of properties of these modules will be particularly useful.

LEMMA 8·1. *Assume that $\mathscr{W}$ is as above.*

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(i) *For any finitely generated module M with $V_G(M) \in \mathscr{W}$ we have that $M \otimes E(\mathscr{W}) \cong M \oplus P$ for some projective module P.*

(ii) *$E(\mathscr{W})$, and every object in $\mathscr{C}_{\mathscr{W}}$ are filtered colimits of finitely generated modules whose varieties are elements of $\mathscr{W}$.*

Proof. The first statement is lemma 5·14 of Rickard [**18**]. Part (ii) follows from the construction preceding lemma 5·3 of that paper. Or viewed another way, lemmas 5·3 and 5·17 of [**18**] assure us that the modules A_i of [**18**] are filtered colimits of objects in $\mathscr{C}_{\mathscr{W}}$, and $E(\mathscr{W})$ is the colimit of the A_is. ∎

If V is a closed homogeneous subvariety of $V_G(k)$, then taking for $\mathscr{W}$ the set of all closed homogeneous subvarieties contained in V, we define $E(V) = E(\mathscr{W})$ and $F(V) = F(\mathscr{W})$, and we get a triangle

$$E(V) \to k \to F(V) \to \Omega^{-1}E(V). \tag{8·1}$$

Similarly, taking for $\mathscr{W}'$ the set of all closed homogeneous subvarieties of V not containing any irreducible component of V, we define $E'(V) = E(\mathscr{W}')$ and $F'(V) = F(\mathscr{W}')$, and we get a triangle

$$E'(V) \to k \to F'(V) \to \Omega^{-1}E'(V). \tag{8·1′}$$

Finally, we write $\kappa(V)$ for the module $E(V) \otimes F'(V)$.

LEMMA 8·2. *If $i\colon H \to G$ is the inclusion of a subgroup H of G, and $\mathscr{W}$ is a collection of closed homogeneous subvarieties of $V_G(k)$ closed under specialization, then the triangle*

$$E(\mathscr{W}) \to k \to F(\mathscr{W}) \to \Omega^{-1}E(\mathscr{W})$$

restricts to

$$E(i_*^{-1}(\mathscr{W})) \to k \to F(i_*^{-1}(\mathscr{W})) \to \Omega^{-1}E(i_*^{-1}(\mathscr{W})).$$

The module $\kappa(V)$ restricts to $\kappa(W)$, where W is the union of the irreducible components of $i_*^{-1}(V)$ whose images under i_* are components of V.

Proof. Let $\mathscr{C}$ be the subcategory of $\mathbf{stmod}(kG)$ consisting of modules whose variety is in $\mathscr{W}$, and let $\mathscr{C}'$ be the subcategory of $\mathbf{stmod}(kH)$ consisting of modules whose variety is in $i_*^{-1}\mathscr{W}$. Then $E(\mathscr{W})$ is in $\mathscr{C}^{\oplus}$, and so $E(\mathscr{W})\downarrow_H$ is in $(\mathscr{C}')^{\oplus}$. If M is a kH-module in $\mathscr{C}'$, then

$$\mathrm{Hom}_{\mathbf{StMod}(kH)}(M, F(\mathscr{W})\downarrow_H) \cong \mathrm{Hom}_{\mathbf{StMod}(kG)}(M\uparrow^G, F(\mathscr{W})) = 0,$$

so that $F(\mathscr{W})\downarrow_H$ is local with respect to $\mathscr{C}'$. The result now follows from the characterization of the triangle for $i_*^{-1}(\mathscr{W})$.

The statement about $\kappa(V)$ follows by examining $i_*^{-1}(\mathscr{W})$ and $i_*^{-1}(\mathscr{W}')$ for the collections $\mathscr{W}$ and $\mathscr{W}'$ used to define the modules $E(V)$ and $F'(V)$. ∎

We end this section with some further properties of the idempotent modules which will be useful in our discussion of the cohomological varieties.

LEMMA 8·3. *Suppose that V and W are closed homogeneous subvarieties of $V_G(k)$.*

(i) *There is a canonical stable isomorphism*

$$E(V) \otimes E(W) \to E(V \cap W).$$

(ii) *Provided that V does not contain any irreducible component of W, there is a canonical stable isomorphism $E(V) \otimes E'(W) \to E(V \cap W)$.*

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(iii) *If $V \subseteq W$ then $E(V) \otimes F(W)$ is projective.*

(iv) *If $V \subseteq W$ and V does not contain any irreducible component of W then $E(V) \otimes F'(W)$ is projective.*

Proof. Statement (i) is part (a) of proposition 6·2 of [**18**]. The proof of statement (ii) is essentially the same. To prove statement (iii), look at the triangle

$$E(V) \otimes E(W) \to E(V) \otimes k \to E(V) \otimes F(W) \to E(V) \otimes \Omega^{-1}E(W)$$

and notice that the first map is the canonical stable isomorphism described in (i). The proof of statement (iv) is similar, using (ii).

PROPOSITION 8·4. *If V and W have the property that no irreducible component of one is contained in any irreducible component of the other (i.e. the intersection contains no irreducible component of either V or W), then we have the following:*

(i) *There is a canonical stable isomorphism $F'(V \cup W) \to F'(V) \otimes F'(W)$.*

(ii) *The module $\kappa(V) \otimes E'(W)$ is projective.*

(iii) *There is a canonical stable isomorphism $E(V) \otimes F'(V \cup W) \to \kappa(V)$.*

(iv) *There is a canonical stable isomorphism $\kappa(V \cup W) \to \kappa(V) \oplus \kappa(W)$.*

Proof. Statement (i) is proved in the same way as Theorem 7·5 of [**18**]. To prove statement (ii), we use part (ii) of the lemma to deduce that there is a stable isomorphism

$$\kappa(V) \otimes E'(W) = E(V) \otimes F'(V) \otimes E'(W) \to E(V \cap W) \otimes F'(W),$$

and part (iv) of the lemma to see that this is projective.

To prove statement (iii), we use statement (i) to see that there is a canonical stable isomorphism

$$E(V) \otimes F'(V \cup W) \to E(V) \otimes F'(V) \otimes F'(W) = \kappa(V) \otimes F'(W).$$

Now apply statement (ii) to the triangle

$$\kappa(V) \otimes E'(W) \to \kappa(V) \otimes k \to \kappa(V) \otimes F'(W) \to \Omega^{-1}\kappa(V) \otimes E'(W)$$

to see that there is a canonical stable isomorphism $\kappa(V) \to \kappa(V) \otimes F'(W)$.

To prove statement (iv), we consider the triangle

$$\begin{aligned} E(V \cap W) \otimes F'(V \cup W) &\to (E(V) \oplus E(W)) \otimes F'(V \cup W) \\ &\to E(V \cup W) \otimes F'(V \cup W) \\ &\to \Omega^{-1}E(V \cap W) \otimes F'(V \cup W) \end{aligned}$$

obtained by tensoring the Mayer–Vietoris triangle (section 8 of [**18**]) with $F'(V \cup W)$. The left hand term is projective by part (iv) of the lemma, the middle term is canonically stably isomorphic to $\kappa(V) \oplus \kappa(W)$ by statement (iii), and the third term is $\kappa(V \cup W)$.

THEOREM 8·5. *Let $V_1, \dots, V_t$ be the irreducible components of a closed homogeneous subvariety $V \subseteq V_G(k)$. Then there is a canonical stable isomorphism between $\kappa(V)$ and $\kappa(V_1) \oplus \dots \oplus \kappa(V_t)$.*

Proof. This follows by repeated application of part (iv) of the proposition.

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9. *Rank varieties for idempotent modules*

An important and possibly confusing point regarding the last section is that the collections $\mathscr{W}$ used to define idempotent modules are collections of closed subvarieties of the *cohomological* variety $V_G(k)$. For this section and the rest of the paper it is necessary to be careful about the distinction between it and the rank variety. For one thing, the rank variety is defined only for elementary abelian p-groups, though we shall see that restrictions to the elementary abelian p-subgroups determine all of the varieties.

THEOREM 9·1. *Let E be an elementary abelian p-group. Suppose that $\mathscr{W}$ is a collection of closed homogeneous subvarieties of $V_E(k)$ and that $\mathscr{W}$ is closed under specialization and finite unions. Then the rank variety $\mathscr{V}_E^r(E(\mathscr{W}))$ is exactly the set of subvarieties $\beta^*(V)$ such that $V \in \mathscr{W}$ and V is irreducible.*

In other words, if K is an algebraically closed extension of k of sufficiently large transcendence degree, then a point $\alpha \in V_E^r(K)$ is in $V_E^r(K \otimes_k E(\mathscr{W}))$ if and only if α is a generic point of $\beta^*(V)$ for some $V \in \mathscr{W}$.

Proof. The equivalence of the two statements is a consequence of Proposition 5·3. Suppose that $V \in \mathscr{W}$ is irreducible. Let M be a finitely generated module with $V_E^r(M) = \beta^*(V)$ (see [**4**], corollary 5·9·2). Then $V_E(M) = V$ and $M \otimes E(\mathscr{W}) \cong M \oplus P$ with P a projective module, by Lemma 8·1. Now by the tensor product theorem (Corollary 7·5), $V \in \mathscr{V}_E^r(E(\mathscr{W}))$.

For the converse suppose that $V \notin \mathscr{W}$. Choose $\alpha \in V_G(k)$ so that $\beta^*(\alpha)$ is generic for $\beta^*(V)$. By Lemma 8·1 (ii), $E(\mathscr{W})$ is a filtered colimit of some system $\{A_i\}_{i \in I}$ with $V_E(A_i) \in \mathscr{W}$ and A_i finitely generated for all i. Let $\langle u_\alpha \rangle$ be the cyclic shifted subgroup of KE determined by $\beta^*(\alpha)$. Because $V \notin \mathscr{W}$ we must have that $\beta^*(\alpha) \notin V_E^r(K \otimes_k A_i) = \beta^*(V_E(K \otimes_k A_i))$. Hence $A_i \downarrow_{\langle u_\alpha \rangle}$ is a projective $K\langle u_\alpha \rangle$-module by the definition of the rank variety. Because the restriction to $\langle u_\alpha \rangle$ commutes with the colimit, we must have that $E(\mathscr{W}) \downarrow_{\langle u_\alpha \rangle}$ is a free $K\langle u_\alpha \rangle$-module, and that $\beta^*(\alpha) \notin V_E^r(K \otimes_k E(\mathscr{W}))$. This implies that $\beta^*(V) \notin \mathscr{V}_E^r(E(\mathscr{W}))$ as desired.

COROLLARY 9·2. *For E and $\mathscr{W}$ as in the Theorem, we have that*

$$\mathscr{V}_E^r(F(\mathscr{W})) = \{\beta^*(V) \in \mathscr{V}_E^r(k) \mid V \notin \mathscr{W}\}.$$

Also, if $V \subseteq V_E(k)$ is a closed homogeneous subvariety then

$$\mathscr{V}_E^r(\kappa(V)) = \{\beta^*(V_1), \ldots, \beta^*(V_t)\},$$

where $V_1, \ldots, V_t$ are the irreducible components of V.

Proof. The first part comes from the fact that $E(\mathscr{W}) \otimes F(\mathscr{W})$ is projective. Hence $\mathscr{V}_E^r(E(\mathscr{W})) \cap \mathscr{V}_E^r(F(\mathscr{W})) = \varnothing$. On the other hand if K is an algebraically closed extension of k of transcendence degree at least the rank of E then there is an exact sequence

$$0 \to K \otimes_k E(\mathscr{W}) \to K \oplus P \to K \otimes_k F(\mathscr{W}) \to 0,$$

where P is a projective module. Hence we must have that

$$\mathscr{V}_E^r(K \oplus P) = \mathscr{V}_E^r(K) = \mathscr{V}_E^r(K \otimes_k E(\mathscr{W})) \cup \mathscr{V}_E^r(K \otimes_k F(\mathscr{W})).$$

Now apply Propositions 5·3 and 3·1.

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The calculation of the variety of $\kappa(V)$ is a straightforward application of the Tensor Product theorem (Corollary 7·5).

10. *Cohomological varieties*

In [5] we formulated a notion of a cohomological variety for an infinitely generated module. The notion had some defects which were pointed out in an appendix to the paper. Among other things, it did not satisfy a tensor product theorem, and it did not behave well on restriction to subgroups. In this section we introduce a different cohomological variety which repairs some of the defects of the old one. We use the notation $\hat{\mathscr{V}}_G(M)$ to denote the collection which was referred to as the cohomological variety in [5]. The symbol $\mathscr{V}_G(M)$ will denote the new variety. We need to recall some basic facts about the complexity of modules.

For M a finitely generated kG-module the complexity of M is the dimension of $V_G(M)$. Several equivalent extensions of the notion to infinitely generated modules were given in [5]. One of the most useful of these is that a kG-module M has complexity c if it is a filtered colimit of finitely generated modules of complexity at most c, but not a filtered colimit of finitely generated modules of complexity less than c. A couple of useful facts about complexity are contained in the following.

LEMMA 10·1. (i) *If M and N are kG-modules then the complexity of $M \otimes N$ is never greater than the complexity of either M or N.*

(ii) *Suppose that $\mathscr{W}$ is a collection of subvarieties of $V_G(k)$ closed under specialization and finite unions. Then the complexity of $E(\mathscr{W})$ is equal to the maximum of* dim V *for $V \in \mathscr{W}$.*

Proof. Part (i) follows directly from the definition, though it is also implicit in proposition 8·2 of [5]. Part (ii) is a consequence of the fact discussed earlier, that $E(\mathscr{W})$ is a filtered colimit of finitely generated modules whose varieties are in $\mathscr{W}$. Hence the complexity of $E(\mathscr{W})$ is at most the maximum value of dim V for $V \in \mathscr{W}$. On the other hand, if N is a finitely generated module with $V_G(N) \in \mathscr{W}$, then $E(\mathscr{W}) \otimes N \cong N \oplus P$ with P a projective kG-module. Hence the complexity of $E(\mathscr{W})$ is at least dim $V_G(N)$ by (i).

Definition 10·2. The cohomological variety $\mathscr{V}_G(M)$ of a kG-module M is the set of all closed homogeneous irreducible subvarieties V of $V_G(k)$ with the property that $M \otimes E(V)$ has complexity equal to the dimension of V.

Notice here that $\mathscr{V}_G(k)$ is the set of all closed homogeneous irreducible subvarieties of $V_G(k)$ by Lemma 10·1(ii). Hence as a set, $\mathscr{V}_G(k)$ is the projective prime ideal spectrum $\mathbf{Proj}\, H^*(G, k)$. For any kG-module M, $\mathscr{V}_G(M)$ is a subset of $\mathscr{V}_G(k)$. It can be checked that if M is finitely generated then $\mathscr{V}_G(M)$ is the set of all closed homogeneous irreducible subvarieties of $V_G(M)$.

LEMMA 10·3. *Suppose that M is a kG-module and that V is a closed homogeneous irreducible subvariety of $V_G(k)$. Then $V \in \mathscr{V}_G(M)$ if and only if $\kappa(V) \otimes M$ is not projective.*

Proof. From the triangle (8·1′) we see that if $M \otimes \kappa(V) = M \otimes E(V) \otimes F'(V)$ is projective then $M \otimes E(V) \otimes E'(V) \cong M \otimes E(V) \oplus$ (projective). But the complexity of $M \otimes E(V) \otimes E'(V)$ is at most equal to the complexity of $E'(V)$ which is dim $V - 1$. On

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the other hand if the complexity of $M \otimes E(V)$ is less than the dimension of V then $M \otimes E(V)$ is a filtered colimit of finitely generated modules whose varieties are proper closed subsets of V (tensor together filtered systems of modules of complexity at most dim $V-1$ for $M \otimes E(V)$ and of modules whose variety is contained in V for $E(V)$ to obtain a filtered system satisfying both conditions for $M \otimes E(V)$). Any such module must have the property that its tensor product with $F'(V)$ is projective. So $M \otimes \kappa(V)$ is a filtered colimit of projective modules and hence is projective.

LEMMA 10·4. *Let V be a closed homogeneous irreducible subvariety of $V_G(k)$. Then the varieties of $\kappa(V)$ and $E(V)$ are given by $\mathscr{V}_G(\kappa(V)) = \{V\}$ and $\mathscr{V}_G(E(V)) = \{W \in \mathscr{V}_G(k) \mid W \subseteq V\}$.*

Proof. The proof of both statements relies on proposition 6·2 of Rickard [**18**], which says that $E(V) \otimes E(W) \cong E(V \cap W) \oplus$ (projective). So $\kappa(V) \otimes \kappa(W) \cong E(V \cap W) \otimes F'(V) \otimes F'(W) \oplus$ (projective), which is projective if $V \cap W$ is a proper subvariety of either V or W. Since both V and W are irreducible, this says that $W \in \mathscr{V}_G(\kappa(V))$ if and only if $V = W$. The proof of the second statement is similar.

THEOREM 10·5. *Suppose that E is an elementary abelian p-group and M is a kE-module. Then $\beta^*(\mathscr{V}_E(M)) = \mathscr{V}^r_E(M)$. Here, $\beta^*(\mathscr{V}_E(M))$ means the set $\{\beta^*(V) \mid V \in \mathscr{V}_E(M)\}$.*

Proof. Notice that $V \in \mathscr{V}_E(M)$ if and only if $\kappa(V) \otimes M$ is not projective. But $\kappa(V) \otimes M$ is not projective if and only if $\mathscr{V}^r_E(\kappa(V)) \cap \mathscr{V}^r_E(M) \neq \varnothing$ by the Tensor Product theorem (Corollary 7·5). Hence by Corollary 9·2 the last condition holds if and only if $\beta^*(V) \in \mathscr{V}^r_E(M)$. ∎

Let $\mathscr{E}A(G)$ denote the set of elementary abelian p-subgroups of G. For $E \in \mathscr{E}A(G)$ we have a restriction map $\mathrm{res}_{G,E}: H^*(G,k) \to H^*(E,k)$, and a corresponding map $\mathrm{res}^*_{G,E}: \mathscr{V}_E(k) \to \mathscr{V}_G(k)$ on projective prime ideal spectra. One of the implications of theorem 6·2 of Quillen [**16**] is that

$$V_G(k) = \bigcup_{E \in \mathscr{E}A(G)} \mathrm{res}^*_{G,E}(V_E(k)).$$

Our next task is to extend this theorem to modules. To do so we should first recall Chouinard's theorem [**12**], which says that a kG-module is projective if and only if its restriction to every elementary abelian p-subgroup of G is projective.

THEOREM 10·6. *Suppose that M is a kG-module. Then*

$$\mathscr{V}_G(M) = \bigcup_{E \in \mathscr{E}A(G)} \mathrm{res}^*_{G,E}(\mathscr{V}_E(M\downarrow_E)).$$

Proof. First suppose that $V \in \mathscr{V}_G(M)$. Then $\kappa(V) \otimes M$ is not projective, so by Chouinard's Theorem, for some $E \in \mathscr{E}A(G)$, $(\kappa(V) \otimes M)\downarrow_E$ is not projective. Now by Lemma 8·2, $\kappa(V)\downarrow_E \cong \kappa(W)$, where $W = \bigcup_{i=1}^m W_i$ is the union of the irreducible subvarieties $W_i \in V_E(k)$ such that $\mathrm{res}^*_{G,E}(W_i) = V$. Now by Theorem 8·5 we know that $\kappa(W)$ is stably isomorphic to $\kappa(W_1) \oplus \ldots \oplus \kappa(W_m)$. So for some W_i, $(\kappa(W_i) \otimes M)\downarrow_E$ is not projective and $W_i \in \mathscr{V}_E(M\downarrow_E)$. Hence $V = \mathrm{res}^*_{G,E}(W_i) \in \mathrm{res}^*_{G,E}(\mathscr{V}_E(M\downarrow_E))$.

Suppose conversely that $V = \mathrm{res}^*_{G,E}(W)$ for some elementary abelian p-subgroup E and some $W \in \mathscr{V}_E(M\downarrow_E)$. Then $(\kappa(W) \otimes M)\downarrow_E$ is not projective and is a direct summand of $(\kappa(V) \otimes M)\downarrow_E$ as in the above argument. Hence $V \in \mathscr{V}_G(M)$ as desired. ∎

We notice at this point that the theorem is not as strong as we would like. It says only that if $V \in \mathscr{V}_G(M)$ then there exists an elementary abelian p-subgroup $E \in \mathscr{E}A(G)$

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such that $V \in \mathrm{res}^*_{G,E}(\mathscr{V}_E(M\downarrow_E))$. It does not tell us what the fibre of the restriction homomorphism $\mathrm{res}^*_{G,E}$ is on the variety of M. For this we need the following. The next theorem is the analogue for infinitely generated modules of the Avrunin–Scott stratification theorem [3] for finitely generated modules.

THEOREM 10·7. *Suppose that* $V \in \mathscr{V}_G(M)$ *and that* $V = \mathrm{res}^*_{G,E}(W)$ *for some* $E \in \mathscr{E}A(G)$ *and* $W \in \mathscr{V}_E(k)$. *Then* $W \in \mathscr{V}_E(M\downarrow_E)$.

Proof. From the previous theorem, we get that there is an elementary abelian p-subgroup E' such that $V = \mathrm{res}^*_{G,E'}(W')$ for $W' \in \mathscr{V}_{E'}(M\downarrow_{E'})$. Now let $E'' \subseteq E'$ be a minimal subgroup of E' such that W' is supported on E''. That is, $W' = \mathrm{res}^*_{E',E''}(W'')$ for some $W'' \in \mathscr{V}_{E''}(k)$, but W' is not in the image of res* from any proper subgroup of E''. Let K be an algebraically closed extension of large transcendence degree over k and let $\alpha \in V^r_{E''}(K)$ be a generic point for $\beta^*(W'')$. Then $\mathrm{res}^*_{E',E''}(\alpha) \in V^r_{E'}(K)$ is a generic point for $\beta^*(W')$.

Let $j: E'' \to E'$ be the inclusion. Let u_α generate a cyclic shifted subgroup of KE'' corresponding to α. Then $j(u_\alpha)$ generates a cyclic shifted subgroup of KE' corresponding to $\mathrm{res}^*_{E',E''}(\alpha)$. In coordinates, and in the notation of Section 5, suppose that $E' = \langle g_1, \dots, g_r\rangle, E'' = \langle g_1, \dots, g_s\rangle$ for $s \leqslant r$, and set $X_i = g_i - 1$. Then $u_\alpha = 1 + \sum_{i=1}^s \lambda_i X_i$ and $j(u_\alpha) = 1 + \sum_{i=1}^r \lambda_i X_i$, where $\lambda_i = 0$ if $s < i \leqslant r$. Since $\alpha \in V^r_{E'}(K \otimes_k M\downarrow_{E'})$, the module $(M\downarrow_{E'})\downarrow_{\langle j(u_\alpha)\rangle} = (M\downarrow_{E''})\downarrow_{\langle u_\alpha\rangle}$ is not projective. Hence $\beta^*(W'') \in V^r_{E''}(M\downarrow_{E''})$ and $W'' \in \mathscr{V}^r_{E''}(M\downarrow_{E''})$.

Now look at the stratification of the variety $V_G(k)$ as given by theorem 10·2 of Quillen [16] (see also Quillen [17]). The point is that there must exist $g \in G$ such that $gE''g^{-1} \subseteq E$ and $W = \mathrm{res}^*_{E,gE''g^{-1}}(c_g(W''))$, where c_g denotes the action of conjugation by g on varieties. Because M is a kG-module, $c_g(W'') \in \mathscr{V}_{gE''g^{-1}}(M\downarrow_{gE''g^{-1}})$ and by the previous theorem $W \in \mathscr{V}_E(M\downarrow_E)$. ∎

THEOREM 10·8 (Tensor Product Theorem). *Suppose that* M *and* N *are* kG*-modules. Then*

$$\mathscr{V}_G(M \otimes N) = \mathscr{V}_G(M) \cap \mathscr{V}_G(N).$$

Proof. First notice that the theorem is true if G is elementary abelian by Corollary 7·5 and Theorem 10·5.

Let $V \in \mathscr{V}_G(M \otimes N)$. Then $\kappa(V) \otimes M \otimes N$ is not projective, and by associativity of tensor product $\kappa(V) \otimes M$ and $\kappa(V) \otimes N$ are also not projective. So $V \in \mathscr{V}_G(M) \cap \mathscr{V}_G(N)$.

Conversely, suppose that $V \in \mathscr{V}_G(M) \cap \mathscr{V}_G(N)$. Let $E \in \mathscr{E}A(G)$ be a subgroup such that $V = \mathrm{res}^*_{G,E}(W)$ for some $W \in \mathscr{V}_E(k)$. Then W is in $\mathscr{V}_E(M\downarrow_E)$ and $\mathscr{V}_E(N\downarrow_E)$ by Theorem 10·7. Because the theorem is true for E we have that $W \in \mathscr{V}_E((M \otimes N)\downarrow_E)$. Therefore, by Theorem 10·6 we are finished.

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