J. Appl. Prob. **55,** 1228–1237 (2018)
*doi:*10.1017/jpr.2018.81
© *Applied Probability Trust* 2018

ON THE CRITICAL THRESHOLD FOR CONTINUUM AB PERCOLATION

DAVID DEREUDRE,* *Université de Lille*
MATHEW PENROSE,** *University of Bath*

Abstract

Consider a bipartite random geometric graph on the union of two independent homogeneous Poisson point processes in d-space, with distance parameter r and intensities λ, μ. For any $\lambda > 0$ we consider the percolation threshold $\mu_c(\lambda)$ associated to the parameter μ. Denoting by λ_c the percolation threshold for the standard Poisson Boolean model with radii r, we show the lower bound $\mu_c(\lambda) \geq c \log(c/(\lambda - \lambda_c))$ for any $\lambda > \lambda_c$ with $c > 0$ a fixed constant. In particular, there is no phase transition in μ at the critical value of λ, that is, $\mu_c(\lambda_c) = \infty$.

Keywords: Continuum percolation; bipartite geometric graph; connectivity threshold; enhancement

2010 Mathematics Subject Classification: Primary 60D05
Secondary 60K35; 82B43

1. Introduction and statement of results

In the continuum AB percolation model, particles of two types A and B are scattered randomly in Euclidean space as two independent Poisson processes, and edges are added between particles of opposite type that are sufficiently close together. This provides a continuum analogue to lattice AB percolation which was discussed in, e.g. [4]. The model was introduced by Iyer and Yogeshwaran [5], where the motivation was discussed in detail; the main motivation comes from wireless communications networks with two types of transmitter. As discussed in [9], a complementary (but distinct) continuum percolation model with two types of particle is the *secrecy random graph* [10], [12].

To describe continuum AB percolation more precisely, we make some definitions. Let $d \in \mathbb{N}$ with $d \geq 2$. Given any two locally finite sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$, and given $r > 0$, let $G(\mathcal{X}, \mathcal{Y}, r)$ be the bipartite graph with vertex sets $\mathcal{X}$ and $\mathcal{Y}$, and with an undirected edge $\{X, Y\}$ included for each $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ with $|X - Y| \leq r$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. Also, let $G(\mathcal{X}, r)$ be the graph with vertex set $\mathcal{X}$ and with an undirected edge $\{X, X'\}$ included for each $X, X' \in \mathcal{X}$ with $|X - X'| \leq r$.

For $\lambda, \mu > 0$, let $\mathcal{P}_\lambda$ and $\mathcal{Q}_\mu$ be independent homogeneous Poisson point processes in $\mathbb{R}^d$ of intensity λ and μ, respectively, where we view each point process as a random subset of $\mathbb{R}^d$. We are here concerned with the bipartite graph $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r)$.

Let $\mathcal{I}$ be the class of graphs having at least one infinite component. By a version of the Kolmogorov zero-one law, given parameters r, λ, μ (and d), we have $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] \in \{0, 1\}$. Provided r, λ, and μ are sufficiently large, we have $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] = 1$; see [5]

Received 15 December 2017; revision received 16 October 2018.
* Postal address: Laboratoire Paul Painlevé, Université de Lille, 59655 Villeneuve d'Ascq, France.
Email address: david.dereudre@univ-lille.fr
** Postal address: Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK.

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

or [9]. Set

$$\mu_c(r, \lambda) := \inf\{\mu : \mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] = 1\},$$

with the infimum of the empty set interpreted as $+\infty$. Also, for the more standard one-type continuum percolation graph $G(\mathcal{P}_\lambda, r)$, define

$$\lambda_c(r) := \inf\{\lambda : \mathbb{P}[G(\mathcal{P}_\lambda, r) \in \mathcal{I}] = 1\}.$$

Thus, $\lambda_c(r)$ is the critical Poisson intensity for percolation of the occupied region for a spherical Poisson Boolean model (see [8]) where the balls are all of radius $\frac{1}{2}r$. In the notation of [8] it would be called $\lambda_c(\frac{1}{2}r)$.

By scaling (see [8, Proposition 2.11]) $\lambda_c(2r) = r^{-d}\lambda_c(2)$. The value of $\lambda_c(2)$ is not known analytically, but is well known to be finite for $d \geq 2$ (see [4] and [8]) and explicit bounds are provided in [8]. Simulation studies indicate that $1 - \mathrm{e}^{-\pi\lambda_c(2)} \approx 0.676\,35$ for $d = 2$ [11] and $1 - \mathrm{e}^{-(4\pi/3)\lambda_c(2)} \approx 0.289\,57$ for $d = 3$ [7].

Obviously, if $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}$ then also $G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}$, and, hence, $\mu_c(r, \lambda) = \infty$ for $\lambda < \lambda_c(2r)$. In [5] and [9], it was proved that $\mu_c(r, \lambda) < \infty$ for $\lambda > \lambda_c(2r)$. Indeed, with π_d denoting the volume of the unit radius ball in d dimensions, we have, from [9],

$$\limsup_{\delta \downarrow 0} \left(\frac{\mu_c(r, \lambda_c + \delta)}{\delta^{-2d} |\log \delta|} \right) \leq \left(\frac{4\lambda_c(2r)^2}{r} \right)^d d^{3d}(d+1)\pi_d. \tag{1.1}$$

It is also indicated in [9] how, for any given $\lambda > \lambda_c(2r)$, one can compute an explicit upper bound for $\mu_c(r, \lambda)$.

As mentioned in [9], it is of interest to give complementary *lower* bounds for $\mu_c(r, \lambda)$. In this paper we make some progress in this direction by showing that there exists $c > 0$ (depending on d) such that for all $\delta > 0$, we have

$$\mu_c\left(\frac{1}{2}, \lambda_c(1) + \delta\right) \geq c \log\left(\frac{c}{\delta}\right). \tag{1.2}$$

We make no attempt here to state an explicit numerical value for c as a function of d. In principle, it should be possible to do this by keeping track of the constants at each step of the proof, but it would undoubtedly give extremely small values for c.

By scaling arguments, the previous lower bound holds for any radius $r > 0$ (after changing the constant c), so, in particular,

$$\lim_{\delta \downarrow 0} \mu_c(r, \lambda_c(2r) + \delta) = +\infty.$$

Immediately, we obtain

$$\mu_c(r, \lambda_c(2r)) = +\infty. \tag{1.3}$$

We note from (1.1) that if $\lambda > \lambda_c(2r)$, we can find finite μ such that $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}$ almost surely. If we were able to prove this under the weaker hypothesis that $G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}$ almost surely, then combining this with (1.3) we would have shown that, in fact, $G(\mathcal{P}_{\lambda_c}, 2r) \notin \mathcal{I}$ almost surely, which would solve the classic open problem of proving nonpercolation at the critical point (in any dimension) for this continuum percolation model.

We will prove (1.2) in the next section. Our strategy of proof goes as follows. We deem all A-particles having no B-particle nearby to be *useless*, since they cannot be used in any

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

percolating AB cluster. Given μ, we use a version of the technique of *enhancement* to show that there exists a value of λ such that $\mathcal{P}_\lambda$ is supercritical for A-percolation (with distance parameter $2r$) but becomes subcritical after removal of all the useless particles (a thinning process with only local dependence).

We will provide an outline of the enhancement argument in the next section. The technique of enhancement has previously been applied to one-type continuum percolation in [2] and [3], and further discussion of enhancement can be found there.

2. Proof of the lower bound

This section is devoted to proving the following theorem.

Theorem 2.1. *There exists $c > 0$ (depending on d) such that for all $\delta > 0$, we have (1.2). In particular,*

$$\lim_{\delta \downarrow 0} \mu_c\left(\tfrac{1}{2}, \lambda_c(1) + \delta\right) = +\infty. \tag{2.1}$$

Fix $\mu > 0$. To prove (2.1), we need to show that there exists $\lambda > \lambda_c(1)$ such that $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, \frac{1}{2}) \notin \mathcal{I}$ almost surely. To obtain (1.2), we need a suitable quantitative lower bound for this λ (or rather, for $\lambda - \lambda_c(1)$) in terms of μ.

We fix some $\lambda_0 > \lambda_c(1)$. Given a realization of $(\mathcal{P}_{\lambda_0}, \mathcal{Q}_\mu)$, we say that a point $x \in \mathbb{R}^d$ is *useless* if no point of $\mathcal{Q}_\mu$ lies within distance $\frac{1}{2}$ of x. Otherwise, we say x is *useful*. We will apply these notions mainly (but not always) in the case where x is itself a point of $\mathcal{P}_{\lambda_0}$.

Given also $p, q \in [0, 1]$, let $\mathcal{P}_{\lambda_0, p, q}$ be a thinned version of $\mathcal{P}_{\lambda_0}$ where each useful point is independently retained with probability p and each useless point is independently retained with probability q. In particular, $\mathcal{P}_{\lambda_0, p, p}$ has the same distribution as $\mathcal{P}_{\lambda_0 p}$.

For $R > 0$, let B_R denote the Euclidean ball of radius R centred at the origin. Let $\theta(p, q)$ be the probability that there exists an infinite component of $G(\mathcal{P}_{\lambda_0, p, q}, 1)$ that includes at least one vertex in B_1, and, for $n \in \mathbb{N}$, let $\theta_n(p, q)$ be the probability that there exists a component of $G(\mathcal{P}_{\lambda_0, p, q}, 1)$ that includes at least one vertex in B_1 and at least one vertex outside B_n. Then, for all p, q, we have $\theta(p, q) = \lim_{n \to \infty} \theta_n(p, q)$.

Before giving further details, we now sketch out our enhancement strategy. By a continuum version of Russo's formula (Lemma 2.1 below), the partial derivative $\partial\theta_n/\partial p$ (respectively, $\partial\theta_n/\partial q$) can be expressed as an integral over $x \in B_n$ of the probability that x is useful and pivotal (respectively, useless and pivotal), where we say x is *pivotal*, loosely speaking, if event $A_{x,n,p,q}$ (defined in Lemma 2.1) occurs.

Let $R > 0$. Suppose, in the event that x is useful and pivotal, that we resample $\mathcal{P}_{\lambda, p, q}$ inside the translated ball $B_R(x) := x + B_R$ (leaving the Poisson process unchanged outside this ball). In Lemma 2.2 we will show there exists a fixed $R > 0$ such that with (conditional) probability bounded away from 0 (uniformly in x and n) the point x is useless and pivotal in the resampled process, given that x is useful and pivotal in the original process. Therefore, $(\partial\theta_n/\partial q)/(\partial\theta_n/\partial p)$ is bounded away from 0, uniformly in n. Using this and setting $p_c := \lambda_c/\lambda_0$, we can show that for some sufficiently small $\varepsilon > 0$ (depending on our initial choice of μ), if we change (p, q) from $(p_c - \varepsilon, p_c - \varepsilon)$ to $(p_c + \varepsilon, 0)$ the 'bad' effect of reducing q to 0 outweighs the 'good' effect of increasing p by 2ε, so that $\theta(p_c + \varepsilon, 0) \le \theta(p_c - \varepsilon, p_c - \varepsilon) = 0$, so removing useless vertices of $\mathcal{P}_{\lambda_c + \varepsilon\lambda_0}$ makes it subcritical.

Lemma 2.1. *Given $x \in \mathbb{R}^d$, denote by $A_{x,n,p,q}$ the event that $G(\mathcal{P}_{\lambda_0, p, q} \cup \{x\}, 1)$ contains a path including vertices both in B_1 and in B_n^c, but $G(\mathcal{P}_{\lambda_0, p, q}, 1)$ does not, and let F_x be the event that x is useful.*

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

Then, for any $n \geq 1$ and $p, q \in (0, 1)$,

$$\frac{\partial \theta_n(p,q)}{\partial p} = \int_{B_{n+1}} \mathbb{P}[A_{x,n,p,q} \cap F_x]\lambda_0 \, \mathrm{d}x, \qquad \frac{\partial \theta_n(p,q)}{\partial q} = \int_{B_{n+1}} \mathbb{P}[A_{x,n,p,q} \cap F_x^{\mathrm{c}}]\lambda_0 \, \mathrm{d}x.$$

Proof. Adapting the proof of [2, Lemma 1], we prove only the first identity since the second is obtained in exactly the same way. We denote by $\mathcal{F}$ the σ-algebra generated by $(\mathcal{P}_{\lambda_0}, \mathcal{Q}_\mu)$. In particular, $\mathcal{F}$ does not contain any information on the thinning procedures for the useful and useless vertices. We denote by A_n the event that there exists a path in $G(\mathcal{P}_{\lambda_0,p,q}, 1)$ from B_1 to outside B_n. We note that the distribution of $\mathcal{P}_{\lambda_0,p,q}$, given the σ-algebra $\mathcal{F}$, consists of a collection of independent Bernoulli variables which indicate whether the vertices are retained or removed by the thinning procedure. Then, applying the standard coupling of Bernoulli variables and Russo's formula (also attributed to Margulis but, in fact, dating back at least to [1, Equation (5.2)]), we obtain, for any $h \in (0, 1 - p]$,

$$0 \leq \mathbb{P}[\mathcal{P}_{\lambda_0,p+h,q} \in A_n \mid \mathcal{F}] - \mathbb{P}[\mathcal{P}_{\lambda_0,p,q} \in A_n \mid \mathcal{F}] \leq h\#(\mathcal{P}_{\lambda_0} \cap B_{n+1})$$

and

$$\lim_{h \downarrow 0} \frac{1}{h}(\mathbb{P}[\mathcal{P}_{\lambda_0,p+h,q} \in A_n \mid \mathcal{F}] - \mathbb{P}[\mathcal{P}_{\lambda_0,p,q} \in A_n \mid \mathcal{F}]) = \mathbb{E}[N_{n,p,q} \mid \mathcal{F}],$$

where $N_{n,p,q}$ is the number of useful vertices in $\mathcal{P}_{\lambda_0}$ that are pivotal for the occurrence of A_n (with the (p, q)-thinning applied to all of the vertices of $\mathcal{P}_{\lambda_0}$ except for the one being counted as pivotal). Recall that a vertex x in a configuration $\mathcal{X}$ is said to be pivotal for an increasing event $\mathcal{A}$ if $\mathcal{X}$ belongs to $\mathcal{A}$ whereas $\mathcal{X} \setminus \{x\}$ does not.

By the dominated convergence theorem, we obtain

$$\begin{aligned}
\frac{\partial^+ \theta_n(p,q)}{\partial p} &= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[\mathbb{P}[\mathcal{P}_{\lambda_0,p+h,q} \in A_n \mid \mathcal{F}] - \mathbb{P}[\mathcal{P}_{\lambda_0,p,q} \in A_n \mid \mathcal{F}]] \\
&= \mathbb{E}\left[\lim_{h \downarrow 0} \frac{1}{h}[\mathbb{P}[\mathcal{P}_{\lambda_0,p+h,q} \in A_n \mid \mathcal{F}] - \mathbb{P}[\mathcal{P}_{\lambda_0,p,q} \in A_n \mid \mathcal{F}]]\right] \\
&= \mathbb{E}[\mathbb{E}[N_{n,p,q} \mid \mathcal{F}]] \\
&= \mathbb{E}[N_{n,p,q}].
\end{aligned}$$

By the Mecke formula (see [6]), it follows that

$$\mathbb{E}[N_{n,p,q}] = \mathbb{E}\left[\sum_{x \in \mathcal{P}_{\lambda_0}} \mathbf{1}_{\{x \text{ is useful and pivotal for } A_n\}}\right] = \int_{B_{n+1}} \mathbb{P}[A_{x,n,p,q} \cap F_x]\lambda_0 \, \mathrm{d}x.$$

One may argue similarly for the left derivative. The lemma is proved. □

Now we want to apply enhancement arguments as in [2] and [3]. For this we need to control the ratio between $\partial\theta_n(p,q)/\partial q$ and $\partial\theta_n(p,q)/\partial p$. The crucial lemma is given here.

Lemma 2.2. *Given $\alpha \in (0, 1)$ and $\lambda_0 > 0$, there exists a constant $c > 0$ such that for any $p \geq \alpha$, $q \geq \alpha$, $\mu > 0$, $n \geq c^{-1}$, and $x \in B_{n+1}$,*

$$\frac{\mathbb{P}[A_{x,n,p,q} \cap F_x^{\mathrm{c}}]}{\mathbb{P}[A_{x,n,p,q} \cap F_x]} \geq c\mathrm{e}^{-\mu/c}.$$

The proof, based on geometrical arguments, can be found at the end of the section.

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

Proof of Theorem 2.1. Set $\lambda_c := \lambda_c(1)$ and $p_c := \lambda_c/\lambda_0$. Choose $\alpha := \frac{1}{2}p_c$. Then, by Lemmas 2.1 and 2.2, there exists $c > 0$ such that for any $p, q \in [\alpha, 1)$, any $\mu > 0$, and any $n \geq c^{-1}$, we have

$$\frac{\partial \theta_n(p,q)}{\partial q} \geq (ce^{-\mu/c})\frac{\partial \theta_n(p,q)}{\partial p}. \tag{2.2}$$

Given $\mu > 0$, let $\delta > 0$ be small enough so that $p_c + \delta < 1$ and $p_c - \delta(1 + (2/c)e^{\mu/c}) \geq \alpha$, that is, $\delta(1 + (2/c)e^{\mu/c}) \leq \frac{1}{2}p_c$. There exists $c' > 0$ such that the choice

$$\delta = c'e^{-\mu/c'} \tag{2.3}$$

is suitable for all $\mu > 0$. By the finite-increments formula on the segment $[(p_c - \delta, p_c - \delta); (p_c + \delta, p_c - \delta(1 + (2/c)e^{\mu/c}))]$ and (2.2),

$$\theta_n\left(p_c + \delta, p_c - \delta\left(1 + \frac{2}{c}e^{\mu/c}\right)\right) \leq \theta_n(p_c - \delta, p_c - \delta).$$

By passing to the limit $n \to +\infty$ and noting that $\theta(p_c - \delta, p_c - \delta) = 0$, we obtain

$$\theta(p_c + \delta, 0) \leq \theta\left(p_c + \delta, p_c - \delta\left(1 + \frac{2}{c}e^{\mu/c}\right)\right) \leq \theta(p_c - \delta, p_c - \delta) = 0,$$

so $G(\mathcal{P}_{\lambda_0, p_c+\delta, 0}, 1) \notin \mathcal{I}$ almost surely and, hence, $G(\mathcal{P}_{\lambda_c+\delta\lambda_0}, \mathcal{Q}_\mu, \frac{1}{2}) \notin \mathcal{I}$ almost surely.

In conclusion, setting $\lambda := \lambda_c + \delta\lambda_0$ with δ given by (2.3), we have $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, \frac{1}{2}) \notin \mathcal{I}$ almost surely. Hence,

$$\mu_c\left(\frac{1}{2}, \lambda\right) \geq \mu = c' \log\left(\frac{\lambda_0 c'}{\lambda - \lambda_c}\right),$$

so the theorem is proved. □

It remains now to show Lemma 2.2. We first state two geometrical lemmas. We denote by $B_k(x)$ the translated ball $x + B_k$ and for each point $a \in \mathbb{R}^d$ and subset $S \subset \mathbb{R}^d$, we denote by $d(a, S)$ the Euclidean distance from a to S.

Lemma 2.3. *There exists $K > 10$ such that for all $R > K$ and all $r, \delta \in (0, 1/K)$, we have: for all $x, y \in B_{R+r}(0) \setminus B_{R-r}(0)$ with $|x - y| > 1$, there exist $x', y' \in B(0, R - \frac{1}{2})$ satisfying*

(i) $|x - x'| \leq 1 - \delta$;

(ii) $|y - y'| \leq 1 - \delta$;

(iii) $|x' - y'| \geq 1 + 2\delta$;

(iv) $d(x', B_R(0)^c \setminus B_1(x)) \geq 1 + \delta$;

(v) $d(y', B_R(0)^c \setminus B_1(y)) \geq 1 + \delta$.

See Figure 1.

Proof. Given $x, y \in B_{R+r}(0) \setminus B_{R-r}(0)$ with $|x - y| > 1$, set $x' := x - (1 - 3\delta)x/|x|$, and

$$y' := \begin{cases} y - \dfrac{(1-3\delta)x}{|x|} + \dfrac{2\delta(y-x)}{|y-x|} & \text{if } |y - x| < 3, \\ y - \dfrac{(1-3\delta)y}{|y|} & \text{otherwise.} \end{cases}$$

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

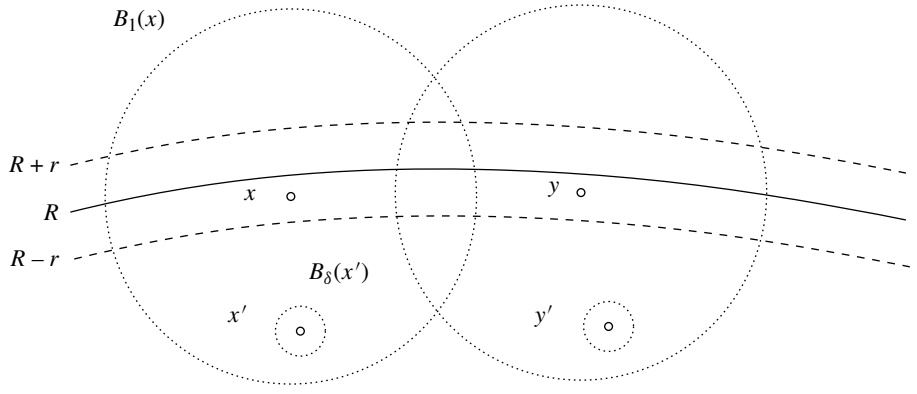


FIGURE 1: An example of configuration x, y, x', and y' in Lemma 2.3.

It is clear that for $R \geq 99$ and r, δ small enough, $x', y' \in B(0, R - \frac{1}{2})$. Moreover, item (i) holds trivially, as do items (ii) and (iii) by the triangle inequality. Finally, provided R is large enough and r, δ small enough, the distances $d(x', B_R(0)^c \setminus B_1(x))$ and $d(y', B_R(0)^c \setminus B_1(y))$ are bounded below by $\sqrt{1.9}$. Therefore, choosing large enough R and small enough r, δ, items (iv) and (v) are satisfied. □

Lemma 2.4. *Let $r > 0$. Then there exists $K' > 10$ such that for any $R > K'$ and any $\delta \in (0, 1/K')$, we have the following: for all $x \in B_R(0) \setminus B_{R-r}(0)$ and $y \in B_{R-r}(0)$ with $|x - y| > 1$ and $d(y, B_{R+r}(0) \setminus B_1(x)) \leq 1$, there exist $x', y' \in B(0, R - \frac{1}{2})$ satisfying*

(i) $|x - x'| \leq 1 - \delta$;

(ii) $|y - y'| \leq 1 - \delta$;

(iii) $|x' - y'| \geq 1 + 2\delta$;

(iv) $|x' - y| \geq 1 + \delta$;

(v) $|y' - x| \geq 1 + \delta$;

(vi) $d(y', B_R(0)^c \setminus B_1(y)) \geq 1 + \delta$;

(vii) $d(x', B_{R+r}(0)^c \setminus B_1(x)) \geq 1 + \delta$.

See Figure 2.

Proof. Let $x \in B_R \setminus B_{R-r}$ and $y \in B_{R-r}$ with $|x - y| > 1$ and $d(y, B_{R+r} \setminus B_1(x)) \leq 1$. Without loss of generality, we assume that $|x - y| < 3$; otherwise, an accurate construction as in the previous lemma is possible. Let $\varepsilon := 10^{-3}$ be fixed, and note that $\sin(\frac{1}{3}\pi - 3\varepsilon) > \frac{1}{2}$. Note also that provided R is large enough, the vector $y - x$ is at an angle at most $\frac{1}{3}\pi + \varepsilon$ with the hyperplane tangent to $B_{|x|}$ at x.

Let u and v be unit vectors in the vector space generated by x and y such that the angle between $y - x$ and u is equal to $\frac{1}{3}\pi + \varepsilon$, the angle between $y - x$ and v is equal to $\frac{1}{3}\pi + 2\varepsilon$, and such that both u and v have negative scalar product with x. (The choice of u and of v is unique provided that R is large enough.) Then we set $x' = x + (1 - 3\delta)v$ and $y' = y + (1 - 3\delta)u$.

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

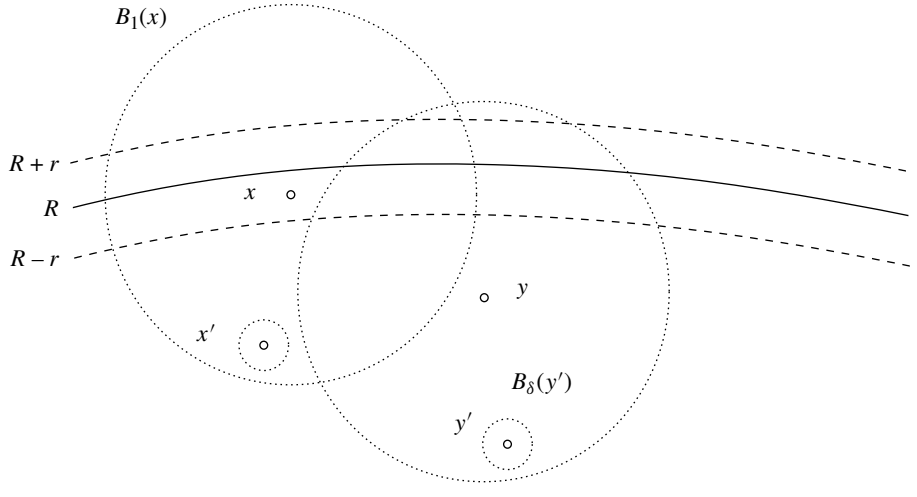


FIGURE 2: An example of configuration x, y, x' and y' in Lemma 2.4.

It is clear that for δ small enough, $x', y' \in B(0, R - \frac{1}{2})$. Moreover, items (i) and (ii) are trivially true, and (iii) holds provided δ is small enough. The quadrilateral x, y, y', x' is almost a parallelogram; the opposite edges (y, y') and (x, x') are the same length, and at an angle of ε to each other. The angle between $y - x$ and $x' - x$ is equal to $\frac{1}{3}\pi + 2\varepsilon$, and all sides of this quadrilateral are of length at least $1 - 3\delta$. Provided δ is small enough, the norms of the diagonals $|y' - x|$ and $|y - x'|$ exceed $1 + \delta$; that is, items (iv) and (v) hold. Also, for sufficiently large R, the angle between $y - y'$ and y is smaller than $\frac{1}{6}\pi + 2\varepsilon$. Moreover, $y \in B_{R-r}(0)$ so $d(y', B_R(0)^c \setminus B_1(y)) \geq 1 + \frac{1}{2}r$. Item (vi) follows for small enough δ. The proof of item (vii) is similar. □

Proof of Lemma 2.2. Assume now that parameters $R > 10$, $r > 0$, and $\delta > 0$ are chosen such that all items in Lemmas 2.3 and 2.4 are satisfied. Assume also that $R \in \mathbb{N}$ and that $\delta < \frac{1}{99}R$. Noting that

$$\frac{\mathbb{P}[A_{x,n,p,q} \cap F_x^c]}{\mathbb{P}[A_{x,n,p,q} \cap F_x]} = \frac{\mathbb{P}[F_x^c \mid A_{x,n,p,q}]}{\mathbb{P}[F_x \mid A_{x,n,p,q}]} \geq \mathbb{P}[F_x^c \mid A_{x,n,p,q}],$$

we have to find a lower bound for $\mathbb{P}[F_x^c \mid A_{x,n,p,q}]$ of the type $ce^{-\mu/c}$. Assume for now that $B_R(x) \subset B_n \setminus B_1$. (We will consider the other cases at the end.)

Consider creating the Poisson processes $\mathcal{P}_{\lambda_0}$, $\mathcal{Q}_\mu$, and $\mathcal{P}_{\lambda_0,p,q}$ in stages, as follows (this is similar to arguments seen in [2] and [3]).

First we generate the points of $\mathcal{P}_{\lambda_0}$ outside $B_R(x)$, the points of $\mathcal{Q}_\mu$ outside $B_{1/2}(x)$, and the retained points $\mathcal{P}_{\lambda_0,p,q}$ outside $B_R(x)$.

At this stage, we let V be the set of vertices of $\mathcal{P}_{\lambda_0,p,q}$ created so far which are connected by a path (in $G(\mathcal{P}_{\lambda_0,p,q} \setminus B_R(x), 1)$) to B_1, and let T be the set of vertices of $\mathcal{P}_{\lambda_0,p,q}$ created so far which are connected by a path to B_n^c. Let V_1 and T_1 denote the 1-neighbourhood of V, T, respectively. Let E_1 be the event that $T \cap V_1 = \varnothing$ and $(T_1 \setminus V_1) \cap B_R \neq \varnothing$ and $(V_1 \setminus T_1) \cap B_R \neq \varnothing$. If $A_{x,n,p,q}$ is realized, x is pivotal and, therefore, E_1 occurs.

Now, assuming that E_1 occurs, build up the point process of retained vertices $\mathcal{P}_{\lambda_0,p,q}$ inwards into $B_R(x) \cap (V_1 \triangle T_1)$ from the boundary of the ball $B_R(x)$ until the appearance of the first new

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

vertex (here, $\triangle$ denotes symmetric set difference). Denote this new vertex X. Let E_2 be the event that such a vertex exists, that is, $E_2 := \{\mathcal{P}_{\lambda_0,p,q} \cap B_R(x) \cap (V_1 \triangle T_1) \neq \varnothing\}$. If $A_{x,n,p,q}$ is realized then E_2 must occur.

Assuming that E_2 occurs, we suppose $X \in T_1$ (the other case, $X \in V_1$, can be treated in the same way). We now have to distinguish several cases.

Case 1: $X \in B_R(x) \setminus B_{R-r}(x)$, which means that X is close to the boundary of $B_R(x)$. Now we have also two sub-cases to consider.

Case 1.1: there exists Y in $V \cap B_{R+r}(x) \setminus B_R(x)$. Note that X and Y are exactly in position for applying Lemma 2.3. By that result, there exist x_1 and y_1 in $B_{R-1/2}(x)$ such that any point in $B_\delta(x_1)$ (respectively, $B_\delta(y_1)$) can be the next point for the path from B_n (respectively, B_1) going to x. These points are now sufficiently far from the boundary of $B_R(x)$ to create two paths $(x_i)_{1\le i\le R+2}$ and $(y_i)_{1\le i\le R+2}$ of $R+2$ points (deterministic, given what has been revealed so far) from T to x and V to x, where each step size in each path is at most $1-2\delta$ (that is, $|x_{i+1}-x_i| < 1-2\delta$ and $|y_{i+1}-y_i| < 1-2\delta$ for each i, and $|x-x_{R+2}| \le 1-2\delta$ and $|x-y_{R+2}| \le 1-2\delta$), and such that these paths remain a distance greater than $1+2\delta$ from each other; see Figure 3.

If each ball $B_\delta(x_i)$ or $B_\delta(y_i)$, $1 \le i \le R+2$, contains a single (retained) point, and there are no other points of $\mathcal{P}_{\lambda_0} \cap B_R(x)$ besides those already considered, and there are no points of $\mathcal{Q}_\mu$ inside $B_{1/2}(x)$, then $A_{x,n,p,q} \cap F_x^c$ occurs. Recall that p and q are at least α. Then the probability that all of the above occur is clearly bounded from below by $ce^{-\pi_d \mu/2^d}$ for a suitable constant $c > 0$ (depending on λ_0, R, r, δ, and α), where π_d is the volume of the unit ball in $\mathbb{R}^d$.

Case 1.2: the set $V \cap B_{R+r}(x) \setminus B_R(x)$ is empty. In this case we continue to build inwards the point process of retained vertices $\mathcal{P}_{\lambda_0,p,q}$, but only in $V_1 \setminus T_1$. Continue until the next new

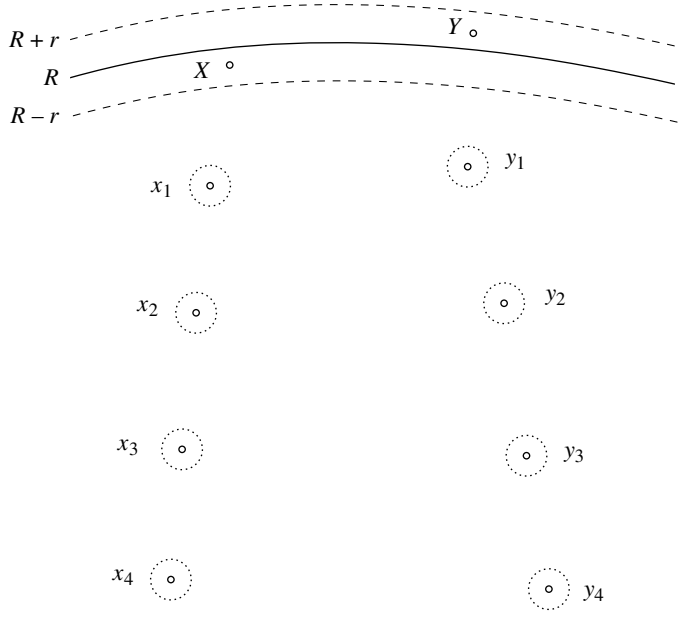


FIGURE 3: Example of paths $(x_i)_{1\le i\le R+2}$ and $(y_i)_{1\le i\le R+2}$.

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

vertex appears, denoted Y and then stop. Let E_3 be the event that such a new vertex Y does indeed appear. If this occurs then two sub-cases are possible.

Case 1.2.1: the point Y is close to the boundary (that is, $Y \in B_R(x) \setminus B_{R-r}(x)$). Then X and Y are exactly in position for applying Lemma 2.3 as in case 1.1 and we conclude this case in the same way.

Case 1.2.2: the point Y is in $B_{R-r}(x)$. Then X and Y are exactly in position for applying Lemma 2.4. So there exist x_1 and y_1 in $B_{R-1/2}(x)$ such that any points in $B_\delta(x_1)$ (respectively, $B_\delta(y_1)$) can be the next point for the path from B_n (respectively, B_1) going to x. These points are now sufficiently far from the boundary of $B_R(x)$ to create two paths $(x_i)_{1\le i\le R+2}$ and $(y_i)_{1\le i\le R+2}$ as in case 1.1.

Case 2: $X \in B_{R-r}(x)$, which means that X is far from the boundary of $B_R(x)$. Then we continue to build inwards the point process of retained vertices $\mathcal{P}_{\lambda_0,p,q}$, but only in $V_1 \setminus T_1$, as in case 1.2. Continue until the next new vertex appears, denoted Y; let E_4 be the event that such a vertex Y does appear, and assume that E_4 occurs. Now X and Y are sufficiently far from the boundary (both in $B_{R-r}(x)$) in order to build $(x_i)_{1\le i\le R+2}$ and $(y_i)_{1\le i\le R+2}$ as in case 1.1.

Let E be the event that events E_1, E_2, and (if in case 1.2) E_3, and (if in case 2) E_4 (along with corresponding events for the case when $X \in V_1$) all occur. If $A_{x,n,p,q}$ occurs then E must occur and, therefore,

$$\mathbb{P}[F_x^{\mathrm{c}} \mid A_{x,n,p,q}] = \frac{\mathbb{P}[F_x^{\mathrm{c}} \cap A_{x,n,p,q} \cap E]}{\mathbb{P}[A_{x,n,p,q}]} \ge \mathbb{P}[F_x^{\mathrm{c}} \cap A_{x,n,p,q} \mid E].$$

Thus, the conclusion of all these cases is that there exists a constant $c > 0$ such that

$$\mathbb{P}[F_x^{\mathrm{c}} \mid A_{x,n,p,q}] \ge c\mathrm{e}^{-\mu/c}.$$

Recall that we have been assuming that $B_R(x) \subset B_n \setminus B_1$. A similar proof can be derived in the other two cases; namely, the case with $B_R(x) \cap B_1 \ne \varnothing$ and the case with $B_R(x) \cap B_n^{\mathrm{c}} \ne \varnothing$. We may assume that $n \ge 99R$. When $B_R(x) \cap B_1 \ne \varnothing$, consider a path $(x_i)_{1\le i\le 2R+2}$ from outside $B_{2R}(x)$ to x and a path $(y_i)_{1\le i\le R+2}$ inside $B_{2R}(x)$ joining x to B_1. When $B_R(x) \cap B_n^{\mathrm{c}} \ne \varnothing$, consider a path $(x_i)_{1\le i\le 2R+2}$ from outside $B_{2R}(x)$ to x and a path $(y_i)_{1\le i\le R+2}$ inside $B_{2R}(x)$ joining x to B_n^{c}. For brevity we omit the details of these cases here (which are similar to the corresponding cases treated in [2]). Lemma 2.2 is proved. □

Acknowledgements

This work was supported in part by the Labex CEMPI (ANR-11-LABX-0007-01), the CNRS GdR 3477 GeoSto and the ANR project PPP (ANR-16-CE40-0016).

We thank the anonymous referees for some very helpful comments.

References

[1] Esary, J. D. and Proschan, F. (1963). Coherent structures of non-identical components. *Technometrics* **5,** 191–209.

[2] Franceschetti, M., Penrose, M. D. and Rosoman, T. (2010). Strict inequalities of critical probabilities on Gilbert's continuum percolation graph. Preprint. Available at https://arxiv.org/abs/1004.1596.

[3] Franceschetti, M., Penrose, M. D. and Rosoman, T. (2011). Strict inequalities of critical values in continuum percolation. *J. Statist. Phys.* **142**, 460–486.

[4] Grimmett, G. (1999). *Percolation*, 2nd edn. Springer, Berlin.

[5] Iyer, S. K. and Yogeshwaran, D. (2012). Percolation and connectivity in AB random geometric graphs. *Adv. Appl. Prob.* **44,** 21–41.

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press

[6] Last, G. and Penrose, M. (2018). *Lectures on the Poisson Process*. Cambridge University Press.
[7] Lorenz, C. D. and Ziff, R. M. (2001). Precise determination of the critical percolation threshold for the three dimensional "Swiss cheese" model using a growth algorithm. *J. Chem. Phys.* **114,** 3659–3661.
[8] Meester, R. and Roy, R. (1996). *Continuum Percolation*. Cambridge University Press.
[9] Penrose, M. D. (2014). Continuum AB percolation and AB random geometric graphs. *J. Appl. Prob.* **51A,** 333–344.
[10] Pinto, P. C. and Win, Z. (2012). Percolation and connectivity in the intrinsically secure communications graph. *IEEE Trans. Inform. Theory* **58,** 1716–1730.
[11] Quintanilla, J. A. and Ziff, R. M. (2007). Asymmetry of percolation thresholds of fully penetrable disks with two different radii. *Phys. Rev. E* **76,** 051115.
[12] Sarkar, A. and Haenggi, M. (2013). Percolation in the secrecy graph. *Discrete Appl. Math.* **161,** 2120–2132.

https://doi.org/10.1017/jpr.2018.81 Published online by Cambridge University Press