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Extending relational algebra with similarities

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In this paper we propose various extensions to the relational model to support similarity-based querying. We build upon the $\mathscr{K}$-relation model, where tuples are assigned values from an arbitrary semiring $\mathscr{K}$, and its associated positive relational algebra $\mathrm{RA}^{+}_{\mathscr{K}}$. We consider a recently proposed extension to $\mathrm{RA}^{+}_{\mathscr{K}}$ using a monus operation on the semiring to support negative queries, and show how, surprisingly, it fails for important 'fuzzy' semirings. Instead, we suggest using a negation operator. We also consider the identities satisfied by the relational algebra $\mathrm{RA}^{+}_{\mathscr{K}}$. We show that moving from a semiring to a particular form of lattice (a De Morgan frame) yields a relational algebra that satisfies all the classical (positive) relational algebra identities. We claim that to support real-world similarity queries realistically, one must move from tuple-level annotations to attribute-level annotations. We show in detail how our De Morgan frame-based model can be extended to support attribute-level annotations and give worked examples of similarity queries in this setting.

1. Introduction

Co-operative systems are a particular form of intelligent information systems that aim to provide more human-friendly results when compared with simple direct answers from a database (Minker 1998; Hajdinjak and Mihelič 2006). A core component of such systems is a treatment of imprecise data. For example, a co-operative system when faced with the query 'What are the names of the professors who taught category theory in the summer semester' might respond 'There were no category theory courses that summer' as opposed to the equally true, but less informative, '`empty set`' response. Or, when there were some other courses with topics from category theory, it might respond 'There were no category theory courses that summer, but the Advanced Algebra and Algebraic Topology courses covered some topics from category theory'.

At the heart of a co-operative system is a database where the data domains come equipped with a *similarity relation* to denote degrees of similarity rather than simply 'equal' and 'not equal'. This notion of similarity leads to an extension of the relational model where data can be *ranked*; for example, a tuple is ranked with the degree to which it matches a query. We shall call such a database a *similarity database*. Analogous to a classical relational database, all stored data (table) in a similarity database is also ranked (initially with some maximal ranking value). Thus ranked tables represent both stored data and the results of queries. We shall refer to rank-aware queries as *similarity queries*.

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For example, the following ranked table could be the result of the query 'show all courses on category theory':

CourseId	Course Name	Lecture Room	*Ranking*
M100.3	Category theory	Maths 701	1.0
M200.6	Advanced algebra	Maths Institute 11	0.6
M200.8	Algebraic topology	Statistics Lab A	0.5
CS300.7	Programming language semantics	Comp Sci 1	0.3
...	...	...	...

There have been many attempts to define extensions of the relational model to deal with similarity querying. Most utilise fuzzy logic (Zadeh 1965) in some way to model the ranking and to provide a framework for the action of the query operators with respect to the ranking. The ranking is typically modelled by a membership function to the unit interval [0, 1] (Schmitt and Schulz 2004; Penzo 2005; Rosado *et al.* 2006; Ma 2006), although there are generalisations where the membership function instead maps to an alebraic structure of some kind, which is typically poset or lattice based (Peeva and Kyosev 2004; Belohlavek and Vychodil 2006). Belohlavek and Vychodil argue that the algebraic approach is a better foundation for a fuzzy relational data model, primarily because it clarifies the choices of connectives and their interpretation. As they argue, much other work appears quite *ad hoc* in comparison. We build on this by showing another advantage: it facilitates a strong connection with other work on generalising the relational algebra.

We begin by observing that a ranked table is simply an annotated relation in the sense of Green, Karvounarakis and Tannen – both attach a value to every tuple. Thus, our starting point is their $\mathscr{K}$-relation model (Green ***et al.*** 2007). In this model, tuples in a relation are annotated with a value taken from a commutative semiring $\mathscr{K}$. A key observation by Green *et al.* was that the familiar positive relational algebra can be lifted naturally over $\mathscr{K}$-relations using the underlying semiring operations. The resulting relational algebra $\mathrm{RA}^{+}_{\mathscr{K}}$ generalises Codd's classic relational algebra (Codd 1970), the bag algebra (Montagna and Sebastiani 2001), the relational algebra on *c*-tables (Imielinski and Lipski 1984), the probabilistic algebra on event tables (Suciu 2008) and the provenance algebra (Cui *et al.* 2000; Buneman *et al.* 2001). One contribution of our work is to show that with relatively little work, the $\mathscr{K}$-relation model is suitable as a basis for modelling data with similarities and simple positive similarity queries. In particular, the typical similarity ranks, including the 'fuzzy' ranks, can be recast as commutative semirings (we refer to these as *similarity semirings*). Moreover, the 'fuzzy logic' appearing in similarity query languages can be clarified and explained simply and elegantly.

Green *et al.* only considered positive queries and left open the problem of supporting negative query operators. Recently, Geerts and Poggi addressed this problem proposing the definition of a *monus operator* on the underlying commutative semiring, which requires restricting the class of commutative semirings to so-called *m-semirings* (Geerts

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and Poggi 2010). One of the surprises of our work is that the monus-based difference operator yields the wrong answer for two important (fuzzy) similarity semirings. Thus we propose a different approach to modelling negative queries in the $\mathscr{K}$-relation model.

Because of the generality of the $\mathscr{K}$-relation model, the corresponding relational algebra does not satisfy all the identities of the classical relational model. In particular, the idempotence of union and self-join do not hold. (These do not hold in the bag algebra, so we would not expect them to hold in a generalisation of that algebra.) The question remains whether there is a simple, abstract algebraic characterisation of generalised relational algebras that satisfies all of the classic relational algebra identities. It turns out that restricting the annotation structure from a commutative semiring $\mathscr{K}$ to a De Morgan frame $\mathscr{L}$ is sufficient. We show how similarity queries fall naturally into the resulting $\mathscr{L}$-relation framework.

Previous attempts to formalise similarity querying and the $\mathscr{K}$-relation model all suffer from an expressivity problem in that there is a single annotation domain. For similarity models involving fuzzy logic (Ma and Yan 2008), most annotate tuples with a value from the real interval $[0, 1]$, and in the $\mathscr{K}$-relation model, the annotation is taken from a single commutative semiring. In other words, even though the domains have their own notion of similarity, a tuple involving more than one must collapse them all into a single rank. For simple examples this may be sufficient, but we quickly run into difficulties. Returning to our example, we might be interested in the query 'show all courses on category theory held near the Engineering Faculty'. Clearly, the similarity relation we wish to use on the lecture room would be some form of distance. Moreover, it is somewhat arbitrary to combine the similarity of course content with the similarity of location. We propose moving to a model where every attribute is annotated with a rank, rather than every tuple. (Or, equivalently, every tuple is annotated with a tuple of ranks, one per attribute, rather than a single rank.) Again we show how similarity queries fall naturally into this extension of the $\mathscr{L}$-relation model and algebra, giving a number of detailed examples.

Organisation of the paper

In Section 2, we recall the definitions of $\mathscr{K}$-relations and the positive relational algebra $\mathrm{RA}^{+}_{\mathscr{K}}$ (Green ***et al.*** 2007), along with its extension $\mathrm{RA}^{+}_{\mathscr{K}}(\setminus)$ to support negative queries, (Geerts and Poggi 2010). In Section 3, we make a small fix to the definition of the relational algebra $\mathrm{RA}^{+}_{\mathscr{K}}$ to handle selection queries dealing with similarities, and we introduce our notion of similarity measures. We consider two important (fuzzy) similarity semirings in which the difference operator from $\mathrm{RA}^{+}_{\mathscr{K}}(\setminus)$ yields the wrong answer. In Section 3.6, we propose a different approach using a negation operator. In Section 4, we consider the problem that $\mathscr{K}$-relational algebra does not support all the classical relational identities. We show how replacing semirings with De Morgan frames, which are a substructure of commutative semirings, leads to the $\mathscr{L}$-relation model, which induces a relational algebra that satisfies all the classical (positive) identities. In Section 5, we move from tuple-based annotations to attribute-based annotations. We define $\mathscr{D}$-relations and an algebra on $\mathscr{D}$-relations, called *relational algebra with similarities*, or $\mathrm{RA}_{\mathscr{D}}$. We also develop the notion of a similarity-based join. In Section 6, we show our model in action and consider a

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scenario of a database representing bus connections in a city. Finally, in Section 7, we give our conclusions, consider some related work and describe some plans for future work.

2. Background: the $\mathscr{K}$-relation model

In this section we recall the definitions of $\mathscr{K}$-relations and the positive relational algebra $\mathrm{RA}^{+}_{\mathscr{K}}$, along with its extension $\mathrm{RA}^{+}_{\mathscr{K}}(\setminus)$ to support negative queries. The aim of the $\mathscr{K}$-relation work was to provide a generalised framework capable of capturing various forms of annotated relations. As similarity can clearly be viewed as a form of annotation, we will use this as our foundation for a model of similarity-based querying.

2.1. *Positive relational algebra* $RA^{+}_{\mathscr{K}}$

We first assume some base domains, or *types*, commonly written as τ, which are simply sets of ground values. We use in our examples common base types such as integers and strings. We adopt the named-attribute approach, so in our model a *schema* U, which is written $\{a_1 : \tau_1, \ldots, a_n : \tau_n\}$, is a finite map from *attribute names* a_i to their types or domains $U(a_i) = \tau_i$. We represent a *U-tuple* as a map $t = \{a_1 : v_1, \ldots, a_n : v_n\}$ from attribute names a_i to values v_i of the corresponding domain, that is, $t(a_i) = v_i$, where $v_i \in \tau_i$ for $i = 1, \ldots, n$. We denote the set of all U-tuples by U-Tup.

Recall that a *semiring* $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is an algebraic structure with two binary operations (sum $\oplus$ and product $\odot$) and two distinguished elements ($\mathbf{0} \neq \mathbf{1}$) such that $(K, \oplus, \mathbf{0})$ is a commutative monoid[†] with identity element $\mathbf{0}$, $(K, \odot, \mathbf{1})$ is a monoid with identity element $\mathbf{1}$, products distribute over sums and $\mathbf{0} \odot a = a \odot \mathbf{0} = \mathbf{0}$ for any $a \in K$ (that is, $\mathbf{0}$ is an annihilating element). A semiring $\mathscr{K}$ is said to be commutative if monoid $(K, \odot, \mathbf{1})$ is commutative.

Definition 2.1 ($\mathscr{K}$-relation (Green *et al.* 2007)). Let $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a commutative semiring. A $\mathscr{K}$-relation over a schema $U = \{a_1 : \tau_1, \ldots, a_n : \tau_n\}$ is a function $A : U\text{-Tup} \to K$ such that its support $\{t \mid A(t) \neq \mathbf{0}\}$ is finite.

Taking this extension of relations, Green *et al.* proposed the following natural lifting of the classical relational operators over $\mathscr{K}$-relations.

Definition 2.2 (positive relational algebra on $\mathscr{K}$-relations (Green *et al.* 2007)). Let $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a commutative semiring. The operations of the positive relational algebra on $\mathscr{K}$, denoted by $\mathrm{RA}^{+}_{\mathscr{K}}$, are defined as follows:

Empty relation: For any set of attributes U, there is $\varnothing_U : U\text{-Tup} \to K$ such that $\varnothing_U(t) \stackrel{\text{def}}{=} \mathbf{0}$ for all U-tuples t[‡].

Union: If $A, B : U\text{-Tup} \to K$, then $A \cup B : U\text{-Tup} \to K$ is defined by

$$(A \cup B)(t) \stackrel{\text{def}}{=} A(t) \oplus B(t).$$

[†] A monoid consists of a set equipped with a binary operation that is associative and has an identity element.
[‡] As usual, we will drop the subscript on the empty relation where it can be inferred by context.

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Projection: If $A : U\text{-Tup} \to K$ and $V \subset U$, we write $f \downarrow V$ for the restriction of the map f to the domain V. The projection $\pi_V A : V\text{-Tup} \to K$ is defined by

$$(\pi_V A)(t) \stackrel{\text{def}}{=} \sum_{(t' \downarrow V) = t \text{ and } A(t') \neq \mathbf{0}} A(t').$$

Selection: If $A : U\text{-Tup} \to K$ and the selection predicate $\mathbf{P}$ maps each U-tuple to either $\mathbf{0}$ or $\mathbf{1}$, then $\sigma_{\mathbf{P}} A : U\text{-Tup} \to K$ is defined by

$$(\sigma_{\mathbf{P}} A)(t) \stackrel{\text{def}}{=} A(t) \odot \mathbf{P}(t).$$

Join: If $A : U_1\text{-Tup} \to K$ and $B : U_2\text{-Tup} \to K$, then $A \bowtie B$ is the $\mathscr{K}$-relation over $U_1 \cup U_2$ defined by

$$(A \bowtie B)(t) \stackrel{\text{def}}{=} A(t \downarrow U_1) \odot B(t \downarrow U_2).$$

Renaming: If $A : U\text{-Tup} \to K$ and $\beta : U \to U'$ is a bijection, then $\rho_\beta A : U'\text{-Tup} \to K$ is defined by

$$(\rho_\beta A)(t) \stackrel{\text{def}}{=} A(t \circ \beta).$$

Note 2.1. The sum is finite in the projection case since A has finite support.

The power of this definition is that it generalises a number of proposals for annotated relations and associated query algebras.

Lemma 2.1 (example algebras on $\mathscr{K}$-relations (Green *et al.* 2007)).

(1) The classical relational algebra with set semantics (Codd 1970) is given by the $\mathscr{K}$-relational algebra on the boolean semiring

$$\mathscr{K}_{\mathbb{B}} = (\mathbb{B}, \vee, \wedge, \mathsf{false}, \mathsf{true}).$$

(2) The relational algebra with bag semantics (Montagna and Sebastiani 2001; Green *et al.* 2007) is given by the $\mathscr{K}$-relational algebra on the semiring of counting numbers

$$\mathscr{K}_{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1).$$

(3) The Fuhr–Rölleke–Zimányi probabilistic relational algebra on event tables (Suciu 2008) is given by the $\mathscr{K}$-relational algebra on the semiring

$$\mathscr{K}_{\text{prob}} = (\mathscr{P}(\Omega), \cup, \cap, \varnothing, \Omega)$$

where Ω is a finite set of events and $\mathscr{P}(\Omega)$ is the powerset of Ω.

(4) The Imielinksi–Lipski algebra on c-tables (Imielinski and Lipski 1984) is given by the $\mathscr{K}$-relational algebra on the semiring

$$\mathscr{K}_{c\text{-table}} = (\mathsf{PosBool}(X), \vee, \wedge, \mathsf{false}, \mathsf{true})$$

where $\mathsf{PosBool}(X)$ is the set of all positive boolean expressions over a finite set of variables X in which any two equivalent expressions are identified.

(5) The provenance algebra of polynomials with variables from X and coefficients from $\mathbb{N}$ (Cui *et al.* 2000; Green ***et al.*** 2007) is given by the $\mathscr{K}$-relational algebra on the

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provenance semiring

$$\mathscr{K}_{\text{prov}} = (\mathbb{N}[X], +, \cdot, 0, 1).$$

In addition, the positive relational algebra $\text{RA}^{+}_{\mathscr{K}}$ satisfies many of the familiar relational equalities (Ullman 1988; 1989).

Proposition 2.1 (identities of $\mathscr{K}$-relations (Green *et al.* 2007)). The following identities hold for the positive relational algebra on $\mathscr{K}$-relations:

— union is associative and commutative, and has identity $\varnothing$;
— selection distributes over union and product;
— join is associative, commutative and distributive over union;
— projection distributes over union and join;
— selections and projections commute with each other;
— selection with boolean predicates gives all or nothing, that is, $\sigma_{\mathsf{false}}(A) = \varnothing$ and $\sigma_{\mathsf{true}}(A) = A$, where $\mathsf{false(t)} = \mathbf{0}$ and $\mathsf{true(t)} = \mathbf{1}$;
— join with an empty relation gives an empty relation, that is, $A \bowtie \varnothing_U = \varnothing_U$ where A is a $\mathscr{K}$-relation over a schema U;
— projection of an empty relation gives an empty relation, that is, $\pi_V(\varnothing) = \varnothing$.

It is important to note that the properties of idempotence of union,

$$A \cup A = A,$$

and self-join,

$$A \bowtie A = A,$$

are missing from this list. These properties fail for the bag semantics and provenance, so it should not be surprising that they fail to hold for the more general model. We will return to this issue in Section 4.

We will find it important in the following sections to consider order on commutative semirings.

Lemma 2.2 (preorder on a semiring). Every commutative semiring

$$\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$$

supports a *preorder* defined as follows:

$$x \preceq y \text{ if and only if there exists a } z \in K \text{ such that } x \oplus z = y$$

2.2. *Relational algebra* $RA^{+}_{\mathscr{K}}(\backslash)$

Geerts and Poggi recently proposed extending the $\mathscr{K}$-relation model by following a standard approach for introducing a monus operator into an additive commutative monoid (Amer 1984). First, they restricted the class of commutative semirings by requiring that every semiring also satisfies the following pair of conditions.

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Definition 2.3 (GP-conditions (Geerts and Poggi 2010)). A commutative semiring $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is said to satisfy the *GP conditions* if the following two conditions hold:

(1) The preorder $x \preceq y$ on K is a *partial order*.
(2) For each pair of elements $x, y \in K$, the set $\{z \in K ; x \preceq y \oplus z\}$ has a smallest element. (As $\preceq$ defines a partial order, this smallest element must be unique, if it exists.)

Definition 2.4 (*m*-semiring (Geerts and Poggi 2010)). Let $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be a commutative semiring that satisfies the GP conditions. For any $x, y \in K$, we define $x \ominus y$ to be the smallest element z such that $x \preceq y \oplus z$. A (commutative) semiring $\mathscr{K}$ that can be equipped with a *monus* operator $\ominus$ is called a *semiring with monus* or *m-semiring*.

Proposition 2.2 (identities in an *m*-semiring (Bosbach 1965)). The notion of an *m*-semiring is characterised by the properties of commutative semirings and the following identities involving $\ominus$:

$$\begin{aligned} x \ominus x &= \mathbf{0} \\ \mathbf{0} \ominus x &= \mathbf{0} \\ x \oplus (y \ominus x) &= y \oplus (x \ominus y) \\ x \ominus (y \oplus z) &= (x \ominus y) \ominus z \\ x \odot (y \ominus z) &= (x \odot y) \ominus (x \odot z). \end{aligned}$$

Geerts and Poggi identified two equationally complete classes in the variety of *m*-semirings, namely:

(1) *m*-semirings that are a boolean algebra (that is, complemented distributive lattice with distinguished elements $\mathbf{0}$ and $\mathbf{1}$), for which the monus behaves like set difference; and
(2) *m*-semirings that are the positive cone of a lattice-ordered commutative ring, for which the monus behaves like the truncated minus of the natural numbers.

Recall that a *lattice-ordered ring* (or *l*-ring) is an algebraic structure

$$\mathscr{K} = (K, \vee, \wedge, \oplus, -, \mathbf{0}, \odot)$$

such that:

— $(K, \vee, \wedge)$ is a lattice;
— $(K, \oplus, -, \mathbf{0}, \odot)$ is a ring;
— the operation $\oplus$ is order-preserving; and
— for $x, y \geqslant \mathbf{0}$, we have $x \odot y \geqslant \mathbf{0}$.

We will refer to the '$-$' operation as *additive inversion*. An *l*-ring is commutative if the multiplication operation $\odot$ is commutative. The set of elements x for which $\mathbf{0} \leqslant x$ is called the *positive cone* of the *l*-ring.

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Example 2.1 (*m*-semirings (Geerts and Poggi 2010)).

— The boolean semiring $\mathscr{K}_{\mathbb{B}} = (\mathbb{B}, \vee, \wedge, \mathsf{false}, \mathsf{true})$ is a boolean algebra. We have

$$\begin{aligned} \mathsf{false} \ominus \mathsf{false} &= \mathsf{false} \\ \mathsf{false} \ominus \mathsf{true} &= \mathsf{false} \\ \mathsf{true} \ominus \mathsf{false} &= \mathsf{true} \\ \mathsf{true} \ominus \mathsf{true} &= \mathsf{false}. \end{aligned}$$

— The semiring of counting numbers,

$$\mathscr{K}_{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1),$$

is the positive cone of the ring of integers, $\mathbb{Z}$. The monus corresponds to the truncated minus:

$$x \ominus y = \max\{0, x - y\}.$$

— The probabilistic semiring,

$$\mathscr{K}_{\text{prob}} = (\mathscr{P}(\Omega), \cup, \cap, \varnothing, \Omega),$$

is a boolean algebra. The monus corresponds to set difference:

$$X \ominus Y = X \setminus Y.$$

— In the case of the semiring of *c*-tables,

$$\mathscr{K}_{c\text{-table}} = (\mathsf{PosBool}(X), \vee, \wedge, \mathsf{false}, \mathsf{true}),$$

the monus cannot be defined unless negated literals are added to the base set, in which case we get a boolean algebra. For any two expressions $\phi_1, \phi_2 \in \text{Bool}(X)$, we then have $\phi_1 \ominus \phi_2 = \phi_1 \wedge \neg\phi_2$, where negation $\neg$ over boolean expressions takes truth to falsity, and *vice versa*, and it interchanges the meet and join operations.

— The provenance semiring,

$$\mathscr{K}_{\text{prov}} = (\mathbb{N}[X], +, \cdot, 0, 1),$$

is the positive cone of the ring of polynomials $\mathbb{Z}[X]$. The monus of two polynomials

$$\begin{aligned} f[X] &= \sum_{\alpha \in I} f_\alpha x^\alpha \\ g[X] &= \sum_{\alpha \in I} g_\alpha x^\alpha, \end{aligned}$$

where I is a finite subset of $\mathbb{N}^n$, corresponds to the polynomial

$$f[X] \ominus g[X] = \sum_{\alpha \in I} (f_\alpha \dot{-} g_\alpha) x^\alpha,$$

where $\dot{-}$ denotes the truncated minus on $\mathbb{N}$.

Given an *m*-semiring, the positive relational algebra $\text{RA}^+_{\mathscr{K}}$ can be extended with the missing difference operator as follows.

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Definition 2.5 (relational algebra on $\mathscr{K}$-relations (Geerts and Poggi 2010)). Let $\mathscr{K}$ be an m-semiring. The algebra $\mathrm{RA}^{+}_{\mathscr{K}}(\setminus)$ is obtained by extending $\mathrm{RA}^{+}_{\mathscr{K}}$ with the operator:

Difference: If $A, B : U\text{-Tup} \to K$, then the difference $A \setminus B : U\text{-Tup} \to K$ is defined by

$$(A \setminus B)(t) \stackrel{\text{def}}{=} A(t) \ominus B(t).$$

3. Modelling similarity using the $\mathscr{K}$-relation model

As mentioned in the introduction, our intention is to model the ranking of tuples in similarity tables using the $\mathscr{K}$-valued annotation in the $\mathscr{K}$-relation model. In this section we explore in detail whether the $\mathscr{K}$-relation model and the $\mathrm{RA}^{+}_{\mathscr{K}}(\setminus)$ algebra are fit for purpose as the underlying model for similarity data and queries.

3.1. *The selection predicate*

Our first step is to make a small change to the original Green *et al.* model. More specifically, we observe in Definition 2.2 that the selection predicate maps U-tuples to either the zero or the unit element of the semiring. Clearly, in a similarity context we would expect the selection predicate to reflect the *degree* of membership of a particular tuple, not just the two possibilities of full membership (**1**) or non-membership (**0**). Thus we propose the following generalisation to the original definition.

Selection: If $A : U\text{-Tup} \to K$ and the selection predicate

$$\mathbf{P} : U\text{-Tup} \to K$$

maps each U-tuple to an element of K (instead of mapping to either **0** or **1**), then $\sigma_{\mathbf{P}} A : U\text{-Tup} \to K$ is (still) defined by

$$(\sigma_{\mathbf{P}} A)(t) = A(t) \odot \mathbf{P}(t).$$

In the rest of this paper when we refer to the relational algebra over $\mathscr{K}$-relations, we will implicitly mean this generalisation.

3.2. *Similarity semirings*

Tuples in similarity databases are typically ranked with either (true or false) or with some value from the real interval $[0, 1]$ reflecting the degree of membership or relevance. If we want the rank to denote the distance to the nearest ideal tuple, tuples could be ranked with values from a real interval $[0, d_{\max}]$, where 0 and $d_{\max}$ mean the zero distance and the greatest possible distance, respectively. All three are the underlying sets of commutative semirings, but because of their use, we shall refer to them as *similarity semirings.*

Lemma 3.1 (common similarity semirings).

(1) The boolean semiring $\mathscr{K}_{\mathbb{B}} = (\mathbb{B}, \vee, \wedge, \mathsf{false}, \mathsf{true})$ with $\mathbb{B} = \{\mathsf{true}, \mathsf{false}\}$ is a commutative semiring.

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(2) The *fuzzy semiring* $\mathscr{K}_{[0,1]} = ([0,1], \max, \min, 0, 1)$ is a commutative semiring.
(3) The *distance semiring* $\mathscr{K}_{[0,d_{\max}]} = ([0, d_{\max}], \min, \max, d_{\max}, 0)$ is a commutative semiring.

3.3. *Similarity measures*

As mentioned earlier, similarity databases typically assume that all data domains come equipped with a similarity relation. We call these *similarity measures* and in terms of our $\mathscr{K}$-relation setting define them as follows.

Definition 3.1 (similarity measures). Given a type τ and a commutative semiring $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$, a *similarity measure* is a function $\rho : \tau \times \tau \to K$ such that ρ is reflexive, that is, $\rho(x, x) = \mathbf{1}$.

We follow earlier work (Shenoi and Melton 1989) and only require reflexivity of the similarity measure. This seems to be the minimal requirement, as other properties simply do not hold in general (Hajdinjak and Bauer 2009). For example, symmetry does not hold when similarity denotes driving distance between two points in a town because of one-way streets. Another property is transitivity, but there are a number of non-transitive similarity measures, for example, when similarity denotes likeness between two colours.

Example 3.1 (common similarity measures). Three common examples of similarity measures are as follows.

(1) An equality measure $\rho : \tau \times \tau \to \mathbb{B}$ where $\rho(x, y) \stackrel{\text{def}}{=} \mathsf{true}$ if x and y are equal and false otherwise.
(2) A fuzzy equality measure $\rho : \tau \times \tau \to [0, 1]$ where $\rho(x, y)$ expresses the degree of equality of x and y; the closer x and y are to each other, the closer $\rho(x, y)$ is to 1.
(3) A distance measure $\rho : \tau \times \tau \to [0, d_{\max}]$ where $\rho(x, y)$ is the distance from x to y.

We assume a predefined environment of similarity measures that can be used for building queries. We assume that $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$ and for every $\mathscr{K}$-relation over a schema $U = \{a_1 : \tau_1, ..., a_n : \tau_n\}$ there are similarity measures $\rho_i : \tau_i \times \tau_i \to K, 1 \leqslant i \leqslant n$. For a given attribute a_i we write ρ_{a_i} to denote this similarity measure. Unfortunately, within a database, all similarity measures must have the same codomain K. We shall return to this issue in Section 5.

Selection queries can now be classified according to whether they are based on the attribute values (as is normal in non-similarity queries) or whether they use the similarity measures. (Clearly, selection queries can use constant values.) We say the former category is primitive predicate.

Definition 3.2 (primitive predicate). Suppose in a schema $U = \{a_1 : \tau_1, \ldots, a_n : \tau_n\}$ the types of attributes a_i and a_j coincide. Then given a commutative semiring

$$\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1}),$$

for a given binary predicate θ, the *primitive predicate*

$$[a_i \, \theta \, a_j] : U\text{-Tup} \to K$$

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is defined as follows:

$$[a_i\,\theta\,a_j](t) \stackrel{\text{def}}{=} \chi_{a_i\theta a_j}(t) = \begin{cases} \mathbf{1} & \text{if } t(a_i)\,\theta\,t(a_j) \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

We can also define similarity predicates that select tuples based on the similarity measures.

Definition 3.3 (similarity predicate). Suppose in a schema $U = \{a_1 : \tau_1, \ldots, a_n : \tau_n\}$ the types of attributes a_i and a_j coincide. Given a commutative semiring

$$\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1}),$$

the *similarity predicate* $[a_i \texttt{ like } a_j] : U\text{-Tup} \to K$ is defined as follows:

$$[a_i \texttt{ like } a_j](t) \stackrel{\text{def}}{=} \rho_{a_i}(t(a_i), t(a_j)).$$

We can also define a symmetric version as follows:

$$[a_i \sim a_j] \stackrel{\text{def}}{=} [a_i \texttt{ like } a_j] \cup [a_j \texttt{ like } a_i],$$

where union ($\cup$) of selection predicates is defined below.

Definition 3.4 (operations on selection predicates). Given a commutative semiring $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$, we define union and intersection of two selection predicates $\mathbf{P}_1, \mathbf{P}_2 : U\text{-Tup} \to K$ as follows:

$$(\mathbf{P}_1 \cup \mathbf{P}_2)(t) \stackrel{\text{def}}{=} \mathbf{P}_1(t) \oplus \mathbf{P}_2(t)$$
$$(\mathbf{P}_1 \cap \mathbf{P}_2)(t) \stackrel{\text{def}}{=} \mathbf{P}_1(t) \odot \mathbf{P}_2(t).$$

3.4. *Example: Levenshtein similarity*

We are now in a position to consider a simple example. This deals with similarity of strings and we select values according to their distance from some given query string. Such queries and similarity measures arise naturally, for example, in the processing of web search queries.

We assume a schema $U = \{\mathsf{name} : \mathsf{String}, \ldots\}$ and take the commutative semiring

$$\mathscr{K}_M = (\{0, 1, \ldots, M\}, \min, \max, M, 0),$$

where M is the maximum string length of the data handled by the database system. We assume that the $\mathscr{K}$-relation function $A : U\text{-Tup} \to K$ maps all tuples in the relation to 0 and those not in the relation to M. We are interested in queries that measure similarity of the name value to some given query string q. We take as an example, $\mathsf{q} = \texttt{"Harry S. Truman"}$.

We consider a predicate

$$\mathbf{P}_{\mathsf{q}} : U\text{-Tup} \to \{0, 1, \ldots, M\}$$

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that maps each tuple $t = \{\mathsf{name}: n, \ldots\} \in U\text{-Tup}$ to the *Levenshtein distance* (Levenshtein 1966) between the name n of length l and the query string `q` (the string `"Harry S. Truman"` of length 15). The value $\mathbf{P}_{\mathsf{q}}(t)$ is thus the minimum number of edits needed to transform the string $t(\mathsf{name})$ into the string `"Harry S. Truman"`, with the allowable edit operations being insertion, deletion or substitution of a single character. It can take any integer value between 0 and $\max\{15, l\}$.

Suppose t_1, t_2 and t_3 are all U-tuples having `name` values of `"Harry S. Truman"`, `"Harry Truman"`, and `"H. S. Truman"`, respectively, and take the selective query $\sigma_{\mathbf{P}_{\mathsf{q}}} A$. The query answer is a $\mathscr{K}_M$-relation with

$$(\sigma_{\mathbf{P}_{\mathsf{q}}} A)(t) = \max\{A(t), \mathbf{P}_{\mathsf{q}}(t)\},$$

where:

— $\mathbf{P}_{\mathsf{q}}(t_1) = 0$ since the name strings coincide,
— $\mathbf{P}_{\mathsf{q}}(t_2) = 3$ from the insertion of 3 characters `"S."` into t_2,
— $\mathbf{P}_{\mathsf{q}}(t_3) = 4$ from the substitution of the character `"."` with `"a"` and the insertion of `"rry"` into t_3.

On the other hand, in the case of a tuple $t_4 = \{\mathsf{name}: \texttt{"Mary Bowman"}, \ldots\}$, we would get $\mathbf{P}_{\mathsf{q}}(t_4) = 8$, and in the case of the tuple $t_5 = \{\mathsf{name}: \texttt{"Elvis Presley"}, \ldots\}$, we would get $\mathbf{P}_{\mathsf{q}}(t_5) = 14$. Smaller values mean a smaller distance to the search string `"Harry S. Truman"`, and thus a greater match with the selection condition.

3.5. *Difference operators over similarity semirings*

We will now test the monus-based extension to the $\mathscr{K}$-relation framework with two semirings that are important in the fuzzy setting of similarity queries. We will see that whilst they support a monus operation in the sense of Geerts and Poggi, the induced difference operator in the relational algebra, somewhat surprisingly, does not behave as desired.

Lemma 3.2. The fuzzy semiring

$$\mathscr{K}_{[0,1]} = ([0,1], \max, \min, 0, 1)$$

satisfies the GP conditions.

Proof. The order relation $\preceq$ coincides with $\leqslant$, so the two conditions hold trivially. □

It is simple to see that the monus operator can be defined directly as follows:

$$x \ominus y = \min\{z \in [0,1]; x \leqslant \max\{y, z\}\} = \begin{cases} 0 & \text{if } x \leqslant y \\ x & \text{if } x > y. \end{cases}$$

This induces the following difference operator in the relational algebra:

$$(A \setminus B)(t) = \begin{cases} 0 & \text{if } A(t) \leqslant B(t) \\ A(t) & \text{if } A(t) > B(t). \end{cases}$$

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Regrettably, this is not the expected definition. First, fuzzy set difference is universally defined as $\min\{A(t), 1 - B(t)\}$ (Rosado *et al.* 2006). Secondly, in similarity settings, only totally irrelevant tuples should be annotated/ranked with 0 and excluded as a possible answer (Hajdinjak and Mihelič 2006). In the case of the fuzzy set difference $A \setminus B$, these are exclusively those tuples t where $A(t) = 0$ or $B(t) = 1$, and certainly not where $A(t) \leqslant B(t)$.

Lemma 3.3. The distance semiring

$$\mathscr{K}_{[0,d_{\max}]} = ([0, d_{\max}], \min, \max, d_{\max}, 0)$$

satisfies the GP-conditions.

Proof. The order relation $\preceq$ coincides with $\geqslant$, which is a partial order relation, so condition (1) holds. Condition (2) holds since every closed subset of the distance semiring has both a smallest and a greatest element. □

The monus operator can be defined directly as follows:

$$x \ominus y = \max\{z \in [0, d_{\max}]; x \geqslant \min\{y, z\}\} = \begin{cases} d_{\max} & \text{if } x \geqslant y \\ x & \text{if } x < y. \end{cases}$$

This induces the following difference operator in the relational algebra:

$$(A \setminus B)(t) = \begin{cases} d_{\max} \text{ if } A(t) \geqslant B(t) \\ A(t) \text{ if } A(t) < B(t). \end{cases}$$

Again, in the fuzzy setting, we would expect the difference operator to be defined as $\max\{A(t), d_{\max} - B(t)\}$. Moreover, this is a continuous function in contrast to the step function behaviour of the operator above resulting from the monus definition.

From the examples above, we conclude that in the fuzzy setting of similarity queries, the monus operation from *m*-semirings does not yield a proper notion of difference in the relational algebra.

3.6. *Difference through negation*

Rather than using a monus-like operator, we propose a different approach involving negation.

Definition 3.5 (negation). Given a set L equipped with a preorder, a *negation* is an operation $\neg : L \to L$ that reverts order, that is,

$$x \leqslant y \implies \neg y \leqslant \neg x,$$

and is involutive, that is,

$$\neg\neg x = x.$$

Note 3.1. It is well known that a preordered set may have more than one negation operation. The complement operation from complemented distributive lattices (that is, boolean algebras) is, however, unique. It is also order-reversing and involutive and thus a negation operation (Davey and Priestley 1990). Recall that a complemented lattice is

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a bounded lattice in which every element x has a complement, that is, an element x' satisfying $x \vee x' = \mathbf{1}$ and $x \wedge x' = \mathbf{0}$.

Note 3.2. For a *bounded* poset (with bounds $\mathbf{0}$ and $\mathbf{1}$), we always have $\neg\mathbf{0} = \mathbf{1}$ and $\neg\mathbf{1} = \mathbf{0}$ (Salii 1983). Our models in Section 4 will satisfy this property.

Example 3.2 (negation operators for common similarity measures). The similarity measures from Example 3.1 all support a negation operation:

— In the boolean semiring $\mathscr{K}_{\mathbb{B}} = (\mathbb{B}, \vee, \wedge, \mathsf{false}, \mathsf{true})$, negation can be defined as complementation:

$$\neg x \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = \mathsf{false} \\ \mathsf{false} & \text{if } x = \mathsf{true}. \end{cases}$$

— In the fuzzy semiring $\mathscr{K}_{[0,1]} = ([0,1], \max, \min, 0, 1)$, ordered by relation $\leqslant$, we can define a negation operator as $\neg x \stackrel{\text{def}}{=} 1 - x$.
(In the generalised fuzzy semiring $\mathscr{K}_{[a,b]} = ([a,b], \max, \min, a, b)$, we can define $\neg x \stackrel{\text{def}}{=} a + b - x$.)

— In the distance semiring $\mathscr{K}_{[0,d_{\max}]} = ([0, d_{\max}], \min, \max, d_{\max}, 0)$ ordered by relation $\geqslant$, we can define a negation operator as $\neg x \stackrel{\text{def}}{=} d_{\max} - x$.

Definition 3.6 (n-semiring). A commutative n-semiring $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1}, \neg)$ is a commutative semiring $(K, \oplus, \odot, \mathbf{0}, \mathbf{1})$ equipped with a negation operator $\neg : K \to K$ (with respect to the preorder on K).

Definition 3.7 (relational algebra on $\mathscr{K}$-relations). Let $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1}, \neg)$ be a commutative n-semiring. The algebra $\mathrm{RA}^{+}_{\mathscr{K}}(\neg)$ is obtained by extending $\mathrm{RA}^{+}_{\mathscr{K}}$ with the operator:
Difference: If $A, B : U\text{-Tup} \to K$, then $A \setminus B : U\text{-Tup} \to K$ is defined by

$$(A \setminus B)(t) \stackrel{\text{def}}{=} A(t) \odot \neg B(t).$$

Example 3.3 (relational difference over common similarity measures). Using the negation operators from Example 3.2, the difference operators in the relational algebra $\mathrm{RA}^{+}_{\mathscr{K}}(\neg)$ for the common similarity measures are as follows:

— In the boolean semiring $\mathscr{K}_{\mathbb{B}} = (\mathbb{B}, \vee, \wedge, \mathsf{false}, \mathsf{true})$, we get

$$(A \setminus B)(t) = \begin{cases} \mathsf{false} & \text{if } B(t) = \mathsf{true} \\ A(t) & \text{if } B(t) = \mathsf{false}. \end{cases}$$

Observe that in this case the monus-based difference operator and the negation-based difference operator coincide.

— In the fuzzy semiring $\mathscr{K}_{[0,1]} = ([0,1], \max, \min, 0, 1)$, we get

$$(A \setminus B)(t) = \min\{A(t), 1 - B(t)\},$$

and in the generalised fuzzy semiring $\mathscr{K}_{[a,b]} = ([a,b], \max, \min, a, b)$, we get

$$(A \setminus B)(t) = \min\{A(t), a + b - B(t)\}.$$

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These coincide with the fuzzy notions of difference on $[0,1]$ and $[a,b]$, respectively, mentioned in Section 3.5.

— In the distance semiring $\mathscr{K}_{[0,d_{\max}]} = ([0,d_{\max}], \min, \max, d_{\max}, 0)$, we get

$$(A \setminus B)(t) = \max\{A(t), d_{\max} - B(t)\}.$$

As required in Section 3.5, this is a continuous function of $A(t)$ and $B(t)$, and it calculates the greatest distance $d_{\max}$ only if $A(t) = d_{\max}$ or $B(t) = 0$.

As mentioned in Section 2.2, there are two equationally complete classes in the variety of *m*-semirings: *m*-semirings that are boolean algebras and *m*-semirings that are the positive cone of a lattice-ordered commutative ring.

Lemma 3.4 (monus in boolean algebras (Amer 1984)). Let $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1})$ be an *m*-semiring. If $(K, \oplus, \odot, \mathbf{0}, \mathbf{1}, ')$ is a boolean algebra where $\oplus$ is the lattice join, $\odot$ is the lattice meet, and $'$ is the complement operation, then the monus is determined by

$$x \ominus y = x \odot y'.$$

Corollary 3.1 (relational difference on $\mathscr{K}$-relations). Suppose $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1}, ')$ is a boolean algebra where $\oplus$ is the lattice join $\vee$, $\odot$ is the lattice meet $\wedge$ and $'$ is the complement operation. If we take the underlying complement operator as the negation operator, the monus-based difference operation and the negation-based difference operation induced in the algebra on $\mathscr{K}$-relations coincide:

$$A(t) \ominus B(t) = A(t) \odot B(t)' = A(t) \wedge \neg B(t).$$

The other semirings considered (the provenance semiring and the semiring of counting numbers) are each a positive cone of a lattice-ordered commutative ring.

Lemma 3.5 (monus in lattice-ordered commutative rings (Amer 1984)). Let $\mathscr{K}$ be the positive cone of a lattice-ordered commutative ring $(K, \vee, \wedge, \oplus, -, \mathbf{0}, \odot)$. Then the monus is determined by

$$x \ominus y = \mathbf{0} \vee (x - y)$$

where $-y$ denotes the additive inverse of y.

Unfortunately, while the provenance semiring $\mathscr{K}_{\text{prov}} = (\mathbb{N}[X], +, \cdot, 0, 1)$ and the semiring of counting numbers $\mathscr{K}_{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$ both contain a monus, neither contain a negation operation. Notice that in the case of an *m*-semiring that is a positive cone of a lattice-ordered commutative ring, even if we had taken a whole lattice-ordered commutative ring and not just its positive cone, we still have no guarantee that it contains a negation operation.

Note 3.3 (additive inversion and negation). The additive inversion operation in a lattice-ordered commutative ring may not be a negation operation. Although it is involutive, that is,

$$-(-x) = x,$$

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it may not be order-reversing, that is, the implication

$$x \leqslant y \Rightarrow -y \leqslant -x$$

may not hold.

Following Definition 3.7, we may define a difference-like operation in a commutative n-semiring by

$$x \div y \stackrel{\text{def}}{=} x \odot \neg y.$$

So

$$\neg x = 1 \div x.$$

Proposition 3.1 (identities in a commutative n-semiring). In a commutative n-semiring $\mathscr{K} = (K, \oplus, \odot, \mathbf{0}, \mathbf{1}, \neg)$, where $x \div y \stackrel{\text{def}}{=} x \odot \neg y$, the following additional identities involving $\div$ hold:

$$\begin{aligned} \mathbf{0} \div x &= \mathbf{0} \\ \mathbf{1} \div (\mathbf{1} \div x) &= x \\ x \div (\mathbf{1} \div y) &= x \odot y \\ (x \div y) \odot y &= x \odot (y \div y). \end{aligned}$$

Notice the differences between the properties of the monus-based difference (Proposition 2.2) and the properties of the negation-based difference (Proposition 3.1). For instance, in a commutative n-semiring, we do *not* have $x \div x = \mathbf{0}$ in general.

4. From semirings to lattices

In this section we return to the fact that the $\mathscr{K}$-relational algebra does not satisfy all of the classical relational algebra identities. The question we will address is whether a substructure of commutative semirings can be identified such that the induced relational algebra satisfes *all* of the classical relational identities.

4.1. *De Morgan frames*

As we observed in Section 2.1, the $\mathscr{K}$-relational algebra does not satisfy the properties of idempotency of union and self-join. By expanding the definitions, it is easy to see that this is because, in general, the sum and product operators of a semiring are not idempotent. The inverse is also true, that is, if the sum and product operators of a semiring $\mathscr{K}$ are idempotent, then the induced $\mathscr{K}$-relational algebra has an idempotent union and self-join.

There are several different substructures of commutative semirings in which union and self-join are idempotent. However, we will take this opportunity to reconsider the *order* on the underlying structure. We noted earlier that all semirings have a preorder and some particular semirings have a partial order (for example, m-semirings). When querying annotated relations, it is often useful to compare rankings. Clearly, for these purposes

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we are looking for something stronger than a pre- or partial order. We could insist on a linear order, but that would preclude many semiring structures of interest, including the probabilistic semiring $\mathscr{K}_{\text{prob}}$ we considered earlier, and other non-linearly ordered posets and lattices from more general variants of fuzzy data models (Peeva and Kyosev 2004). Thus we impose a weaker condition, but one that is still of practical use: we require that any two ranks must have a least upper bound and a greatest lower bound. In other words, we restrict the commutative semiring to be a *lattice*, with the lattice join defined as sum and the lattice meet as product (Davey and Priestley 1990). Note that lattices satisfy all the identities from Proposition 2.1 if they are bounded and distributive, and they have idempotent join (sum) and meet (product). To support relational difference, we need to consider lattices that also have a negation operation.

To summarise, at this point we are proposing to move from $\mathscr{K}$-relations, where $\mathscr{K}$ is a commutative semiring, to $\mathscr{L}$-relations, where $\mathscr{L}$ is a bounded, distributive lattice with a negation operation. However, whilst this would work, such structures are not very well known in the world of mathematics, where it is more natural to drop the bound requirements, and allow infinite sums. Such structures are known as De Morgan frames.

Definition 4.1 (De Morgan frame (Salii 1983)). A *De Morgan frame*

$$\mathscr{L} = (L, \bigvee, \wedge, \mathbf{0}, \mathbf{1}, \neg)$$

is a complete lattice $(L, \bigvee, \wedge, \mathbf{0}, \mathbf{1})$ where finite meets distribute over arbitrary joins, that is,

$$x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i),$$

and $\neg : L \to L$ is a negation operation.

Whilst our motivations are theoretical, infinite relations have been used in foundational studies of data models before: for example, in the context of constraint databases (Kuper *et al.* 2000).

Lemma 4.1. Every boolean algebra is a De Morgan frame. Moreover, the *n*-semirings defined in Lemma 3.1 can be extended to De Morgan Frames:

$$\begin{aligned}
\mathscr{L}_{\mathbb{B}} &= (\mathbb{B}, \vee, \wedge, \mathsf{false}, \mathsf{true}, \neg) \\
\mathscr{L}_{[0,1]} &= ([0,1], \max, \min, 0, 1, \neg) \\
\mathscr{L}_{[0,d_{\max}]} &= ([0,d_{\max}], \min, \max, d_{\max}, 0, \neg).
\end{aligned}$$

We refer to these as the boolean, fuzzy and distance De Morgan frames, respectively.

There are some other lattice structures with a form of negation in the literature, such as De Morgan lattices (Hutton 1975) and fuzzy lattices (Wang 1986). The former are too restrictive because not all semiring structures of interest, including the fuzzy semiring, support a complementation operation. Fuzzy lattices are a completely distributive substructure of De Morgan frames, and we do not see a need for this additional property.

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Proposition 4.1 (De Morgan laws (Salii 1983)). Given a De Morgan frame $\mathscr{L} = (L, \bigvee, \wedge, \mathbf{0}, \mathbf{1}, \neg)$, the following laws hold:

$$\neg\mathbf{0} = \mathbf{1}$$
$$\neg\mathbf{1} = \mathbf{0}$$
$$\neg(x \vee y) = \neg x \wedge \neg y$$
$$\neg(x \wedge y) = \neg x \vee \neg y.$$

Note 4.1. It is important to note that negation behaves differently from the familiar notion of complements in a lattice (Davey and Priestley 1990, Chapter 7). In particular, given a De Morgan frame $\mathscr{L} = (L, \bigvee, \wedge, \mathbf{0}, \mathbf{1}, \neg)$, it is not necessarily the case that $x \vee \neg x = \mathbf{1}$ or $x \wedge \neg x = \mathbf{0}$. Indeed, these properties would preclude the fuzzy models necessary for similarity querying because the fuzzy semiring is not complemented, that is, these properties do not hold for $\neg x = 1 - x$.

4.2. *The $\mathscr{L}$-relation model*

Definition 4.2 ($\mathscr{L}$-relation). Let $\mathscr{L} = (L, \bigvee, \wedge, \mathbf{0}, \mathbf{1}, \neg)$ be a De Morgan frame. An *$\mathscr{L}$-relation* over a schema $U = \{a_1 : \tau_1, \ldots, a_n : \tau_n\}$ is a function $A : U\text{-Tup} \to L$.

Definition 4.3 (relational algebra on $\mathscr{L}$-relations). Let $\mathscr{L} = (L, \bigvee, \wedge, \mathbf{0}, \mathbf{1}, \neg)$ be a De Morgan frame. The operations of the relational algebra on $\mathscr{L}$, denoted by $\mathrm{RA}_{\mathscr{L}}$, are defined as follows:

Empty relation: For any set of attributes U there is $\varnothing_U : U\text{-Tup} \to L$ such that $\varnothing(t) \stackrel{\text{def}}{=} \mathbf{0}$ for all U-tuples t.

Union: If $A, B : U\text{-Tup} \to L$, then $A \cup B : U\text{-Tup} \to L$ is defined by

$$(A \cup B)(t) \stackrel{\text{def}}{=} A(t) \vee B(t).$$

Projection: If $A : U\text{-Tup} \to L$ and $V \subset U$, the projection of A on attributes V is defined by

$$(\pi_V A)(t) \stackrel{\text{def}}{=} \bigvee_{(t' \downarrow V) = t \text{ and } A(t') \neq \mathbf{0}} A(t').$$

Selection: If $A : U\text{-Tup} \to L$ and the selection predicate $\mathbf{P} : U\text{-Tup} \to L$ maps each U-tuple to an element of $\mathscr{L}$, then $\sigma_{\mathbf{P}} A : U\text{-Tup} \to L$ is defined by

$$(\sigma_{\mathbf{P}} A)(t) \stackrel{\text{def}}{=} A(t) \wedge \mathbf{P}(t).$$

Join: If $A : U_1\text{-Tup} \to L$ and $B : U_2\text{-Tup} \to L$, then $A \bowtie B$ is the $\mathscr{L}$-relation over $U_1 \cup U_2$ defined by

$$(A \bowtie B)(t) \stackrel{\text{def}}{=} A(t \downarrow U_1) \wedge B(t \downarrow U_2).$$

Difference: If $A, B : U\text{-Tup} \to L$, then $A \setminus B : U\text{-Tup} \to L$ is defined by

$$(A \setminus B)(t) \stackrel{\text{def}}{=} A(t) \wedge \neg B(t).$$

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Renaming: If $A : U\text{-Tup} \to L$ and $\beta : U \to U'$ is a bijection, then $\rho_\beta A : U'\text{-Tup} \to L$ is defined by

$$(\rho_\beta A)(t) \stackrel{\text{def}}{=} A(t \circ \beta).$$

Note 4.2. Recall that, unlike the case for $\mathscr{K}$-relations, we did not require $\mathscr{L}$-relations to have finite support. As De Morgan frames are *complete* lattices, we are guaranteed the existence of the join arising from the definition of projection, so requiring finite support is redundant.

As we observed earlier, the lattice supremum $\vee$ and infimum $\wedge$ operators from De Morgan frames are idempotent and as a consequence the union and self-join in $\text{RA}_{\mathscr{L}}$ are idempotent, which was not the case in $\text{RA}_{\mathscr{K}}$. Consequently, $\text{RA}_{\mathscr{L}}$ satisfies *all* the main positive relational algebra identities, which, in terms of query optimisation, means that all algebraic rewrites familiar from the classical (positive) relational algebra apply to $\text{RA}_{\mathscr{L}}$ without restriction.

Proposition 4.2 (identities of $\mathscr{L}$-relations). The following identities hold for the relational algebra on $\mathscr{L}$-relations:

— union is associative, commutative and idempotent, and has identity $\varnothing$;
— selection distributes over union and difference;
— join is associative and commutative, and distributes over union;
— projection distributes over union and join;
— selections and projections commute with each other;
— difference has identity $\varnothing$ and distributes over union and intersection;
— selection with boolean predicates gives all or nothing, that is, $\sigma_{\mathsf{false}}(A) = \varnothing$ and $\sigma_{\mathsf{true}}(A) = A$;
— join with an empty relation gives an empty relation, that is, $A \bowtie \varnothing_U = \varnothing_U$ where A is an $\mathscr{L}$-relation over a schema U;
— projection of an empty relation gives an empty relation, that is, $\pi_V(\varnothing) = \varnothing$.

Proof. It is an easy exercise to see that projection distributes over union because De Morgan frames are complete. Moreover, selections and projections commute with each other because in the De Morgan frames finite meets distribute over arbitrary joins. □

Matters are a little different for the negative identities. In fuzzy relations (Rosado *et al.* 2006) many of the familiar laws concerning difference do not hold. For example, it is not the case that $A \setminus A = \varnothing$, so it is not the case in general for the $\mathscr{L}$-relational algebra. By expanding the definition of difference in the case of the common fuzzy semiring $\mathscr{L} = ([0, 1], \max, \min, 0, 1)$, we get

$$(A \setminus A)(t) = \min\{A(t), 1 - A(t)\}.$$

Clearly, this is only equal to 0 when $A(t) = 0$ or $A(t) = 1$. Consequently, some identities from the classical relational algebra no longer hold.

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Proposition 4.3. In $\mathrm{RA}_{\mathscr{L}}$ the following identities hold:

$$\begin{aligned} A \cap \neg A &= A \setminus A \\ A \cup \neg A &= \neg(A \setminus A) \\ (A \setminus B) \cap B &= A \cap (B \setminus B) \\ (A \setminus B) \cup B &= (A \cup B) \cap \neg(B \setminus B) \\ A \cap B &= A \setminus (1 \setminus B) \end{aligned}$$

where

$$\begin{aligned} (\neg A)(t) &\stackrel{\text{def}}{=} \neg A(t) \\ (A \cap B)(t) &\stackrel{\text{def}}{=} A(t) \wedge B(t) \\ 1(t) &\stackrel{\text{def}}{=} \mathbf{1}. \end{aligned}$$

Proof. These all hold by simple expansion of definitions and use of the De Morgan laws given in Proposition 4.1. For example, the second identity is verified as follows.

$$\begin{aligned} (A \cup \neg A)(t) &= A(t) \vee \neg A(t) \\ &= \neg(\neg A(t) \wedge A(t)) \\ &= \neg(A \setminus A)(t). \end{aligned}$$

□

Lemma 4.2. If $\mathscr{L}$ is a boolean algebra, the identities from Proposition 4.3 are simplified as follows.

$$\begin{aligned} A \cap \neg A &= \varnothing \\ A \cup \neg A &= 1 \\ (A \setminus B) \cap B &= \varnothing \\ (A \setminus B) \cup B &= A \cup B \\ A \cap B &= A \setminus (A \setminus B). \end{aligned}$$

These coincide with the identities known from classical relational algebra.

Example 4.1. Let U be a schema with an attribute $\mathsf{loc} : \mathsf{Loc}$, where Loc ranges over names of locations in a city, and let

$$\mathscr{L}_{[0,d_{\max}]} = ([0, d_{\max}], \min, \max, d_{\max}, 0, \neg)$$

be the distance De Morgan frame (see Lemma 4.1), where $d_{\max}$ is the road distance between the two most extreme locations in the city. We also assume a predefined similarity measure, $\rho_{\mathsf{loc}}(x, y)$, which denotes the shortest road distance from x to y.

Let $A : U\text{-Tup} \to [0, d_{\max}]$ be a $\mathscr{L}_{[0,d_{\max}]}$-relation over schema U. We assume initially that every U-tuple t has either desirability $A(t) = 0$ or $A(t) = d_{\max}$, reflecting whether the tuple t is considered in the city centre or not. Let q denote the city centre, and take the

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following selection predicates, which both map from U-Tup to $[0, d_{\max}]$:

$$\mathbf{P}_{\text{road}}(t) \stackrel{\text{def}}{=} \rho_{\mathsf{loc}}(t(\mathsf{loc}), \mathsf{q})$$
$$\mathbf{P}_{\text{air}}(t) \stackrel{\text{def}}{=} \text{the air distance from } t(\mathsf{loc}) \text{ to } \mathsf{q}.$$

Take the queries

$$Q_1 \stackrel{\text{def}}{=} \sigma_{\mathbf{P}_{\text{road}}} A$$
$$Q_2 \stackrel{\text{def}}{=} (\sigma_{\mathbf{P}_{\text{air}}} A) \setminus (\sigma_{\mathbf{P}_{\text{road}}} A).$$

The first query,

$$Q_1(t) = \begin{cases} \mathbf{P}_{\text{road}}(t) & \text{if } A(t) = 0 \\ d_{\max} & \text{if } A(t) = d_{\max} \end{cases}$$

is a relation in which tuples are ranked according to their road distances to the city centre $\mathbf{P}_{\text{road}}(t)$.

The second query,

$$Q_2(t) = \begin{cases} \max\{\mathbf{P}_{\text{air}}(t), d_{\max} - \mathbf{P}_{\text{road}}(t)\} & \text{if } A(t) = 0 \\ d_{\max} & \text{if } A(t) = d_{\max}, \end{cases}$$

is a relation in which a tuple, t, gets a small annotation value (high desirability) whenever $t(\mathsf{loc})$ is near the city centre by air, that is, $\mathbf{P}_{\text{air}}(t)$ is near 0, and the road distance from $t(\mathsf{loc})$ to the city centre is great, that is, $\mathbf{P}_{\text{road}}(t)$ is near $d_{\max}$.

5. From tuple-based annotations to attribute-based annotations

In this section we highlight a weakness of both the $\mathscr{K}$-relation and $\mathscr{L}$-relation models. Whilst both are sufficient for very simple examples of similarity querying, there is a serious complication when considering more involved examples. This stems from the fact that there is only a *single* annotation semiring/De Morgan frame. In other words, all tuples across all the tables in the database and intermediate relations in queries must be annotated with a value from the same De Morgan frame. As we would like to support several different similarity measures simultaneously (for example, similarity of strings, driving distance between cities, likelihood of objects being equal), and use these different measures in our queries (even within the same query), this is clearly insufficient. We could, of course, combine all the semirings/De Morgan frames required into one semiring/De Morgan frame, but this would quickly become quite cumbersome. Moreover, it is highly non-compositional, as every query needs to involve all the similarity domains in the database, whether they are needed in the query or not. We feel that it is conceptually cleaner to move from a tuple-annotated model to an attribute-annotated model; in other words, every attribute is associated with its own De Morgan frame.

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5.1. *The $\mathscr{D}$-relation model*

In this section we generalise the $\mathscr{L}$-relation model by annotating all attributes in a relation individually. There are a number of ways of setting up the technical machinery, but for ease of comparison, we shall again mirror the definitions of the $\mathscr{K}$-relation and $\mathscr{L}$-relation models. We generalise an $\mathscr{L}$-relation, which is a map from a tuple to an annotation value from a De Morgan frame, to a *$\mathscr{D}$-relation*, which is a map from a tuple to a corresponding tuple containing an annotation value for every element in the source tuple (what we will refer to as a De Morgan frame tuple).

Definition 5.1 (De Morgan frame schema, De Morgan frame tuple and $\mathscr{D}$-relation).

— A *De Morgan frame schema* $\mathscr{D} = \{a_1 : \mathscr{L}_1, ..., a_n : \mathscr{L}_n\}$ maps an attribute name a_i to a De Morgan frame $\mathscr{L}_i = (L_i, \bigvee_i, \wedge_i, \mathbf{0}_i, \mathbf{1}_i, \neg_i)$.
— A *De Morgan frame tuple* $s = \{a_1 : l_1, ..., a_n : l_n\}$ maps an attribute name a_i to a De Morgan frame element l_i.
— Given a De Morgan frame schema $\mathscr{D}$, a tuple s is said to be a *De Morgan frame tuple matching* $\mathscr{D}$ if s and $\mathscr{D}$ have the same domain $dom(s) = dom(\mathscr{D})$.
— Given a De Morgan frame schema $\mathscr{D}$ and a schema U, a tuple s is said to be a *De Morgan frame tuple matching $\mathscr{D}$ over U* if $dom(s) = dom(U) = dom(\mathscr{D})$.
— We write $\mathscr{D}(U)$-Tup to denote the set of all De Morgan frame tuples matching $\mathscr{D}$ over U.
— A *$\mathscr{D}$-relation over U* is a finite map from U-Tup to $\mathscr{D}(U)$-Tup. *Its support need* not *be finite.*

We are now ready to define an algebra on $\mathscr{D}$-relations.

Definition 5.2 (relational algebra with similarities). The operations of the *relational algebra with similarities* $\mathrm{RA}_{\mathscr{D}}$ are defined as follows:

Empty relation: For any set of attributes U and corresponding De Morgan frame schema $\mathscr{D}$, the empty $\mathscr{D}$-relation $\varnothing_U$ over U is defined by $\varnothing_U(t)(a) \stackrel{\text{def}}{=} \mathbf{0}_a$ where t is a U-tuple and

$$\mathscr{D}(a) = (L_a, \bigvee_a, \wedge_a, \mathbf{0}_a, \mathbf{1}_a, \neg_a).$$

Union: If $A, B : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$, then $A \cup B : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$ is defined by

$$(A \cup B)(t)(a) \stackrel{\text{def}}{=} A(t)(a) \vee_a B(t)(a)$$

where

$$\mathscr{D}(a) = (L_a, \bigvee_a, \wedge_a, \mathbf{0}_a, \mathbf{1}_a, \neg_a).$$

Projection: If $A : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$ and $V \subset U$, the projection of A on attributes V is defined by

$$(\pi_V A)(t)(a) \stackrel{\text{def}}{=} \bigvee_{(t' \downarrow V) = t \text{ and } A(t')(a) \neq \mathbf{0}_a} A(t')(a)$$

where

$$\mathscr{D}(a) = (L_a, \bigvee_a, \wedge_a, \mathbf{0}_a, \mathbf{1}_a, \neg_a).$$

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Selection: If $A : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$ and the selection predicate

$$\mathbf{P} : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$$

maps each U-tuple to an element of $\mathscr{D}(U)$-Tup, then

$$\sigma_{\mathbf{P}} A : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$$

is defined by

$$(\sigma_{\mathbf{P}} A)(t)(a) \stackrel{\text{def}}{=} A(t)(a) \wedge_a \mathbf{P}(t)(a)$$

where

$$\mathscr{D}(a) = (L_a, \bigvee_a, \wedge_a, \mathbf{0}_a, \mathbf{1}_a, \neg_a).$$

Join: Let

$$\mathscr{D}_1 = \{a_1 : \mathscr{L}_1, ..., a_n : \mathscr{L}_n\}$$
$$\mathscr{D}_2 = \{b_1 : \mathscr{L}'_1, ..., b_m : \mathscr{L}'_m\}$$

be De Morgan frame schemata. Let their union $\mathscr{D}_1 \cup \mathscr{D}_2$ contain an attribute $c_i : \mathscr{L}_i$, when $c_i : \mathscr{L}_i$ is in $\mathscr{D}_1$ or $\mathscr{D}_2$ or both. (If there is an attribute with different corresponding De Morgan frames in $\mathscr{D}_1$ and $\mathscr{D}_2$, a renaming of attributes is needed.) If

$$A : U_1\text{-Tup} \to \mathscr{D}_1(U_1)\text{-Tup}$$
$$B : U_2\text{-Tup} \to \mathscr{D}_2(U_2)\text{-Tup},$$

then $A \bowtie B$ is the $(\mathscr{D}_1 \cup \mathscr{D}_2)$-relation over $U_1 \cup U_2$ defined as follows:

$$(A \bowtie B)(t)(a) \stackrel{\text{def}}{=} \begin{cases} A(t \downarrow U_1)(a) & \text{if } a \in U_1 - U_2 \\ B(t \downarrow U_2)(a) & \text{if } a \in U_2 - U_1 \\ A(t \downarrow U_1)(a) \wedge_a B(t \downarrow U_2)(a) & \text{if } a \in U_1 \cap U_2. \end{cases}$$

Difference: If $A, B : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$, then

$$A \setminus B : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$$

is defined by

$$(A \setminus B)(t)(a) \stackrel{\text{def}}{=} A(t)(a) \wedge_a (\neg_a B(t)(a))$$

where

$$\mathscr{D}(a) = (L_a, \bigvee_a, \wedge_a, \mathbf{0}_a, \mathbf{1}_a, \neg_a).$$

Renaming: If $A : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$ and $\beta : U \to U'$ is a bijection, then

$$\rho_\beta A : U'\text{-Tup} \to \mathscr{D}(U')\text{-Tup}$$

is defined by

$$(\rho_\beta A)(t)(a) \stackrel{\text{def}}{=} A(t)(\beta(a)).$$

As in the case of $\mathscr{L}$-relations, we require that every tuple outside of a similarity database is ranked with the minimal De Morgan frame tuple, $\{a_1 : \mathbf{0}_1, \ldots, a_n : \mathbf{0}_n\}$, and every other tuple is ranked either with the maximal De Morgan frame tuple, $\{a_1 : \mathbf{1}_1, \ldots, a_n : \mathbf{1}_n\}$, or a

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smaller De Morgan frame tuple expressing a lower degree of containment of the tuple in the database.

The relational algebra with similarities $\mathrm{RA}_{\mathscr{D}}$ satisfies the following relational algebra identities.

Proposition 5.1 (identities of $\mathscr{D}$-relations). The following identities hold for the relational algebra on $\mathscr{D}$-relations:

— union is associative, commutative and idempotent, and has identity $\varnothing$;
— selection distributes over union and difference;
— join is associative and commutative, and distributes over union;
— projection distributes over union and join;
— selections and projections commute with each other;
— difference has identity $\varnothing$ and distributes over union and intersection;
— selection with boolean predicates gives all or nothing, that is,

$$\sigma_{\mathsf{false}}(A) = \varnothing$$
$$\sigma_{\mathsf{true}}(A) = A,$$

where

$$\mathsf{false(t)(a)} = \mathbf{0}_a$$
$$\mathsf{true(t)(a)} = \mathbf{1}_a$$

for

$$\mathscr{D}(a) = (L_a, \bigvee_a, \wedge_a, \mathbf{0}_a, \mathbf{1}_a, \neg_a);$$

— join with an empty relation gives an empty relation, that is, $A \bowtie \varnothing_U = \varnothing_U$ where A is a $\mathscr{D}$-relation over a schema U;
— projection of an empty relation gives an empty relation, that is, $\pi_V(\varnothing) = \varnothing$.

Example 5.1. We take the schema $U = \{\mathsf{name} : \mathsf{String}, \mathsf{location} : \mathsf{String}\}$, which records pairs of customer names and their current location, and the De Morgan frame schema $\mathscr{D} = \{\mathsf{name} : \mathscr{L}_M, \mathsf{location} : \mathscr{L}_{[0,d_{\max}]}\}$ where $\mathscr{L}_M$ is the De Morgan Frame corresponding to the Levenshtein similarity semiring from Example 3.4 and $\mathscr{L}_{[0,d_{\max}]}$ is the distance De Morgan frame.

Let us imagine that we have a relation A of customers and their location in which it is anticipated that several rows, whilst different, actually denote the same information[†]. In the real world this is common; people use variants of their name (for example, 'Bill Clinton', 'W. J. Clinton') and of their city ('Manhattan', 'New York'). Within the similarity setting, such data can be naturally queried. Imagine a select query $\sigma_{\mathbf{P}_{\mathsf{N,L}}} A$ asking for a certain customer with name N at location L, and let the selection predicate

$$\mathbf{P}_{\mathsf{N,L}} : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$$

[†] Initially, all tuples in A are annotated with $\{\mathsf{name} : 0, \mathsf{location} : 0\}$, and those not in the relation with $\{\mathsf{name} : M, \mathsf{location} : d_{\max}\}$.

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map a U-tuple $t = \{\mathsf{name} : n, \mathsf{location} : l\}$ to a De Morgan frame tuple

$$\mathbf{P}_{\mathsf{N,L}}(t) = \{\mathsf{name} : k, \mathsf{location} : d\}$$

where k is the Levenshtein distance between the names n and N, and d is the distance between locations l and L. Using these values, it is simple to see that the query $\sigma_{\mathbf{P}_{\mathsf{N,L}}} A$ satisfies the following:

$$(\sigma_{\mathbf{P}_{\mathsf{N,L}}} A)(t)(a) = \begin{cases} \max\{A(t)(a), k\} & \text{if } a = \mathsf{name} \\ \max\{A(t)(a), d\} & \text{if } a = \mathsf{location}. \end{cases}$$

Then the above query succeeds in identifying the tuples that are closest to the customer with name N at location L by giving those the smallest annotation values in the associated De Morgan frame tuple.

5.2. *The selection predicate*

Each of the similarity measures associated with the attributes now maps to its own De Morgan frame. We assume that for every $\mathscr{D}$-relation over U, where

$$\begin{aligned} \mathscr{D} &= \{a_1 : \mathscr{L}_1, ..., a_n : \mathscr{L}_n\} \\ \mathscr{L}_i &= (L_i, \bigvee_i, \wedge_i, \mathbf{0}_i, \mathbf{1}_i, \neg_i) \\ U &= \{a_1 : \tau_1, ..., a_n : \tau_n\}, \end{aligned}$$

there is a similarity measure $\rho_i : \tau_i \times \tau_i \to L_i, 1 \leqslant i \leqslant n$. For a given attribute a_i, we will simply write ρ_{a_i} to denote this similarity measure, rather than fold it into the definition of $\mathscr{D}$-relations.

In the $\mathscr{D}$-relation model, we will now redefine the primitive and similarity predicates.

Definition 5.3. Suppose in a schema $U = \{a_1 : \tau_1, \ldots, a_n : \tau_n\}$ the types of attributes a_i and a_j coincide. Then for a given binary predicate θ, the *primitive predicate*

$$[a_i \,\theta\, a_j] : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$$

is defined by

$$[a_i \,\theta\, a_j](t)(a_k) \stackrel{\text{def}}{=} \begin{cases} \chi_{a_i \theta a_j}(t) & \text{if } k = i \text{ or } k = j \\ \mathbf{1}_k & \text{otherwise.} \end{cases}$$

In words, $[a_i \,\theta\, a_j]$ has value $\mathbf{1}$ in every attribute except a_i and a_j, where it behaves as the characteristic map of θ defined by

$$\chi_{a_i \theta a_j}(t) \stackrel{\text{def}}{=} \begin{cases} \mathbf{1}_k & \text{if } t(a_i) \,\theta\, t(a_j) \\ \mathbf{0}_k & \text{otherwise.} \end{cases}$$

Similarity predicates rank tuples based on the similarity measures.

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Definition 5.4. Suppose in a schema $U = \{a_1 : \tau_1, \ldots, a_n : \tau_n\}$ the types of attributes a_i and a_j coincide. The similarity predicate $[a_i \texttt{ like } a_j] : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$ is defined by

$$[a_i \texttt{ like } a_j](t)(a_k) \stackrel{\text{def}}{=} \begin{cases} \rho_{a_i}(t(a_i), t(a_j)) & \text{if } a_k = a_i \\ \rho_{a_j}(t(a_i), t(a_j)) & \text{if } a_k = a_j \\ \mathbf{1}_k & \text{otherwise.} \end{cases}$$

In words, $[a_i \texttt{ like } a_j]$ measures the similarity of attributes a_i and a_j, each with its own similarity measure. The symmetric version is defined by

$$[a_i \sim a_j] \stackrel{\text{def}}{=} [a_i \texttt{ like } a_j] \cup [a_j \texttt{ like } a_i].$$

Now union and intersection of selection predicates are computed component-wise.

The selection criterion from Example 5.1, for instance, may be expressed by the selection predicate $[\mathsf{name} \sim a] \cap [\mathsf{location} \sim b]$, where a and b are the name of a chosen customer and where they live, respectively.

5.3. *Additional algebraic operations*

There are some other useful operations on $\mathscr{D}$-relations that can be expressed in terms of the basic operations.

Definition 5.5 (intersection). We define intersection $A \cap B : \mathscr{D}\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$ of relations $A, B : U\text{-Tup} \to \mathscr{D}(U)\text{-Tup}$ by

$$(A \cap B)(t)(a) \stackrel{\text{def}}{=} A(t)(a) \wedge_a B(t)(a)$$

where $\mathscr{D}(a) = (L_a, \bigvee_a, \wedge_a, \mathbf{0}_a, \mathbf{1}_a, \neg_a)$.

Proposition 5.2. Intersection $A \cap B$ is equivalent to selection $\sigma_{\mathbf{P}} A$ with an appropriately chosen selection predicate. That is, if $\mathbf{P}(t) \stackrel{\text{def}}{=} B(t)$, for all $a \in U$, we have

$$(A \cap B)(t)(a) = (\sigma_{\mathbf{P}} A)(t)(a).$$

Proof. Clearly, if $\mathbf{P}(t) \stackrel{\text{def}}{=} B(t)$, we have $(\sigma_{\mathbf{P}} A)(t)(a) = A(t)(a) \wedge_a B(t)(a)$. □

Definition 5.6 (product). We define the product of

$$A : U_1 \to \mathscr{D}_1(U_1)$$

and

$$B : U_2 \to \mathscr{D}_2(U_2)$$

with $U_1 \cap U_2 = \varnothing$ as a $(\mathscr{D}_1 \cup \mathscr{D}_2)$-relation over $U_1 \cup U_2$ by

$$(A \times B)(t) \stackrel{\text{def}}{=} A(t \downarrow U_1) \cup B(t \downarrow U_2).$$

If the joining relations have no common attributes, product and join coincide.

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Proposition 5.3. If

$$A : U_1 \to \mathscr{D}_1(U_1)$$
$$B : U_2 \to \mathscr{D}_2(U_2)$$

and $U_1 \cap U_2 = \varnothing$, then product is equivalent to join:

$$(A \times B)(t)(a) = (A \bowtie B)(t)(a).$$

Proof. For the proof, we just need to observe that

$$(A \times B)(t)(a) = \begin{cases} A(t \downarrow U_1)(a) & \text{if } a \in U_1 \\ B(t \downarrow U_2)(a) & \text{if } a \in U_2. \end{cases}$$ □

5.4. *Similarity-based joins*

Given the similarity measures associated with attributes, it is possible to define similarity-based variants of other familiar relational operators. In this section we develop the notion of a similarity-based join. The intuition behind this operator is clear: we join two rows not only when their join-attributes have equal associated values, but also when the values are similar.

Definition 5.7 (similarity-based joins). Consider a schema U containing attribute $a : \tau$, and a schema V containing attribute $b : \tau$. Note that the types of the attributes a and b must be the same, but their supporting similarity measures ρ_a and ρ_b need not be. We *join* $\mathscr{D}$-relations A (over U) and B (over V) with respect to attributes a and b in different ways:

Equality join:

$$A \bowtie_{a=b} B \stackrel{\text{def}}{=} \sigma_{[a=b]}(A \times B).$$

Similarity join:

$$A \bowtie_{a \texttt{ like } b} B \stackrel{\text{def}}{=} \sigma_{[a \texttt{ like } b]}(A \times B).$$

Symmetric similarity join:

$$A \bowtie_{a \sim b} B \stackrel{\text{def}}{=} \sigma_{[a \sim b]}(A \times B).$$

For simplicity, we have only considered the case of a binary join: the generalisation to a join of several pairs of attributes is straightforward – the selection predicate is defined as the intersection ($\cap$) of several primitive and/or similarity predicates. Similar notions of similarity-based joins have been developed (but only) on a fixed structure of truth degrees for all attributes (Adali *et al.* 1998; Penzo 2005; Belohlavek and Vychodil 2006).

Since $=$ and $\sim$ are symmetric, we may commute the compared attributes.

Proposition 5.4. We have

$$A \bowtie_{a=b} B = A \bowtie_{b=a} B$$
$$A \bowtie_{a \sim b} B = A \bowtie_{b \sim a} B.$$

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This, of course, does not hold for the (ordinary) similarity joins since [a `like` b] and [b `like` a] are in general different similarity predicates.

6. Extended example

We consider a simplified scenario of a database representing bus connections in a city. A co-operative database system should include consideration of journeys made up of several connecting rides as well as possible routes that include (small!) sections of walking between stations. Clearly, such a system requires knowledge of the distance between arbitrary locations in a city. Moreover, a notion of similarity between arrival and departure times allows us to consider interconnection timings.

Consider the relational schema

$$U = \{\, \mathsf{from} : \mathsf{Loc}, \mathsf{to} : \mathsf{Loc}, \mathsf{dep} : \mathsf{Time}, \mathsf{arr} : \mathsf{Time}, \mathsf{bus} : \mathsf{Bus_id} \,\},$$

where Loc, Time and Bus_id range over names of bus stations, bus departure/arrival times in minutes, and bus ids, respectively. Assume also the De Morgan frame schema

$$\mathscr{D} = \{\, \mathsf{from} : \mathscr{L}_{[0,d_{\max}]}, \mathsf{to} : \mathscr{L}_{[0,d_{\max}]}, \mathsf{dep} : \mathscr{L}_{[0,t_{\max}]}, \mathsf{arr} : \mathscr{L}_{[0,t_{\max}]}, \mathsf{bus} : \mathscr{L}_{\mathbb{B}} \,\},$$

where $d_{\max}$ is the walking time distance between two furthest locations in the city and $t_{\max} = 24 \cdot 60 = 1440$ is the number of minutes in a day, together with a predefined environment of similarity measures:

$$\rho_{\mathsf{from}}(x, y) = \rho_{\mathsf{to}}(x, y) \stackrel{\text{def}}{=} \text{time needed to walk from } x \text{ to } y$$

$$\rho_{\mathsf{dep}}(x, y) \stackrel{\text{def}}{=} \begin{cases} x - y & \text{if } x \geqslant y \\ t_{\max} & \text{otherwise} \end{cases}$$

$$\rho_{\mathsf{arr}}(x, y) \stackrel{\text{def}}{=} \rho_{\mathsf{dep}}(y, x)$$

$$\rho_{\mathsf{bus}}(x, y) \stackrel{\text{def}}{=} \mathsf{true}.$$

The similarity $\rho_{\mathsf{dep}}(x, y)$ between the bus departure time x and the time we arrive at the station y denotes how long we have to wait for the bus to depart. If we arrive too late at the station, we miss the bus. Similarly, the similarity $\rho_{\mathsf{arr}}(x, y)$ is defined as the penalty when we arrive at time x and would like to arrive at time y, where coming too late is deemed catastrophic. Because we (usually) do not care which bus we take, all bus Ids are declared similar by ρ_{bus}.

Let buses be a bus timetable relation, which is a $\mathscr{D}$-relation over schema U. Prior to any querying, we assume that each timetable entry, tuple t, has desirability

$$1 = \{\mathsf{from} : 0, \mathsf{to} : 0, \mathsf{dep} : 0, \mathsf{arr} : 0, \mathsf{bus} : \mathsf{true}\},$$

and all other tuples have desirability

$$0 = \{\mathsf{from} : d_{\max}, \mathsf{to} : d_{\max}, \mathsf{dep} : t_{\max}, \mathsf{arr} : t_{\max}, \mathsf{bus} : \mathsf{false}\}.$$

We will now develop several ways in which the question

'How can I get from station a to station b now?'

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can be expressed as a query, assuming that the current time is given by a constant now.

(1) We first consider a selective query,

$$Q \stackrel{\text{def}}{=} \sigma_{\mathbf{P}}\mathsf{buses},$$

and vary the selection predicate **P** as follows:

— If we define

$$\mathbf{P} \stackrel{\text{def}}{=} [\mathsf{from} = a] \cap [\mathsf{to} = b] \cap [\mathsf{dep} = \mathsf{now}],$$

the query Q corresponds exactly to the text of the question. It is unlikely to succeed because it asks for a bus that departs at exactly this moment.

— We now relax the departure time by replacing the equality $=$ by the similarity predicate `like` :

$$\mathbf{P} \stackrel{\text{def}}{=} [\mathsf{from} = a] \cap [\mathsf{to} = b] \cap [\mathsf{dep}\ \texttt{like}\ \mathsf{now}].$$

The answer to the resulting query consists of rows t together with their desirabilities $Q(t)$, which are De Morgan frame tuples of the form

$$Q(t) = \{\mathsf{from} : x, \mathsf{to} : y, \mathsf{dep} : z, \mathsf{arr} : 0, \mathsf{bus} : \mathsf{true}\}.$$

The desirability components $Q(t)(\mathsf{from}) = x$ and $Q(t)(\mathsf{to}) = y$ are either 0 or $d_{\max}$, depending on whether we have $t(\mathsf{from}) = a$ and $t(\mathsf{to}) = b$. The number $Q(t)(\mathsf{dep}) = z$ conveys how long we have to wait for the bus to depart. The attributes arr and bus receive the greatest value because they do not appear in the selection condition.

— If we are willing to walk from a to a different starting station, but we want to arrive exactly at b, we can use the selection predicate

$$\mathbf{P} \stackrel{\text{def}}{=} [\mathsf{from}\ \texttt{like}\ a] \cap [\mathsf{to} = b] \cap [\mathsf{dep}\ \texttt{like}\ \mathsf{now}].$$

In this case each answer t is annotated with a De Morgan frame tuple

$$Q(t) = \{\mathsf{from} : x, \mathsf{to} : y, \mathsf{dep} : z, \mathsf{arr} : 0, \mathsf{bus} : \mathsf{true}\},$$

where y and z are as before, but $Q(t)(\mathsf{from}) = x$ tells us how long it takes to walk from a to the departure station. However, it may happen that the bus will depart from $\mathsf{from}(t)$ before we get there.

— We can also support the case where the starting and ending locations a and b are not necessarily locations of bus stations. The selection predicate

$$\mathbf{P} \stackrel{\text{def}}{=} [\mathsf{from}\ \texttt{like}\ a] \cap [\mathsf{to}\ \texttt{like}\ b] \cap [\mathsf{dep}\ \texttt{like}\ \mathsf{now}]$$

gives useful answers for arbitrary locations a and b. The desirability $Q(t)$ of a row t contains information about the time needed to get from location a to the departure station and from the arrival station to the final destination b.

(2) Now suppose we are interested in indirect connections between a and b, and let buses_1 and buses_2 be two instances of the $\mathscr{D}$-relation buses. The query

$$Q \stackrel{\text{def}}{=} \sigma_{\mathbf{P}}(\mathsf{buses}_1 \bowtie_{\mathsf{to}_1 \sim \mathsf{from}_2} \mathsf{buses}_2)$$

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with the selection predicate being defined as

$$\mathbf{P} \stackrel{\text{def}}{=} [\mathsf{from}_1 \texttt{ like } a] \cap [\mathsf{to}_2 \texttt{ like } b] \cap [\mathsf{dep}_1 \texttt{ like now}]$$

computes connecting rides from stations near a to stations near b. The schema of relation Q is $U_1 \cup U_2$, which is obtained (after a renaming of attributes) as the union of two instances of schema U. Its De Morgan frame schema, $\mathscr{D}_1 \cup \mathscr{D}_2$, is obtained similarly. In this case each answer tuple $t \in (U_1 \cup U_2)$-Tup contains information about a pair of rides. It is annotated with a De Morgan frame tuple

$$\begin{aligned} Q(t) = \{&\mathsf{from}_1 : x, \mathsf{to}_1 : y, \mathsf{dep}_1 : u, \mathsf{arr}_1 : 0, \mathsf{bus}_1 : \mathsf{true}, \\ &\mathsf{from}_2 : y, \mathsf{to}_2 : z, \mathsf{dep}_2 : 0, \mathsf{arr}_2 : 0, \mathsf{bus}_2 : \mathsf{true}\}. \end{aligned}$$

The main components of desirability $Q(t)$ are:

— x is walking time from a to the departure station;
— y is walking time between the connecting stations $t(\mathsf{to}_1)$ and $t(\mathsf{from}_2)$;
— z is walking time from the arrival station to b;
— u is waiting time until the first bus departs.

(3) Finally, if we only want to know which buses drive from station a to station b, we can simply combine a projection and a selection:

$$Q \stackrel{\text{def}}{=} \pi_{\mathsf{bus}}(\sigma_{[\mathsf{from}=a] \cap [\mathsf{to}=b]} \mathsf{buses}).$$

The answers now only contain the attribute bus, and the desirability $Q(t)$ has value true for buses that pass station a before they pass station b, and false otherwise.

7. Related work and conclusions

In this paper we have considered the problem of defining a formal model suitable for modelling similarity querying. After observing that ranked tables are simply annotated relations in the sense of Green *et al.* we have considered the suitability of their $\mathscr{K}$-relation model for encoding similarity information and similarity queries. We have also considered a recent extension of this model to handle negative queries due to Geerts and Poggi. We have shown that when considering the fuzzy commutative semirings that are important in the similarity setting, their extension does not behave well. We have shown that using a negation operation, rather than a monus operation, offers a solution that works in the similarity settings. We then moved on to consider the identities induced in the relational algebra by the underlying commutative semiring. To handle all the familiar identities from the relational algebra, we propose moving from commutative semirings with negation to De Morgan frames. Our final contribution is to propose moving from row-level annotations from a single semiring/De Morgan frame to attribute-level annotations, where each attribute supports its own annotation domain.

There have been a huge number of extensions to the relational model to support imprecise/uncertain data. Much of that work models probabilistic uncertainty of data, which is quite different from the kind of deterministic, fuzzy uncertainty we are interested in. The field of 'fuzzy' extensions to the relational model is also huge – see Ma and

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Yan (2008) for a comprehensive overview. Rather than simply repeating this overview, we shall just mention two closely related works. The approach to modelling similarity querying taken in Belohlávek and Vychodil (2006) is similar to ours in taking an algebraic approach and proposing complete lattices to encode the degree of truth. There are some small differences of detail (for example, they require similarity measures to be symmetric), but there are also three major differences between our work and theirs:

(1) We relate our model to existing other work on annotated relations.
(2) We consider negation whereas they focus only on positive queries.
(3) We propose attribute-level ranking, whereas Belohlávek and Vychodil rank all tuples in the database using a single, fixed lattice structure, but, as we have argued, we find this to be extremely limiting in practice.

The work reported in Schmitt and Schulz (2004) is also close to ours. They define a similarity relational calculus that combines the handling of imprecise truth values together with a traditional relational domain calculus, and show how to map the similarity relational calculus to a similarity algebra. Putting aside their use of a calculus rather than an algebra, the major difference between our work is that they consider a single, concrete rank domain (the interval $[0, 1]$), whereas we take an abstract, algebraic approach. This enables us to derive properties that *all* models must satisfy. They define a tuple rank based on a specific aggregation of the specific truth values (with similarities all taken from the unit interval), whereas we rank each tuple with a tuple of similarity values (taken from possibly different similarity domains). They do not develop important features such as similarity joins and do not attempt any comparison with the classical relational identities.

Although there are many advantages to our proposal, there are also a number of disadvantages. First, it is clear that asking all attributes to be annotated requires more storage than simple row-level annotation. Another problem is that even if each De Morgan frame used in a De Morgan frame schema is linearly ordered, it is not the case that there is a linear order on the De Morgan frame tuples. Hence, it may not be possible to list query answers in a (decreasing) order of relevance. But this simply reflects a fact about ordered structures as opposed to a specific flaw in our model.

However, an ordering with decreasing relevance is guaranteed when the product of the De Morgan frames from the De Morgan frame schema is *graded* (Stanley 1997). A *graded* or *ranked poset* is a partially ordered set P equipped with a monotone rank function $\rho : P \to \mathbb{Z}$ compatible with the ordering (so $\rho(x) < \rho(y)$ whenever $x < y$) such that whenever y covers x, then $\rho(y) = \rho(x) + 1$. Examples of graded posets are the natural numbers with the usual order, the Cartesian product of two or more sets of natural numbers with the product order being the sum of the coefficients, and the boolean lattice of finite subsets of a set ordered according to the number of elements in the subset. On the other hand, neither the integers (with the usual order) nor any interval (with more than one element) of rational or real numbers is a graded poset. Hence, we cannot expect to have a rank function when querying similarity information in the $\mathscr{D}$-relation model. The De Morgan frame tuples in the answer space may be pairwise incomparable, and there may not exist a 'most' desirable tuple. However, the most relevant tuples (which should be offered to the user first) are those that are not dominated by any other tuple on all

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dimensions of desirability. Such tuples build the so-called *skyline* (Papadias *et al.* 2005). The final decision about the best choice from all the skyline tuples should be left to the user – it cannot be a part of the database system. If, nevertheless, a top-1 object (tuple) is chosen using (as a concession to performance) any approximate top-k method based on any monotone ranking function, then it is known that it must be one of the skyline objects (Ilyas *et al.* 2008).

As we have already mentioned, our relational algebra with similarities does not generalise the provenance algebra because the provenance semiring does not have a negation operation. However, this is the only reason, and our proposal to move to attribute-based annotations is still applicable and provides a more fine-grained provenance model. We leave as interesting future work the development of a notion of 'similarity provenance' that captures the similarity between a query answer and the source tuples that have influence on it.

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