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Axiomatizability in inclusive equational logics

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A categorical framework for equational logics is presented, together with axiomatizability results in the style of Birkhoff. The distinctive categorical structures used are inclusion systems, which are an alternative to factorization systems in which factorization is required to be unique rather than unique 'up to an isomorphism'. In this framework, models are any objects, and equations are special epimorphisms in $\mathscr{C}$, while satisfaction is injectivity. A first result says that equations-as-epimorphisms define exactly the quasi-varieties, suggesting that epimorphisms actually represent conditional equations. In fact, it is shown that the projectivity/freeness of the domain of epimorphisms is what makes the difference between unconditional and conditional equations, the first defining the varieties, as expected. An abstract version of the axiom of choice seems to be sufficient for free objects to be projective, in which case the definitional power of equations of projective and free domain, respectively, is the same. Connections with other abstract formulations of equational logics are investigated, together with an organization of our logic as an institution.

1. Introduction

This paper presents axiomatizability results for *inclusive equational logic*, a categorical generalization of equational logic that further generalizes local equational logic (Căzănescu 1993). The word 'inclusive' is motivated by our pervasive use of inclusion systems, invented by Diaconescu, Goguen and Stefaneas (Diaconescu *et al.* 1993), and further developed in Hilberdink (1996) and Căzănescu and Roşu (1997); these are categories with extra structure to capture the notion of 'inclusion'. Inclusion systems can be very useful in computing science, for example, in specification theory (Diaconescu *et al.* 1993).

Birkhoff-like axiomatizability results say that a class of models (usually universal algebras) is definable by a set of equations iff that class is closed under certain operations, such as products, quotients, and submodels. Such results have been proved separately for many different formulations of equational logics, beginning with Birkhoff (1935). The aim of this paper is to prove general results that give as many special cases as possible; so far as we know, all special cases are included in the results presented here.

Given a suitable category $\mathscr{C}$, a *conditional* (*semantic*) *equation* is a triple $(\forall A)\ e$ `if` e' where A is an object in $\mathscr{C}$ called the *quantifier*, and e, e' are special morphisms with domain A; here e' corresponds to the hypotheses of the equation and e to its conclusion. Following Căzănescu (1993), quantifiers of equations as objects play a central role in this paper.

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Without restrictions on quantifiers, the definitional power of conditional equations is the same as that of unconditional equations: they both define the quasi-varieties, because the constraints given by the hypothesis of a conditional equation can be translated into the quantifier object. However, this translation does not preserve projectivity or freeness of quantifiers. In fact, we show that the definitional powers of conditional equations and unconditional equations are distinct when the quantifiers of equations are required to be projective or free objects: the first case yields quasi-varieties, and the second yields varieties.

Although free objects are projective in many special cases, this is not true in general (see Section 4.4). Therefore, we try to keep them distinct as long as possible in this paper.

A different categorical generalization of equational logic, called *Category-Based Equational Logic* (abbreviated *CBEL*), was given by Diaconescu in his thesis (Diaconescu 1994; Goguen and Diaconescu 1995). The relationship between our framework and *CBEL* is discussed in Section 6.3.

The purpose of this paper is not to develop a new logic including new relevant examples for computer science, but to create a general framework suitable for Birkhoff-like theorems, including all the equational approaches that we are aware of. We believe that the results in this paper can be easily dualized and thus transformed into axiomatizability results for the coalgebra.

2. Preliminaries

This section presents basic notions of category theory and inclusion systems; the reader already familiar with these concepts can skip it.

2.1. *Categories*

This paper takes category theory as its basic language; it is the expressive power of category theory that gives inclusive equational logic its generality. We will assume the reader is familiar with basic concepts of category theory, such as functor, product, pushout, and so on; see Mac Lane (1971) for this and other background. This section reviews some notation and results that are used later.

We denote categories and functors by calligraphic letters. Let $\mathscr{C}$ be a category and let $f : A \to B$, $g : B \to C$ be two morphisms in $\mathscr{C}$. Then we let $f;g : A \to C$ denote the composition of f and g in diagrammatic order. If the codomain of a morphism f having the domain A is not important, we write $f : A \to \bullet$, and similarly for domains. A morphism h is f-**factorizable** iff there exists g such that $h = f;g$.

A functor $\mathscr{F} : \mathscr{D} \to \mathscr{C}$ is the **left adjoint** of a functor $\mathscr{U} : \mathscr{C} \to \mathscr{D}$ iff for any object D in $\mathscr{D}$ there is a morphism $\eta_D : D \to \mathscr{U}(\mathscr{F}(D))$, called the **unit** of the adjunction, such that for any C in $\mathscr{C}$ and any $f : D \to \mathscr{U}(C)$ there exists a unique morphism, denoted $f^\sharp : \mathscr{F}(D) \to C$, having the property $f = \eta_D;\mathscr{U}(f^\sharp)$. Given an object C in $\mathscr{C}$, we let ϵ_C denote the morphism $1^\sharp_{\mathscr{U}(C)}$; it is called the **counit** of the adjunction. This morphism ϵ_C has the couniversal property, that for any D in $\mathscr{D}$ and any $f : \mathscr{F}(D) \to C$ there exists

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a unique morphism $f^\flat : D \to \mathscr{U}(C)$ such that $\mathscr{F}(f^\flat); \epsilon_C = f$. Many useful properties of adjoint functors are known (Mac Lane 1971; Goguen and Burstall 1984; Goguen and Burstall 1984b), including the following proposition.

Proposition 2.1. Let $\mathscr{U} : \mathscr{C} \to \mathscr{D}$ be a functor with a left adjoint $\mathscr{F}$, and let f, g be morphisms such that the expressions below make sense, then

1 $f^{\flat\sharp} = f$
2 $f^{\sharp\flat} = f$
3 $(f;g)^\flat = f^\flat; \mathscr{U}(g)$
4 $(f;\mathscr{U}(g))^\sharp = f^\sharp; g$
5 $(\mathscr{F}(f);g)^\flat = f;g^\flat$.

2.2. *Inclusion systems*

Inclusion systems formalize the intuitive notion of 'inclusion' in a categorical setting, using factorization as a technical tool. The potential value of inclusion systems was suggested in Goguen and Burstall (1992), and a first definition was given in Diaconescu *et al.* (1993) in the context of modularization. The papers Hilberdink (1996), Căzănescu and Roşu (1997), and Căzănescu and Roşu (2000) further simplify and generalize inclusion systems. We now review definitions and properties used later, omitting the proofs, which may be found with other details in Căzănescu and Roşu (1997) and Căzănescu and Roşu (2000).

Definition 2.1. $\langle \mathscr{I}, \mathscr{E} \rangle$ is a **weak inclusion system** for a category $\mathscr{C}$ if $\mathscr{I}$ and $\mathscr{E}$ are two subcategories of $\mathscr{C}$ having the same objects as $\mathscr{C}$ such that:

1 $\mathscr{I}$ is a *partial order*, that is,
   — $\mathscr{I}(A,B)$ contains at most one morphism for any objects A, B in $\mathscr{C}$, and
   — $\mathscr{I}(A,B) \neq \phi$ and $\mathscr{I}(B,A) \neq \phi$ imply that $A = B$.
2 Every morphism $f \in \mathscr{C}$ can be factored *uniquely* as $e;i$ with $e \in \mathscr{E}$ and $i \in \mathscr{I}$.

The morphisms in $\mathscr{I}$ are called **inclusions**, and we may write $A \hookrightarrow B$ for an inclusion with domain A and codomain B. The morphisms in $\mathscr{E}$ are called **$\mathscr{E}$-morphisms**. If $e : A \to B$ is an $\mathscr{E}$-morphism, then B is called an **$\mathscr{E}$-quotient** of A. We may write $e_f;i_f$ for the factorization of a morphism f with domain A, and $f(A)$ for the codomain of e_f. A weak inclusion system $\mathscr{I}$ is called an **inclusion system** if $\mathscr{E}$ only contains epics. The category *calC* is often called a (**weak**) **inclusive category** if it admits a weak inclusion system.

Inclusion systems are an alternative to factorization systems (Herrlich and Strecker 1973; Németi 1982). We prefer them as a technical tool because proofs tend to be smoother than using factorization systems and because they are more intuitive in many situations in computer science (for example, a module imports another module *as it is*, not 'up to an isomorphism').

Example 2.1. *Algebras*: The category $Alg_{(S,\Sigma)}$ of many sorted algebras over a signature (S,Σ) has a natural inclusion system with $\mathscr{I}$ the inclusions of (S,Σ)-algebras and $\mathscr{E}$ the

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epimorphisms of (S, Σ)-algebras. The category $Alg_{(S,\leqslant,\Sigma)}$ of order-sorted algebras (Goguen 1978) also has a natural inclusion system with $\mathscr{E}$ the surjective morphisms; however, the reader is warned that in this case some epimorphisms are also inclusions.

Example 2.2. *Topological Spaces*: Two different inclusion systems for topological spaces are given in Căzănescu and Roşu (1997); this shows that a category may admit more than one inclusion system.

Let Top be the category with topological spaces as objects and continuous functions as morphisms. A continuous function $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is **$\mathscr{U}$-final** iff for any continuous function $h : (X, \sigma_X) \to (Z, \sigma_Z)$ and any function $g : Y \to Z$ such that $f; g = h$, there is a continuous function $g^\sharp : (Y, \sigma_Y) \to (Z, \sigma_Z)$ that extends g (in the sense that $\mathscr{U}(g^\sharp) = g$, where $\mathscr{U} : Top \to Set$ is the forgetful functor).

The first inclusion system for Top is $\langle \mathscr{I}, \mathscr{E}_{\mathscr{I}} \rangle$, where

$$\mathscr{I} = \{f : (T, \sigma) \to (T'\sigma') \mid T \subseteq T' \text{ and } f \text{ is a continuous inclusion}\}$$

and $\mathscr{E}_{\mathscr{I}}$ consists of the $\mathscr{U}$-final continuous surjections.

The second inclusion system for Top is $\langle \mathscr{J}, \mathscr{E}_{\mathscr{J}} \rangle$, where

$$\mathscr{J} = \{(T, \sigma) \hookrightarrow (T'\sigma') \mid T \subseteq T' \text{ and } \sigma \text{ is the induced topology}^\dagger\}$$

and $\mathscr{E}_{\mathscr{J}}$ consists of the continuous surjections.

The following results are useful technical tools in this paper.

Fact 2.1. If $\langle \mathscr{I}, \mathscr{E} \rangle$ is a weak inclusion system for $\mathscr{C}$, then:

1 $\mathscr{I}$ contains only monics.
2 $\mathscr{I} \cap \mathscr{E}$ contains only identities.
3 If $f; g$ is an $\mathscr{E}$-morphism, then g is an $\mathscr{E}$-morphism.
4 All isomorphisms in $\mathscr{C}$ are $\mathscr{E}$-morphisms.
5 $\mathscr{E}$ contains all the coequalizers.

The following is also often used.

Lemma 2.1. (Diagonal fill-in) Given morphisms $f, g \in \mathscr{C}$, $e \in \mathscr{E}$, $i \in \mathscr{I}$, if $f; i = e; g$, then there is a unique morphism $h \in \mathscr{C}$ such that $e; h = f$ and $h; i = g$.

Lemma 2.2. If $\langle \mathscr{I}_1, \mathscr{E}_1 \rangle$ and $\langle \mathscr{I}_2, \mathscr{E}_2 \rangle$ are two weak inclusion systems for a category $\mathscr{C}$ and if $\mathscr{I}_1 \subseteq \mathscr{I}_2$, then $\mathscr{E}_2 \subseteq \mathscr{E}_1$. In addition, if $\mathscr{I}_1 \subsetneq \mathscr{I}_2$, then $\mathscr{E}_2 \subsetneq \mathscr{E}_1$.

This implies that $\mathscr{I}$ unambiguously determines $\mathscr{E}$. For this reason, we sometimes denote a (weak) inclusion system by just $\mathscr{I}$. A construction of $\mathscr{E}$ from just $\mathscr{I}$ is given in Căzănescu and Roşu (1997).

Example 2.2. *Topological Spaces – continued*: $\mathscr{J} \subsetneq \mathscr{I}$ and $\mathscr{E}_{\mathscr{I}} \subsetneq \mathscr{E}_{\mathscr{J}}$, where $\langle \mathscr{I}, \mathscr{E}_{\mathscr{I}} \rangle$ and $\langle \mathscr{J}, \mathscr{E}_{\mathscr{J}} \rangle$ are the two inclusion systems of topological spaces.

A useful property of weak inclusion systems is that pushouts preserve $\mathscr{E}$-morphisms.

† That is, σ is the restriction of σ' to T.

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Proposition 2.2. Let $\mathscr{C}$ be a weak inclusive category, let $u : C \to A$ be a morphism in $\mathscr{C}$, and let $e : C \to B$ be an $\mathscr{E}$-morphism. If $\langle f : A \to D, g : B \to D \rangle$ is a pushout for these morphisms, then f is an $\mathscr{E}$-morphism.

Definition 2.2. A **subobject** of an object B is an object A for which there is an inclusion $A \hookrightarrow B$. A class of objects is closed under subobjects if for any object in the class all its subobjects are also in the class. A class is closed under $\mathscr{E}$-morphisms if for any $\mathscr{E}$-morphism having its domain in the class, its codomain is also in the class. An object P is **$\mathscr{E}$-projective** (see also, Mitchell (1965) and Herrlich and Strecker (1973)) iff for any $\mathscr{E}$-morphism $e : A \to B$ and any $f : P \to B$, there is some $g : P \to A$ such that $g;e = f$. The notion of **$\mathscr{E}$-injective** object is dual to that of $\mathscr{E}$-projective.

3. Equations and satisfiability

From now on, we think of $\mathscr{C}$ as a category of *models*, and its morphisms as *interpretations*. Then we can think of $\mathscr{E}$-morphisms as *constraints* on their domains. For example, if $\mathscr{C}$ is the category of (S, Σ)-algebras and $e : A \to B$ is an epimorphism, then e gives a set of constraints on A, namely the pairs belonging to its kernel.

Definition 3.1. A **conditional (semantic) equation** is a triple $(\forall A)\ e\ \texttt{if}\ e'$ where A is an object (called its **quantifier**) and e and e' are $\mathscr{E}$-morphisms having domain A. e' is the **hypothesis** of the equation, and e is its **conclusion** (or **body**). If the quantifier is $\mathscr{E}$-projective, we call the equation **$\mathscr{E}$-projective**.

The idea of quantifying equations by objects that are not necessarily free is due to Căzănescu in his local equational logic (Căzănescu 1993). More about this logic can be found in Section 6.1.

Definition 3.2. An object M **satisfies** a conditional equation $(\forall A)\ e\ \texttt{if}\ e'$ iff any interpretation $h : A \to M$ that is e'-factorizable is also e-factorizable. In this case, we write $M \models (\forall A)\ e\ \texttt{if}\ e'$. If Γ is a set of equations, we say that M satisfies Γ or that M is a **Γ-object** iff M satisfies each equation in Γ.

It is not difficult to see that if e' is an isomorphism, an object M satisfies $(\forall A)\ e\ \texttt{if}\ e'$ iff any $h : A \to M$ is e-factorizable. In such a case, we say that the equation is unconditional and denote it by $(\forall A)\ e$. For an unconditional equation, the notion of satisfaction above reduces to the notion of injectivity (see Section 6.2).

These definitions of equation and satisfaction are central to what we call *Inclusive Equational Logic* (abbreviated *IEL*). Therefore, *IEL* is nothing other than a category that admits a weak inclusion system. All the other ingredients (equation, satisfaction) are built over this simple structure and do not require special hypotheses. *IEL* is not going to be a new equational logic with miraculous properties. It is only a very general framework that preserves the abstract properties of the other equational logics. Our intention was to keep it minimal but still rich enough to prove the standard Birkhoff axiomatizability results. Connections of *IEL* with other abstract approaches are developed in Section 6.

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Example 3.1. *Algebras – continued*: Definition 3.2 has the usual (S, Σ)-algebra notion of satisfaction as a special case. Suppose $(\forall X)\ t = t' \ \texttt{if}\ u_1 = u'_1, \ldots, u_n = u'_n$ is an ordinary conditional equation, where X is a set of variables and $t, t', u_1, u'_1, ..., u_n, u'_n$ are Σ-terms over X. Now let $\equiv'$ be the congruence generated by the pairs $(u_1, u'_1), ..., (u_n, u'_n)$ on $T_\Sigma(X)$ and let $e' : T_\Sigma(X) \to T_\Sigma(X)/_{\equiv'}$ be the quotient morphism. Also, let $\equiv$ be the congruence generated by the pair (t, t'), and let e be the quotient morphism. Then what we are looking for is the conditional semantic equation $(\forall T_\Sigma(X))\ e\ \texttt{if}\ e'$, since a Σ-algebra B satisfies $(\forall T_\Sigma(X))\ e\ \texttt{if}\ e'$ iff any $h : T_\Sigma(X) \to B$ that is e'-factorizable (that is, such that $Ker(e') \subseteq Ker(h)$) is also e-factorizable. This means that any h satisfying $h(u_1) = h(u'_1), ..., h(u_n) = h(u'_n)$ also satisfies $h(t) = h(t')$, which is the classical definition of satisfaction.

Example 3.2. *Topological Spaces – continued*: Under the first inclusion system $\langle \mathscr{I}, \mathscr{E}_{\mathscr{I}} \rangle$ for topological spaces, (B, σ_B) satisfies the equation $(\forall (A, \sigma_A))\ e\ \texttt{if}\ e'$ where e, e' are $\mathscr{U}$-final continuous surjections iff any continuous function $h : (A, \sigma_A) \to (B, \sigma_B)$ with $Ker(e') \subseteq Ker(h)$ (as subsets of $A \times A$) also has $Ker(e) \subseteq Ker(h)$.

However, for the second inclusion system $\langle \mathscr{J}, \mathscr{E}_{\mathscr{J}} \rangle$, the definition of satisfaction cannot be simplified in such a way.

4. Axiomatizability

This section is concerned with investigating both the definitional power of semantic equations and the role played by quantifiers. First, we introduce the notions of variety and quasi-variety, which are basic in what follows.

Definition 4.1. A **quasi-variety** in a category $\mathscr{C}$ is a class of objects closed under products and subobjects, and a **variety** in $\mathscr{C}$ is a class closed under products, subobjects and $\mathscr{E}$-morphisms.

Varieties and quasi-varieties appear in many places in universal algebra. Sometimes they are defined in a different manner, as in the following definition.

Definition 4.2. A class of objects is **definable** by (conditional) equations iff there is a class of (conditional) equations such that the class of objects contains exactly all the objects that satisfy the class of (conditional) equations.

From now on, $\mathscr{C}$ is a category and $\langle \mathscr{I}, \mathscr{E} \rangle$ is a (weak) inclusion system for it, and when we say equation, we will mean semantic equation (as given in Definition 3.1).

4.1. *Definability with conditional and projective equations*

This subsection shows, under the weak assumptions in Framework 1 below, that a class of objects definable by conditional equations is a quasi-variety, and that a class of objects definable by projective equations is a variety.

Framework 1:

— $\mathscr{I}$ is a weak inclusion system.
— $\mathscr{C}$ has products.

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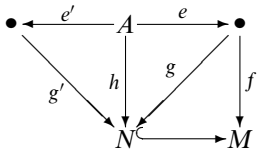
The following lemma presents some relatively basic results.

Proposition 4.1. In Framework 1:

1 Let $N \hookrightarrow M$ be an inclusion and let $(\forall A)\ e\ \texttt{if}\ e'$ be a conditional equation. If M satisfies $(\forall A)\ e\ \texttt{if}\ e'$, then N also satisfies $(\forall A)\ e\ \texttt{if}\ e'$.
2 Let $\{M_j \mid j \in J\}$ be a set of objects satisfying the conditional equation $(\forall A)\ e\ \texttt{if}\ e'$. Then $\prod_{j\in J} M_j$ also satisfies $(\forall A)\ e\ \texttt{if}\ e'$.
3 Let $e' : M \to N$ be a morphism in $\mathscr{E}$ and let $(\forall P)\ e$ be an $\mathscr{E}$-projective (unconditional) equation. If M satisfies $(\forall P)\ e$, then N also satisfies $(\forall P)\ e$.

Proof.

1 Let $h : A \to N$ be a morphism such that there is g' with $e'; g' = h$.



Since $M \models (\forall A)\ e\ \texttt{if}\ e'$, there is a morphism f such that $e; f = h; N \hookrightarrow M$. By the diagonal-fill lemma, there is a morphism g with $e; g = h$ and $g; N \hookrightarrow M = f$. Therefore, we have $N \models (\forall A)\ e\ \texttt{if}\ e'$.
2 Let π_j denote the projection of $\prod_{j\in J} M_j$ to M_j. Let $h : A \to \prod_{j\in J} M_j$ be a morphism such that there exists g' having the property $e'; g' = h$. Since $M_j \models (\forall A)\ e\ \texttt{if}\ e'$ for any $j \in J$, it follows that there exists a family of morphisms $\{\alpha_j \mid j \in J\}$ such that $e; \alpha_j = h; \pi_j$. Therefore, by the universal property of products, there is a morphism g such that $g; \pi_j = \alpha_j$. So, for any $j \in J$, we have $h; \pi_j = e; g; \pi_j$, and hence $h = e; g$.
3 Let $h : P \to N$ be a morphism. By the $\mathscr{E}$-projectivity of P, there is $h' : P \to M$ such that $h'; e' = h$. Since $M \models (\forall P)\ e$, there is a morphism g' with $e; g' = h'$. It follows that $e; (g'; e') = h$, that is, that $N \models (\forall P)\ e$. □

This gives us the following, which is essential in the rest of the paper.

Corollary 4.1. In Framework 1:

1 A class of objects definable by conditional equations is a quasi-variety.
2 A class of objects definable by $\mathscr{E}$-projective (unconditional) equations is a variety.

4.2. *Definability of quasi-varieties*

Our treatment of definability for quasi-varieties and varieties uses the following technical concept (Mitchell 1965; Herrlich and Strecker 1973; Andréka and Németi 1981).

Definition 4.3. If $\mathscr{E}$ is a class of morphisms from $\mathscr{C}$, then the category $\mathscr{C}$ is **$\mathscr{E}$-co-well-powered** iff for any object B and for any class of morphisms from $\mathscr{E}$ having domain B (let us say $\{e_j \mid j \in J\}$) there is a **subset** M of J such that for any j in J there is some m in M such that e_j and e_m are isomorphic, that is, there is an isomorphism α with $e_j = e_m; \alpha$.

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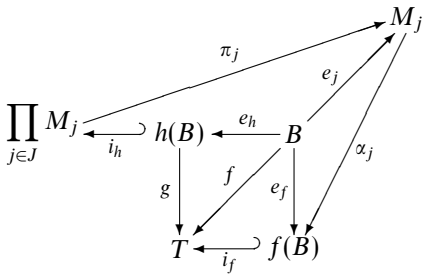
Mitchell (1965) called such a category (but with $\mathscr{E}$ containing just the epics) **co-locally small**. The next proposition was inspired by similar results in Andréka and Németi (1981) and Tarlecki (1986). We use the following framework.

Framework 2:

— $\mathscr{I}$ is an inclusion system.
— $\mathscr{C}$ has products.
— $\mathscr{C}$ is $\mathscr{E}$-co-well-powered.

Proposition 4.2. In Framework 2, any quasi-variety is definable by (unconditional) equations.

Proof. Let Q be a quasi-variety and let Γ be the class of all equations satisfied by every object in Q. We must show that any Γ-object belongs to Q. Let B be a Γ-object, and let $\{e_j : B \to M_j \mid j \in J\}$ be a *set* of morphisms from $\mathscr{E}$ having codomains in Q, such that for any other morphism $e : B \to M$ from $\mathscr{E}$ with M in Q there exist an index $j \in J$ and an *isomorphism* α_j such that $e_j ; \alpha_j = e$ (such a set exists because $\mathscr{C}$ is $\mathscr{E}$-co-well-powered). Let $\prod_{j\in J} M_j$ be a product of the set $\{M_j \mid j \in J\}$, let $\{\pi_j \mid j \in J\}$ be the set of projections, and let $h : B \to \prod_{j\in J} M_j$ be the unique morphism such that $h ; \pi_j = e_j$. Factor h as $e_h ; i_h$.



We now show that $(\forall B)\, e_h$ is in Γ. Let T be an object in Q and let $f : B \to T$. Factor f as $e_f ; i_f$. Since Q is closed under subobjects, it follows that $f(B) \in Q$. Thus there is a morphism $\alpha_j : M_j \to f(B)$ such that $e_j ; \alpha_j = e_f$. We use g to denote the morphism $i_h ; \pi_j ; \alpha_j ; i_f$. Then $e_h ; g = f$, and so $T \models (\forall B)\, e_h$.

Since B is a Γ-object, it follows that $B \models (\forall B)\, e_h$. So 1_B is e_h-factorizable, and thus there exists t such that $1_B = e_h ; t$. Moreover, $e_h ; t ; e_h = e_h$, and because (by hypothesis) e_h is an epimorphism, it follows that $t ; e_h = 1_{h(B)}$, that is, that e_h is an isomorphism.

Since Q is closed under products and subobjects, we obtain that $\prod_{j\in J} M_j$ and $h(B)$ are in Q, and, furthermore, that B is in Q. □

Since the definitional power of unconditional equations is less than that of conditional equations (any unconditional equation can be seen as a conditional equation), Corollary 4.1 yields the following corollary.

Corollary 4.2. A class of objects is a quasi-variety iff it is definable by (unconditional) equations iff it is definable by conditional equations.

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This result is definitely surprising and it seems to be in contradiction with the standard axiomatizability results for universal algebra. This is because the form of our equations is too general. We will see that there is no semantic distinction between unconditional equations and conditional equations as long as we accept any objects as quantifiers.

4.3. *Eliminating conditional equations*

We have seen that the definitional power of conditional equations is the same as that of the unconditional equations. Thus, it is natural to replace conditional equations with unconditional ones in such a way that an object satisfies the conditional equations iff it satisfies the unconditional ones.

4.3.1. *Local equational logic* We show first how to translate a conditional equation into an unconditional equation in the case of local equational logic (the reader might consult Section 6.1 for more details about this logic), where a conditional equation has the form

$$(\forall A)\ u = v \ \texttt{if}\ u_1 = v_1, \ldots, u_n = v_n\ ,$$

and is satisfied by an algebra B iff every interpretation (morphism) $h : A \to B$ that satisfies the identities $h(u_i) = h(v_i)$ for $i \in [n]$, also satisfies the identity $h(u) = h(v)$.

We can consider a conditional equation as a constraint problem, namely to solve $u = v$, subject to the constraints $u_1 = v_1, \ldots, u_n = v_n$, in an algebra A, which may be a term algebra, or more generally a projective algebra; thus the constraints on the problem appear in the algebra A itself.

This suggests putting the constraints into the quantification object, which can be done in the following two steps:

1 Build the congruence $\equiv$ on A, generated by the pairs $(u_1, v_1), \ldots, (u_n, v_n)$.
2 Factor A by the congruence $\equiv$ and build the unconditional equation $(\forall A/_{\equiv})\ \widehat{u} = \widehat{v}$, where $\widehat{u}, \widehat{v}$ are the cosets associated to the elements u and v in algebra $A/_{\equiv}$.

It is now relatively easy to prove the following fact.

Fact 4.1. An algebra B satisfies a conditional equation $(\forall A)\ u = v \ \texttt{if}\ u_1 = v_1, \ldots, u_n = v_n$ iff it satisfies the unconditional equation $(\forall A/_{\equiv})\ \widehat{u} = \widehat{v}$.

4.3.2. *The general case* We can generalize the previous construction and Fact 4.1 to the categorical case. For this purpose, we work in the following framework.

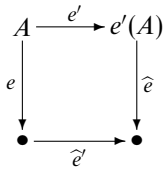
Framework 3:

— $\mathscr{I}$ is a weak inclusion system.
— $\mathscr{C}$ has pushouts.

The construction is as follows:

1 Let $(\forall A)\ e \ \texttt{if}\ e'$ be a conditional equation. We take the pushout of the morphisms $\langle e, e' \rangle$, yielding a pair of morphisms denoted $\langle \widehat{e}, \widehat{e'} \rangle$.

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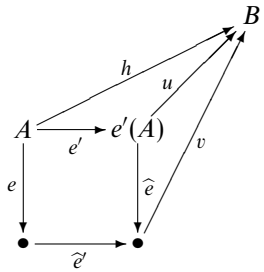
Since morphisms in $\mathscr{E}$ are preserved by pushouts (Proposition 2.2), it follows that $\widehat{e}$ and $\widehat{e}'$ belong to $\mathscr{E}$.

2 We construct the unconditional equation $(\forall e'(A))\,\widehat{e}$.

Then the following proposition holds.

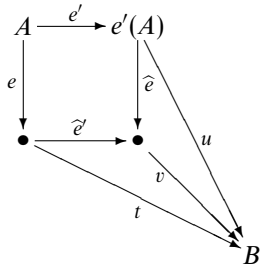
Proposition 4.3. An object B satisfies the conditional equation $(\forall A)\ e$ `if` e' iff it satisfies the unconditional equation $(\forall e'(A))\,\widehat{e}$.

Proof. First, we suppose $B \models (\forall e'(A))\,\widehat{e}$, and will prove that $B \models (\forall A)\ e$ `if` e'. Let $h : A \to B$ be a morphism such that there exists a morphism u having the property $e'; u = h$.



Since $B \models (\forall e'(A))\,\widehat{e}$, there exists a morphism v such that $\widehat{e}; v = u$. Thus $h = e'; \widehat{e}; v$, that is, $h = e; (\widehat{e}'; v)$.

Conversely, we suppose that $B \models (\forall A)\ e$ `if` e' and will prove that $B \models (\forall e'(A))\,\widehat{e}$. Consider a morphism $u : e'(A) \to B$.



Since $B \models (\forall A)\ e$ `if` e', there is a morphism t such that $e; t = e'; u$, so that $\langle u, t\rangle$ is a cocone for the morphisms $\langle e, e'\rangle$. But $\langle \widehat{e}', \widehat{e}\rangle$ is a pushout, so there exists a morphism v having the property $\widehat{e}; v = u$. □

Unfortunately, the above construction destroys the projectivity of quantifiers. We will see that, in fact, the projectivity of quantifiers is the key point that distinguishes varieties from quasi-varieties.

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4.4. $\mathscr{E}$-Extensions

The notion of *$\mathscr{E}$-extension* will allow us to translate quantifiers from one class of objects to another, provided any object from the first class is a quotient of an object from the second. Such a translation is useful for axiomatizability results in which the equations must have their quantifiers in specified classes. We use a single method to give axiomatizability results for projective and free equations, working in the following framework.

Framework 0:

— $\mathscr{I}$ is an inclusion system.

We warn the reader that even if in many particular cases the free objects are projective (in some cases they just conflate, as in groups or abelian groups), this is not true in any context. For example, the discrete Hausdorff spaces are free objects in Top_2, the category of Hausdorff spaces and continuous applications, but they are not projective with regards to epimorphisms (this is because every morphism $e: A \to B$ with $e(A)$ dense in B is an epic in Top_2). For this reason, we prefer to keep the notions of projective and free object distinct, and to get results that hold for both of them. The reader interested in more about free and projective objects is referred to Herrlich and Strecker (1973).

Fact 4.2. In Framework 0, let $(\forall J)\ e\ \texttt{if}\ e'$ be a conditional equation and let $e_J : K \to J$ be a morphism in $\mathscr{E}$. Then an object satisfies $(\forall J)\ e\ \texttt{if}\ e'$ iff it satisfies $(\forall K)\ e_J ; e\ \texttt{if}\ e_J ; e'$.

Proof. Let M be an object in $\mathscr{C}$.

First, suppose $M \models (\forall J)\ e\ \texttt{if}\ e'$, and pick a morphism $h: K \to M$ for which there exists u such that $e_{\mathscr{J}} ; e' ; u = h$. Since $M \models (\forall J)\ e\ \texttt{if}\ e'$, there is a morphism v such that $e ; v = e' ; u$. Then $e_{\mathscr{J}} ; e ; v = e_{\mathscr{J}} ; e' ; u = h$. Therefore, $M \models (\forall K)\ e_{\mathscr{J}} ; e\ \texttt{if}\ e_{\mathscr{J}} ; e'$.

Conversely, suppose $M \models (\forall K)\ e_{\mathscr{J}} ; e\ \texttt{if}\ e_{\mathscr{J}} ; e'$, and pick $h: J \to M$ such that there is some u such that $e' ; u = h$. Since $M \models (\forall K)\ e_{\mathscr{J}} ; e\ \texttt{if}\ e_{\mathscr{J}} ; e'$, the existence of a morphism v with $e_{\mathscr{J}} ; e ; v = e_{\mathscr{J}} ; h$ follows. Since $\mathscr{I}$ is an inclusion system, $\mathscr{E}$ contains only epics, so $e ; v = h$. Therefore, $M \models (\forall J)\ e\ \texttt{if}\ e'$. □

This provides our mechanism for translating quantifiers of conditional equations.

Definition 4.4. Let $\mathscr{K}$ and $\mathscr{J}$ be two classes of objects in $\mathscr{C}$. We say that class $\mathscr{K}$ is an **$\mathscr{E}$-extension** of class $\mathscr{J}$ iff for any object J in $\mathscr{J}$ there is an object K in $\mathscr{K}$ such that $\mathscr{E}(K, J) \neq \emptyset$.

In other words, the class $\mathscr{K}$ is an $\mathscr{E}$-extension of the class $\mathscr{J}$ iff any object in $\mathscr{J}$ is an $\mathscr{E}$-quotient of an object in $\mathscr{K}$; in this case we will write $\mathscr{K} \overset{\mathscr{E}}{\to} \mathscr{J}$. The following lemma is a consequence of Fact 4.2.

Lemma 4.1. Let $\mathscr{K} \overset{\mathscr{E}}{\to} \mathscr{J}$ be an $\mathscr{E}$-extension, and let $\Gamma_{\mathscr{J}}$ be a class of conditional equations having all quantifiers in $\mathscr{J}$. Then there is a class of conditional equations $\Gamma_{\mathscr{K}}$ having their quantifiers in $\mathscr{K}$ such that an object is a $\Gamma_{\mathscr{J}}$-object iff it is a $\Gamma_{\mathscr{K}}$-object.

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Proof. By Fact 4.2, any conditional equation with its quantifier in $\mathscr{J}$ can be replaced by a conditional equation with its quantifier in $\mathscr{K}$ such that the objects satisfying the conditional equations before and after replacement are the same. □

4.4.1. *$\mathscr{E}$-Projectivity* When (S,Σ)-algebras are objects, for any class $\mathscr{J}$ one can choose a class $\mathscr{K}$ as above that contains only $\mathscr{E}$-projective objects. Indeed, if $\mathscr{U} : Alg_{(S,\Sigma)} \to Set^S$ is the forgetful functor and $\mathscr{T}_{(S,\Sigma)} : Set^S \to Alg_{(S,\Sigma)}$ is the free functor, then for any (S,Σ)-algebra A there is a morphism $\epsilon_A : A\mathscr{U}\mathscr{T}_{(S,\Sigma)} \to A$ (the counit morphism for A) that is epic. In addition, it is well known and easy to prove that any free algebra is projective.

Definition 4.5. A category $\mathscr{C}$ **has enough $\mathscr{E}$-projectives** iff any object is the $\mathscr{E}$-quotient of an $\mathscr{E}$-projective object.

If $\mathscr{C}$ has enough $\mathscr{E}$-projectives, there exists an $\mathscr{E}$-extension $\mathscr{P} \xrightarrow{\mathscr{E}} \mathscr{C}$ such that $\mathscr{P}$ contains only $\mathscr{E}$-projective objects. In fact, this holds for any subclass $\mathscr{J}$ of objects of $\mathscr{C}$.

Proposition 4.4. If the category $\mathscr{C}$ has enough $\mathscr{E}$-projectives, a class of objects is definable by conditional equations iff it is definable by projective conditional equations.

Proof. The statement follows from Lemma 4.1 by replacing $\mathscr{J}$ by $\mathscr{C}$ and $\mathscr{K}$ by $\mathscr{P}$. □

Therefore the projective conditional equations have the same definitional power as the other conditional equations, assuming that $\mathscr{C}$ has enough $\mathscr{E}$-projectives.

4.4.2. *Freeness* Based on the same example as above, we can give another extension that is useful for free quantifiers. Let $\mathscr{C} \underset{\mathscr{F}}{\overset{\mathscr{U}}{\rightleftarrows}} \mathscr{D}$ be an adjunction, with $\mathscr{F}$ left adjoint to $\mathscr{U}$. Then an object in $\mathscr{C}$ is **free** iff it is the image under $\mathscr{F}$ of an object in $\mathscr{D}$.

Definition 4.6. An object C in $\mathscr{C}$ is **$\mathscr{U}$-generated (with respect to $\langle \mathscr{I}, \mathscr{E} \rangle$)** if it is the $\mathscr{E}$-quotient of a free object. The category $\mathscr{C}$ is $\mathscr{U}$-generated iff each object in it is $\mathscr{U}$-generated.

One could also say that '$\mathscr{C}$ has enough $\mathscr{F}$-free objects', but we prefer to say 'C is $\mathscr{U}$-generated' because it is more compact. It is relatively easy to see that an object C is $\mathscr{U}$-generated iff its counit ϵ_C is in $\mathscr{E}$, and in this case, there is an $\mathscr{E}$-extension $\mathscr{F}(\mathscr{U}(\mathscr{C})) \xrightarrow{\mathscr{E}} \mathscr{C}$. There is a result similar to that for projective quantifiers.

Proposition 4.5. If a category $\mathscr{C}$ is $\mathscr{U}$-generated, a class of objects in $\mathscr{C}$ is definable by conditional equations iff it is definable by free conditional equations.

Proof. If $\mathscr{C}$ is $\mathscr{U}$-generated, there is an $\mathscr{E}$-extension $\mathscr{F}(\mathscr{U}(\mathscr{C})) \xrightarrow{\mathscr{E}} \mathscr{C}$, by Lemma 4.1, replacing $\mathscr{J}$ by $\mathscr{C}$ and $\mathscr{K}$ by $\mathscr{F}(\mathscr{U}(\mathscr{C}))$. □

It is not obvious when a category is $\mathscr{U}$-generated. In what follows in this section, we investigate conditions under which an object is $\mathscr{U}$-generated. The following definition is equivalent to the previous one (see Proposition 4.6), and it is easier to check in practical cases than the previous definition.

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Definition 4.7. An object C in $\mathscr{C}$ is $\mathscr{U}$**-reachable** (**with respect to** $\langle \mathscr{I}, \mathscr{E} \rangle$) iff it admits no proper subobject $B \hookrightarrow C$, that is, non-identity member of $\mathscr{I}$, such that $\mathscr{U}(B \hookrightarrow C)$ is an isomorphism. A category $\mathscr{C}$ is $\mathscr{U}$-reachable iff any object belonging to it is $\mathscr{U}$-reachable.

In most examples, the functors $\mathscr{U}$ are forgetful functors that respect the definition above.

Example 4.1. *Algebras – continued*: Let C be a (S, Σ)-algebra and let B be a subalgebra of C. If the inclusion of sets $\mathscr{U}(B) \subseteq \mathscr{U}(C)$ is a bijection, then $\mathscr{U}(B) = \mathscr{U}(C)$ and $B = C$. Therefore $Alg_{(S,\Sigma)}$ is $\mathscr{U}$-reachable.

Example 4.2. *Topological Spaces – continued*: Using the first inclusion system $\langle \mathscr{I}, \mathscr{E}_{\mathscr{I}} \rangle$ for Top, let $(T, \sigma) \hookrightarrow (T', \sigma')$ be an inclusion, that is, let $T \subseteq T'$ be an inclusion of sets that is continuous, that is, such that $T \cap \sigma' \subseteq \sigma$. If $T = T'$ and σ is a topology finer than σ', we obtain a proper inclusion $(T, \sigma) \hookrightarrow (T, \sigma')$ such that $\mathscr{U}(T, \sigma) = T = \mathscr{U}(T, \sigma')$. Therefore, Top is not $\mathscr{U}$-reachable with respect to $\langle \mathscr{I}, \mathscr{E}_{\mathscr{I}} \rangle$.

Under the second inclusion system $\langle \mathscr{J}, \mathscr{E}_{\mathscr{J}} \rangle$ for Top, let $(T, \sigma) \hookrightarrow (T', \sigma')$ be an inclusion, that is, $T \subseteq T'$ and σ is the induced topology, that is, $T \cap \sigma' = \sigma$. If $T = T'$, then $\sigma = \sigma'$, so $(T, \sigma) = (T', \sigma')$. Therefore, Top is $\mathscr{U}$-reachable with respect to $\langle \mathscr{J}, \mathscr{E}_{\mathscr{J}} \rangle$.

The following proposition shows that the previous two concepts are equivalent. Similar results, but for factorization systems, are given in Goguen and Burstall (1992) and Tarlecki (1986).

Proposition 4.6. An object is $\mathscr{U}$-reachable iff it is $\mathscr{U}$-generated.

Proof. Let $B \hookrightarrow C$ be an inclusion such that $\mathscr{U}(B \hookrightarrow C)$ is an isomorphism. Since $\epsilon : \mathscr{U}; \mathscr{F} \Rightarrow 1_{\mathscr{C}}$ is a natural transformation, we have $\epsilon_B; B \hookrightarrow C = \mathscr{F}(\mathscr{U}(B \hookrightarrow C)); \epsilon_C$. But $\mathscr{F}(\mathscr{U}(B \hookrightarrow C))$ is an isomorphism, so $\epsilon_B; B \hookrightarrow C$ belongs to $\mathscr{E}$. Now, Parts 2 and 3 of Fact 2.1 imply that $B \hookrightarrow C$ is an identity.

Conversely, suppose that C is a $\mathscr{U}$-reachable object. If one factors ϵ_C as $e_{\epsilon_C}; i_{\epsilon_C}$, then, by Proposition 2.1, one gets that $1_{\mathscr{U}(C)} = \epsilon_C^\flat = e_{\epsilon_C}^\flat; \mathscr{U}(i_{\epsilon_C})$. On the other hand, one can easily prove that $\mathscr{U}(i_{\epsilon_C}); e_{\epsilon_C}^\flat = 1_{\mathscr{U}(B)}$, where B is $\epsilon_C(\mathscr{F}(\mathscr{U}(C)))$. Couniversality gives $\mathscr{F}(\mathscr{U}(i_{\epsilon_C}); e_{\epsilon_C}^\flat); \epsilon_B; i_{\epsilon_C} = \mathscr{F}(\mathscr{U}(i_{\epsilon_C})); e_{\epsilon_C}; i_{\epsilon_C}$. Since ϵ is a natural transformation and i_{ϵ_C} is a monic, we get $\mathscr{F}(\mathscr{U}(i_{\epsilon_C}); e_{\epsilon_C}^\flat); \epsilon_B = \epsilon_B$, and by uniqueness of the morphism $1_{\mathscr{U}(B)}$ satisfying $\mathscr{F}(1_{\mathscr{U}(B)}); \epsilon_B = \epsilon_B$, we get $\mathscr{U}(i_{\epsilon_C}); e_{\epsilon_C}^\flat = 1_{\mathscr{U}(B)}$. □

In this section we have seen that under reasonable assumptions, the conditional equations can be independently replaced either by projective conditional equations or by free conditional equations. Assuming, in addition, that $\mathscr{C}$ has products (this is not required in the framework of this subsection), we can conclude that a class of objects is a quasi-variety iff it is defined by projective conditional equations iff it is defined by free conditional equations.

4.5. *The definability of varieties*

In the traditional Birkhoff theorem (Birkhoff 1935), a class of objects is a variety iff it is definable by ordinary equations, quantified by variables, which means a term algebra in

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our approach. We will prove a more general result, that a class of objects is a variety iff it is definable by projective equations, iff it is definable by free equations. We will work in the following framework.

Framework 4:

— $\mathscr{I}$ is a weak inclusion system.
— $\mathscr{C}$ has products.
— $\mathscr{C}$ is $\mathscr{E}$-co-well-powered.

Lemma 4.2. Let $\mathscr{K} \stackrel{\mathscr{E}}{\to} \mathscr{C}$ be an $\mathscr{E}$-extension. In Framework 4, any variety is definable by (unconditional) equations having quantifiers in $\mathscr{K}$.

Proof. Let V be a variety, and let $\Gamma_{\mathscr{K}}$ be the class of all equations that have their quantifiers in $\mathscr{K}$ and are satisfied by every object in V. We must prove that any $\Gamma_{\mathscr{K}}$-object belongs to V. Let C be a $\Gamma_{\mathscr{K}}$-object and let B be an object in $\mathscr{K}$ such that there is an $\mathscr{E}$-morphism $e_C : B \to C$.

We use the same construction as in Proposition 4.2. Let $\{e_j : B \to M_j \mid j \in J\}$ be a *set* of morphisms in $\mathscr{E}$ (with codomains in V) given by the $\mathscr{E}$-co-well-power property and $h : B \to \prod_{j \in J} M_j$ be the unique morphism such that $h;\pi_j = e_j$ for any $j \in J$. As in Proposition 4.2, we can prove that the equation $(\forall B)\, e_h$ belongs to $\Gamma_{\mathscr{K}}$.

Since C is a $\Gamma_{\mathscr{K}}$-object, it follows that $C \models (\forall B)\, e_h$. Therefore there is a morphism $u : h(B) \to C$ such that $e_h;u = e_C$. By Fact 2.1, we get $u \in \mathscr{E}$. But since V is a variety, $\prod_{j \in J} M_j$ is in V, and $h(B)$ is in V, and C is in V. □

We warn the reader that even though the lemma above is similar in spirit to Lemma 4.2, we do not need an inclusion system; a weak inclusion system is enough.

4.5.1. *$\mathscr{E}$-Projectivity*

Proposition 4.7. Assuming that $\mathscr{C}$ has enough $\mathscr{E}$-projectives, a class of objects is a variety iff it is definable by projective (unconditional) equations.

Proof. There is an $\mathscr{E}$-extension $\mathscr{P} \stackrel{\mathscr{E}}{\to} \mathscr{C}$ having only $\mathscr{E}$-projective objects in $\mathscr{P}$. Lemma 4.2 implies that any variety is definable by projective (unconditional) equations.

Conversely, any class of objects definable by projective equations is a variety by Corollary 4.1. □

4.5.2. *Freeness* Let $\mathscr{C} \underset{\mathscr{F}}{\overset{\mathscr{U}}{\rightleftarrows}} \mathscr{D}$ be an adjunction. To give a result for free quantifiers like that for the $\mathscr{E}$-projective case above, we need to know that any free object is $\mathscr{E}$-projective.

Proposition 4.8. If $\mathscr{C}$ is $\mathscr{U}$-generated (iff it is $\mathscr{U}$-reachable), and if each free object in $\mathscr{C}$ is $\mathscr{E}$-projective, then a class of objects is a variety iff it is definable by free (unconditional) equations.

Proof. From subsection 4.4.2, we know that $\mathscr{F}(\mathscr{U}(\mathscr{C})) \stackrel{\mathscr{E}}{\to} \mathscr{C}$ is an $\mathscr{E}$-extension. Lemma 4.2 implies that any variety is definable by free (unconditional) equations.

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Conversely, any class of objects definable by free equations is a variety, because, in view of Corollary 4.1, any free object is projective. □

The following will give some conditions under which a free object is $\mathscr{E}$-projective. Suppose $\mathscr{D}$ admits a weak inclusion system $\langle \mathscr{I}_{\mathscr{D}}, \mathscr{E}_{\mathscr{D}} \rangle$. Then we call $\mathscr{U}$ an **$\mathscr{E}$-functor** iff it preserves $\mathscr{E}$-morphisms, that is, $\mathscr{U}(e)$ is in $\mathscr{E}_{\mathscr{D}}$ for each $e \in \mathscr{E}$.

Fact 4.3. Suppose $\mathscr{U}$ is an $\mathscr{E}$-functor and D is an $\mathscr{E}_{\mathscr{D}}$-projective object in $\mathscr{D}$. Then $\mathscr{F}(D)$ is an $\mathscr{E}$-projective object.

Proof. Let $e: M \to N$ be a morphism in $\mathscr{E}$ and $h: \mathscr{F}(D) \to N$. Since D is $\mathscr{E}_{\mathscr{D}}$-projective, there is a morphism $f: D \to \mathscr{U}(M)$ such that $f; \mathscr{U}(e) = h^\flat$. Fact 2.1 implies that $h = (f; \mathscr{U}(e))^\sharp = f^\sharp; e$, where $f^\sharp: \mathscr{F}(D) \to M$. Therefore $\mathscr{F}(D)$ is $\mathscr{E}$-projective. □

Other similar results in which a free object is projective appeared in Herrlich and Strecker (1973) and Diaconescu (1994).

Example 4.1. *Algebras – continued*: If we let $\mathscr{C}$ be $Alg_{(S,\Sigma)}$, let $\mathscr{D}$ be Set^S, and let $\mathscr{U}$ and $\mathscr{F}$ be the forgetful and free functor, respectively, then it follows that any free algebra is projective.

Example 4.2. *Topological Spaces – continued*: If we let $\mathscr{C}$ be Top, let $\mathscr{D}$ be Set, and let $\mathscr{U}$ and $\mathscr{F}$ be the forgetful and free functor, respectively, then it follows that any free topological space (that is, any topological space of the form $(X, 2^X)$) is projective in Top.

In practical examples, the fact that any free object is projective is a direct consequence of the axiom of choice. The importance of (an abstract form of) the axiom of choice is also shown in Diaconescu (1994) for equational logics, especially for completeness. Assuming the axiom of choice, it can be shown that any set is projective. For this reason, we give the following definition.

Definition 4.8. A category $\mathscr{D}$ **respects the axiom of choice** iff any object in $\mathscr{D}$ is $\mathscr{E}_{\mathscr{D}}$-projective.

5. Putting them all together

By putting together the above results, a general characterization theorem can be obtained.

In this theorem, $\mathscr{Q}$ denotes the class of quasi-varieties, $\mathscr{V}$ denotes the class of varieties and $\mathscr{E}q$, $\mathscr{CE}q$, $\mathscr{PE}q$, $\mathscr{PCE}q$, $\mathscr{FE}q$, $\mathscr{FCE}q$ denote the classes of objects definable, respectively, by equations, conditional equations, projective equations, projective conditional equations, free equations, and free conditional equations. We suppose the existence of an adjunction wherever free equations appear.

Theorem 5.1. Let $\mathscr{C}$ be a category and $\langle \mathscr{I}, \mathscr{E} \rangle$ be a weak inclusion system for it. Then:

1 If $\mathscr{C}$ has products, then

$\mathscr{CE}q \subseteq \mathscr{Q}$,

$\mathscr{PE}q \subseteq \mathscr{V}$.

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2 If $\mathscr{I}$ is an inclusion system and $\mathscr{C}$ has products and is $\mathscr{E}$-co-well-powered, then
$\mathscr{Q} = \mathscr{C}\mathscr{E}q = \mathscr{E}q$.

3 If $\mathscr{I}$ is an inclusion system, then
$\mathscr{C}\mathscr{E}q = \mathscr{P}\mathscr{C}\mathscr{E}q$ if $\mathscr{C}$ has enough $\mathscr{E}$-projectives,
$\mathscr{C}\mathscr{E}q = \mathscr{F}\mathscr{C}\mathscr{E}q$ if $\mathscr{C}$ is $\mathscr{U}$-reachable.

4 If $\mathscr{C}$ has products and it is $\mathscr{E}$-co-well-powered, then
$\mathscr{V} = \mathscr{P}\mathscr{E}q$ if $\mathscr{C}$ has enough $\mathscr{E}$-projectives,
$\mathscr{V} = \mathscr{F}\mathscr{E}q$ if $\mathscr{C}$ is $\mathscr{U}$-reachable and free objects are $\mathscr{E}$-projective.

Proof.

1 This is exactly Corollary 4.1.
2 This is Corollary 4.2.
3 The first part follows from Proposition 4.4 and the second part from Propositions 4.6 and 4.5.
4 The first part is from Proposition 4.7 and the second from Propositions 4.6 and 4.8. □

Unifying all the hypotheses in the theorem above, we get the following corollary.

Corollary 5.1. If $\mathscr{I}$ is an inclusion system, $\mathscr{C}$ has products, is $\mathscr{E}$-co-well-powered and is $\mathscr{U}$-reachable, and if $\mathscr{U}$ is an $\mathscr{E}$-functor, and $\mathscr{D}$ respects the axiom of choice, then

$$\mathscr{Q} = \mathscr{E}q = \mathscr{C}\mathscr{E}q = \mathscr{P}\mathscr{C}\mathscr{E}q = \mathscr{F}\mathscr{C}\mathscr{E}q$$
$$\mathscr{V} = \mathscr{P}\mathscr{E}q = \mathscr{F}\mathscr{E}q.$$

Proof. We first observe that if $\mathscr{U}$ is an $\mathscr{E}$-functor and $\mathscr{D}$ respects the axiom of choice, then by Fact 4.3 any free object is projective. Because $\mathscr{C}$ is $\mathscr{U}$-reachable (and so it is $\mathscr{U}$-generated by Proposition 4.6), we deduce that any object in $\mathscr{C}$ is an $\mathscr{E}$-quotient of a free object. Therefore $\mathscr{C}$ has enough $\mathscr{E}$-projectives.

The rest follows from the theorem above. □

Example 5.1. *Algebras – continued*: Since Set^S respects the axiom of choice and since $\mathscr{U} : Alg_{(S,\Sigma)} \to Set^S$ is an $\mathscr{E}$-functor (because any surjective morphism is a surjective function), the classical Birkhoff theorem follows: a class of algebras is a variety iff it is definable by equations, and is a quasi-variety iff it is definable by conditional equations. In addition, for local equational logic (Căzănescu 1993) (where equations may be quantified by any algebra), the definitional power of projective quantified equations is the same as that of classical free equations, and the definitional power of conditional equations is the same as that of the unconditional equations (without any other hypotheses).

Using the second inclusion system $\langle \mathscr{J}, \mathscr{E}_{\mathscr{J}} \rangle$ for topological spaces, one can obtain characterization results for quasi-varieties and varieties in Top.

6. Connections with other logics

In this section we will explore the relationships between *IEL* and three other approaches. We chose these approaches because we consider them as being conceptually representative

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for equational logic. Each of them generalizes the standard equational logic in an original way and were the main sources of inspiration for *IEL*.

6.1. *Local equational logic*

The idea of quantifying equations by objects that are not necessary free is due to Căzănescu in his local equational logic (Căzănescu 1993), where a conditional equation has the form

$$(\forall A)\ u = v \ \texttt{if}\ u_1 = v_1, \ldots, u_n = v_n$$

where A is a Σ-algebra and $u, v, u_1, v_1, \ldots, u_n, v_n$ are elements of the carrier of A. In the standard case, that is, non-local, A is a term algebra over a set of variables. Let us denote by $\models_{LEL}$ the satisfaction relation in local equational logic. For any Σ-algebra B we say that $B \models_{LEL} (\forall A)\ u = v \ \texttt{if}\ u_1 = v_1, \ldots, u_n = v_n$ iff for any $h : A \to B$ such that $h(u_1) = h(v_1), \ldots, h(u_n) = h(v_n)$, it is the case that $h(u) = h(v)$.

The following steps let us view such an equation as one of our conditional semantic equations:

— Let $\equiv$ be the congruence generated by the pairs $(u_1, v_1), ..., (u_n, v_n)$ on A, and let $e' : A \to A/_{\equiv}$ be the quotient morphism.

— Let $\sim$ denote the congruence generated by (u, v), and let $e : A \to A/_{\sim}$ be the quotient morphism.

Then $(\forall A)\ e \ \texttt{if}\ e'$ is the conditional semantic equation we are looking for. Without proof, we claim the following fact.

Fact 6.1. For any Σ-algebra B, we have
$B \models_{IEL} (\forall A)\ e \ \texttt{if}\ e'$ iff $B \models_{LEL} (\forall A)\ u = v \ \texttt{if}\ u_1 = v_1, \ldots, u_n = v_n$

6.2. *Injectivity*

Our definition of satisfaction is very similar to the definition of *injectivity*, which, according to Németi and Sain (1981) was first used for representing validity in Banaschewski and Herrlich (1976).

Basically, an object M is injective with respect to. a morphism $f : A \to \bullet$ iff for any morphism $h : A \to M$ there is some g such that $f ; g = h$. In its unconditional case, our *IEL* satisfaction says nothing else but $M \models_{IEL} (\forall A)\ e$ iff M is injective with respect to. e.

This injectivity approach was intensively used by Andréka, Németi and Sain in their work on axiomatizability (for example, see Andréka and Németi (1981) and Németi and Sain (1981)). They first generalized the standard injectivity by introducing the notion of *cone injectivity*, and then they proved some Birkhoff-like theorems making use of very sophisticated technical tools. Remarkably, cone injectivity is strong enough to have first order formulas as a special case.

Because our main goal was to explore equational logics and the role of quantifiers on the power of expressiveness, we decided to generalize the injectivity along a more

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equational direction, as we did in Section 3. Even so, much of our work was inspired by Andréka, Németi and Sain's results.

6.3. *Category based equational logic*

As we have already said, *CBEL* (category based equational logic) (Diaconescu 1994; Diaconescu 1995) is another categorical generalization of equational logics. There are many interesting results related to this logic, especially in connection with constraint logic (Diaconescu 1996a; Diaconescu 1996c).

In this section we establish some relations between *CBEL* and inclusive logics, showing that each of them may be seen semantically as a special case of the other under some hypotheses. Inclusive equational logic can be seen as a bridge between *CBEL* and the logic based on injectivity as validity of formulas, for which axiomatizability results are given in Andréka and Németi (1981) and Németi and Sain (1981).

We first remind the reader of the basic notions of *CBEL*.

Definition 6.1. Given a functor $\mathscr{U} : \mathscr{A} \to \mathscr{X}$ (the objects in category $\mathscr{A}$ are named *models* and the objects in category $\mathscr{X}$ are named *domains*), a *$\mathscr{U}$-equation* is a triple $(\forall A)\ l = r$ where A is an object in $\mathscr{A}$ and $l, r : k \to \mathscr{U}(A)$ are morphisms in $\mathscr{X}$ having the same domain k and codomain $\mathscr{U}(A)$. A model B *satisfies* a $\mathscr{U}$-equation $(\forall A)\ l = r$ iff for each $h : A \to B$, $l; \mathscr{U}(h) = r; \mathscr{U}(h)$. Let $B \models_{CBEL} (\forall A)\ l = r$ denote this satisfaction relation. If the functor $\mathscr{U}$ admits a left adjoint, we say that we work in an *adjointness framework*.

In this subsection we will assume that we are working in an adjointness framework. The following proposition shows that under some hypotheses, *CBEL* may be seen semantically as a special case of *IEL*.

Proposition 6.1. Let $\mathscr{A}$ be a weak inclusive category admitting coequalizers, $\mathscr{U} : \mathscr{A} \to \mathscr{X}$ be a functor admitting a left adjoint $\mathscr{F}$ and $(\forall A)\ l = r$ be a $\mathscr{U}$-equation with $l, r : k \to \mathscr{U}(A)$. Let $l^\sharp, r^\sharp : \mathscr{F}(k) \to A$ be the free extensions of l and r, respectively, and $e_{l,r}$ be the coequalizer of $l^\sharp$ and $r^\sharp$. Then $B \models_{CBEL} (\forall A)\ l = r$ iff $B \models_{IEL} (\forall A)\ e_{l,r}$ for each model B in $\mathscr{A}$.

Proof. Let $\eta : 1_{\mathscr{X}} \Rightarrow \mathscr{F}; \mathscr{U}$ be the unit of the adjunction. Then

$B \models_{CBEL} (\forall A)\ l = r$	iff
$l; \mathscr{U}(h) = r; \mathscr{U}(h)$ for each $h : A \to B$	iff
$\eta_k; \mathscr{U}(l^\sharp); \mathscr{U}(h) = \eta_k; \mathscr{U}(r^\sharp); \mathscr{U}(h)$ for each $h : A \to B$	iff
$\eta_k; \mathscr{U}(l^\sharp; h) = \eta_k; \mathscr{U}(r^\sharp; h)$ for each $h : A \to B$	iff
$l^\sharp; h = r^\sharp; h$ for each $h : A \to B$	iff
there is (a unique) g such as $e_{l,r}; g = h$, for each $h : A \to B$	iff
$B \models_{IEL} (\forall A)\ e_{l,r}$.	

□

The proposition above says that any particular logic that can be organized as *CBEL* can also be organized as *IEL* as long as the forgetful functor admits a left adjoint and the category of models is weak inclusive and has coequalizers. From the definability point of view (see Definition 4.2), any class of models that can be defined by equations in *CBEL*

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can also be defined by equations in the associated *IEL*. As shown in Diaconescu (1994), many sorted algebras, order sorted algebras, Horn clause logic and equational logic modulo axioms are all *CBEL*s; since the associated forgetful functors admit left adjoints and the model categories are inclusive and have coequalizers, we deduce that all of them are also *IEL*s.

Definition 6.2. A **maximal weak inclusion system** is one in which $\mathscr{E}$ contains exactly the coequalizers.

Notice that if $\mathscr{E}$ contains exactly the coequalizers, then, by Fact 2.1 and Lemma 2.2, $\mathscr{I}$ is maximal as a class of morphisms. The following proposition shows how inclusive logic may be organized as *CBEL*. This result is due to Răzvan Diaconescu (personal communication).

Proposition 6.2. Suppose that $\mathscr{A}$ is an inclusive category that admits a maximal weak inclusive system. Consider the identity functor $\mathscr{U} : \mathscr{A} \to \mathscr{A}$ and a semantic equation $(\forall A)\ e$ in $\mathscr{A}$, and let $l, r : \bullet \to A$ be two morphisms such that e is their coequalizer. Then for each model B in $\mathscr{A}$, we have $B \models_{IEL} (\forall A)\ e$ iff $B \models_{CBEL} (\forall A)\ l = r$.

Proof. First, observe that the $\mathscr{U}$-equation $(\forall A)\ l = r$ is well-defined because l and r have the same domain and the codomain $\mathscr{U}(A)$, that is A. Furthermore,

$$
\begin{array}{ll}
B \models_{IEL} (\forall A)\ e & \text{iff} \\
\text{for any } h : A \to B \text{ there is some } g \text{ such that } e;g = h & \text{iff} \\
\text{for any } h : A \to B,\ l;h = r;h & \text{iff} \\
\text{for any } h : A \to B,\ l;\mathscr{U}(h) = r;\mathscr{U}(h) & \text{iff} \\
B \models_{CBEL} (\forall A)\ l = r &
\end{array}
$$

□

There are examples of *CBEL*s that do not deal well with inclusion systems, but we consider that they also do not deal well with Birkhoff-like axiomatizability results, and so we are not interested in them within this paper.

To apply our Birkhoff-like theorem for *CBEL*, we actually need a stronger result, as can be seen in the following proposition. But let us first define the concept of a free maximal inclusion system.

Definition 6.3. Let $\mathscr{A} \underset{\mathscr{F}}{\overset{\mathscr{U}}{\rightleftarrows}} \mathscr{X}$ be an adjunction and $\langle \mathscr{I}, \mathscr{E} \rangle$ be a weak inclusion system for $\mathscr{A}$. Then $\langle \mathscr{I}, \mathscr{E} \rangle$ is called **free maximal** if for each $e : A \to \bullet$ in $\mathscr{E}$ there are some $k \in |\mathscr{X}|$ and $u, v : \mathscr{F}(k) \to A$ such that e is a coequalizer for u and v.

It is straightforward to see that any free maximal inclusion system is a maximal inclusion system.

Proposition 6.3. Let $\mathscr{A} \underset{\mathscr{F}}{\overset{\mathscr{U}}{\rightleftarrows}} \mathscr{X}$ be an adjunction and $\langle \mathscr{I}, \mathscr{E} \rangle$ be a free maximal inclusion system for $\mathscr{A}$. Then, for every $e : A \to \bullet$ in $\mathscr{E}$ there are some $k \in |\mathscr{X}|$ and $l, r : k \to \mathscr{U}(A)$ such that $B \models_{IEL} (\forall A)\ e$ iff $B \models_{CBEL} (\forall A)\ l = r$

Proof. There are some k and $u, v : \mathscr{F}(k) \to A$ such that e is a coequalizer for u and v. We take $l = u^\flat$ and $r = v^\flat$, and then we can reuse the proof of Proposition 6.1. □

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The propositions above were proved for unconditional equations, but they also hold for conditional equations. Notice that factorization, the heart of inclusion and factorization systems, was not used at all within these propositions. Therefore, they could have been reformulated in a weaker framework.

Corollary 6.1. Corollary 5.1 can be fully applied for *CBEL* if we require that $\mathscr{I}$ is a free maximal inclusion system and $\mathscr{C}$ has coequalizers.

Proof. The statement follows from Propositions 6.1 and 6.3. □

7. The institution of inclusive equational logic

The concept of institution was introduced by Goguen and Burstall (Goguen and Burstall 1992) to formalize the abstract notion of 'logical system'. Having as basic idea Tarski's classic semantic definition of truth, institutions have the possibility of translating sentences and models along signature morphisms, with respect to an axiom called the satisfaction condition, which says that *truth is invariant under change of notation.*

An **institution** consists of a category **Sign** whose objects are called signatures, a functor $Sen : \mathbf{Sign} \to \mathbf{Set}$ giving for each signature a set whose elements are called sentences over that signature, a functor $Mod : \mathbf{Sign} \to \mathbf{Cat}^{op}$ giving for each signature Σ a category of Σ-models, and a Σ-indexed relation $\models = \{\models_\Sigma | \ \Sigma \in \mathbf{Sign}\}$, where $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ such that for each morphism $\phi : \Sigma_1 \to \Sigma_2$ the following **Satisfaction Condition**

$$m' \models_{\Sigma_2} Sen(\phi)(e) \ \text{ iff } \ Mod(\phi)(m') \models_{\Sigma_1} e$$

holds for each $m' \in |Mod(\Sigma_2)|$ and each $e \in Sen(\Sigma_1)$.

We build the institution of *IEL* as follows:

— **Sign** is the category having the *small* weak inclusive categories as objects, and functors that preserve $\mathscr{E}$-morphisms and admit right adjoints as morphisms. For each $\phi : \mathscr{A} \to \mathscr{A}'$ we consider a fixed right adjoint $_\restriction_\phi : \mathscr{A}' \to \mathscr{A}$ called the reduct. Also, we consider that the right adjoint functors verify $_\restriction_{1_\mathscr{A}} = 1_\mathscr{A}$ for each $\mathscr{A}$ and $(_\restriction_{\phi_2}); (_\restriction_{\phi_1}) = _\restriction_{\phi_1;\phi_2}$ for each composable pair of morphisms ϕ_1, ϕ_2.

— *Sen* maps each weak inclusive category to the *set*

$$\{(\forall A)\ e \ \texttt{if}\ e' \mid A \in |\mathscr{A}| \text{ and } e, e' \text{ have the domain } A\}$$

For a functor $\phi : \mathscr{A} \to \mathscr{A}'$, define $Sen(\phi)((\forall A)\ e \ \texttt{if}\ e') = (\forall \phi(A))\ \phi(e) \ \texttt{if}\ \phi(e')$. Obviously, *Sen* is a functor.

— *Mod* is the identity on objects, and $Mod(\phi) = _\restriction_\phi$ for each morphism ϕ in **Sign**.

— The satisfaction relation is defined as earlier in the paper, that is, $B \models_\mathscr{A} (\forall A)\ e \ \texttt{if}\ e'$ in $\mathscr{A}$ if any interpretation $h : A \to B$ that is e'-factorizable is also e-factorizable.

To prove that what we have defined is an institution, we need to check the satisfaction condition. But first, we prove the following lemma.

Lemma 7.1. Let $\phi : \mathscr{A} \to \mathscr{A}'$ be a morphism in **Sign**, and $f : A \to B$ and $h : A \to A'\restriction_\phi$ be two morphisms in $\mathscr{A}$ with $A \in \mathscr{A}$ and $A' \in \mathscr{A}'$. Then h is f-factorizable if and only if $h^\sharp$ is $\phi(f)$-factorizable.

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Proof. Let us assume that h is f-factorizable, that is, there is some t such that $h = f;t$. Then

$$\begin{aligned} \eta_A;(\phi(f);t^\sharp)\restriction_\phi &= \\ \eta_A;\phi(f)\restriction_\phi;t^\sharp\restriction_\phi &= \\ f;\eta_B;t^\sharp\restriction_\phi &= \\ f;t &= h \end{aligned}$$

that is, $\phi(f);t^\sharp$ verifies the commutativity diagram, which makes $h^\sharp$ unique, and so $h^\sharp = \phi(f);t^\sharp$.

Conversely, let u be a morphism such that $h^\sharp = \phi(f);u$. Then $h^{\sharp\flat} = (\phi(f);u)^\flat$, that is, $h = f;u^\flat$, by Proposition 2.1. □

Now, we can prove the following proposition.

Proposition 7.1. (Satisfaction condition) Let $\phi : \mathscr{A} \to \mathscr{A}'$ be a morphism in **Sign** and $(\forall A)\ e\ \texttt{if}\ e'$ be a sentence in $Sen(\mathscr{A})$ and $A' \in |\mathscr{A}'|$. Then

$$A'\restriction_\phi \models_{\mathscr{A}} (\forall A)\ e\ \texttt{if}\ e' \ \text{ iff } \ A' \models_{\mathscr{A}'} (\forall \phi(A))\ \phi(e)\ \texttt{if}\ \phi(e')$$

Proof. First, let us suppose that $A'\restriction_\phi \models (\forall A)\ e\ \texttt{if}\ e'$ and let $u : \phi(A) \to A'$ be a $\phi(e')$-factorizable morphism. Applying the lemma above for e' instead of f and $u^\flat$ instead of h, we deduce that $u^\flat$ is e'-factorizable. Because $A'\restriction_\phi \models (\forall A)\ e\ \texttt{if}\ e'$, it follows that $u^\flat$ is e-factorizable, and, furthermore, u is $\phi(e)$-factorizable.

Conversely, let us suppose that $A' \models (\forall \phi(A))\ \phi(e)\ \texttt{if}\ \phi(e')$ and let $h : A \to A'\restriction_\phi$ be an e'-factorizable morphism. Again we apply the lemma above and obtain that $h^\sharp$ is $\phi(e')$-factorizable. This implies that $h^\sharp$ is $\phi(e)$-factorizable and, furthermore, h is e-factorizable. □

The institution of *CBEL* is given in Diaconescu (1996b). According to the definition of a morphism of institutions (Goguen and Burstall 1992), Proposition 6.2 gives a translation from *IEL* to *CBEL*, which is a morphism of institutions, whereas Proposition 6.1 gives a translation that is not a morphism.

8. Conclusions and future work

We have developed a framework suitable for axiomatizability results in the spirit of Birkhoff. We call it *inclusive equational logic* because it is heavily based on inclusion systems as technical tools. We have shown that it is general enough to embed *category based equational logic*, which is already a well-developed abstract equational framework, and so our results can be applied to all logics that are *CBEL*s.

First, we proved that there are no differences between conditional semantical equations and unconditional semantical equations without assuming projectivity/freeness of quantifiers. Moreover, we showed that quantifier projectivity/freeness is what makes the difference between varieties and quasi-varieties from the definitional point of view.

We introduced the notion of *$\mathscr{E}$-extension* to help us translate quantifiers from one class to another. This let us develop results for projective/free quantifiers in a unified manner,

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while keeping them distinct. Finally, we gave conditions under which free objects are projective and then proved that projective and free (unconditional) semantical equations have the same definitional power: they both yield the varieties.

Section 6 showed some connections between inclusive equational logic and other abstract approaches to equational logics, and, finally, Section 7 showed how inclusive equational logic can be organized as an institution.

One further direction of research is to apply our results to *constraint logic*. The fact that this logic is a sub-institution of *CBEL* (Diaconescu 1996a) gives hope that it admits nice axiomatizability characterizations. Another interesting direction would be to dualize the results in this paper and obtain axiomatizability for categorical formulations of coalgebra or hidden algebra, such as those in Roşu (2001).

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