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A note on the entropy of factors of random dynamical systems

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Abstract. In this paper we show that two semi-conjugate random dynamical systems (RDSs) on Polish spaces have the same entropy and pressure if the cardinal number of the pre-image of a point under the semi-conjugacy is finite almost everywhere.

1. *The results for deterministic dynamical systems*

In this paper a measure-theoretic dynamical system (DS) $(X, \mathcal{A}, \mu, f)$ is a measure-preserving map f on a probability space $(X, \mathcal{A}, \mu)$. A measure-theoretic DS $(Y, \mathcal{B}, \nu, g)$ is said to be a *factor* of $(X, \mathcal{A}, \mu, f)$ if there is a measure-preserving map $\pi : (X, \mathcal{A}, \mu) \to (Y, \mathcal{B}, \nu)$ such that the following diagram is commutative μ-almost everywhere:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{g} & Y \end{array}$$

Let $h_\mu(f)$ denote the (measure-theoretic) entropy of a measure-theoretic DS $(X, \mathcal{A}, \mu, f)$ (see Walters [**23**] for the notion). Our main result in this paper for deterministic DSs is the following.

THEOREM 1.1. *Assume that* $(Y, \mathcal{B}, \nu, g)$ *is a factor of* $(X, \mathcal{A}, \mu, f)$ *with factor map* π. *For each* $y \in Y$ *let* $k(y)$ *be the cardinal number of* $\pi^{-1}\{y\}$. *If* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *are Lebesgue spaces and* $k(y)$ *is measurable and* ν*-almost everywhere finite, then*

$$h_\nu(g) = h_\mu(f).$$

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Remark 1.2. If X is a Polish space (i.e. a separable complete metric space), then $(X, \mathcal{B}_\mu(X), \mu)$ constitutes a Lebesgue space, where μ is a Borel probability on X and $\mathcal{B}_\mu(X)$ is the completion of the Borel σ-algebra $\mathcal{B}(X)$ with respect to μ (see, e.g., Walters [**23**, Theorem 2.1]). Thus, Theorem 1.1 applies in the case where X, Y are Polish spaces and f, g, h are continuous. In this case, if h is a local homeomorphism and $k(y)$ is finite for each $y \in Y$, then $k(y)$ is lower semi-continuous and hence is Borel measurable.

Remark 1.3. The problem if such a result as Theorem 1.1 holds was proposed to the author by De-Jun Feng and Jun Wu. Such a result seems not to be new, but we do not know if it is the case†, so we present here Theorem 1.1 which is possibly the strongest result. There are a number of cases where Theorem 1.1 can find its applications. Among them we mention the symbolic representations of a Smale space via Markov partitions (see Bowen [**5**, Proposition 10] and Ruelle [**19**, Theorem 7.6]), a finite group extension of a deterministic map (see Ruelle and Wilkinson [**21**]) and Liao's problem concerning the entropy of a hyperbolic state flow and its orthonormal n-frame flow (see Sun [**22**]).

Theorem 1.1 essentially follows from the Abramov–Rokhlin formula (Bogenschütz and Crauel [**4**]). We state it below for convenience of the reader. Before doing this, we first recall the notion of the conditional entropy of a measure-theoretic DS $(X, \mathcal{A}, \mu, f)$ (Kifer [**10**, Ch. 2]). Suppose that $\mathcal{A}_0 \subset \mathcal{A}$ is a σ-subalgebra satisfying $f^{-1}\mathcal{A}_0 \subset \mathcal{A}_0$. The conditional entropy of $(X, \mathcal{A}, \mu, f)$ with respect to $\mathcal{A}_0$ is defined as

$$h_\mu^{\mathcal{A}_0}(f) := \sup_\xi \lim_{n\to+\infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} f^{-k}\xi \,\Big|\, \mathcal{A}_0\Big),$$

where the supremum is taken over the set of all finite measurable partitions of X and $H_\mu(\eta|\mathcal{A}_0) := -\int \sum_{C\in\eta} E(\chi_C|\mathcal{A}_0) \log E(\chi_C|\mathcal{A}_0)\, d\mu$ for a finite partition η of $(X, \mathcal{A}, \mu)$, with $E(\chi_C|\mathcal{A}_0)$ denoting the conditional expectation of the characterization function χ_C of C with respect to $\mathcal{A}_0$. When $(X, \mathcal{A}, \mu)$ is a Lebesgue space, there exists a unique (mod 0) measurable partition ξ_0 of $(X, \mathcal{A}, \mu)$ which generates $\mathcal{A}_0$ (see Rokhlin [**17**]). In this case, if $\mathcal{A}_0$ satisfies $f^{-1}\mathcal{A}_0 = \mathcal{A}_0$, and then

$$h_\mu^{\mathcal{A}_0}(f) = \sup_\xi H_\mu\Big(\xi \,\Big|\, \bigvee_{k=1}^{+\infty} f^{-k}\xi \vee \xi_0\Big),$$

where the supremum is taken over the set of all finite measurable partitions of $(X, \mathcal{A}, \mu)$ (see Liu and Qian [**16**, Ch. 0]).

LEMMA 1.4. (Abramov–Rokhlin formula, [**4**]) *Suppose that $(X, \mathcal{A}, \mu, f)$ is a measure-theoretic DS with a factor $(Y, \mathcal{B}, \nu, g)$, where $\mathcal{B}$ is countably generated* (ν-mod 0). *Then*

$$h_\mu(f) = h_\nu(g) + h_\mu^{\pi^{-1}\mathcal{B}}(f)$$

(both sides may be $+\infty$), where π is the factor map.

† After the paper was completed, the author noticed that, when $f : X \to X$ and $g : Y \to Y$ are continuous maps on compact metric spaces, Theorem 1.1 follows from a general result of Ledrappier and Walters [**12**, Theorem 2.1]. The approaches are different.

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Proof of Theorem 1.1. We first assume that f, g are measurably invertible and $(Y, \mathcal{B}, \nu, g)$ is ergodic. Since f is invertible, one has $k(y) = k(gy)$ for ν-a.e. y. This, together with the ergodicity of ν, implies that $k(y)$ is equal to a constant k_0 for ν-a.e. y. Let ϵ be the measurable partition of $(Y, \mathcal{B}, \nu)$ into single points. $\pi^{-1}\epsilon$ gives then a measurable partition of $(X, \mathcal{A}, \mu)$ which generates $\pi^{-1}\mathcal{B}$. Let $n > 0$ be an arbitrarily fixed integer. Note that for each finite measurable partition ξ of $(X, \mathcal{A}, \mu)$

$$\begin{aligned} H_\mu\left(\xi \,\middle|\, \bigvee_{k=1}^{+\infty} (f^n)^{-k}\xi \vee \pi^{-1}\epsilon\right) &\le H_\mu(\xi|_{\pi^{-1}\epsilon}) \\ &= \int_{X/\pi^{-1}\epsilon} H_{\mu_C}(\xi|_C)\, d\mu_{\pi^{-1}\epsilon}(C) \\ &= \int_Y H_{\mu_{\pi^{-1}\{y\}}}(\xi|_{\pi^{-1}\{y\}})\, d\nu(y) \\ &\le \log k_0, \end{aligned}$$

where $\{\mu_C\}_{C\in\pi^{-1}\epsilon}$ is a canonical system of conditional measures of μ associated with $\pi^{-1}\epsilon$, $\xi|_C$ is the restriction of ξ to C and $\mu_{\pi^{-1}\epsilon}$ is the measure induced by μ on the factor space $X/\pi^{-1}\epsilon$, the last inequality being due to the fact that $\xi|_{\pi^{-1}\{y\}}$ consists of k_0 elements for ν-a.e. y. Since $g^{-n}\mathcal{B} = \mathcal{B}$ and $\pi \circ f^n = g^n \circ \pi$, one has $(f^n)^{-1}(\pi^{-1}\mathcal{B}) = \pi^{-1}\mathcal{B}$, and hence

$$h_\mu^{\pi^{-1}\mathcal{B}}(f^n) \le \log k_0.$$

This, together with the fact that $h_\mu^{\pi^{-1}\mathcal{B}}(f^n) = nh_\mu^{\pi^{-1}\mathcal{B}}(f)$ and the arbitrariness of n, implies that $h_\mu^{\pi^{-1}\mathcal{B}}(f) = 0$ and hence, by the Abramov–Rokhlin formula,

$$h_\mu(f) = h_\nu(g). \tag{1.1}$$

We now consider possibly non-invertible f and g. Put

$$\bar{X} = \{\bar{x} = (\dots, x_{-1}, x_0, x_1, \dots) : x_i \in X,\, fx_i = x_{i+1},\, i \in \mathbb{Z}\}$$

and let $(\bar{X}, \bar{\mathcal{A}}, \bar{\mu}, \bar{f})$ be the natural extension of $(X, \mathcal{A}, \mu, f)$ which constitutes a Lebesgue system (Rokhlin [**17**]). Define $(\bar{Y}, \bar{\mathcal{B}}, \bar{\nu}, \bar{g})$ similarly and they are measurably invertible [**17**, p. 3]. Since we are assuming that $(Y, \mathcal{B}, \nu, g)$ is ergodic, $(\bar{Y}, \bar{\mathcal{B}}, \bar{\nu}, \bar{g})$ is also ergodic. Define $\bar{\pi} : \bar{X} \to \bar{Y}$ by $\bar{\pi}\bar{x} = (\dots, \pi x_{-1}, \pi x_0, \pi x_1, \dots)$ for $\bar{x} = (\dots, x_{-1}, x_0, x_1, \dots) \in \bar{X}$, which clearly gives a measure-preserving map $\bar{\pi} : (\bar{X}, \bar{\mathcal{A}}, \bar{\mu}) \to (\bar{Y}, \bar{\mathcal{B}}, \bar{\nu})$ satisfying

$$\bar{\pi} \circ \bar{f} = \bar{g} \circ \bar{\pi}.$$

Let $\bar{k}(\bar{y})$ be the cardinal number of $\bar{\pi}^{-1}\{\bar{y}\}$ for $\bar{y} \in \bar{Y}$. Since $k(y)$ is measurable and finite for ν-a.e. y, there exists an integer $K > 0$ such that $D = \{y : k(y) \le K\}$ has positive ν measure. Define $\psi : \bar{Y} \to \mathbb{R}$, $\bar{y} \mapsto \chi_D(y_0)$, where χ_D is the characterization function of D. By the Birkhoff ergodic theorem, for $\bar{\nu}$-a.e. $\bar{y}$ one has

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(\bar{g}^{-k}\bar{y}) = \nu(D),$$

and hence for $\bar{\nu}$-a.e. $\bar{y}$ one has $y_{-n} \in D$ for infinitely many $n \in \mathbb{N}$ and this implies that $\bar{k}(\bar{y}) \le K$ for $\bar{\nu}$-a.e. $\bar{y}$. The reasoning above in the case of f and g being measurably

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invertible shows that $h_{\bar{\mu}}(\bar{f}) = h_{\bar{\nu}}(\bar{g})$ and hence (1.1) still holds since $h_{\bar{\mu}}(\bar{f}) = h_\mu(f)$ and $h_{\bar{\nu}}(\bar{g}) = h_\nu(g)$.

Finally, we suppress the ergodicity assumption on $(Y, \mathcal{B}, \nu, g)$. According to Rokhlin [**17**], there exists a measurable partition ζ of $(Y, \mathcal{B}, \nu)$ such that $g^{-1}C = C$ and $g_C := g|_C : (C, \mathcal{B}|_C, \nu_C) \hookleftarrow$ is ergodic for ν_ζ-a.e. $C \in Y/\zeta$, where ν_C is the conditional measure of ν on C. $\hat{\zeta} := \pi^{-1}\zeta$ gives a measurable partition of $(X, \mathcal{A}, \mu)$ which has the property that $f^{-1}\hat{C} = \hat{C}$ and $f_{\hat{C}} := f|_{\hat{C}} : (\hat{C}, \mathcal{A}|_{\hat{C}}, \mu_{\hat{C}}) \hookleftarrow$ and $\pi|_{\hat{C}} : (\hat{C}, \mathcal{A}|_{\hat{C}}, \mu_{\hat{C}}) \to (C, \mathcal{B}|_C, \nu_C)$ are both measure-preserving for $\mu_{\hat{\zeta}}$-a.e. $\hat{C} := \pi^{-1}C \in X/\hat{\zeta}$. For ν_ζ-a.e. $C \in Y/\zeta$, or equivalently for $\mu_{\hat{\zeta}}$-a.e. $\hat{C} = \pi^{-1}C \in X/\hat{\zeta}$, one has

$$h_{\mu_{\hat{C}}}(f_{\hat{C}}) = h_{\nu_C}(g_C),$$

and hence (see Rokhlin [**17**, Theorem 9.8])

$$h_\mu(f) = \int_{X/\hat{\zeta}} h_{\mu_{\hat{C}}}(f_{\hat{C}})\, d\mu_{\hat{\zeta}}(\hat{C}) = \int_{Y/\zeta} h_{\nu_C}(g_C)\, d\nu_\zeta(C) = h_\nu(g).$$

The proof of Theorem 1.1 is completed. □

COROLLARY 1.5. *Let $f : X \to X$, $g : Y \to Y$ be continuous maps on compact metric spaces X and Y, and assume that there exists a surjective continuous map $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. If the cardinal number $k(y)$ of $\pi^{-1}\{y\}$ is Borel measurable and is finite for every $y \in \Omega(g)$, the non-wandering set of g, then for every continuous $\phi : Y \to \mathbb{R}$ one has*

$$P_g(\phi) = P_f(\phi \circ \pi),$$

where $P_g(\phi)$ is the topological pressure of g with respect to ϕ and $P_f(\phi \circ \pi)$ is defined similarly; in particular, $h_{\text{top}}(f) = h_{\text{top}}(g)$, where $h_{\text{top}}(\cdot)$ denotes the topological entropy.

Proof. It is a standard fact that for every g-invariant measure ν there is an f-invariant measure μ such that $\pi\mu = \nu$. The corollary follows then from Theorem 1.1 and the variational principle. □

2. *Extensions to random dynamical systems*

Let $(\Omega, \mathcal{A}, P)$ be a probability space with a measure-preserving transformation τ on it and let X be a Polish space with its Borel σ-algebra $\mathcal{B}(X)$. Assume that $\{E_\omega\}_{\omega\in\Omega} \subset \mathcal{B}(X)$ is a family of subsets of X such that $E = \bigcup_{\omega\in\Omega}\{\omega\} \times E_\omega$ constitutes a measurable subset of $(\Omega \times X, \mathcal{A} \times \mathcal{B}(X))$, and let $\{f(\omega) : E_\omega \to E_{\tau\omega}\}_{\omega\in\Omega}$ be a family of maps which makes the skew product transformation

$$\Phi : E \to E, \quad (\omega, x) \mapsto (\tau\omega, f(\omega)x)$$

measurable (with respect to the restriction of $\mathcal{A} \times \mathcal{B}(X)$ to E). The iterates $f(\omega, n) : E_\omega \to E_{\tau^n\omega}$, $n \in \mathbb{Z}^+$, $\omega \in \Omega$ defined by

$$f(\omega, n) = \begin{cases} \text{id} & \text{for } n = 0, \\ f(\tau^{n-1}\omega) \circ \cdots \circ f(\omega) & \text{for } n > 0, \end{cases}$$

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give a *bundle RDS* over $(\Omega, \mathcal{A}, P, \tau)$ which will be denoted by $\mathcal{F}_E$. For a general theory of RDSs we refer to Kifer [**10**], Arnold [**1**] and the survey paper Liu [**15**].

Let $\mathcal{F}_E$ be as defined above. An *invariant measure* μ of $\mathcal{F}_E$ is defined to be a probability on $(\Omega \times X, \mathcal{A} \times \mathcal{B}(X))$ which satisfies $\mu(E) = 1$, $\Phi\mu = \mu$ and which has marginal P on Ω, and the set of all $\mathcal{F}_E$-invariant measures will be denoted by $\mathcal{M}(\mathcal{F}_E)$. It constitutes a case of particular interest when P almost everywhere E_ω is compact, depends measurably on ω (with respect to the Hausdorff topology on the space of compact subsets of X) and $f(\omega) : E_\omega \to E_{\tau\omega}$ is continuous. In this case $\mathcal{M}(\mathcal{F}_E) \neq \emptyset$ (Crauel [**6**], see also Arnold [**1**, Theorem 1.6.13]), though $E_{\tau^n\omega}$ may leave with positive probability any bounded domain as n increases, and such bundle RDSs have been shown to occur as random attractors of some stochastic partial differential equations, whose study now constitutes a significant research topic of the theory of RDSs (see Ruelle [**20**], Crauel and Flandoli [**7**], and see also Liu [**15**, §3.1] for a brief review).

Let $\mu \in \mathcal{M}(\mathcal{F}_E)$ and let $d\mu(\omega, x) = d\mu_\omega(x)\, dP(\omega)$ be the decomposition of μ into its *sample measures* μ_ω, $\omega \in \Omega$ (see Arnold [**1**]). For a finite measurable partition ξ of X the limit

$$h_\mu(\mathcal{F}_E, \xi) := \lim_{n\to+\infty} \frac{1}{n} \int H_{\mu_\omega}\left(\bigvee_{k=0}^{n-1} f(\omega, k)^{-1}(\xi|_{E_{\tau^k\omega}}) \right) dP(\omega)$$

always exists and $h_\mu(\mathcal{F}_E) := \sup h_\mu(\mathcal{F}_E, \xi)$ is called the *entropy* of $(\mathcal{F}_E, \mu)$, where the supremum is taken over the set of all finite measurable partitions of X (see Bogenschütz [**2**]).

$(\Omega, \mathcal{A}, P, \tau)$ is said to be a *Polish system* if Ω is a Polish space and $\mathcal{A}$ is its Borel σ-algebra. In what follows we will make this assumption since we need to work with Lebesgue spaces. This seems to be an unpleasant assumption but it covers the canonical cases where $\Omega = C^r(M, M)^{\mathbb{Z}^+}$ or $C^r(M, M)^{\mathbb{Z}}$ (M a C^r-manifold, $r \geq 0$ integer) endowed with the product weak topology (see Hirsch [**9**]), or Ω is the classical Winner space. An extension of Theorem 1.1 is the following.

THEOREM 2.1. *Assume that $(\Omega, \mathcal{A}, P, \tau)$ is a Polish system. Let $(\mathcal{F}_E, \mu)$, $(\mathcal{G}_F, \nu)$ with $\mu \in \mathcal{M}(\mathcal{F}_E)$, $\nu \in \mathcal{M}(\mathcal{G}_F)$ be two RDSs over $(\Omega, \mathcal{A}, P, \tau)$ on Polish spaces X and Y respectively. Assume that there exists a family of measure-preserving maps $\{\pi_\omega : (E_\omega, \mu_\omega) \to (F_\omega, \nu_\omega)\}_{\omega\in\Omega}$ such that for P-a.e. ω the following diagram is commutative μ_ω-almost everywhere:*

$$\begin{array}{ccc} E_\omega & \xrightarrow{f(\omega)} & E_{\tau\omega} \\ {\scriptstyle \pi_\omega}\downarrow & & \downarrow{\scriptstyle \pi_{\tau\omega}} \\ F_\omega & \xrightarrow{g(\omega)} & F_{\tau\omega} \end{array}$$

and $\Pi : E \to F$, $(\omega, x) \mapsto (\omega, \pi_\omega x)$ constitutes a measurable map. For each $y \in F_\omega$ let $k_\omega(y)$ be the cardinal number of $\pi_\omega^{-1}\{y\}$. If $k_\omega(y)$ is measurable in (ω, y) and is finite ν-almost everywhere, then

$$h_\mu(\mathcal{F}_E) = h_\nu(\mathcal{G}_F). \tag{2.1}$$

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An example of situations where Theorem 2.1 can be applied is symbolic representations of bundle RDSs arising from random perturbations of Axiom A basic sets of diffeomorphisms (see Liu [**13**]) or of invariant sets of expanding maps. The basic idea of the proof of Theorem 2.1 is similar to that of the proof of Theorem 1.1. We present below the necessary modifications. First, by slightly modifying the proof in Bogenschütz and Crauel [**4**], Lemma 1.4 can be generalized to the following.

LEMMA 2.2. (Generalized Abramov–Rokhlin formula) *Suppose that* $(X, \mathcal{C}, \mu, T)$ *is a measure-theoretic DS with a factor* $(Y, \mathcal{D}, \nu, S)$, *where* $\mathcal{D}$ *is countably generated* (ν-mod 0). *Then for any* σ*-subalgebra* $\mathcal{D}_0 \subset \mathcal{D}$ *with* $S^{-1}\mathcal{D}_0 \subset \mathcal{D}_0$ *one has*

$$h_\mu^{\pi^{-1}\mathcal{D}_0}(T) = h_\nu^{\mathcal{D}_0}(S) + h_\mu^{\pi^{-1}\mathcal{D}}(T)$$

(both sides may be $+\infty$*), where* π *is the factor map.*

Proof of Theorem 2.1 Since Ω, X and Y are Polish spaces, $(E, \mathcal{C}, \mu)$ and $(F, \mathcal{D}, \nu)$ are Lebesgue spaces, where $\mathcal{C}$ and $\mathcal{D}$ are, respectively, the completion of $(\mathcal{A} \times \mathcal{B}(X))|_E$ and $(\mathcal{A} \times \mathcal{B}(Y))|_F$ with respect to μ and ν. We first assume that $f(\omega) : E_\omega \to E_{\tau\omega}$ and $g(\omega) : F_\omega \to F_{\tau\omega}$ are measurably invertible for P-a.e. ω, $\tau : (\Omega, \mathcal{A}, P) \hookleftarrow$ is measurably invertible and

$$\Psi : (F, \nu) \to (F, \nu), \quad (\omega, y) \mapsto (\tau\omega, g(\omega)y)$$

is ergodic. Then $k_\omega(y) = k_{\tau\omega}(g(\omega)y)$ for ν-a.e. (ω, y). An argument similar to the proof of Theorem 1.1 shows that $h_\mu^{\Pi^{-1}\mathcal{D}}(\Phi) = 0$. Then (2.1) follows from Lemma 2.2 since, by Bogenschütz [**2**],

$$h_\mu(\mathcal{F}_E) = h_\mu^{\Pi^{-1}\mathcal{D}_0}(\Phi) \quad \text{and} \quad h_\nu(\mathcal{G}_F) = h_\nu^{\mathcal{D}_0}(\Psi), \tag{2.2}$$

where $\mathcal{D}_0$ is the restriction of $\{A \times Y : A \in \mathcal{A}\}$ to F.

We now suppress the invertibility assumption on $f(\omega)$ and $g(\omega)$. Using Liu and Qian [**16**, §0.5] and following the proof of Rokhlin [**17**, Theorem 9.9], one can prove the following fact. Let $(X, \mathcal{C}, \mu, T)$ be a Lebesgue measure-theoretic DS and let $(\bar{X}, \bar{\mathcal{C}}, \bar{\mu}, \bar{T})$ be its natural extension; if $\mathcal{C}_0$ is a σ-subalgebra of $\mathcal{C}$ satisfying $T^{-1}\mathcal{C}_0 \subset \mathcal{C}_0$, then

$$h_\mu^{\mathcal{C}_0}(T) = h_{\bar{\mu}}^{\pi_X^{-1}\mathcal{C}_0}(\bar{T}),$$

where $\pi_X : \bar{X} \to X$, $\bar{x} = (x_i)_{i=-\infty}^{+\infty} \mapsto x_0$ is the natural projection. Since $(\Omega, \mathcal{A}, P, \tau)$ is assumed now to be measurably invertible, one has $\Psi^{-1}\mathcal{D}_0 = \mathcal{D}_0$ and $\Phi^{-1}(\Pi^{-1}\mathcal{D}_0) = \Pi^{-1}\mathcal{D}_0$. Then

$$h_\nu^{\mathcal{D}_0}(\Psi) = h_{\bar{\nu}}^{\pi_F^{-1}\mathcal{D}_0}(\bar{\Psi}), \quad h_\mu^{\Pi^{-1}\mathcal{D}_0}(\Phi) = h_{\bar{\mu}}^{\pi_E^{-1}\Pi^{-1}\mathcal{D}_0}(\bar{\Phi}), \tag{2.3}$$

where $\pi_E : \bar{E} \to E, \pi_F : \bar{F} \to F$ are the natural projections. By arguments similar to the proof of Theorem 1.1, one has

$$h_{\bar{\nu}}^{\pi_F^{-1}\mathcal{D}_0}(\bar{\Psi}) = h_{\bar{\mu}}^{\bar{\Pi}^{-1}\pi_F^{-1}\mathcal{D}_0}(\bar{\Phi}) = h_{\bar{\mu}}^{\pi_E^{-1}\Pi^{-1}\mathcal{D}_0}(\bar{\Phi}),$$

where $\bar{\Pi} : \bar{E} \to \bar{F}$ is the natural projection induced by $\Pi : E \to F$. Hence, by (2.2) and (2.3), $h_\nu(\mathcal{G}_F) = h_\mu(\mathcal{F}_E)$.

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We now suppress the ergodicity assumption on (F, ν, Ψ) but require $(\Omega, \mathcal{A}, P, \tau)$ to be ergodic and measurably invertible. According to the ergodic decomposition theorem (Rokhlin [**17**], Kifer [**10**, Appendix A]), there exists a measurable partition ζ of $(F, \mathcal{D}, \nu)$ such that $\Psi^{-1}C = C$ and $\Psi_C : (C, \mathcal{D}|_C, \nu_C) \hookleftarrow$ is ergodic for ν_ζ-a.e. $C \in F/\zeta$. Since P is ergodic and ν has marginal P on Ω, one can show that ν_C also has marginal P on Ω for ν_ζ-a.e. C. The result follows then from Liu and Qian [**16**, Theorem 0.5.5] and an argument similar to the last paragraph of the proof of Theorem 1.1.

If $(\Omega, \mathcal{A}, P, \tau)$ is not ergodic, the result (2.1) follows easily by using its ergodic decomposition.

Finally, we prove the result (2.1) without requiring $(\Omega, \mathcal{A}, P, \tau)$ to be measurably invertible. Let $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P}, \bar{\tau})$ be the natural extension of $(\Omega, \mathcal{A}, P, \tau)$. Put $E^* = \bigcup_{\bar{\omega}\in\bar{\Omega}}\{\bar{\omega}\} \times E_{\omega_0} \subset \bar{\Omega} \times X$ and define

$$\Phi^* : E^* \to E^*, \quad (\bar{\omega}, x) \mapsto (\bar{\tau}\bar{\omega}, f(\omega_0)x).$$

It is easy to show that there is a unique probability μ^* on $(E^*, (\bar{\mathcal{A}} \times \mathcal{B}(X))|_{E^*})$ such that $\Phi^*\mu^* = \mu^*$ and μ^* projects to μ under the natural projection $E^* \to E$, $(\bar{\omega}, x) \mapsto (\omega_0, x)$. Moreover, by arguments similar to the proof of Liu [**14**, Proposition 2.2] one has $h_\mu^{\Pi^{-1}\mathcal{D}_0}(\Phi) = h_{\mu^*}^{\mathcal{D}_{E^*}}(\Phi^*)$, where $\mathcal{D}_{E^*} = \{\bar{A} \times X : \bar{A} \in \bar{\mathcal{A}}\}|_{E^*}$. Similar things hold for similarly defined (F^*, ν^*, Ψ^*). Note that $\bar{\Omega}$ constitutes a Borel subset of the Polish space $\Omega^{\mathbb{Z}}$. To see this, we first state a simple fact: if T is a Borel map on a metric space (Z, d), then $\{z \in Z : Tz = z\}$ is equal to $\{z \in Z : d(z, Tz) = 0\}$ and hence is a Borel subset of Z. Define

$$T : \Omega^{\mathbb{Z}} \to \Omega^{\mathbb{Z}}, \quad (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \mapsto (\ldots, \tau\omega_{-2}, \tau\omega_{-1}, \tau\omega_0, \ldots).$$

Then $\bar{\Omega}$ is equal to the set of fixed points of T. This implies that $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P}, \bar{\tau})$ constitutes a Polish system. The result follows then from the arguments in the last two paragraphs. □

Let $\mathcal{F}_E$ be an RDS on a Polish space X over $(\Omega, \mathcal{A}, P, \tau)$, as introduced at the beginning of this section. We now assume that P-almost everywhere E_ω is compact, depends measurably on ω and $f(\omega) : E_\omega \to E_{\tau\omega}$ is continuous. Denote by $L_E^1(\Omega, C(X))$ the space of families $\phi = \{\phi_\omega(\cdot) \in C(E_\omega)\}_{\omega\in\Omega}$ which are such that $\phi_\omega(x)$ depends measurably on (ω, x) and $\|\phi\| := \int \max_{x\in E_\omega} |\phi_\omega(x)|\, dP(\omega) < +\infty$. Identifying ϕ and ψ with $\|\phi - \psi\| = 0$, one can verify that $L_E^1(\Omega, C(X))$ constitutes a Banach space with the norm $\|\cdot\|$. Let $\phi \in L_E^1(\Omega, C(X))$. Putting for $\omega \in \Omega$, $\varepsilon > 0$ and $n \geq 1$,

$$P_{\mathcal{F}_E}(\phi)(\omega, \varepsilon, n) = \sup\left\{\sum_{x\in Z} \exp\sum_{k=0}^{n-1} \phi_{\tau^k\omega}(f(\omega, k)x) : Z \text{ an } (\omega, \varepsilon, n)\text{-separated subset of } E_\omega\right\},$$

we call

$$P_{\mathcal{F}_E}(\phi) := \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \int \log P_{\mathcal{F}_E}(\phi)(\omega, \varepsilon, n)\, dP(\omega)$$

the *topological pressure* of $\mathcal{F}_E$ with respect to ϕ (see Bogenschütz [**2**] and Kifer [**11**] for related notions in the case of X being compact).

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THEOREM 2.3. *Let $\mathcal{F}_E$ and $\mathcal{G}_F$ be two RDSs, over a Polish system $(\Omega, \mathcal{A}, P, \tau)$, on Polish spaces X and Y respectively such that P-almost everywhere E_ω and F_ω are compact, they depend measurably on ω and $f(\omega) : E_\omega \to E_{\tau\omega}$ and $g(\omega) : F_\omega \to F_{\tau\omega}$ are continuous. If for P-a.e. ω there is a surjective continuous map $\pi_\omega : E_\omega \to F_\omega$ such that $\pi_{\tau\omega} \circ f(\omega) = g(\omega) \circ \pi_\omega$, $\Pi : E \to F$, $(\omega, x) \mapsto (\omega, \pi_\omega x)$ constitutes a Borel map and the cardinal number $k_\omega(y)$ of $\pi_\omega^{-1}\{y\}$ is measurable in (ω, y) and is finite for each $y \in F_\omega$, then for each $\psi \in L_F^1(\Omega, C(Y))$ one has*

$$P_{\mathcal{G}_F}(\psi) = P_{\mathcal{F}_E}(\psi \circ \Pi),$$

where $\psi \circ \Pi = \{\psi_\omega \circ \pi_\omega \in C(E_\omega)\}_{\omega\in\Omega}$.

The inequality $P_{\mathcal{G}_F}(\psi) \geq P_{\mathcal{F}_E}(\psi \circ \Pi)$ follows from Theorem 2.1 and the proposition that follows.

PROPOSITION 2.4. (Variational principle) *Let $\mathcal{F}_E$ be an RDS as given in Theorem 2.3. Then, for any $\phi \in L_E^1(\Omega, C(X))$,*

$$P_{\mathcal{F}_E}(\phi) = \sup\left\{h_\mu(\mathcal{F}_E) + \iint \phi_\omega(x)\, d\mu_\omega(x)\, dP(\omega) : \mu \in \mathcal{M}(\mathcal{F}_E)\right\}.$$

When X is compact, Proposition 2.4 was first proved by Ledrappier and Walters [**12**] in the case when Ω is compact, E is a compact subset of $\Omega \times X$, Φ is continuous and only continuous functions on E are considered. The result was then proved by Bogenschütz [**3**] for the trivial bundle case $E = \Omega \times X$ and by Kifer [**11**] for the non-trivial bundle case $E = \bigcup_{\omega\in\Omega}\{\omega\} \times E_\omega$. The proof of Kifer [**11**] can be carried over to the present case where X and Ω are assumed to be Polish but E_ω is compact almost everywhere. To see this, note that Kifer [**11**, Lemma 2.1], the essential facts used in his proof but formulated in the compact metric phase space case, hold true in the present case: see Crauel [**6**, Corollary 6.13] for the much more difficult facts corresponding to Kifer [**11**, Lemma 2.1 (i) and (ii)]; the fact corresponding to [**11**, Lemma 2.1(iii)] can be proved similarly due to $\Omega \times X$ being Polish; other details hold in a parallel way.

The inequality $P_{\mathcal{G}_F}(\psi) \leq P_{\mathcal{F}_E}(\psi \circ \Pi)$ is a general result (see Corollary 2.7 below). For a proof we need the following.

PROPOSITION 2.5. *Let $\mathcal{F}_E$ and $\mathcal{G}_F$ be as given in Theorem 2.3. Assume that for P-a.e. ω there is a surjective continuous map $\pi_\omega : E_\omega \to F_\omega$ such that $\pi_{\tau\omega} \circ f(\omega) = g(\omega) \circ \pi_\omega$ and $\Pi : E \to F$, $(\omega, x) \mapsto (\omega, \pi_\omega x)$ constitutes a Borel map. Then for each $\mathcal{G}_F$-invariant measure ν there exists an $\mathcal{F}_E$-invariant measure μ such that $\Pi\mu = \nu$.*

To prove this proposition, we need the following Proposition 2.6. Its proof is a slight modification of the usual proof of the Hahn–Banach theorem (see, e.g., Rudin [**18**]) but we include it here for completeness.

PROPOSITION 2.6. (A modification of the Hahn–Banach theorem) *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $E = \bigcup_{\omega\in\Omega}\{\omega\} \times E_\omega$ where P-almost everywhere E_ω is a compact subset of a Polish space X which depends measurably on ω. If V is a subspace of $L_E^1(\Omega, C(X))$ which contains $\mathbf{1}$, the element of $L_E^1(\Omega, C(X))$ which takes the constant*

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value 1, *and if* $L : V \to \mathbb{R}$ *is a linear functional which satisfies* $\|L\| = 1$ *and is positive and normalized (i.e.* $\phi \geq 0$ *implies* $L(\phi) \geq 0$ *and* $L(\mathbf{1}) = 1$*), then* L *can be extended to a bounded linear functional* $\tilde{L}$ *on* $L^1_E(\Omega, C(X))$ *such that* $\|\tilde{L}\| = \|L\|$ *and it is still positive.*

Proof. Choose $\phi_0 \in L^1_E(\Omega, C(X))$ with $\phi_0 \notin V$, and let $V_1 = \{\phi + \lambda\phi_0 : \phi \in V, \lambda \in \mathbb{R}\}$. We now show that one can choose a suitable real number α such that, if we define $L_1(\phi + \lambda\phi_0) = L(\phi) + \lambda\alpha$, then L_1 is a positive linear extension of L to V_1 with $\|L_1\| = \|L\|$.

Define $A_\phi = L(\phi) - \|\phi - \phi_0\|$, $B_\phi = L(\phi) + \|\phi - \phi_0\|$ and put

$$A = \sup_{\phi \in V} A_\phi, \qquad B = \inf_{\phi \in V} B_\phi.$$

Clearly $A \leq B$ since $A_\phi \leq B_\psi$ for any $\phi, \psi \in V$. Take $\alpha = A$. Then it is easy to see that for any $\phi \in V$,

$$|L(\phi) - \alpha| \leq \|\phi - \phi_0\|,$$

which implies, by replacing ϕ with $-\phi/\lambda$ for $\lambda \neq 0$,

$$|L_1(\phi + \lambda\phi_0)| = |L(\phi) + \lambda\alpha| \leq \|\phi + \lambda\phi_0\|.$$

Hence $\|L_1\| = 1$ (note that $L_1(\mathbf{1}) = 1$). We now show that L_1 is positive.

Assume that $\phi + \lambda\phi_0 \geq 0$ and $\lambda \neq 0$. If $\lambda > 0$, then

$$\begin{aligned}
L_1(\phi + \lambda\phi_0) &= L(\phi) + \lambda\alpha \\
&= L(\phi) + \sup_{\psi \in V} \{L(\psi) - \|\psi - \lambda\phi_0\|\} \\
&= \sup_{\psi \in V} \{L(\psi) - \|\psi - (\phi + \lambda\phi_0)\|\} \\
&\geq \lim_{n \to +\infty} \{n - \|n - (\phi + \lambda\phi_0)\|\} \\
&= \lim_{n \to +\infty} \int \inf_{x \in E_\omega} \{n - |n - (\phi + \lambda\phi_0)_\omega(x)|\}\, dP(\omega) \\
&= \int \inf_{x \in E_\omega} (\phi + \lambda\phi_0)_\omega(x)\, dP(\omega) \\
&\geq 0.
\end{aligned}$$

If $\lambda < 0$, then

$$\begin{aligned}
L_1(\phi + \lambda\phi_0) &= L(\phi) + \lambda\alpha \geq L(\phi) + \lambda B \\
&= L(\phi) + \lambda \inf_{\psi \in V} \{L(\psi) + \|\psi - \phi_0\|\} \\
&= L(\phi) + \sup_{\psi \in V} \{L(\psi) - \|\psi - \lambda\phi_0\|\} \\
&= \sup_{\psi \in V} \{L(\psi) - \|\psi - (\phi + \lambda\phi_0)\|\} \\
&\geq 0,
\end{aligned}$$

by the same reasoning as in the case $\lambda > 0$. L_1 is thus positive. We have now proved that there exists a norm-preserving and positive linear extension L_1 of L on V_1.

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Let $\mathcal{P}$ be the collection of all pairs (V', L'), where V' is a subspace of $L^1_E(\Omega, C(X))$ which contains V and L' is a norm-preserving and positive linear extension of L to V'. Define a partial order $\preceq$ on $\mathcal{P}$ by declaring $(V', L') \preceq (V'', L'')$ to mean that $V' \subset V''$ and $L''(\phi) = L'(\phi)$ for all $\phi \in V'$. By the Hausdorff maximality theorem, there exists a maximal totally ordered subcollection $\mathcal{Q}$ of $\mathcal{P}$.

Let Γ be the collection of all V' such that $(V', L') \in \mathcal{Q}$. Since $\mathcal{Q}$ is totally ordered, the union $\tilde{V}$ of all members of Γ is a subspace of $L^1_E(\Omega, C(X))$. For $\phi \in \tilde{V}$, choose $V' \in \Gamma$ which contains ϕ and define $\tilde{L}(\phi) = L'(\phi)$. Then it is easy to check that $\tilde{L}$ is a well-defined positive linear extension of L to $\tilde{V}$ with $\|\tilde{L}\| = 1$. Clearly, $\tilde{V} = L^1_E(\Omega, C(X))$ since otherwise the first part of the proof would give a further extension of $\tilde{L}$ and this would contradict the maximality of $\mathcal{Q}$. The proof is thus completed. □

Proof of Proposition 2.5. For $\psi = \{\psi_\omega \in C(F_\omega)\} \in L^1_F(\Omega, C(Y))$, define $\psi \circ \Pi = \{\psi_\omega \circ \pi_\omega \in C(E_\omega)\} \in L^1_E(\Omega, C(X))$. Then $V = \{\psi \circ \Pi : \psi \in L^1_F(\Omega, C(Y))\}$ is a subspace of $L^1_E(\Omega, C(X))$ which clearly contains the element $\mathbf{1}$. Define

$$L : V \to \mathbb{R}, \quad \psi \circ \Pi \mapsto \int \psi \, d\nu := \int \psi_\omega(y) \, d\nu(\omega, y).$$

By the surjectivity of π_ω, $\omega \in \Omega$, L is well defined and it is clearly a positive and normalized linear functional on V with $\|L\| = 1$. By Proposition 2.6, L can be extended to a linear functional $\tilde{L}$ on $L^1_E(\Omega, C(X))$ such that $\|\tilde{L}\| = 1$ and it is still positive. $L^1_E(\Omega, C(X))$ is clearly a Stone vector lattice (see Dudley [**8**] for the notion) and $\tilde{L}$ is a pre-integral on $L^1_E(\Omega, C(X))$, i.e. $\tilde{L}$ is a positive linear functional such that $\tilde{L}(\phi_n) \downarrow 0$ whenever $\phi_n \in L^1_E(\Omega, C(X))$ and $(\phi_n)_\omega(x) \downarrow 0$ for all (ω, x). By the Stone-Daniel theorem [**8**, Theorem 4.5.2], there is a unique measure $\tilde{\mu}$ on $(E, (\mathcal{A} \times \mathcal{B}(X))|_E)$ such that

$$\tilde{L}(\phi) = \int \phi_\omega(x) \, d\tilde{\mu}(\omega, x), \quad \forall \phi \in L^1_E(\Omega, C(X)).$$

Since $\tilde{L}$ is normalized and $\int h \, d\tilde{\mu} = \int h \, dP$ for each bounded measurable function $h : \Omega \to \mathbb{R}$ (regarding h as a function on E for the first integral), one knows that $\tilde{\mu}$ is a probability which has marginal P on Ω. Clearly $\Pi\tilde{\mu} = \nu$ since $\int \psi \circ \Pi \, d\tilde{\mu} = \int \psi \, d\nu$ for all $\psi \in L^1_F(\Omega, C(Y))$. The set of limit points of $\{(1/n) \sum_{k=0}^{n-1} \Phi^k \tilde{\mu} : n \in \mathbb{N}\}$ is non-empty and any element μ of this set gives an $\mathcal{F}_E$-invariant measure (Crauel [**6**, Corollary 6.13]) which satisfies $\Pi\mu = \nu$. □

COROLLARY 2.7. *Under the assumptions of Proposition 2.5, one has for any* $\psi \in L^1_F(\Omega, C(Y))$,

$$P_{\mathcal{F}_E}(\psi \circ \Pi) \geq P_{\mathcal{G}_F}(\psi).$$

Proof. By (2.2), for any $\mu \in \mathcal{M}(\mathcal{F}_E)$

$$h_\mu(\mathcal{F}_E) = h_\mu^{\Pi^{-1}\mathcal{D}_0}(\Phi) \geq h_{\Pi\mu}^{\mathcal{D}_0}(\Psi) = h_{\Pi\mu}(\mathcal{G}_F).$$

The result then follows from Propositions 2.4 and 2.5. □

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