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A non-constant invariant function for certain ergodic flows

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Abstract. Let U be the vector space of uniformly continuous real-valued functions on the real line $\mathbb{R}$ and let U_0 denote the subspace of U consisting of all bounded uniformly continuous functions. If X is a compact differentiable manifold and we are given a flow on X, then we associate with the flow a function $F : X \to H_1(X, U/U_0)$ that is invariant under the flow. We give examples for which the flow on X is ergodic but there is no $\lambda \in H_1(X, U/U_0)$ such that $F(p) = \lambda$ for almost all points p.

1. *Introduction*

Let X be a compact Riemannian manifold and let f be a continuous function from the real line $\mathbb{R}$ to X. For each $T > 0$ choose a curve A_T joining $f(T)$ to $f(0)$, ensuring that the lengths of these curves are uniformly bounded. For each $T > 0$, we form a closed curve from the arc $\{f(t) : 0 \le t \le T\}$ by concatenating A_T. The element of $H_1(X, \mathbb{R})$ specified by this closed curve is denoted h_T, and if $\lim_{T\to\infty}(1/T)h_T$ exists, then it does so independently of the Riemannian metric and the choice of curves A_T. If this limit exists we call it the *asymptotic cycle* determined by f.

Given a continuous flow on X we can associate with $p \in X$ the function $f_p : t \mapsto t.p$. The set of points p for which there is an asymptotic cycle associated with f_p is invariant under the flow and the function sending each p in this set into the corresponding asymptotic cycle is invariant. In [**2**] the author showed that the set of points p for which the asymptotic cycle is defined has measure one with respect to any normalized invariant measure.

Now suppose that we have a function $f : \mathbb{R} \to X$ that is uniformly continuous. Let U be the vector space of uniformly continuous real-valued functions on $\mathbb{R}$ and let U_0 be the subspace of bounded uniformly continuous functions. In [**3**] the author showed how to associate with f an element of $H_1(X, U/U_0)$. We call this element of $H_1(X, U/U_0)$ the *uniform homology class* of f.

Let U_1 be the subspace of U consisting of all functions $g \in U$ such that $g(t) = o(t)$. Then $U_0 \le U_1 \le U$ so there is a natural homomorphism P of $H_1(X, U/U_0)$ into $H_1(X, U/U_1)$. If $f : \mathbb{R} \to X$ is uniformly continuous and λ is the uniform homology

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class of f, then if f determines an asymptotic cycle, then the asymptotic cycle can be determined from λ.

To see the connection, let i be the identity function on $\mathbb{R}$ and let $\bar{\imath}$ be the element of U/U_1 determined by i. Then using the notation of the preceding paragraphs, if μ is an element of $H_1(X, \mathbb{R})$ a necessary and sufficient condition for there to be an asymptotic cycle equal to μ associated with f is that $P(\lambda) = \bar{\imath} \otimes_R \mu$, where we have identified $H_1(X, U/U_1)$ with $U/U_1 \otimes_R H_1(X, \mathbb{R})$.

Suppose now that we have a flow on X arising from a smooth vector field V and that this flow is ergodic with respect to a normalized invariant measure m. If for each $p \in X$ for which an asymptotic cycle exists we denote this element of $H_1(X, \mathbb{R})$ by μ_p then the function sending p to μ_p is defined almost everywhere and is measurable. Since the flow is ergodic there exists a $\mu \in H_1(X, \mathbb{R})$ such that $\mu_p = \mu$ for almost all p.

It follows from what was done in [**2**] that μ can be computed without any knowledge of the individual orbits under the flow. In fact, for any continuous closed 1-form w it can be seen that $\int w(V)\, dm$ depends only on the element of the de Rham cohomology group $H^1(X, \mathbb{R})$ determined by w. Thus we get an element of $\mathrm{Hom}(H^1(X, \mathbb{R}), \mathbb{R})$. The corresponding element of $H_1(X, \mathbb{R})$ is the μ referred to above.

For each $p \in X$ let $F(p)$ be the uniform homology class of the function f_p. Then F is invariant under the flow. It therefore seems reasonable to hope that there is a $\lambda \in H_1(X, U/U_0)$ for which $F(p) = \lambda$ for almost all $p \in X$ and that in this case too it is possible to compute this λ. However, it turns out that, even though our flow is ergodic, F does not have to be constant almost everywhere.

To see this, for each integer $g > 1$ we are going to display a compact two-dimensional Riemannian manifold of constant negative curvature and genus g and consider the geodesic flow on its unit tangent bundle E_g, which is known to be ergodic. If F_g is the function from E_g to $H_1(E_g, U/U_0)$, we will prove that there is no $\lambda_g \in H_1(E_g, U/U_0)$ such that $F_g(p) = \lambda_g$ for almost all $p \in E_g$.

2. *Uniform homology classes*

Firstly we have to define the uniform homology class associated with a uniformly continuous map f from $\mathbb{R}$ into a compact differentiable manifold X. In [**3**] we actually did this in the case where X is any compact polyhedron. We are restricting our attention to the situation where X is a compact differentiable manifold because in this case we can take advantage of the de Rham treatment of cohomology over the reals. Our exposition will differ somewhat from that given in [**3**] but the interested reader should have no difficulty in seeing the connection.

Recall that if f_L is a continuous map of an interval $[0, L]$ into the differentiable manifold X and w is a continuous closed form on X then $\int_{f_L} w$ can be defined. Here L can be positive or negative and we put the obvious orientation on the interval. We need only remark that there exists a smooth map $\gamma : [0, L] \to X$ homotopic to f_L under a homotopy that fixes $f_L(0)$ and $f_L(L)$ and that if γ_1 and γ_2 are two such smooth maps $\int_{\gamma_1} w = \int_{\gamma_2} w$. We define $\int_{f_L} w$ to equal this common value. Now suppose that f is a continuous map $\mathbb{R} \to X$. Let C be the vector space of continuous real-valued functions on $\mathbb{R}$ and let C_0 be the subspace of bounded continuous functions.

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If α is any continuous closed 1-form on X and $t \in \mathbb{R}$ define $\theta_\alpha^f(t)$ to equal $\int_{f_t} \alpha$ where f_t is the restriction of f to $[0, t]$. Then $\theta_\alpha^f(t) \in C$, while if α is a bounding form $\theta_\alpha^f(t) \in C_0$. Thus f determines an element of $\mathrm{Hom}(H^1(X, \mathbb{R}), C/C_0)$. However, if f_1 and f_2 are homotopic the corresponding elements of $H_1(X, C/C_0)$ need not be equal.

To overcome this difficulty, in [**3**] we restricted our attention to uniformly continuous functions from $\mathbb{R}$ to X. If α is a continuous closed 1-form on X and we define θ_α^f as before, then $\theta_\alpha^f \in U$. Thus we can associate with f an element of $H_1(X, U/U_0)$, which we call the *uniform homology class* of f.

We say that two uniformly continuous functions f_1 and f_2 are *uniformly homotopic* provided there exists a uniformly continuous function $H : [0, 1] \times \mathbb{R} \to \mathbf{X}$ such that $H(0, t) = f_1(t)$ and $H(1, t) = f_2(t)$; the relation of being uniformly homotopic is an equivalence relation. It is shown in [**3**] that if f_1 and f_2 are uniformly homotopic then they determine the same element of $H_1(X, U/U_0)$, so we get a map of uniform homotopy classes of functions from $\mathbb{R}$ to X into uniform homology classes. In [**3**] it is proved that this map is always surjective, while if X is an n-dimensional torus, it is bijective.

3. *Main theorem*

We are now ready to prove the basic result we will need. Let M be a compact Riemannian manifold whose unit tangent bundle is E. Suppose that the geodesic flow on E is ergodic and that we are given an isometry $\overline{T} : M \to M$ such that $\overline{T}^2$ equals the identity map. Suppose further that we are given g continuous maps $p_1, \ldots, p_g$ of the unit circle in the complex plane into M such that, if $e_1, \ldots, e_g$ denote the elements of $H_1(M, \mathbb{R})$ corresponding to these closed curves, $\{e_1, \ldots, e_g, \overline{T}^*(e_1), \ldots, \overline{T}^*(e_g)\}$ is a basis for $H_1(M, \mathbb{R})$, where $\overline{T}^*$ is the automorphism of $H_1(M, \mathbb{R})$ induced by $\overline{T}$.

THEOREM 1. *Let F be the map of E into $H_1(E, U/U_0)$ that sends $p \in E$ into the uniform homology class λ_{f_p} associated with the $f_p : t \mapsto t.p$. Then there is no $\lambda \in H_1(E, U/U_0)$ such that $F(p) = \lambda$ for almost all $p \in E$.*

Proof. Let Π be the projection of E onto M and let $\Pi^* : H_1(E, U/U_0) \to H_1(M, U/U_0)$ be the map induced by Π. Suppose $\lambda \in H_1(E, U/U_0)$ is such that $F(p) = \lambda$ for almost all $p \in E$. Then for almost all $p \in E$, $\Pi^*(F(p)) = \Pi^*(\lambda)$.

If we identify $H_1(M, U/U_0)$ with $U/U_0 \otimes_{\mathbb{R}} H_1(M, \mathbb{R})$ then for any $p \in E$ we can write $\Pi^*(F(p))$ in the form

$$\alpha_1^p \otimes_R e_1 + \cdots + \alpha_g^p \otimes_R e_g + \beta_1^p \otimes_R \overline{T^*}(e_1) + \cdots + \beta_g^P \otimes_R \overline{T}^*(e_g). \tag{1}$$

If we let $\bar{S} : E \to E$ be the map induced by $\overline{T}$ then $\overline{S^2}$ equals the identity and $\overline{S}$ commutes with the geodesic flow. For any $p \in E$ we see that $\Pi^*(F(\overline{S}(p))$ equals

$$\alpha_1^p \otimes_R \overline{T}^*(e_1) + \cdots + \alpha_g^p \otimes_R \overline{T}^*(e_g) + \beta_1^p \otimes_R e_1 + \cdots + \beta_g^p \otimes_R e_g. \tag{2}$$

Since $\overline{S}$ is measure-preserving, if $F(p) = \lambda$ for almost all $p \in E$ and $\Pi^*(\lambda)$ equals

$$\alpha_1 \otimes_R e_1 + \cdots + \alpha_g \otimes_R e_g + \beta_1 \otimes_R \overline{T^*}(e_1) + \cdots + \beta_g \otimes_R \overline{T}^*(e_g), \tag{3}$$

then it follows that for all i we must have $\alpha_i = \beta_i$.

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Identify $H^1(M, \mathbb{R})$ with $\mathrm{Hom}(H_1(M, \mathbb{R}), \mathbb{R})$ and let w be a 1-form on M such that $\overline{w}(e_i) = \overline{w}(\overline{T}^*(e_i)) = 0$ for $i > 1$ while $\overline{w}(e_1) = 1$ and $\overline{w}(\overline{T}^*(e_1)) = -1$ where $\overline{w}$ is the element of $H^1(M, \mathbb{R})$ determined by w. Then $\overline{w}(\Pi^*(\lambda))$ equals zero so $\overline{w}(\Pi^*(F(p)))$ equals zero for almost all $p \in E$.

For any closed 1-form α on M and any uniformly continuous function $f : \mathbb{R} \to E$ let $\theta_\alpha^{\Pi f}$ be the element of U we use in determining the element of $\mathrm{Hom}(H^1(M, \mathbb{R}), U/U_0)$ associated with Πf. For any $p \in E$ let $f_p : \mathbb{R} \to E$ be the map such that $f_p(t) = t.p$, the point into which p is sent after time t by the geodesic flow. Then $\theta_w^{\Pi f_p}$ is an element of U that we will denote by θ_p. Since $\overline{w}(\Pi^*(F(p)) = 0$ for almost all $p \in E$ it follows that for almost all $p \in E$ we must have $\theta_p \in U_0$.

For each $p \in E$ let $\triangle(p) = \sup |\theta_p(t_1) - \theta_p(t_2)|$ and let K_N for each positive integer N be the set of all $p \in E$ such that $\triangle(p) \leq N$. Then K_N is a closed subset of E invariant under the geodesic flow and the union of all the K_N is almost all of E. Therefore there must be an N such that the measure of K_N is positive. Since we are assuming the geodesic flow is ergodic it follows that K_N must be almost all of E. But from this we can conclude that K_N is dense in E, and since K_N is closed it must be all of E. Thus θ_p is a bounded function for all $p \in E$.

It is known that every non-trivial free homotopy class of closed curves in M contains a geodesic. Therefore there is a closed geodesic on M such that $\int w$ over this closed geodesic is non-zero. If we let p_0 be a point in E corresponding to a unit tangent to this closed geodesic it follows that θ_{p_0} is unbounded. Thus we get a contradiction, so F cannot equal a constant almost everywhere. □

4. *Example flows*

Next we want to see that for each even integer $g > 0$ there is a two-dimensional compact Riemannian manifold of genus g satisfying the hypothesis of our theorem. Let $\mathbb{H}$ be the interior of the unit circle about the origin in the complex plane endowed with the usual Riemannian metric of constant negative curvature. We will refer to $\mathbb{H}$ as the hyperbolic plane.

For any ρ, $0 < \rho < 1$, locate on the circle about the origin with Euclidean radius ρ the collection of $4g$ equally spaced points starting with the point with complex coordinates denoted ρ, where g is any even integer greater than zero. Connect each of these $4g$ points to the point following it, going counterclockwise around the circle of radius ρ by a segment of a hyperbolic geodesic. Going counterclockwise around the origin starting with the hyperbolic geodesic segment originating at the point with complex coordinate ρ, associate with each of these hyperbolic segments an integer modulo $4g$, so that the segments have associated with them in order the residue classes $\overline{0}, \overline{1}, \ldots, \overline{4g-1}$. For any one of these hyperbolic segments I we will denote the corresponding residue class by $\alpha(I)$.

If λ is any residue class mod $4g$ we will denote the corresponding residue class mod 4 by $\overline{\lambda}$. Thus the four residue classes mod 4 can be denoted by $\overline{\overline{0}}, \overline{\overline{1}}, \overline{\overline{2}}, \overline{\overline{3}}$, while the residue classes mod $4g$ are denoted by $\overline{0}, \overline{1}, \ldots, \overline{4g-1}$.

With any one of our hyperbolic segments I such that $\overline{\alpha(I)}$ is either $\overline{\overline{0}}$ or $\overline{\overline{1}}$ we associate the unique orientation-preserving isometry $\mathbb{H} \to \mathbb{H}$ that sends our segment onto

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the segment J such that $\alpha(J) = \alpha(I) + \overline{2}$ in such a way as to reverse the orientation that each segment inherits from the orientation of the circle of radius ρ about the origin.

It is known [**1**] that the collection of these isometries generates a group Γ of orientation-preserving isometries $\mathbb{H} \to \mathbb{H}$ which is actually a group of deck transformations. The quotient $\mathbb{H}/\Gamma$ is a compact oriented two-dimensional manifold endowed with a Riemannian metric of constant negative curvature. The $4g$ points on the circle of radius ρ about the origin all get identified by Γ so that each of our hyperbolic segments determines a closed curve on $\mathbb{H}/\Gamma$ on which we can give the orientation inherited from the orientation of the circle of radius ρ.

The $2g$ hyperbolic geodesic segments I for which $\overline{\alpha(I)}$ equals either $\overline{\overline{0}}$ or $\overline{\overline{1}}$ together with their orientations determine a basis for $H_1(\mathbb{H}/\Gamma, \mathbb{R})$, so $\mathbb{H}/\Gamma$ is a compact surface of genus g. Let T be the map of $\mathbb{H}$ onto itself sending z into $\overline{z}$. Then $T : \mathbb{H} \to \mathbb{H}$ is an isometry and T^2 equals the identity map, so $T^{-1} = T$. It is easy to see that T induces an isometry $\overline{T}$ of $\mathbb{H}/\Gamma \to \mathbb{H}/\Gamma$.

Next let $\overline{T}^*$ be the automorphism of $H_1(\mathbb{H}/\Gamma, \mathbb{R})$ induced by $\overline{T}$. If g is even let $\{\overline{e_1}, \ldots, \overline{e_g}\}$ be the elements of $H_1(\mathbb{H}/\Gamma, \mathbb{R})$ determined by the oriented closed curves on $\mathbb{H}/T$ arising from hyperbolic segments for which the residue classes mod $4g$ are $\{\overline{0}, \overline{1}, \overline{4}, \overline{5}, \ldots, \overline{2g-4}, \overline{2g-3}\}$. Then $\{\overline{e}_1, \ldots, \overline{e}_g, \overline{T}^*(\overline{e}_1), \ldots, \overline{T}^*(\overline{e}_g)\}$ is a basis for $H_1(\mathbb{H}/\Gamma, \mathbb{R})$.

It follows from our previous discussion that if g is even then the geodesic flow on the unit tangent bundle E_g to $\mathbb{H}/\Gamma$ is such that our invariant function from E_g to $H_1(E_g, U/U_0)$ is not equal to a constant almost everywhere even though this geodesic flow is ergodic.

A slight modification to our main theorem suffices to show that the same conclusion holds when g is odd. We omit the details.

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