TWO-DIMENSIONAL ELASTIC INCLUSION PROBLEMS

By M. A. JASWON and R. D. BHARGAVA

Communicated by J. G. Oldroyd

Received 13 August 1960

ABSTRACT. An account is given of Eshelby's point-force method for solving elastic inclusion problems, and of his equations relating an inhomogeneity to its equivalent inclusion. The introduction of complex variable formalism enables explicit solutions to be found in various two-dimensional cases. Strain energies are calculated. The equilibrium shape of an elliptic inclusion exhibits an interesting feature not previously expected. A fresh analysis of stress magnification effects is developed.

Introduction. This paper deals with a class of problems in classical elasticity theory known as 'inclusion' problems. The inclusion may be a cavity or inhomogeneity within an elastic medium, perturbing an otherwise uniform stress field. Alternatively, it could be some limited region which tends to undergo a homogeneous deformation consequent upon thermal effects, plastic flow or phase transition, thereby generating locked-up accommodation stresses within itself and the surrounding matrix. Such problems arise naturally in various fields of physical and technological interest. At the same time they are of considerable theoretical interest, in that the elastic displacement field does not remain continuous on crossing the inclusion boundary. Only the net displacement, made up of an elastic plus a non-elastic contribution, is everywhere continuous. Accordingly, we are concerned with states of elastic strain which do not everywhere satisfy St Venant's compatibility relations and which are, nevertheless, realized without the material being ruptured.

The simple problem of a spherical inclusion in an infinite isotropic elastic continuum was examined by Frenkel (1) in connexion with his theory of liquids, and by Mott and Nabarro (2) in connexion with their theory of precipitation hardening. A systematic investigation of the ellipsoidal inclusion was undertaken in 1957 by Eshelby (3). This work forms the starting point for the present analysis.

Although Eshelby has proved some general theorems of great interest, using elegant methods, his solutions involve analytically intractable integrals of a formidable nature. This applies even in two-dimensional situations, i.e. where the inclusion has the shape of a long cylinder of elliptic cross-section under conditions of plane strain, or appears as an elliptic region of a thin plate under conditions of generalized plane stress. We obtain explicit solutions in these cases by an approach based upon complex variable formalism. Detailed analysis thus becomes possible of stress and displacement behaviour anywhere outside an elliptic inclusion, of stress magnification effects at the boundary, and of the energy-density field. For non-elliptic inclusions, e.g. a square, complex variable methods lead to contour integrals difficult to evaluate analytically, though the analysis goes much further than would be possible using a real variable procedure. All the integrals can be handled numerically without undue labour.

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

It may be remarked that the word 'inclusion', as used in the opening paragraph, was taken to include cavities and inhomogeneities as well as regions that undergo irreversible dimensional changes. Whilst this usage has certain advantages, confusion could sometimes arise from identifying too closely the three cases enumerated. Accordingly, we henceforth restrict 'inclusion' to the third case only, distinguishing the first two by name when necessary. The possibility of treating elliptic cavities and inhomogeneities stems essentially from the fact that the stress field is uniform within an elliptic inclusion, thus often enabling a valuable transformation to be effected from problems of the latter type to those of the former.

The stress field generated by a distribution of inclusions, whether or not dilutely dispersed, is obtained by straightforward superposition of the stress fields generated by each independently.

The point-force method. An infinite isotropic elastic medium contains a specified region (the inclusion) which spontaneously tends to undergo a prescribed homogeneous deformation E^0_{ij}. However, owing to the elastic constraints of the surrounding material (the matrix) it assumes an equilibrium configuration E^q_{ij}. In effect, the matrix equilibrium reactions T_i generate an elastic deformation $E^q_{ij} - E^0_{ij}$ that combines with the free deformation E^0_{ij} to produce a net deformation E^q_{ij}. Corresponding to these three deformations there exist three displacement fields $U^q_i - U^0_i, U^0_i, U^q_i$ having the same respective significance. As regards the matrix, it contains an internal boundary subject to tractions $-T_i$ that generate an elastic strain field e_{ij} and displacement field u_i. The latter is necessarily continuous with U^q_i, and hence cannot be continuous with the purely elastic field $U^q_i - U^0_i$. Finally, we introduce stress fields P_{ij}, p_{ij} related by Hooke's law to $E^q_{ij} - E^0_{ij}, e_{ij}$, respectively, i.e. in the usual tensor symbolism

$$\left.\begin{aligned} P_{ij} &= 2\mu(E^q_{ij} - E^0_{ij}) + \lambda(E^q - E^0)\,\delta_{ij}, \\ p_{ij} &= 2\mu e_{ij} + \lambda e \delta_{ij}. \end{aligned}\right\} \tag{1}$$

The normal and tangential components of P_{ij} are of course continuous with those of p_{ij} at the boundary; as a corollary, the hoop stress components must be discontinuous since otherwise the boundary would have no physical significance.

The mathematical problem here may be posed in very simple terms: what is E^q_{ij} and the corresponding equilibrium elastic state of the system? An ingenious attack on this problem, utilizing the point-force concept, has been devised by Eshelby. In view of the novelty and importance of his approach, we briefly go over the argument, which invokes a sequence of hypothetical operations. First, cut out the inclusion from the matrix and allow it to achieve E^0_{ij}. Next, impress upon it surface tractions that restore its original dimensions, i.e. that generate an elastic deformation $-E^0_{ij}$. Insert the stressed inclusion into the cavity left behind and rejoin the material across the cut; at this stage no stresses appear in the matrix. Finally, introduce a distribution of point-forces, equal and opposite to the impressed surface tractions. If the matrix were absent these forces would evidently nullify the latter, i.e. they would generate an elastic deformation E^0_{ij} that exactly cancels the already impressed deformation $-E^0_{ij}$. However, since they are actually acting within an infinite elastic solid, they generate a deformation that does not cancel $-E^0_{ij}$, thereby producing the equilibrium configura-

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

tion in question. This being recognized, a complete formal solution to the problem may be written down at once in terms of integrals taken round the boundary.

A point-force F_i, acting at a point y_i, generates a known displacement at any other point x_i. For a continuous distribution of forces over an inclusion surface S, the cumulative effect at x_i is obtained by integration over S. If x_i lies in the matrix, this provides the elastic displacement u_i already introduced. If x_i lies in the inclusion, however, it provides only part of the elastic displacement, the 'constrained' part U_i^c, with derivative constrained fields E_{ij}^c, P_{ij}^c. The actual elastic fields necessarily appear as

$$U_i^c - U_i^0, \quad E_{ij}^c - E_{ij}^0, \quad P_{ij}^c - P_{ij}^0 \equiv P_{ij} \tag{2}$$

on bearing in mind the already existing elastic fields $-U_i^0$, $-E_{ij}^0$, $-P_{ij}^0$. Expressions (2) show that U_i^c, E_{ij}^c have an important formal significance: they yield directly U_i^q, E_{ij}^q introduced at the outset. It may be noted that P_{ij} now appears as the sum of two contributions calculated in entirely different ways; hence the fulfilment of its required continuity properties provides a useful check on the analysis.

The main limitation of Eshelby's solution concerns the formidable nature of the point-force integrals. Only for problems exhibiting spherical or circular symmetry can these be handled anaytically. It was in an effort to overcome this practical limitation, at least for two-dimensional systems, that the complex variable approach was initiated.

Complex variable formalism. Any biharmonic function can be written in the form (4)

$$\mathrm{X} = \mathscr{R}\{\bar{x}\phi(x) + \psi(x)\},$$

where ϕ, ψ are analytic functions of the complex variable $x \equiv x_1 + \iota x_2$ ($\iota^2 = -1$) throughout the relevant domain; $\mathscr{R}$ denotes 'the real part of', and a bar indicates the complex conjugate operation. Stress components are most conveniently calculated from the relations

$$p_{11} + p_{22} = \nabla^2 \mathrm{X} = 4\mathscr{R}\{\phi'(x)\}, \tag{3}$$

$$p_{22} - p_{11} + 2\iota p_{12} = 2\{\bar{x}\phi'(x) + \psi''(x)\}, \tag{4}$$

where dashes signify differentiation. Similarly, the displacement components may be found from

$$2\mu(u_1 + \iota u_2) = \kappa\phi(x) - x\phi'(x) - \psi'(x), \tag{5}$$

where $\kappa = 3 - 4\nu$ under conditions of plane strain, and $\kappa = (3-\nu)/(1+\nu)$ under conditions of generalized plane stress. All these expressions possess immediate physical significance: (3) provides the hydrostatic pressure; (4) provides the maximum shear stress field $|p_{22} - p_{11} + 2\iota p_{12}|$ in the plane; (5) provides the displacement vector field in the plane. Another feature concerns the ease with which they enable coordinate transformations to be effected. Thus, for a counter-clockwise rotation of local axes through an angle θ, the new components $p_{\xi\xi}, p_{\eta\eta}, p_{\xi\eta}$; u_ξ, u_η corresponding respectively to p_{11}, p_{22}, p_{12}; u_1, u_2 appear at once from the formulae

$$\left.\begin{aligned} p_{\xi\xi} + p_{\eta\eta} &= p_{11} + p_{22}, \\ p_{\eta\eta} - p_{\xi\xi} + 2\iota p_{\xi\eta} &= (p_{22} - p_{11} + 2\iota p_{12})\, e^{2\iota\theta}, \\ u_\xi + \iota u_\eta &= (u_1 + \iota u_2)\, e^{-\iota\theta}. \end{aligned}\right\} \tag{6}$$

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

In the case of a finite domain, subject to given boundary tractions or displacements, the usual procedure consists in setting up functional equations that define ϕ, ψ on the boundary, whence they are continued into the interior by using Cauchy's theorem or power series expansions. Such analysis plays no role here. If a point-force F_1, F_2 acts at $y \equiv y_1 + \iota y_2$ it may be shown (5) that

$$\phi'(x) = -\frac{1}{2\pi(1+\kappa)} \frac{F_1 + \iota F_2}{x-y},$$

$$\psi''(x) = \frac{\kappa}{2\pi(1+\kappa)} \frac{F_1 - \iota F_2}{x-y} + \frac{1}{2\pi(1+\kappa)} \frac{d}{dx}\left\{\frac{\bar{y}(F_1 + \iota F_2)}{x-y}\right\},$$

at x. The success of the complex variable approach hinges on these key results. For a continuous distribution of forces round a contour Γ, we introduce the integrals

$$\left.\begin{aligned} \phi'(x) &= -\frac{1}{2\pi(1+\kappa)} \int_\Gamma \frac{(F_1 + \iota F_2)\, ds}{x-y}, \\ \psi''(x) &= \frac{\kappa}{2\pi(1+\kappa)} \int_\Gamma \frac{(F_1 - \iota F_2)\, ds}{x-y} + \frac{1}{2\pi(1+\kappa)} \frac{d}{dx} \int_\Gamma \frac{\bar{y}(F_1 + \iota F_2)\, ds}{x-y}, \end{aligned}\right\} \tag{7}$$

where $F_1 \pm \iota F_2, y, \bar{y}$ vary along Γ and ds denotes the arc differential (moving in the counter-clockwise direction) along Γ. To evaluate these integrals as functions of x, ds must in the first place be eliminated in favour of dy: write

$$dy = \left(\frac{dy_1}{ds} + \iota \frac{dy_2}{ds}\right) ds, \quad d\bar{y} = \left(\frac{dy_1}{ds} - \iota \frac{dy_2}{ds}\right)$$

so that

$$2\frac{dy_1}{ds} ds = dy + d\bar{y}, \quad 2\frac{dy_2}{ds} ds = -\iota(dy - d\bar{y}), \tag{8}$$

and then remove $d\bar{y}$ by throwing the equation of Γ into the form $\bar{y} = f(y)$. This last relation, for the ellipse $y_1^2/a^2 + y_2^2/b^2 = 1$, is

$$\bar{y} = \frac{a^2+b^2}{c^2} y - \frac{2ab}{c^2}(y^2 - c^2)^{\frac{1}{2}} \quad (c^2 = a^2 - b^2). \tag{9}$$

At the point y_1, y_2 of an inclusion boundary Γ, the outward normal to Γ has direction-cosines dy_2/ds, $-dy_1/ds$. Hence, corresponding to Eshelby's hypothetical stress field P^0_{ij}, the point force components per unit length are

$$F_1 = P^0_{11}(dy_2/ds) + P^0_{12}(-dy_1/ds),$$
$$F_2 = P^0_{12}(dy_2/ds) + P^0_{22}(-dy_1/ds).$$

Accordingly, making use of (8), we arrive at the expressions

$$\left.\begin{aligned} (F_1 + \iota F_2)\, ds &= -\tfrac{1}{2}\iota\{(P^0_{11} + P^0_{22})\, dy - (P^0_{11} - P^0_{22})\, d\bar{y}\} + P^0_{12} d\bar{y}, \\ (F_1 - \iota F_2)\, ds &= -\tfrac{1}{2}\iota\{(P^0_{11} - P^0_{22})\, dy - (P^0_{11} + P^0_{22})\, d\bar{y}\} + P^0_{12} dy, \end{aligned}\right\} \tag{10}$$

valid for a contour of any shape, that must be introduced into the integrals (7).

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

The elliptic inclusion: principal strains. For an elliptic inclusion of semi-axes a, b tending to deform into a similarly situated ellipse, of semi-axes $a(1+E^0_{11})$, $b(1+E^0_{22})$: $P^0_{12}=0$ and, under conditions of plane strain,

$$\left.\begin{aligned} P^0_{11} &= 2\mu E^0_{11}+\lambda(E^0_{11}+E^0_{22}) \equiv \frac{2\mu}{\kappa-1}(E^0_{11}+E^0_{22})+\mu(E^0_{11}-E^0_{22}),\\ P^0_{22} &= 2\mu E^0_{22}+\lambda(E^0_{11}+E^0_{22}) \equiv \frac{2\mu}{\kappa-1}(E^0_{11}+E^0_{22})-\mu(E^0_{11}-E^0_{22}),\end{aligned}\right\} \tag{11}$$

on writing $\lambda = 4\mu\nu/(\kappa-1)$. Accordingly, substituting into (10) and eliminating $\bar{y}, d\bar{y}$ by virtue of (9), we are enabled to set up the relevant contour integrals (7). These may be evaluated explicitly (6) as functions of the pole x. In the case of an exterior pole, i.e. for a point within the matrix,

$$\left.\begin{aligned} \phi'(x) &= \frac{2\mu}{\kappa+1}\frac{ab}{c^2}\left[1-\frac{x}{(x^2-c^2)^{\frac{1}{2}}}\right](E^0_{11}-E^0_{22}),\\ \psi''(x) &= -\frac{2\mu}{\kappa+1}\frac{2ab}{c^2}\left[1-\frac{x}{(x^2-c^2)^{\frac{1}{2}}}\right](E^0_{11}+E^0_{22})\\ &\quad -\frac{2\mu}{\kappa+1}\frac{ab(a^2+b^2)}{c^4}\left[2\left(1-\frac{x}{(x^2-c^2)^{\frac{1}{2}}}\right)+\frac{c^2x}{(x^2-c^2)^{\frac{3}{2}}}\right](E^0_{11}-E^0_{22}).\end{aligned}\right\} \tag{12}$$

Expressions (12) are most conveniently handled by working with confocal elliptic coordinates ξ, η defined by the transformation

$$x \equiv x_1+\iota x_2 = c\cosh(\xi+\iota\eta) \quad (c^2=a^2-b^2).$$

The appropriate stress components are related to p_{ij} by (6), where θ now denotes the angle between the x_1-axis and the ellipse-normal at ξ, η (in the direction of ξ increasing), it being noted that $e^{2\iota\theta} = \sinh(\xi+\iota\eta)/\sinh(\xi-\iota\eta)$. Substituting from (12) into (3), (4) and eliminating Cartesian coordinates

$$\left.\begin{aligned} p_{\xi\xi}+p_{\eta\eta} &= \frac{8\mu}{\kappa+1}\frac{ab}{c^2}\left(1-\frac{\sinh 2\xi}{\cosh 2\xi-\cos 2\eta}\right)(E^0_{11}-E^0_{22}),\\ p_{\eta\eta}-p_{\xi\xi}+2\iota p_{\xi\eta} &= \frac{8\mu}{\kappa+1}\frac{ab}{c^2}(AE^0_{11}+BE^0_{22})/(\cosh 2\xi-\cos 2\eta),\end{aligned}\right\} \tag{13}$$

where
$$A = \left(\cosh 2\xi-\frac{a^2+b^2}{c^2}\right)\coth(\xi+\iota\eta)-\sinh 2\xi+\frac{2a^2}{c^2}[1-e^{-2(\xi+\iota\eta)}],$$

$$B = -\left(\cosh 2\xi-\frac{a^2+b^2}{c^2}\right)\coth(\xi+\iota\eta)+\sinh 2\xi-\frac{2b^2}{c^2}[1-e^{-2(\xi+\iota\eta)}].$$

Along the inclusion boundary, ξ has a value ξ_0 defined by $\cosh 2\xi_0 = (a^2+b^2)/c^2$, thereby simplifying A, B and hence the expressions for the boundary stresses $p^b_{\xi\xi}, p^b_{\eta\eta}, p^b_{\xi\eta}$. However, more convenient formulae for these will be presented below (see Appendix). Another simplification appears when $E^0_{11}=E^0_{22}=\frac{1}{2}E^0$, the hydrostatic pressure vanishes, and $AE^0_{11}+BE^0_{22} = E^0[1-e^{-2(\xi+\iota\eta)}]$. Finally, letting $\xi\to\infty$ shows, as expected, that the individual stress components fall off as $1/|x|^2$ at a large distance $|x|$ from the inclusion centre.

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

In the case of an interior pole, i.e. for a point within the inclusion, the functions ϕ', ψ'' turn out to be constants and yield the simple results

$$\left.\begin{aligned} P_{11}+P_{22} &= -\frac{4\mu}{\kappa+1}\left[(E^0_{11}+E^0_{22})+(E^0_{11}-E^0_{22})\frac{a-b}{a+b}\right], \\ P_{22}-P_{11}+2\iota P_{12} &= \frac{4\mu}{\kappa+1}\left[(E^0_{11}+E^0_{22})\frac{a-b}{a+b}+(E^0_{11}-E^0_{22})\frac{a^2+b^2}{(a+b)^2}\right]. \end{aligned}\right\} \tag{14}$$

The stresses are seen to be uniform, and $P_{12} = 0$, as expected on general grounds. On transforming to confocal elliptic components

$$\left.\begin{aligned} P_{\xi\xi} &= \tfrac{1}{2}(P_{11}+P_{22})-\tfrac{1}{2}(P_{22}-P_{11})\cos 2\theta, \\ P_{\eta\eta} &= \tfrac{1}{2}(P_{11}+P_{22})+\tfrac{1}{2}(P_{22}-P_{11})\cos 2\theta, \\ P_{\xi\eta} &= \tfrac{1}{2}(P_{22}-P_{11})\sin 2\theta, \end{aligned}\right\} \tag{15}$$

it may be proved that

$$p^b_{\xi\eta} = P_{\xi\eta}, \quad p^b_{\xi\xi} = P_{\xi\xi}, \quad p^b_{\eta\eta} = P_{\eta\eta}+\frac{4\mu}{\kappa+1}[(E^0_{11}+E^0_{22})-(E^0_{11}-E^0_{22})\cos 2\theta] \tag{16}$$

along the boundary. Equations (16) provide, in effect, formulae for $p^b_{\xi\xi}, \ldots$ that involve θ as a coordinate instead of η. These formulae are particularly useful in examining stress behaviour round an elongated ellipse, i.e. $a/b \geqslant 10$: we readily find

$$p^b_{\xi\xi} \approx -\frac{4\mu}{\kappa+1}E^0_{11}(1+\cos 2\theta), \quad p^b_{\eta\eta} \simeq \frac{4\mu}{\kappa+1}E^0_{22}(1+\cos 2\theta), \quad p^b_{\xi\eta} \simeq \frac{4\mu}{\kappa+1}E^0_{11}\sin 2\theta \tag{17}$$

on neglecting terms in (16) of order b/a compared with 1. The components $p^b_{\xi\xi}$, $p^b_{\eta\eta}$ attain their maxima at $\theta = 0$, i.e. at the ends of the major axis; $p^b_{\xi\eta}$ attains its maximum at $\theta = \frac{1}{4}\pi$, which for $a/b \geqslant 10$ defines points very near the ends of the major axis.

The net displacement field in the inclusion has components

$$U^c_1 = E^c_{11}x_1, \quad U^c_2 = E^c_{22}x_2 \tag{18}$$

where

$$\left.\begin{aligned} (\kappa+1)E^c_{11} &= \frac{2b}{a+b}(E^0_{11}+E^0_{22})+\frac{b}{a+b}\left(\kappa+\frac{a-b}{a+b}\right)(E^0_{11}-E^0_{22}), \\ (\kappa+1)E^c_{22} &= \frac{2a}{a+b}(E^0_{11}+E^0_{22})-\frac{a}{a+b}\left(\kappa-\frac{a-b}{a+b}\right)(E^0_{11}-E^0_{22}). \end{aligned}\right\} \tag{19}$$

To the first order, therefore, the equilibrium shape of the boundary is an ellipse of semi-axes $a(1+E^c_{11})$, $b(1+E^c_{22})$ similarly situated to the initial ellipse a, b and the free ellipse $a(1+E^0_{11})$, $b(1+E^0_{22})$. Some interesting features emerge on examining the ratios E^c_{11}/E^0_{11}, E^c_{22}/E^0_{22} in various particular cases. Taking $E^0_{11} = E^0_{22}$, and $\nu = \frac{1}{3}$, they have been calculated for a series of axial ratios as shown in the following table and as depicted in Figs. 1. We see that the major axis of the free ellipse suffers a steadily increasing relative compression when a/b increases until, eventually, it is forced completely back to the initial length. The minor axis at first suffers compression, but

a/b	1	2	3	10	∞
E^c_{11}/E^0_{11}	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{22}$	0
E^c_{22}/E^0_{22}	$\frac{3}{4}$	1	$\frac{9}{8}$	$\frac{15}{11}$	$\frac{3}{2}$

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

this becomes entirely eliminated at $a/b = 2$; thereafter, the minor axis actually extends beyond its free length, though not exceeding a definite limiting value. Intuitively this surprising effect appears as a direct result of the intense compression suffered by the major axis as the ellipse becomes more elongated.

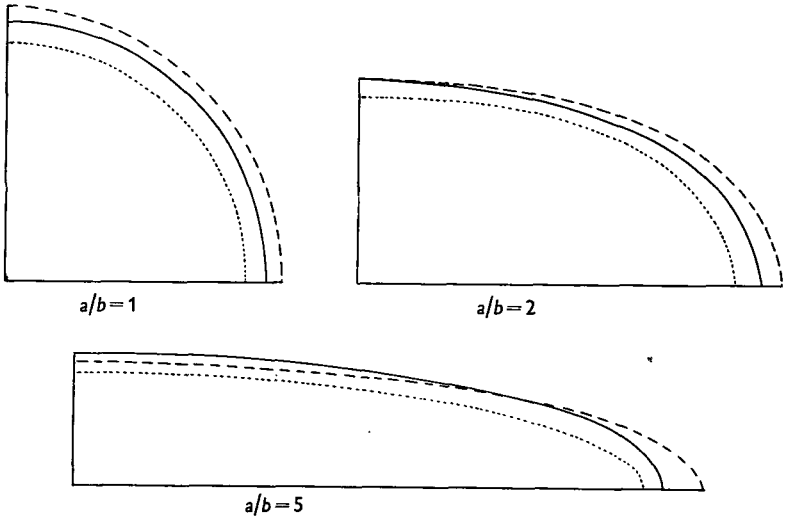


Fig. 1. Schematic illustration of relation between initial ellipse (dotted line), free ellipse (broken line) and equilibrium ellipse (full line), at different axial ratios.

As regards the matrix displacement field, expressions (12) integrate without difficulty to yield the functions ϕ, ψ' that must be introduced into (5).

Most of the results of this and the following sections may be readily adapted to conditions of generalized plane stress.

The elliptic inclusion: shear strains. For an elliptic inclusion tending to undergo a pure shear deformation E^0_{12}, i.e. in which any point x_1, x_2 undergoes displacement components $U^0_1 = E^0_{12}x_2$, $U^0_2 = E^0_{12}x_1$ parallel to the semi-axes a, b we have

$$P^0_{11} = P^0_{22} = 0, \quad P^0_{12} = 2\mu E^0_{12}. \tag{20}$$

Proceeding as in the last section, we find that

$$\left.\begin{aligned} p_{\xi\xi} + p_{\eta\eta} &= -\frac{8\mu E^0_{12}}{\kappa+1}\frac{2ab}{c^2}\frac{\sin 2\eta}{\cosh 2\xi - \cos 2\eta}, \\ p_{\eta\eta} - p_{\xi\xi} + 2\iota p_{\xi\eta} &= \iota\frac{8\mu E^0_{12}}{\kappa+1}\frac{2ab}{c^2}\frac{A+\iota B}{\cosh 2\xi - \cos 2\eta}, \end{aligned}\right\} \tag{21}$$

where $A + \iota B = \left(\cosh 2\xi - \dfrac{a^2+b^2}{c^2}\right)\coth(\xi+\iota\eta) - \sinh 2\xi + \dfrac{a^2+b^2}{c^2}[1 - e^{-2(\xi+\iota\eta)}]$.

The inclusion stresses turn out to be

$$P_{11} = P_{22} = 0, \quad P_{12} = -\frac{8\mu E^0_{12}}{\kappa+1}\frac{ab}{(a+b)^2}, \tag{22}$$

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

and, in confocal elliptic components,

$$P_{\xi\xi} = -P_{\eta\eta} = -\frac{8\mu E_{12}^0}{\kappa+1}\frac{ab}{(a+b)^2}\sin 2\theta, \quad P_{\xi\eta} = -\frac{8\mu E_{12}^0}{\kappa+1}\frac{ab}{(a+b)^2}\cos 2\theta. \tag{23}$$

It may be proved that

$$p_{\xi\eta}^b = P_{\xi\eta}, \quad p_{\xi\xi}^b = P_{\xi\xi}, \quad p_{\eta\eta}^b = P_{\eta\eta} - \frac{8\mu E_{12}^0}{\kappa+1}\sin 2\theta \tag{24}$$

along the boundary. For an elongated ellipse, the stresses attain their maxima at or near the ends of the major axis; however, in contrast to (17), $p_{\eta\eta}^b$ dominates the situation since $p_{\xi\eta}^b, p_{\xi\xi}^b$ are now only of order b/a.

The net displacement field in the inclusion has components

$$U_1^c = (E_{12}^c + \omega)\, x_2, \quad U_2^c = (E_{12}^c - \omega)\, x_1 \tag{25}$$

where

$$E_{12}^c = E_{12}^0\left[1 - \frac{4ab}{(\kappa+1)\,(a+b)^2}\right], \quad \omega = \frac{a-b}{a+b}E_{12}^0. \tag{26}$$

It is clear that the equilibrium shape of the boundary, to the first order, is an ellipse, but nothing very useful can be stated in general about its dimensions or orientation. Two particular limiting cases are of interest. When $a/b = 1$, the equilibrium ellipse has semi-axes $a(1 \pm E_{12}^c)$ oriented at an angle $\frac{1}{4}\pi$ relative to the coordinate axes chosen in setting up (20). When $a/b \to \infty$, the equilibrium ellipse exhibits no first-order deviation from the initial ellipse, apart from a rigid-body rotation of amount $\omega = E_{12}^0$.

The matrix displacement field is sufficiently defined by the complex functions

$$\phi(x) = \iota\frac{\mu E_{12}^0}{\kappa+1}\frac{4ab}{c^2}[x - (x^2-c^2)^{\frac{1}{2}}],$$

$$\psi'(x) = \iota\frac{\mu E_{12}^0}{\kappa+1}\frac{4ab(a^2+b^2)}{c^4}\left[\frac{c^2}{(x^2-c^2)^{\frac{1}{2}}} - 2(x-(x^2-c^2)^{\frac{1}{2}})\right],$$

to be introduced into (5).

Elliptic cavities and inhomogeneities. We now extend the analysis to embrace an elliptic inhomogeneity, of elastic constants μ_1, λ_1, different from those of the matrix. If such an inhomogeneity tends to undergo a uniform free deformation Ω_{ij}^0, what is its net deformation E_{ij}^c, and the accompanying equilibrium elastic state of the system? Following Eshelby, we proceed by introducing an equivalent inclusion. This is a hypothetical inclusion which assumes the same E_{ij}^c as does the inhomogeneity, which therefore exerts the same actions on the matrix, and which is therefore subject to the same equilibrium stresses, thus enabling us to write

$$P_{ij} \equiv 2\mu(E_{ij}^c - E_{ij}^0) + \lambda(E^c - E^0)\,\delta_{ij} = 2\mu_1(E_{ij}^c - \Omega_{ij}^0) + \lambda_1(E^c - \Omega^0)\,\delta_{ij}. \tag{27}$$

The second equality defines three linear equations for the three unknown inclusion parameters E_{ij}^0, bearing in mind that E_{ij}^c may be eliminated by virtue of (19) and (26). With E_{ij}^0 known, the problem has been reduced to a standard inclusion problem and hence solved.

How does an elliptic inhomogeneity perturb an otherwise uniform stress field P_{ij}^A? Regard the matrix and inhomogeneity as two separate interacting systems, into each of which P_{ij}^A is introduced. If P_{ij}^A generates an elastic deformation E_{ij}^A in the former, it

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

evidently generates a different, though uniform, deformation E^B_{ij} in the latter. Accordingly, the inhomogeneity tends to undergo a uniform deformation $E^B_{ij} - E^A_{ij}$ relative to the matrix, and we have reduced this problem to the last. Putting $\Omega^0_{ij} = E^B_{ij} - E^A_{ij}$ in (27), and noting that $2\mu_1 E^B_{ij} + \lambda_1 E^B \delta_{ij} = P^A_{ij}$, we arrive at equations

$$P_{ij} \equiv 2\mu(E^c_{ij} - E^0_{ij}) + \lambda(E^c - E^0)\,\delta_{ij} = 2\mu_1(E^c_{ij} + E^A_{ij}) + \lambda_1(E^c + E^A)\,\delta_{ij} - P^A_{ij} \tag{28}$$

for E^0_{ij}. With E^0_{ij} known, the inhomogeneity fields P_{ij}, E^c_{ij} are known; these exist relative to P^A_{ij}, E^A_{ij} and are hence the perturbations required. Accompanying perturbation fields exist in the matrix, these of course vanishing at infinity. It may be remarked that Eshelby's treatment of the problem does not bring out the close connexion demonstrated here between (27) and (28).

Putting $\mu_1 = \lambda_1 = 0$ in (26) yields the expected cavity equations $P_{ij} = -P^A_{ij}$, i.e. implying that no stresses appear within the relevant domain. These equations are solved directly for E^0_{ij} by inverting relations (14) and (22)

$$\left.\begin{aligned} E^0_{22} \pm E^0_{11} &= \frac{\kappa+1}{4\mu}\left[\frac{a^2 \pm b^2}{ab}(P^A_{11} + P^A_{22}) + \frac{(a+b)(a \mp b)}{ab}(P^A_{22} - P^A_{11})\right], \\ E^0_{12} &= \frac{\kappa+1}{8\mu}\frac{(a+b)^2}{ab} P^A_{12}. \end{aligned}\right\} \tag{29}$$

Bearing in mind (16) and (24), it follows that

$$p^b_{\eta\eta} + P^A_{\eta\eta} = \frac{4\mu}{\kappa+1}[(E^0_{11} + E^0_{22}) - (E^0_{11} - E^0_{22})\cos 2\theta - 2E^0_{12}\sin 2\theta] \tag{30}$$

for the boundary hoop stress, with E^0_{ij} given by (29). This advantageous expression does not seem to have emerged in previous solutions of the problem. For an elongated ellipse we find

$$p^b_{\eta\eta} \approx \frac{a}{b}[P^A_{22}(1 + \cos 2\theta) - P^A_{12}\sin 2\theta], \tag{31}$$

on retaining only the terms in (30) that are of order a/b compared with 1. The stress magnification effect and its independence of P^A_{11} are very clearly revealed in (31).

Putting $\mu_1 = \lambda_1 = \infty$ in (28) yields the expected rigid inhomogeneity equations $E^c_{ij} = -E^A_{ij}$, i.e. implying that no strains appear within the relevant domain. These equations are solved directly for E^0_{ij} by inverting relations (19) and (26). We then calculate the inhomogeneity stresses from

$$P_{ij} + P^A_{ij} = -(2\mu E^0_{ij} + \lambda E^0 \delta_{ij}),$$

and the jump in boundary hoop stress, $\Delta P_{\eta\eta}$, from the right-hand side of (30). An immediate deduction from (26) is

$$E^0_{12} = -E^A_{12}\Big/\left[1 - \frac{4ab}{(\kappa+1)(a+b)^2}\right] = -\frac{P^A_{12}}{2\mu}\Big/\left[1 - \frac{4ab}{(\kappa+1)(a+b)^2}\right],$$

giving

$$P_{12} + P^A_{12} = P^A_{12}\Big/\left[1 - \frac{4ab}{(\kappa+1)(a+b)^2}\right],$$

$$\omega = -\frac{a-b}{a+b}\frac{P^A_{12}}{2\mu}\Big/\left[1 - \frac{4ab}{(\kappa+1)(a+b)^2}\right].$$

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

It is apparent that the component P_{12}^A produces no stress magnification effects at an elongated rigid inhomogeneity. Solving equations (19) for this case, i.e. retaining only terms of order a/b in the expression for E_{ij}^0, we find

$$P_{11} \approx \mu \frac{\kappa+1}{\kappa} \frac{a}{b} E_{11}^A, \quad P_{22} \approx 0; \qquad p_{\eta\eta}^b = P_{\eta\eta} + \Delta P_{\eta\eta} \approx \mu \frac{3-\kappa}{2\kappa} \frac{a}{b} E_{11}^A (1+\cos 2\theta). \quad (32)$$

Significant magnification, depending entirely on the applied strain component E_{11}^A, is thus revealed. The pattern differs completely from that existing at a cavity, as has previously been noted by Taylor (7) and Jeffreys (8). Particularly critical regions are at the ends of the major axis, for $p_{\eta\eta}^b$ reaches its maximum there and combines with $p_{\xi\xi}^b \equiv P_{11}$.

Our analysis has an interesting application to phase transformations of martensite type that take place in steels. Usually the transformed region has the shape of a thin lenticular plate, of approximately elliptic cross-section normal to its plane, and may be regarded as approximately rigid compared with the surrounding austenitic matrix. As a well-marked feature of such phenomena, new plates nearly always form at the tips of already existing plates when temperature or loading are suitably altered. This behaviour becomes readily understandable on bearing in mind that regions of stress concentration, such as must exist at the tips of a plate, constitute preferred sites for the nucleation of physical effects. It may be remarked that the formation of a martensite plate could produce stress magnification even in the absence of an applied field: putting $\mu_1 = \lambda_1 = \infty$ in (27) yields the solution (32), with E_{11}^A replaced by $-\Omega_{11}^0$. It is probably to avoid this contingency that in practice $\Omega_{11}^0 = 0$, the main dilatation component being Ω_{22}^0.

A general analysis of equations (28) shows that no choice of μ_1, λ_1 apart from $\mu_1 = \lambda_1 = 0$ and $\mu_1 = \lambda_1 = \infty$ can lead to stress magnification effects at an elongated boundary.

Strain–energy considerations. The strain–energy density of a two-dimensional elastic system is provided by

$$\begin{aligned} W = \tfrac{1}{2} p_{ij} e_{ij} &= \tfrac{1}{2}(p_{11}e_{11} + p_{22}e_{22} + 2p_{12}e_{12}) \\ &= \tfrac{1}{4}[(p_{11}+p_{22})(e_{11}+e_{22}) + (p_{22}-p_{11})(e_{22}-e_{11}) + 4p_{12}e_{12}] \\ &= \tfrac{1}{4}[(p_{11}+p_{22})(e_{11}+e_{22}) + \mathscr{R}\{(p_{22}-p_{11}+2\iota p_{12})\,\overline{(e_{22}-e_{11}+2\iota e_{12})}\}], \end{aligned} \quad (33)$$

this last expression being perfectly adapted to the complex variable formalism.

For the strain energy of an elliptic inclusion, per unit height, we have at once

$$W_i = \tfrac{1}{2} P_{ij}(E_{ij}^c - E_{ij}^0) A \quad (A = \pi ab). \quad (34)$$

As regards the matrix, it may be readily proved that

$$W_m = -\tfrac{1}{2} P_{ij} E_{ij}^c A \quad (35)$$

on computing the work done by its internal boundary tractions $-T_i$ in producing boundary displacements U_i^c. Adding (34) and (35) yields

$$W = W_i + W_m = -\tfrac{1}{2} P_{ij} E_{ij}^0 A \quad (36)$$

for the total energy of the system per unit height. All these expressions are preferably cast into the form (33).

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

In the principal strains case, we find

$$W = \frac{2\mu}{(\kappa+1)(a+b)^2}[a(2a+b)(E^0_{11})^2 + b(2b+a)(E^0_{22})^2 + 2abE^0_{11}E^0_{22}]\pi ab,$$

$$W_i = \frac{4\mu}{(\kappa+1)^2(a+b)^2}[A(E^0_{11})^2 + B(E^0_{22})^2 + 2CE^0_{11}E^0_{22}]\pi ab,$$

where

$$A = (\kappa+1)a^2 + \frac{2a^4}{(a+b)^2}, \quad B = (\kappa-1)b^2 + \frac{2b^4}{(a+b)^2}, \quad C = (\kappa-1)ab - \frac{2a^2b^2}{a^2+b^2}.$$

An interesting theorem can now be proved: for a prescribed free dilatation $E^0_{11} + E^0_{22} = E^0$, W attains its minimum value when $E^0_{11}/E^0_{22} = b^2/a^2$, at which ratio

$$P_{11} = P_{22} = -\frac{4\mu}{\kappa+1}\frac{ab}{a^2+b^2}E^0.$$

This accords with a very general principle of Kröner (9). In the shear strain case, we find

$$W = \frac{8\mu}{\kappa+1}\frac{ab}{(a+b)^2}(E^0_{12})^2\pi ab, \quad W_i = \frac{32\mu}{(\kappa+1)^2}\frac{a^2b^2}{(a+b)^4}(E^0_{12})^2\pi ab.$$

Again an interesting theorem can be proved: for an ellipse of given area πab, W attains its maximum when $a/b = 1$, and steadily decreases as a/b increases. This accords with a well-known conclusion of Nabarro (10).

An elliptic inhomogeneity tending to undergo a free deformation Ω^0_{ij}, and which develops stresses P_{ij}, generates a total strain energy per unit height

$$W = -\tfrac{1}{2}P_{ij}\Omega^0_{ij}A. \tag{37}$$

This result is exactly analogous to (36), and may be proved in the same way: replace E^0_{ij} by Ω^0_{ij} in (34) and add to (35).

APPENDIX

From $e^{2i\theta} = \sinh(\xi+i\eta)/\sinh(\xi-i\eta)$, we deduce the following relations for eliminating η in favour of θ:

$$\sinh 2\xi/(\cosh 2\xi - \cos 2\eta) = (\cosh 2\xi + \cos 2\theta)/\sinh 2\xi,$$

$$\sin 2\eta/(\cosh 2\xi - \cos 2\eta) = \sin 2\theta/\sinh 2\xi,$$

$$[1 - e^{-2(\xi+i\eta)}]/(\cosh 2\xi - \cos 2\eta) = (1 - e^{-2\xi}e^{2\theta i})/\sinh 2\xi,$$

$$\coth(\xi+i\eta) = \frac{\sinh 2\xi - i\sin 2\eta}{\cosh 2\xi - \cos 2\eta} = (\cosh 2\xi + e^{-2\theta i})/\sinh 2\xi,$$

$$(\cosh 2\xi - \cos 2\eta)^{-1} = (\cosh 2\xi + \cos 2\theta)/\sinh^2 2\xi.$$

Along the boundary,

$$\cosh 2\xi_0 = \frac{a^2+b^2}{c^2}, \quad \sinh 2\xi_0 = \frac{2ab}{c^2}, \quad e^{-2\xi_0} = \frac{a-b}{a+b}.$$

Introducing these results into formulae (13), (21) enables the important boundary identifications (16), (24) to be proved without any difficulty.

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press

Thanks are due to Professor F. R. N. Nabarro and Dr J. D. Eshelby for helpful comments on the manuscript, and to Dr A. N. Gordon for useful discussions. One of us (R.D.B.) was in receipt of a maintenance grant from the Government of India during the course of the work.

REFERENCES

(1) FRENKEL, J. *Kinetic theory of liquids* (Oxford, 1946).
(2) MOTT, N. F. and NABARRO, F. R. N. *Proc. Phys. Soc.* 52 (1940), 90.
(3) ESHELBY, J. D. *Proc. Roy. Soc.* A, 241 (1957), 376; 252 (1959), 561.
(4) MUSHKELISHVILI, N. I. *Some basic problems of the mathematical theory of elasticity* (Groningen, 1953).
(5) GREEN, A. E. and ZERNA, W. *Theoretical elasticity* (Oxford, 1954).
(6) BHARGAVA, R. D. Ph.D. Thesis (London, 1959).
(7) TAYLOR, G. I. *Proc. Roy. Soc.* A, 145 (1934), 1.
(8) JEFFREYS, H. *Proc. Roy. Soc.* A, 252 (1959), 431.
(9) KRÖNER, E. *Acta Met.* 2, (1954), 301.
(10) NABARRO, F. R. N. *Proc. Roy. Soc.* A, 175 (1940), 519.

IMPERIAL COLLEGE
LONDON

https://doi.org/10.1017/S0305004100035702 Published online by Cambridge University Press