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A LOWER BOUND FOR THE GONALITY CONJECTURE

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Abstract. For every integer $k \geqslant 3$ we construct a k-gonal curve C along with a very ample divisor of degree $2g + k - 1$ (where g is the genus of C) to which the vanishing statement from the Green–Lazarsfeld gonality conjecture does not apply.

The gonality conjecture due to Green and Lazarsfeld [5] states that for C a smooth complex projective curve of genus $g \geqslant 1$ and gonality $k \geqslant 2$, and L a globally generated divisor on C of sufficiently large degree, one has the following vanishing criterion in Koszul cohomology:

$$K_{i,1}(C, L) \neq 0 \Leftrightarrow 1 \leqslant i \leqslant h^0(C, L) - k - 1. \tag{1}$$

This conjecture was proved in 2015 by Ein and Lazarsfeld [2], who moreover provided the sufficient lower bound $\deg L \geqslant g^3$. In the meantime this was improved to $\deg L \geqslant 4g - 3$ by Rathmann [6].

It is likely that this bound can be improved further, although Green and Lazarsfeld already noted [5, p. 86] that one at least needs

$$\deg L \geqslant 2g + k - 1 \tag{2}$$

because of the non-vanishing of $K_{g-1,1}(K_C + D)$ for any divisor D of rank 1 and degree k on C. Very recently, Farkas and Kemeny [3] showed that if C is sufficiently generic inside the moduli space of k-gonal curves of genus g, then the bound (2) is sufficient for the vanishing criterion (1) to hold. Moreover, they conjectured that this should be true for *all* curves whose genus is large enough when compared to the gonality, a statement which they called the effective gonality conjecture. Results due to Green [4, Theorem 3.c.1] respectively Teixidor i Bigas [8, Proposition 3.8] imply that this is indeed the case for trigonal respectively tetragonal curves of genus $g > 3$ respectively $g > 6$.

In this note we aim for an improved delimitation of the foregoing considerations, through the following result.

THEOREM. *For each $k \geqslant 3$ there exists a curve C of genus $g = k(k-1)/2$ along with a very ample divisor L of degree $2g + k - 1$ such that*

$$K_{h^0(C,L)-k,1}(C, L) \neq 0. \tag{3}$$

In particular, the bound (2) is not sufficient for the gonality conjecture to apply.

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The main conclusions to be drawn are that on the one hand, at most, one can hope to improve Rathmann's bound to $\deg L \geqslant 2g + k$, and that on the other hand $g > k(k-1)/2$ is a necessary lower bound in the statement of Farkas and Kemeny's effective gonality conjecture. In particular, one observes that the special cases implied by the works of Green and Teixidor i Bigas are sharp.

The construction is very short and explicit. Namely we let C be a smooth projective plane curve of degree $k+1$ and let L be the effective divisor cut out by a curve of degree $k-1$. Note that the gonality of C equals k by [7, Proposition 3.13] and that the degree of L equals $(k-1)(k+1) = 2g + k - 1$, as announced. This automatically entails that L is very ample. Now the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-C) \longrightarrow \mathcal{O}_{\mathbf{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

can be easily converted into

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-2) \longrightarrow \mathcal{O}_{\mathbf{P}^2}(k-1) \longrightarrow \mathcal{O}_C(L) \longrightarrow 0.$$

Taking cohomology and using that H^0 and H^1 vanish in the case of $\mathcal{O}_{\mathbf{P}^2}(-2)$, we end up with an isomorphism

$$H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k-1)) \longrightarrow H^0(C, L),$$

which shows that $h^0(C, L) = \binom{k+1}{2}$ and that the embedding

$$C \overset{|L|}{\hookrightarrow} \mathbf{P}^{\binom{k+1}{2}-1}$$

lands inside the $(k-1)$th Veronese surface. This implies that we have an injection

$$K_{\binom{k+1}{2}-k,1}(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(k-1)) \hookrightarrow K_{\binom{k+1}{2}-k,1}(C, L) = K_{h^0(C,L)-k,1}(C, L).$$

But the former space is non-trivial, from which our theorem follows. One reference for this last fact is [4, Appendix], which can be applied to the decomposition $\mathcal{O}_{\mathbf{P}^2}(k-1) = \mathcal{O}_{\mathbf{P}^2}(k-2) \otimes \mathcal{O}_{\mathbf{P}^2}(1)$. Alternatively, we refer to [1, Theorem 1.8] for a more direct statement.

Remark. We work over $\mathbf{C}$ because for instance [2] does so as well, but the theorem presented above is valid over any algebraically closed field, with the same proof.

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