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103.41 A spatial characterisation of Pascal limaçons

In $\mathbb{R}^2$, let α be a circle of diameter a, A a fixed point on α and b a positive real number. For any point B on α, let P and Q be the points on the straight line AB which are at distance b from B (for $A = B$ take AB to be the tangent to α at A). When B moves along α, P and Q move along a Pascal limaçon. In other words, the limaçon is a conchoid of the circle with respect to its point A (see Figure 1).

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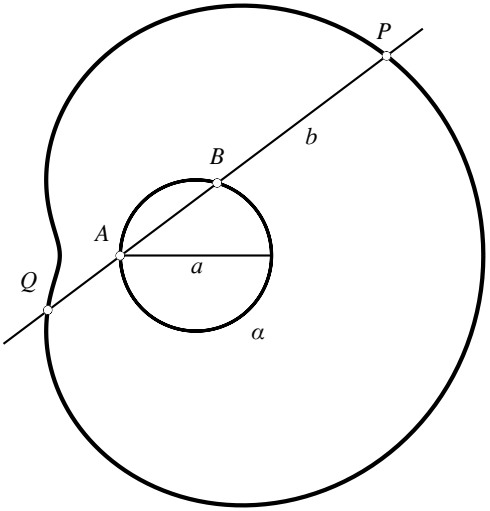


FIGURE 1: A Pascal limaçon as a conchoid

It is usual to put A at the origin $(0, 0)$ and the centre of α at $\left(\frac{a}{2},\ 0\right)$; then the polar equation of the limaçon is

$$\rho = a\cos\theta + b \tag{1}$$

and its cartesian equation is

$$\left(x^2 + y^2 - ax\right)^2 = b^2\left(x^2 + y^2\right). \tag{2}$$

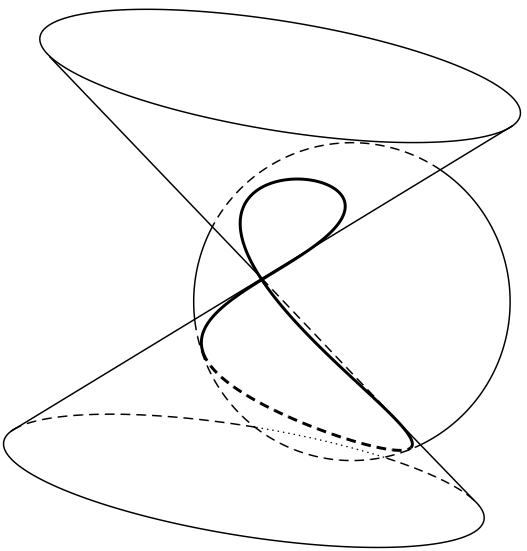


FIGURE 2: A sphere and a cone whose vertex lies on the sphere

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Many other definitions of the limaçons are known (see [1], [2, pp. 148-151], [3, pp. 41-51], [4, pp. 68-69]), but the following one seems to have gone largely unnoticed.

In $\mathbb{R}^3$, let Σ be a sphere, Γ a (double) cone of revolution whose vertex is on the sphere and γ their intersection (see Figure 2). The orthogonal projection of γ on a plane normal to the axis of Γ is a Pascal limaçon, and every Pascal limaçon can be obtained in this way.

We may suppose that the axis of Γ is the z-axis and its vertex the origin $(0, 0, 0)$. We may also take the centre of the sphere to be of the form $(c, 0, -d)$ with $c, d \geqslant 0$ and $c^2 + d^2 = r^2 \neq 0$. We must find the orthogonal projection of γ on the xy-plane.

The equation of Γ is

$$e^2x^2 + e^2y^2 - z^2 = 0 \tag{3}$$

where e is a fixed positive number controlling the angle φ between the axis and a generatrix of Γ: $\varphi = \arctan\left(\frac{1}{e}\right)$. The equation of Σ is

$$(x - c)^2 + y^2 + (z + d)^2 = r^2$$

or

$$x^2 - 2cx + y^2 + z^2 + 2dz = 0 \tag{4}$$

Adding (3) to (4) we get

$$\left(1 + e^2\right)x^2 + \left(1 + e^2\right)y^2 - 2cx + 2dz = 0$$

hence

$$x^2 + y^2 - \frac{2cx}{1 + e^2} = \frac{-2dz}{1 + e^2}$$

then

$$\left(x^2 + y^2 - \frac{2cx}{1 + e^2}\right)^2 = \frac{4d^2z^2}{(1 + e^2)^2}$$

and finally, using (3) again,

$$\left(x^2 + y^2 - \frac{2cx}{1 + e^2}\right)^2 = \frac{4d^2e^2}{(1 + e^2)^2}\left(x^2 + y^2\right)$$

which is the cartesian equation (2) of the limaçon with

$$a = \frac{2c}{1 + e^2} \qquad \text{and} \qquad b = \frac{2de}{1 + e^2}.$$

We have thus proved that the projection of γ on the xy-plane is included in the limaçon. However, this is not enough.

Indeed, consider the orthogonal projection of γ on the plane containing the axis of the cone and the centre of the sphere—here the xz-plane. We obtain it by subtracting from the equation (4) of the sphere the equation (3)

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of the cone written $x^2 + y^2 - \frac{z^2}{e^2} = 0$ to get $-2cx + z^2 + \frac{z^2}{e^2} + 2dz = 0$. Hence this projection is included in a parabola, but clearly it cannot be the whole parabola (see Figures 3 and 4—this was observed in [5, p. 138] for the case $e = 1$).

In fact, in such a problem it is our mind's eye (with the aid, for example, of Figures 3 and 4 or of a 3D-plotting program) which tells us that the projection of γ on the xy-plane is actually the whole limaçon. We can also prove that γ must consist of closed loops in the following way. The sphere divides the cone into regions inside and outside the sphere. To pass from one region to the other, one has to cross the curve of intersection γ. If there were gaps in this curve it would be possible to pass along the cone from the inside of the sphere to the outside without passing through the sphere, which is impossible.

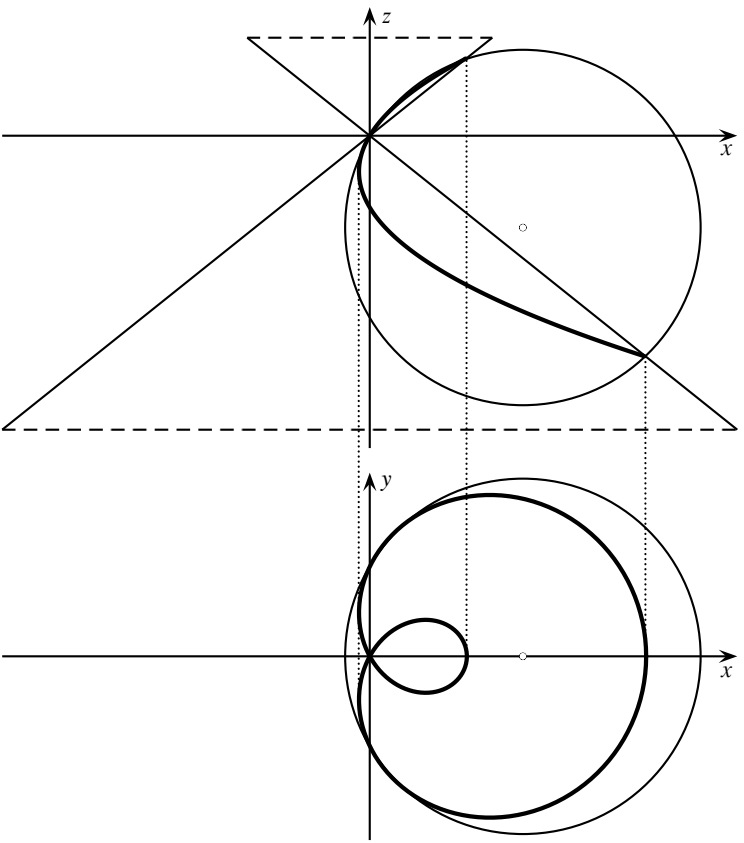


FIGURE 3: The case $c = 2.5, d = 1.5, e = 0.8$

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In Figure 4, and in general when $\frac{c}{d} < e$, observe that the origin, being the projection of the vertex of the cone, is part of the limaçon as an isolated point, in accordance to the cartesian equation (2), but not to the polar equation (1).

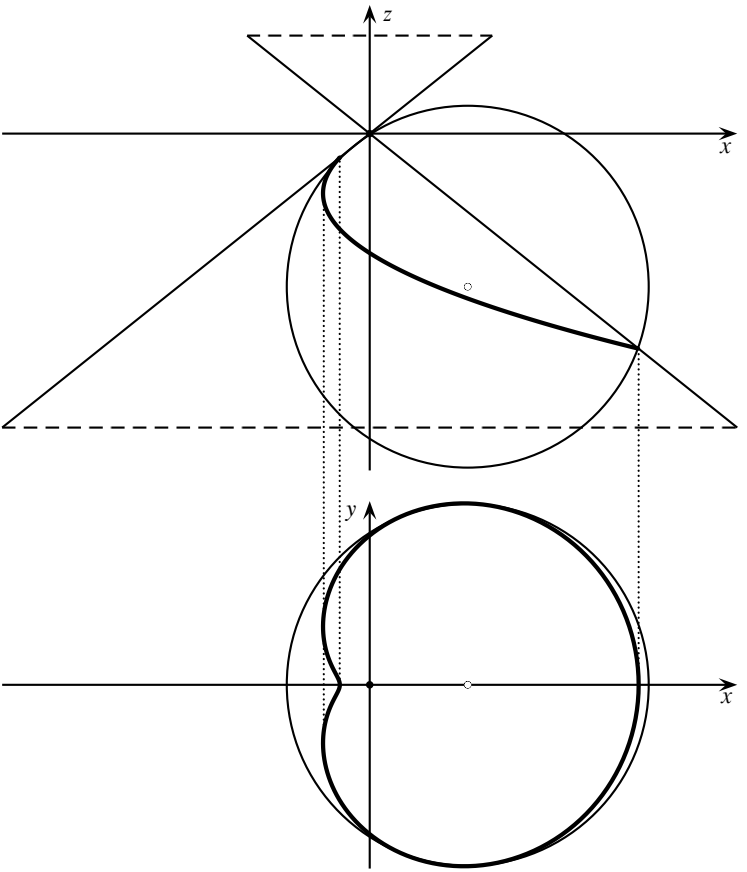


FIGURE 4: The case $c = 1.6, d = 2.5, e = 0.8$

Finally, let us mention two particular cases. When $d = 0$, the centre of the sphere is on the plane normal to the axis of the cone which contains its vertex, the curve γ is symmetric with respect to this plane, and the limaçon reduces to a (double) circle, its equation being here

$$\left(x^2 + y^2 - \frac{2cx}{1 + e^2}\right)^2 = 0 \tag{5}$$

(circle with centre $\left(\frac{c}{1 + e^2}, 0\right)$ and diameter $\frac{2c}{1 + e^2}$). The curve γ is therefore also on the cylinder with equation (5). We deduce that, in the still

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more special case $d = 0$ and $e = 1$, the curve γ is the Viviani curve (see [6]). This shows that the Viviani curve can also be defined as the intersection of a sphere with a cone of revolution.

When $\frac{c}{d} = e$, the sphere is tangent to the cone at its vertex and $a = b$, so the limaçon is a cardioid (see [2, pp. 4-7], [3, pp. 34-43]).

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103.42 The Feuerbach hyperbola revisited

For a scalene triangle ABC, it may be shown by means of areal coordinates that the Nagel point E and the incentre I lie on the circumscribing Feuerbach rectangular hyperbola ([1]).

In this Note we will use a Cartesian framework to locate the excentres I_A, I_B, I_C and also the Bevan point.

The Cartesian framework

Taking the asymptotes of the hyperbola $xy = 1$ for our coordinate axes, we assign the coordinates $A\left(a, \frac{1}{a}\right)$, $B\left(b, \frac{1}{b}\right)$, $C\left(c, \frac{1}{c}\right)$, $E\left(e, \frac{1}{e}\right)$ and $I\left(i, \frac{1}{i}\right)$. The orthocentre $H\left(h, \frac{1}{h}\right)$ also lies on $xy = 1$ subject to the condition $abch = -1$. The parameters e, h, i and two of a, b, c may be taken as positive.

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