Ergod. Th. & Dynam. Sys. (2010), **30**, 1419–1430 doi:10.1017/S0143385709000637
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Printed in the United Kingdom

On sums of indicator functions in dynamical systems

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(*Received* 16 *October* 2008 *and accepted in revised form* 31 *May* 2009)

Abstract. In this paper, we are interested in the limit theorem question for sums of indicator functions. We show that in every invertible ergodic dynamical system, for every increasing sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $a_n \nearrow \infty$ and $a_n/n \to 0$ as $n \to \infty$, there exists a dense G_δ of measurable sets A such that the sequence of the distributions of the partial sums $(1/a_n)\sum_{i=0}^{n-1}(\mathbb{1}_A - \mu(A)) \circ T^i$ is dense in the set of the probability measures on $\mathbb{R}$.

1. *Introduction*

In [**1**], Burton and Denker showed that in every aperiodic dynamical system there exists a process $(f \circ T^i)$ for which the central limit theorem holds and they posed the question how big is the subset of $f \in L^2$ with this property. Clearly, we have to study the space L^2_0 of f with $E(f \mid \mathcal{I}) = 0$. As already observed by Burton and Denker, because the coboundaries are dense in L^2_0, this set is dense. Volný [**7**] proved that for any sequence $a_n \to \infty$, $(a_n/n) \to 0$, there exists a dense G_δ part G of L^2_0 such that for any $f \in G$ and any probability law ν there exists a sequence $n_k \to \infty$ such that $(1/a_{n_k})S_{n_k}(f)$ converges in law to ν. The same result occurs for all spaces L^p, $1 \le p \le \infty$. Liardet and Volný [**6**] obtained the same result for the space of continuous functions for a uniquely ergodic continuous homeomorphism of a metrizable compact space and similar results for spaces of smooth functions for irrational rotations of a circle. As a corollary we get that generically, the rate of convergence in the ergodic theorems (of Birkhoff and of von Neumann) may be arbitrarily slow. This gave a new proof of a result of del Junco and Rosenblatt [**3**]. In the paper of del Junco and Rosenblatt a similar result on the rate of convergence in the ergodic theorems was found for functions $\mathbb{1}_A - \mu(A)$, where the genericity was studied in the space of $A \in \mathcal{A}$ equipped with the (pseudo)metric of the measure of symmetric difference.

In the present paper we shall study the distributional convergence for the functions $\mathbb{1}_A - \mu(A)$. The research was motivated by the study of the invariance principle of the empirical process of strictly stationary sequences $(X_i)_{i\in\mathbb{N}}$ in Dehling *et al* [**2**].

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2. *Result*

Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic Lebesgue probability space and T be an invertible measurable transformation from Ω to Ω. We say that T is measure preserving if for all $A \in \mathcal{A}$, $\mu(T^{-1}A) = \mu(A)$. In the sequel, we will often say that $(\Omega, \mathcal{A}, \mu, T)$ is a dynamical system when $(\Omega, \mathcal{A}, \mu)$ is a non-atomic Lebesgue probability space and T is an invertible measure preserving transformation of Ω.

Further, the transformation T is ergodic if $T^{-1}A = A$ implies that $\mu(A) = 0$ or 1.

On $\mathcal{A}$ we consider the pseudo-metric Θ defined by

$$\Theta(A, B) = \mu(A \triangle B), \quad A, B \in \mathcal{A}.$$

Our main result is the following, where $X_n \xrightarrow[n\to\infty]{\mathcal{D}} \nu$ means that the sequence of real random variables X_n converges in distribution to a real random variable having distribution ν.

THEOREM 1. *Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic Lebesgue probability space and T be an ergodic invertible measure preserving transformation of Ω. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $a_n/n \to 0$ as $n \to \infty$.*

There exists a Θ-dense G_δ of sets $A \in \mathcal{A}$ having the property that for every probability ν on $\mathbb{R}$, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ satisfying

$$\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_A - \mu(A)) \xrightarrow[k\to\infty]{\mathcal{D}} \nu.$$

For completeness' sake, notice that, in the non-ergodic case, to find a set which satisfies the conclusion of Theorem 1, we have to consider the sets A such that $\mathbb{E}(\mathbb{1}_A|\mathcal{I})$ is almost surely constant. The class of such sets is not, in general, dense in $\mathcal{A}$. So, in the non-ergodic case, we cannot expect the result of genericity.

Nevertheless, we can prove the existence of such sets by an explict construction. This is the purpose of the paper by Durieu and Volný [**5**].

3. *Some preliminary results*

Let $\mathcal{M}$ be the set of all probability measures on $\mathbb{R}$ and $\mathcal{M}_0$ be the set of all probability measures on $\mathbb{R}$ which have zero mean. Recall that $\mathcal{M}_0$ is dense in $\mathcal{M}$ for the weak convergence topology. We denote by d the Lévy metric on $\mathcal{M}$. For all μ and ν in $\mathcal{M}$ with distribution functions F and G,

$$d(\mu, \nu) = \inf\{\varepsilon > 0 \mid G(t-\varepsilon) - \varepsilon \le F(t) \le G(t+\varepsilon) + \varepsilon, \forall t \in \mathbb{R}\}.$$

The space $(\mathcal{M}, d)$ is a complete separable metric space and convergence with respect to d is equivalent to weak convergence of distributions (see Dudley [**4**, pp. 394–395]).

If $X : \Omega \longrightarrow \mathbb{R}$ is a random variable, we denote by $\mathcal{L}_\Omega(X)$ the distribution of X on $\mathbb{R}$.

LEMMA 3.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a Lebesgue probability space and ν be a probability on $\mathbb{R}$. Then, there exists a random variable $X : \Omega \longrightarrow \mathbb{R}$, such that*

$$\mathcal{L}_\Omega(X) = \nu.$$

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Proof. It is well known that $(\Omega, \mathcal{A}, \mu)$ is isomorphic to $([0, 1], \mathcal{B}[0, 1], \lambda)$, where λ is the Lebesgue measure on $[0, 1]$. If Q_ν denotes the pseudo-inverse of the distribution function of ν and U is the identity on $[0, 1]$, it is a classical result that $\mathcal{L}_{[0,1]}(Q_\nu(U)) = \nu$. □

Let ν be a probability on $\mathbb{R}$. For $B \in \mathcal{B}(\mathbb{R})$ with $\nu(B) > 0$, ν_B denotes the probability on $\mathbb{R}$ defined by

$$\nu_B(A) = \nu(B)^{-1}\nu(A \cap B).$$

For $x \in \mathbb{R}$, ν_x denotes the probability on $\mathbb{R}$ defined by

$$\nu_x(B) = \nu(xB)$$

where $xB = \{xb \mid b \in B\}$.

Here are some properties of the Lévy metric which will be used in the sequel.

LEMMA 3.2.

(i) *For each probability ν on $\mathbb{R}$, for all Borel sets B,*

$$d(\nu_B, \nu) \leq \nu(\mathbb{R}\backslash B).$$

(ii) *For all probabilities ν and η on $\mathbb{R}$, for all $x \geq 1$,*

$$d(\nu_x, \eta_x) \leq d(\nu, \eta).$$

(iii) *For all probability ν on $\mathbb{R}$, for all mesurable functions f and g from Ω to $\mathbb{R}$,*

$$d(\mathcal{L}_\Omega(f + g), \nu) \leq (\mathcal{L}_\Omega(f), \nu) + d(\mathcal{L}_\Omega(g), \delta_0)$$

where δ_0 is the Dirac measure at 0.

(iv) *For all probability ν on $\mathbb{R}$,*

$$d(\nu, \delta_0) \leq A \quad \text{if and only if } \nu((-\infty, -A)) \leq A \text{ and } \nu((A, \infty)) \leq A.$$

The proof is left to the reader as an exercise.

LEMMA 3.3. *Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be an increasing sequence such that $a_n \nearrow \infty$ as $n \to \infty$. For each probability $\nu \in \mathcal{M}_0$ and $\varepsilon > 0$, there exist $C \geq 1$ and $n_0 \in \mathbb{N}$, and for all $n \geq n_0$, there exists a probability η on $\mathbb{R}$ with support $S \subset [-a_nC, a_nC] \cap \mathbb{Z}$ such that*

$$d(\eta_{a_n}, \nu) \leq \varepsilon$$

and

$$\mathbb{E}(\eta) := \int x \, d\eta = 0.$$

Proof. Let $\nu \in \mathcal{M}_0$ and $\varepsilon > 0$ be fixed and choose $\alpha > 0$ such that $6\alpha \leq \varepsilon$ and $\alpha < \frac{1}{2}$. There exists $C \geq 1$ such that

$$\int_{\mathbb{R}\backslash[-C,C]} |x| \, d\nu(x) \leq \alpha.$$

In particular, $\nu(\mathbb{R}\backslash[-C, C]) \leq \alpha$. Define $\tau = \nu_{[-C,C]}$. By Lemma 3.2,

$$d(\tau, \nu) \leq \nu(\mathbb{R}\backslash[-C, C]) \leq \alpha$$

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and we have

$$|\mathbb{E}(\tau)| \le \nu([-C, C])^{-1}\left|\mathbb{E}(\nu) - \int_{\mathbb{R}\setminus[-C,C]} x\, d\nu(x)\right| \le \frac{\alpha}{1-\alpha}.$$

Now, choose $n_0 \in \mathbb{N}$ such that $1/a_{n_0} < \alpha$ and fix $n \ge n_0$. Then we define the probability η' on $\mathbb{R}$ with support in $\mathbb{Z}$, by $\eta'(\{k\}) := \tau([k/a_n, (k+1)/a_n))$, $k \in \mathbb{Z}$.

We have, for all $t \in \mathbb{R}$,

$$\begin{aligned}\eta'_{a_n}((-\infty, t]) &= \eta'((-\infty, \lfloor ta_n \rfloor]) \\ &= \tau\left(\left(-\infty, \frac{\lfloor ta_n \rfloor + 1}{a_n}\right)\right) \\ &\le \tau\left(\left(-\infty, t + \frac{1}{a_n}\right]\right)\end{aligned}$$

and

$$\tau((-\infty, t]) \le \eta'_{a_n}((-\infty, t]).$$

Thus $d(\eta'_{a_n}, \tau) \le 1/a_n \le \alpha$ and $d(\eta'_{a_n}, \nu) \le 3\alpha$.

So, if $\mathbb{E}(\eta') = 0$, η' verifies the conclusion of the proposition. If it is not the case, we proceed as follows. Observe that

$$a_n\mathbb{E}(\tau) - 1 \le \mathbb{E}(\eta') \le a_n\mathbb{E}(\tau),$$

and thus, since $\alpha < \frac{1}{2}$ and $a_n\alpha > 1$,

$$|\mathbb{E}(\eta')| \le a_n|\mathbb{E}(\tau)| + 1 \le a_n\frac{\alpha}{1-\alpha} + 1 \le 3a_n\alpha.$$

We denote by s the sign of $\mathbb{E}(\eta')$ and we set $p = 1 + (|\mathbb{E}(\eta')|/\lfloor a_nC \rfloor)$.

Now we denote by η the probability on $\mathbb{R}$ with support in $\{-\lfloor a_nC \rfloor, \dots, \lfloor a_nC \rfloor\}$ defined by

$$\eta(\{i\}) = \begin{cases} \dfrac{1}{p}\left(\eta'(\{i\}) + \dfrac{|\mathbb{E}(\eta')|}{\lfloor a_nC \rfloor}\right) & \text{if } i = -s\lfloor a_nC \rfloor, \\ \dfrac{1}{p}\eta'(i) & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}(\eta) = 0$ and by Lemma 3.2(ii),

$$d(\eta'_{a_n}, \eta_{a_n}) \le d(\eta', \eta) \le \frac{|\mathbb{E}(\eta')|}{\lfloor a_nC \rfloor} \le 3\alpha.$$

Therefore $d(\eta_{a_n}, \nu) \le \varepsilon$. □

PROPOSITION 3.4. *Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic dynamical system, $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $a_n/n \to 0$ as $n \to \infty$ and $\varepsilon > 0$. Let $A \in \mathcal{A}$ be a set such that $\mu(A) < 1$ and ν be a probability in $\mathcal{M}_0$. There exists $N \in \mathbb{N}$ such that for any $n \ge N$, there exists a set $B_n \in \mathcal{A}$ such that $\mu(B_n) \le \varepsilon$, $A \cap B_n = \emptyset$ and*

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n}S_n(\mathbb{1}_{B_n} - \mu(B_n))\right), \nu\right) \le \varepsilon.$$

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Proof. Fix $\varepsilon > 0$ and $A \in \mathcal{A}$ such that $\mu(A) < 1$. Let α be a positive constant such that $\alpha \leq \varepsilon/5$ and $\mu(A) + 2\alpha < 1$.

By Lemma 3.3 applied to ν and α we get the constants $C \geq 1$ and $n_0 \geq 1$ for which the conclusion of the lemma holds. Set $\gamma := \alpha/(C+1)$. Let n_1 be an integer such that, for all $n \geq n_1$,

$$2\frac{a_n C + 1}{n} \leq \alpha. \tag{3.1}$$

Applying Birkhoff's ergodic theorem and Egorov's theorem, we get that there exist a set $E \in \mathcal{A}$ of measure greater than $1 - (\gamma/2)$ and an integer $n_2 \geq 1$ such that for all $n \geq n_2$, for all $x \in E$,

$$\left|\frac{1}{n} S_n(\mathbb{1}_A - \mu(A))(x)\right| \leq \alpha. \tag{3.2}$$

We denote by $\overline{n}$ the maximum of n_0, n_1 and n_2 and we choose $N \in \mathbb{N}$ such that

$$\frac{\overline{n}}{N} \leq \alpha.$$

For any $n \geq N$, there exists a Rokhlin tower of height n with base $F \subset E$ and junk set of measure smaller than γ.

Indeed, let G be the base of a Rokhlin tower of height n and of measure greater than $1 - (\gamma/2)$. Because $\mu(\Omega \backslash E) \leq \gamma/2$, there exists an integer $i_0 \in \{0, \ldots, n-1\}$ such that

$$\mu((T^{i_0} G) \cap E) \geq \frac{1}{n}\left(1 - \frac{\gamma}{2} - \frac{\gamma}{2}\right) = \frac{1-\gamma}{n}.$$

If $F = T^{i_0} G \cap E$, then $F \subset E$ and the sets $F, TF, \ldots, T^{n-1}F$ are disjoint. So, F is the base of a Rokhlin tower of height n with a junk set of measure smaller than γ.

From now on, n is fixed. By Lemma 3.3, there exists a centered probability η with support in $S \subset [-a_n C, a_n C] \cap \mathbb{Z}$ such that

$$d(\eta_{a_n}, \nu) \leq \alpha.$$

By Lemma 3.1, there exists a function $h : F \longrightarrow \mathbb{Z}$ such that $\mathcal{L}_F(h) = \eta$. In particular, $\mathbb{E}_F(h) := \int_F h \, d\mu_F = 0$. We set $d = \lfloor a_n C \rfloor + 1$ and

$$\begin{aligned} g : F &\longrightarrow \mathbb{Z} \\ x &\longmapsto h(x) + d. \end{aligned}$$

Note that $1 \leq g \leq 2d$ almost surely and $\mathbb{E}_F(g) = d$. We now set

$$F_i := g^{-1}(\{i\}), \quad i = 1, \ldots, 2d.$$

Note that the F_i depend on ν, α, C and n. Furthermore, $\{F_1, F_2, \ldots, F_{2d}\}$ is a partition of the set F.

By (3.2), for each $x \in F$, the sub-orbit $\{x, Tx, \ldots, T^{\overline{n}-1}x\}$ hits A at most $\overline{n}(\mu(A) + \alpha)$ times (i.e. $\#(\{x, Tx, \ldots, T^{\overline{n}-1}x\} \cap A) \leq \overline{n}(\mu(A) + \alpha)$). Since by (3.1), $2d/\overline{n} \leq \alpha$ and $\mu(A) + 2\alpha < 1$, we can find $2d$ points in this sub-orbit which are not in A.

Then, for each $i = 1, \ldots, 2d$ and for each $x \in F$, we can define the set $b_i(x)$ consisting of the i first points of the sub-orbit $\{x, Tx, \ldots, T^{\overline{n}-1}x\}$ which are not in A.

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We now set, for each $i = 1, \dots, 2d$,

$$B_i = \bigcup_{x \in F_i} b_i(x).$$

Thus the B_i are disjoint measurable sets. For each $i = 1, \dots, 2d$, $B_i \subset \bigcup_{j=0}^{\overline{n}-1} T^j F_i$, $B_i \cap A = \emptyset$, $\mu(B_i) = i\mu(F_i)$ and for any $x \in F_i$,

$$S_{\overline{n}}(\mathbb{1}_{B_i})(x) = i.$$

Finally, we set

$$B = \bigcup_{i=1}^{2d} B_i.$$

We have $B \in \mathcal{A}$ and $A \cap B = \emptyset$.

From the construction of B and (3.1),

$$\mu(B) = \mathbb{E}_F(g)\mu(F) = d\mu(F) \le \frac{a_n C + 1}{n} \le \alpha \le \varepsilon.$$

We define

$$\Omega_k = \bigcup_{i=0}^{n-\overline{n}-1} T^{-i} F_k, \quad k = 1, \dots, 2d,$$

and

$$\overline{\Omega} = \bigcup_{k=1}^{2d} \Omega_k.$$

Since the $T^{-i}F_k$ are disjoint, using (3.1) and the fact that $\gamma \le \alpha$, we have

$$\mu(\overline{\Omega}) = (n - \overline{n})\mu(F) \ge 1 - \gamma - \frac{\overline{n}}{n} \ge 1 - 3\alpha. \tag{3.3}$$

For $x \in \overline{\Omega}$, by construction and by disjointness of the $T^{-i}F_k$, we have

$$S_n(\mathbb{1}_B)(x) = k \quad \text{if and only if } x \in \Omega_k.$$

Therefore, for all $k = 1, \dots, 2d$,

$$\mu_{\overline{\Omega}}(S_n(\mathbb{1}_B) = k) = \mu_{\overline{\Omega}}(\Omega_k) = \mu_F(F_k).$$

Thus $\mathcal{L}_{\overline{\Omega}}(S_n(\mathbb{1}_B)) = \mathcal{L}_F(g)$ and by (3.3) and Lemma 3.2(i),

$$d(\mathcal{L}_\Omega(S_n(\mathbb{1}_B)),\, \mathcal{L}_F(g)) \le 3\alpha.$$

So, by Lemma 3.2(ii),

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n} S_n(\mathbb{1}_B)\right),\, \mathcal{L}_F\left(\frac{g}{a_n}\right)\right) \le 3\alpha$$

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and

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n}(S_n(\mathbb{1}_B)-d)\right),\ \mathcal{L}_F\left(\frac{g-d}{a_n}\right)\right)\le 3\alpha. \tag{3.4}$$

Now observe that

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n}(S_n(\mathbb{1}_B)-d)\right),\ \mathcal{L}_\Omega\left(\frac{1}{a_n}S_n(\mathbb{1}_B-\mu(B))\right)\right)\le \alpha. \tag{3.5}$$

Indeed, since $\mu(B)=d\mu(F)$, we have

$$-\gamma d\le n\mu(B)-d\le 0$$

and then

$$\left|\frac{1}{a_n}S_n(\mathbb{1}_B-\mu(B))-\frac{1}{a_n}(S_n(\mathbb{1}_B)-d)\right|\le\frac{\gamma d}{a_n}\le\alpha.$$

To conclude, using (3.4), (3.5) and the fact that $d(\mathcal{L}_F(h/a_n),\ \nu)\le\alpha$, we get

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n}S_n(\mathbb{1}_B-\mu(B))\right),\ \nu\right)\le 5\alpha\le\varepsilon. \qquad \square$$

PROPOSITION 3.5. *Let $(\Omega,\mathcal{A},\mu,T)$ be an ergodic dynamical system, and $(a_n)_{n\in\mathbb{N}}\subset\mathbb{R}_+$ be an increasing sequence such that $a_n\nearrow\infty$ and $a_n/n\to 0$ as $n\to\infty$ and $\varepsilon>0$. For any set $A\in\mathcal{A}$ such that $\mu(A)<1$, there exists a set $B\in\mathcal{A}$ such that:*

(i) $\mu(A\triangle B)\le\varepsilon$;

(ii) *there exists a sequence $(n_k)_{k\ge1}$ such that for all $k\ge1$,*

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_B-\mu(B))\right),\ \delta_0\right)\le\varepsilon.$$

This proposition will be proved as a corollary of the following lemma.

LEMMA 3.6. *Let $(\Omega,\mathcal{A},\mu,T)$ be an ergodic dynamical system, and $(a_n)_{n\in\mathbb{N}}\subset\mathbb{R}_+$ be an increasing sequence such that $a_n\nearrow\infty$ and $a_n/n\to0$ as $n\to\infty$ and $\varepsilon>0$. For any set $A\in\mathcal{A}$ such that $\mu(A)<1$, for any $N\in\mathbb{N}$, there exist $n\ge N$ and a set $C\in\mathcal{A}$ such that $\mu(A\triangle C)\le\varepsilon$ and*

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n}S_n(\mathbb{1}_C-\mu(C))\right),\ \delta_0\right)\le\varepsilon.$$

Proof. Let $\varepsilon>0$, $A\in\mathcal{A}$ such that $\mu(A)<1$ and $N\in\mathbb{N}$ be fixed and let α be a positive constant such that $5\alpha\le\varepsilon$ and $\mu(A)+2\alpha<1$.

By Birkhoff's ergodic theorem and Egorov's theorem, there exist a set $G\in\mathcal{A}$ with $\mu(G)>1-\alpha$ and an integer M such that for all $k\ge M$ and for all $x\in G$,

$$\left|\frac{1}{k}S_k(\mathbb{1}_A-\mu(A))(x)\right|\le\alpha. \tag{3.6}$$

Furthermore, we can choose M such that $(1-\alpha)/M\le\alpha$.

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There exists an integer $n \geq N$, such that

$$\frac{M}{a_n} \leq \alpha. \tag{3.7}$$

Let F be the base of a Rokhlin tower of height Mn with a junk set of measure smaller than α. For $k = 0, \ldots, M-1$, we define the sets

$$F_k = \bigcup_{i=0}^{n-1} T^{iM+k} F.$$

Since $\mu(G) > 1 - \alpha$, there exists $k_0 \in \{0, \ldots, M-1\}$ such that

$$\mu(F_{k_0} \backslash G) \leq \frac{2\alpha}{M}.$$

Further, $H = T^{k_0} F$ is the base of a Rokhlin tower of height Mn with a junk set J such that $\mu(J) \leq \alpha$.

For $x \in \Omega$, we denote by $s_l(x) = \{x, Tx, \ldots, T^{l-1}x\}$ the sub-orbit of length l which begins at x. The set A will be modified in a way that for each orbit $s_M(x)$, $x \in F_{k_0}$, the average of visits of the set along the orbit is close to the measure $\mu(A)$.

For $x \in H$, we modify the set A along the sub-orbit $s_M(x)$ in the following way. We write $a(x) := A \cap s_M(x)$. There are two situations.

- If $x \in G$, by (3.6), along the sub-orbit $s_M(x)$, the number of visits of the set A belongs to $[M(\mu(A) - \alpha), M(\mu(A) + \alpha)]$ (i.e. $\#a(x) \in [M(\mu(A) - \alpha), M(\mu(A) + \alpha)]$). Further, $M(\mu(A) + \alpha) \leq (1 - \alpha)M$. So by adding to or removing from $a(x)$ at most αM points, we can modify the set $a(x)$ to get a set $a_0(x)$ (with $a_0(x) \subset s_M(x)$) such that along the sub-orbit $s_M(x)$, the number of visits of the set $a_0(x)$ belongs to $[M\mu(A) - 1, M\mu(A) + 1]$. A way to do that is to remove or add the $|\#a(x) - \lfloor M\mu(A) \rfloor|$ first points of $a(x)$ depending on whether $\#a(x) - \lfloor M\mu(A) \rfloor$ is positive or not.
- If $x \notin G$, we can also modify $a(x)$ in order to have that the number of visits of the set $a_0(x)$ along the segment $s_M(x)$ belongs to $[M\mu(A) - 1, M\mu(A) + 1]$. Here, to do that we possibly need to add to or remove from $a(x)$ M points along $s_M(x)$.

To summarize, for each $x \in H$, we can modify the set $A \cap s_M(x)$ to get a set $a_0(x) \subset s_M(x)$ such that $\#a_0(x) \in [M\mu(A) - 1, M\mu(A) + 1]$; if $x \in G$, $\#(a_0(x) \triangle a(x)) \leq \alpha M$, and if $x \notin G$, $\#(a_0(x) \triangle a(x)) \leq M$.

We then have a set $A_0 = \bigcup_{x \in H} a_0(x)$ having the property that for all $x \in H$,

$$|S_M(\mathbb{1}_{A_0} - \mu(A))(x)| \leq 1 \tag{3.8}$$

(the problem of measurability for A_0 is not discussed, but we can see that the modifications can be done in a measurable way).

Now we will do almost the same modifications on sub-orbits of length M starting from $T^M H$.

For each $x \in T^M H$, we modify the set $a(x) = A \cap s_M(x)$ to get a set $a_1(x) \subset s_M(x)$ such that $\#a_1(x) \in [M\mu(A) - 1, M\mu(A) + 1]$.

Notice that for each $x \in T^M H$, considering the number $\#a_0(T^{-M}x)$, we can further define $a_1(x)$ in such a way that $\#(a_0(T^{-M}x) \cup a_1(x)) \in [2M\mu(A) - 1, 2M\mu(A) + 1]$.

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We retain the property that if $x \in T^M H \cap G$, $\#(a_1(x) \triangle a(x)) \leq \alpha M$, and if $x \in T^M H \backslash G$, $\#(a_1(x) \triangle a(x)) \leq M$.

We can define $A_1 = \bigcup_{x \in T^M H} a_1(x)$. Then for all $x \in T^M H$,

$$|S_M(\mathbb{1}_{A_1} - \mu(A))(x)| \leq 1 \tag{3.9}$$

and for all $x \in H$,

$$|S_{2M}(\mathbb{1}_{A_0 \cup A_1} - \mu(A))(x)| \leq 1.$$

Now we do the same modifications for all points of $T^{2M} H$, $T^{3M} H$, $\ldots$, $T^{(n-1)M} H$. Finally, we can get a measurable set $B = \bigcup_{i=0}^{n-1} A_i$ deduced from A and having the property that for all $k \in \{1, \ldots, n\}$ and for each $x \in T^{kM} H$,

$$|S_M(\mathbb{1}_B - \mu(A))(x)| \leq 1. \tag{3.10}$$

Further, for all $x \in H$ and for all $k \in \{1, \ldots, n\}$,

$$|S_{kM}(\mathbb{1}_B - \mu(A))(x)| \leq 1.$$

Note that we did not change the set A on the junk set J, so $A \cap J = B \cap J$. Recall that we did at most αM changes for points of $F_{k_0} \cap G$ and at most M for points of $F_{k_0} \backslash G$, so

$$\begin{aligned} \mu(A \triangle B) &\leq \alpha M \mu(F_{k_0} \cap G) + M \mu(F_{k_0} \backslash G) \\ &\leq 3\alpha. \end{aligned} \tag{3.11}$$

We also have

$$\begin{aligned} |\mu(B) - \mu(A)| &\leq \left| \int_H S_{Mn}(\mathbb{1}_B)(x)\, d\mu(x) + \mu(B \cap J) - \mu(A) Mn \mu(H) - \mu(A)\mu(J) \right| \\ &\leq \int_H |S_{Mn}(\mathbb{1}_B - \mu(A))(x)|\, d\mu(x) + |\mu(B \cap J) - \mu(A)\mu(J)| \\ &\leq \mu(H) + |\mu(B \cap J) - \mu(A)\mu(J)| \\ &\leq \mu(H) + \mu(J). \end{aligned}$$

Then, by changing the set B on only one level of the tower and on the junk set, we can obtain a new set $C \in \mathcal{A}$ such that $\mu(A) = \mu(C)$ and $\mu(B \triangle C) \leq \mu(H) + \mu(J) \leq 2\alpha$.

Thus, by (3.11),

$$\mu(A \triangle C) \leq 5\alpha \leq \varepsilon$$

and we have the following property: for all $x \in H$, for all $k, k' \in \{1, \ldots, n\}$,

$$|S_{kM}(\mathbb{1}_C - \mu(C))(x) - S_{k'M}(\mathbb{1}_C - \mu(C))(x)| \leq 3. \tag{3.12}$$

Let $\tilde{\Omega} = \bigcup_{i=0}^{n(M-1)} T^i H$. We have

$$\mu(\Omega \backslash \tilde{\Omega}) \leq n\mu(H) + \mu(J) \leq 2\alpha$$

and, by (3.12) and (3.7), for all $x \in \tilde{\Omega}$,

$$\frac{1}{a_n} |S_n(\mathbb{1}_C - \mu(C))(x)| \leq \frac{3 + 2M}{a_n} \leq 3\alpha.$$

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From these two inequalities, by Lemma 3.2(iv), we deduce that

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n}S_n(\mathbb{1}_C-\mu(C))\right),\delta_0\right)\le 3\alpha\le\varepsilon. \qquad \square$$

Proof of Proposition 3.5. Let $\varepsilon>0$ and $A\in\mathcal{A}$ such that $\mu(A)<1$ is fixed and choose $\varepsilon_1\le\varepsilon/2$.

By Lemma 3.6, there exist $n_1\in\mathbb{N}$ and a set $C_1\in\mathcal{A}$ such that $\mu(A\triangle C_1)\le\varepsilon_1$ and

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_1}}S_{n_1}(\mathbb{1}_{C_1}-\mu(C_1))\right),\delta_0\right)\le\varepsilon_1.$$

We proceed by induction. After step $k-1$, we choose $\varepsilon_k=\varepsilon_{k-1}/2n_{k-1}$. By application of Lemma 3.6, there exist an integer $n_k\ge n_{k-1}$ and a set $C_k\in\mathcal{A}$ such that $\mu(C_{k-1}\triangle C_k)\le\varepsilon_k$ and

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_{C_k}-\mu(C_k))\right),\delta_0\right)\le\varepsilon_k.$$

Note that for all $i,\ j>0$, $\varepsilon_{i+j}\le\varepsilon_i/2^j$.

Finally, we set $B=\bigcup_{n\ge1}\bigcap_{k\ge n}C_k$. Noticing that $D\triangle(E\cap F)\subset(D\triangle E)\cup(E\triangle F)$ for all sets $D,\ E,\ F$, we get

$$\mu(A\triangle B)\le\sum_{k\ge1}\varepsilon_k\le\varepsilon$$

and for all $k\ge1$, using Lemma 3.2(iii) and (iv),

$$\begin{aligned}
d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_B-\mu(B))\right),\delta_0\right)&\le d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_{C_k}-\mu(C_k))\right),\delta_0\right)\\
&\quad+d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_{B\setminus C_k}-\mu(B\setminus C_k))\right),\delta_0\right)\\
&\quad+d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_{C_k\setminus B}-\mu(C_k\setminus B))\right),\delta_0\right)\\
&\le\varepsilon_k+n_k\mu(B\triangle C_k)\\
&\le\varepsilon_k+n_k\sum_{i\ge k+1}\mu(C_{i-1}\triangle C_i)\\
&\le\varepsilon_k+n_k\sum_{i\ge k+1}\varepsilon_i\\
&\le\varepsilon. \qquad \square
\end{aligned}$$

4. *Proof of Theorem 1*

Let $(\Omega,\mathcal{A},\mu,T)$ be an ergodic dynamical system. Let $(a_n)_{n\in\mathbb{N}}\subset\mathbb{R}_+$ be an increasing sequence such that $a_n\nearrow\infty$ and $a_n/n\to0$ as $n\to\infty$.

Let $(\varepsilon_k)_{k\ge1}$ be a decreasing sequence of positive reals such that ε_k goes to 0 as k goes to ∞.

For each $\nu\in\mathcal{M}_0$ and for each $k\ge1$, we define

$$H_k^\nu=\left\{A\in\mathcal{A}\;\middle|\;\exists n\ge k\text{ such that }d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n}S_n(\mathbb{1}_A-\mu(A))\right),\nu\right)<\varepsilon_k\right\}.$$

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For each $\nu \in \mathcal{M}_0$ and for each $k \geq 1$, it is clear that H_k^ν is an open set in $\mathcal{A}$. We now prove that it is dense.

Assume that ν and k are fixed and let $\varepsilon > 0$ and $A \in \mathcal{A}$. By Proposition 3.5, there exists a set $B \in \mathcal{A}$ such that $\mu(A \triangle B) < \varepsilon/2$ and there exists a sequence $(n_i)_{i\geq 1}$ such that for all $i \geq 1$,

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_i}} S_{n_i}(\mathbb{1}_B - \mu(B))\right), \delta_0\right) \leq \frac{\varepsilon_k}{2}. \tag{4.1}$$

By Proposition 3.4, there exists an integer i_0 such that, for the integer $n = n_{i_0} \geq k$, there exists a set $C \in \mathcal{A}$ satisfying $\mu(C) < \varepsilon/2$, $C \cap B = \emptyset$ and

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n} S_n(\mathbb{1}_C - \mu(C))\right), \nu\right) < \frac{\varepsilon_k}{2}. \tag{4.2}$$

Hence, $\mu((B \cup C)\triangle A) < \varepsilon$ and since B and C are disjoint, by Lemma 3.2(iii) and by (4.1) and (4.2), we get

$$\begin{aligned} &d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n} S_n(\mathbb{1}_{B\cup C} - \mu(B \cup C))\right), \nu\right) \\ &\quad \leq d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n} S_n(\mathbb{1}_C - \mu(C))\right), \nu\right) + d\left(\mathcal{L}_\Omega\left(\frac{1}{a_n} S_n(\mathbb{1}_B - \mu(B))\right), \delta_0\right) \\ &\quad < \varepsilon_k, \end{aligned}$$

i.e. $B \cup C$ belongs to H_k^ν. Therefore H_k^ν is dense in $\mathcal{A}$ for the pseudo-metric Θ.

Let M be a countable subset of $\mathcal{M}_0$ which is dense in $\mathcal{M}$ and set

$$H = \bigcap_{\nu\in M} \bigcap_{k=1}^{\infty} H_k^\nu.$$

By Baire's theorem, H is a dense G_δ (for the metric of the measure of the symmetric difference).

Further, for each $A \in H$, the sequence $(\mathcal{L}_\Omega((1/a_n)S_n(\mathbb{1}_A - \mu(A))))_{n\geq 1}$ is dense in $\mathcal{M}$ for the Lévy metric d.

Indeed, let $A \in H$, $\eta \in \mathcal{M}$ and $\varepsilon > 0$. By density of M, there exist $\nu \in M$ such that

$$d(\nu, \eta) < \frac{\varepsilon}{2}.$$

But $A \in H_k^\nu$ for all $k \geq 1$, then there exists an increasing sequence $(n_k)_{k\geq 1}$ such that

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_A - \mu(A))\right), \nu\right) \leq \varepsilon_k.$$

Thus, there exists $K \in \mathbb{N}$ such that

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_K}} S_{n_K}(\mathbb{1}_A - \mu(A))\right), \nu\right) \leq \frac{\varepsilon}{2}$$

and

$$d\left(\mathcal{L}_\Omega\left(\frac{1}{a_{n_K}} S_{n_K}(\mathbb{1}_A - \mu(A))\right), \eta\right) \leq \varepsilon.$$

https://doi.org/10.1017/S0143385709000637 Published online by Cambridge University Press

Acknowledgements. We would like to thank Jean-Pierre Conze who found an error in an earlier version of this paper and Sébastien Gouëzel who suggested a simplification in the proof which is set out in §4. We also thank the referee for his advice.

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