Ergod. Th. & Dynam. Sys. (2004), **24**, 1443–1450
DOI: 10.1017/S0143385703000683
© *2004 Cambridge University Press*
Printed in the United Kingdom

On some approximation of the 3D Euler system

E. I. DINABURG† and YA G. SINAI‡

† *The Institute of Physics of Earth, Russian Academy of Sciences,*
B. Gruzinskaya 10, Moscow 123995, Russia
‡ *Mathematics Department of Princeton University,*
Princeton, NJ 08544, USA and
Landau Institute of Theoretical Physics, Russian Academy of Sciences,
Institutskii pr., 12 Chernogolovka, Moscow 142432, Russia
(e-mail: sinai@math.princeton.edu)

(*Received* 18 *December* 2001 *and accepted in revised form* 19 *November* 2003)

Dedicated to the memory of Michael Herman

Abstract. In this paper we consider a quasi-linear approximation for the 3D Euler system as well as for the Navier–Stokes system in our previous paper (*Moscow Math. J.* **1**(3) (2001), 381–388). We define finite-dimensional versions of the approximation and study properties of their solutions.

In our joint paper [**1**], we proposed a quasi-linear approximation for the incompressible 3D Navier–Stokes system on the whole space R^3. It was constructed for Fourier transforms

$$\tilde{u}(k,t) = \{\tilde{u}_1(k,t), \tilde{u}_2(k,t), \tilde{u}_3(k,t)\}$$

of the incompressible vector fields $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, div $u = 0$. Here $x = (x_1, x_2, x_3) \in R^3$, $k = (k_1, k_2, k_3) \in R^3$ and $\tilde{u}$ is pure imaginary and odd. Putting $\tilde{u}(k,t) = iv(k,t)$ with $v(-k,t) = -v(-k,t)$ we can write our approximation as a system of quasi-linear equations (see [**1**]):

$$\frac{\partial v_i(k,t)}{\partial t} + \sum_{s=1}^{3} B_s \frac{\partial v_i(k,t)}{\partial k_s} = A_i^{(v)}, \quad i = 1,2,3 \tag{1}$$

$$\sum_{i=1}^{3} k_i v_i(k,t) = 0 \tag{2}$$

where

$$A_i^{(v)} = A_i^{(v)}(k,v,t) = -\nu|k|^2 v_i(k,t) + \sum_{j=1}^{3} v_j(k,t) a_{ji}(t) - \frac{2k_i}{|k|^2} \sum_{j=1}^{3} \sum_{\ell=1}^{3} v_j(k,t)\, k_\ell a_{j\ell}(t),$$

https://doi.org/10.1017/S0143385703000683 Published online by Cambridge University Press

and

$$a_{ij}(t) = \int_{R^3} k_i v_j(k,t)\,dk$$

are the first moments of v_j with respect to the space variables, $\nu > 0$ is the viscosity and

$$B_s = B_s(k,t) = \sum_{j=1}^{3} a_{sj}(t)k_j.$$

By putting $\nu = 0$ we get a similar approximation for the Euler system describing the dynamics of the free ideal fluid in R^3.

In what follows we consider systems (1) and (2) for $\nu \geq 0$ satisfying, at $t = 0$, the incompressibility condition (2). As was shown in [**1**], any solution $v(k,t)$ satisfies the incompressibility condition (2) for all t for which a solution exists.

System (1) has characteristics which are described by the system of equations

$$\frac{dk_i(t)}{dt} = B_i, \quad i = 1,2,3. \tag{3}$$

Along solutions of (3), system (1) becomes a system of ordinary differential equations (ODEs):

$$\begin{aligned}\frac{dv_i(t)}{dt} &= -\nu|k(t)|^2 v_i(t) + \sum_{j=1}^{3} v_j(t)a_{ji}(t) \\ &\quad - \frac{2k_i}{|k|^2}\sum_{j=1}^{3}\sum_{\ell=1}^{3} v_j(t)k_\ell(t)a_{j\ell}(t), \quad i = 1,2,3 \end{aligned} \tag{4}$$

$$\sum_{j=1}^{3} k_j(t)v_j(t) = 0.$$

Systems (3) and (4) have finite-dimensional versions if we assume that the initial conditions $v(k,0)$ are non-zero only for a finite set of points $k^{(n)} = k^{(n)}(0)$, $1 \leq n \leq N$, and the integrals giving $a_{ij}(t)$ are replaced by the Riemannian sums

$$a_{ij}(t) = \frac{1}{N}\sum_{n=1}^{N} k_i^{(n)}(t)v_j^{(n)}(t).$$

Consider the 3×3 matrix $A(t) = \{a_{ij}(t)\}$. Then our finite-dimensional system of ODEs which is a system of $6N$ equations can be written in the following form:

$$\frac{dk^{(m)}(t)}{dt} = A(t)k^{(m)}(t) \tag{5}$$

$$\frac{dv^{(m)}(t)}{dt} = -\nu|k^{(m)}(t)|^2 v^{(m)}(t) + A^t(t)v^{(m)}(t) - \frac{2k^{(m)}(t)}{|k^{(m)}(t)|^2}(v^{(m)}(t), A(t)k^{(m)}(t)) \tag{6}$$

$$(k^{(m)}(t), v^{(m)}(t)) = 0, \quad 1 \leq n \leq N, \tag{7}$$

where A^t is the transpose of A.

https://doi.org/10.1017/S0143385703000683 Published online by Cambridge University Press

Some properties of systems (5) and (6):

(1) the matrix elements of $A(t)$ do not depend explicitly on ν;
(2) $\operatorname{tr} A(t) = 0$;
(3) for any three numbers $1 \le p, q, r \le N$, the oriented volume $D(p, q, r)$ of the parallelepiped generated by $k^{(p)}(t)$, $k^{(q)}(t)$, $k^{(r)}(t)$ is the first integral of (5) and (6);
(4) $\det A(t)$ depends only on $v^{(m)}(t)$, $1 \le m \le N$.

Property (1) is obvious, properties (2) and (3) are proven in [**1**]. To prove property (4), we note that

$$\begin{aligned}\det A(t) &= \frac{1}{N^3} \sum_{1\le \ell,m,n\le N} \sum_{(v_{i_1}, v_{i_2}, v_{i_3})} k_1^{(\ell)} v_{i_1}^{(\ell)} k_2^{(m)} v_{\ell_2}^{(m)} k_3^{(n)} v_{i_3}^{(n)} (-1)^{\sigma(i_1,i_2,i_3)} \\ &= \frac{1}{N^3} \sum_{1\le \ell,m,n\le N} D(\ell, m, n) v_1^{(\ell)} v_2^{(m)} v_3^{(n)}\end{aligned}$$

where the second sum is taken over all permutations of $(1, 2, 3)$ and $\sigma(i_1, i_2, i_3) = 0$ for even permutations and 1 for odd. Thus, property (4) is proven.

In [**2**] the case of two particles ($N = 2$) was studied for $\nu > 0$. The main result in [**2**] was the following theorem.

THEOREM 1. *Let $\nu > 0$. For any initial condition $k^{(1)}(0) \ne 0$, $k^{(2)}(0) \ne 0$, $v^{(1)}(0)$, $v^{(2)}(0)$ satisfying (6), the solution of (5) exists for all $t > 0$ and there exist non-zero vectors $k^{(1)}(\infty)$, $k^{(2)}(\infty)$, depending on the initial conditions, such that $\lim_{t\to\infty} k^{(i)}(t) = k^{(i)}(\infty)$, $i = 1, 2$; and $\lim_{t\to\infty} v^{(i)}(t) = 0$, $i = 1, 2$.*

Beginning with this moment, we consider $N = 2$ and $\nu = 0$. We shall now show that the behavior of the solutions can be quite different. System (5) can be written in a more compact form:

$$\begin{aligned}\frac{dk_i^{(1)}(t)}{dt} &= k_i^{(2)}(t) V^{(2)}(t), \\ \frac{dk_i^{(2)}(t)}{dt} &= k_i^{(1)}(t) V^{(1)}(t), \\ \frac{dv_i^{(1)}(t)}{dt} &= v_i^{(2)}(t) V^{(1)}(t) - \frac{4k_i^{(1)}(t)}{|k^{(1)}(t)|^2} V^{(1)}(t) V^{(2)}(t) \\ \frac{dv_i^{(2)}(t)}{dt} &= v_i^{(1)}(t) V^{(2)}(t) - \frac{4k_i^{(2)}(t)}{|k^{(2)}(t)|^2} V^{(1)}(t) V^{(2)}(t), \quad i = 1, 2, 3\end{aligned} \tag{8}$$

where

$$V^{(1)}(t) = \tfrac{1}{2}(k^{(2)}(t), v^{(1)}(t)), \quad V^{(2)}(t) = \tfrac{1}{2}(k^{(1)}(t), v^{(2)}(t))$$

and the incompressibility condition

$$\sum_{i=1}^{3} k_i^{(j)}(t) v_i^{(j)}(t) = 0, \quad j = 1, 2. \tag{9}$$

LEMMA 1. *The components of the vector $J = (J_1, J_2, J_3) = [k^{(1)}(t), k^{(2)}(t)]$ and $I_1 = |k^{(1)}(t)|^2 V^{(1)}(t)$, $I_2 = |k^{(2)}(t)|^2 V^{(2)}(t)$ are the first integrals of the systems (8) and (9).*

https://doi.org/10.1017/S0143385703000683 Published online by Cambridge University Press

The first three integrals are also the first integrals for $\nu > 0$ and the proof remains the same (see [**2**]). I_1 and I_2 are the first integrals only for $\nu = 0$. This statement follows from direct checking.

LEMMA 2. *There exists a function C_0 of initial conditions such that*

$$(k^{(1)}(t), k^{(2)}(t)) = (I_1 + I_2)t + C_0.$$

Proof. It is clear that

$$\frac{d(k^{(1)}(t), k^{(2)}(t))}{dt} = \left(k^{(1)}(t), \frac{dk^{(2)}(t)}{dt}\right) + \left(\frac{dk^{(1)}(t)}{dt}, k^{(2)}(t)\right)$$
$$= |k^{(1)}|^2 V^{(1)}(t) + |k^{(2)}(t)|^2 V^{(2)}(t) = I_1 + I_2. \qquad \square$$

LEMMA 3. *Let $I_1 \neq 0$, $I_2 \neq 0$. There exists another function C_1 of initial conditions such that*

$$|k^{(2)}(t)| = C_1 |k^{(1)}(t)|^{I_1/I_2}.$$

Proof. It is easy to see that

$$\frac{dk_i^{(1)}(t)}{dt} = k_i^{(2)}(t) \frac{I_2}{|k^{(2)}(t)|^2} = I_2 \frac{\partial}{\partial k_2^{(2)}} \ln |k^{(2)}(t)|.$$

Therefore,

$$\sum_{i=1}^{3} \frac{dk_i^{(1)}(t)}{dt} \frac{dk_i^{(2)}(t)}{dt} = I_2 \sum_{i=1}^{3} \frac{\partial \ln |k^{(2)}(t)|}{\partial k_i^{(2)}} \frac{dk_i^{(2)}}{dt}$$
$$= I_2 \frac{d}{dt} \ln |k^{(2)}(t)| = \frac{d}{dt} I_2 \ln |k^{(2)}(t)|$$

because I_2 is the first integral. Changing the indices 1 and 2, we have

$$\sum_{i=1}^{3} \frac{dk_i^{(1)}(t)}{dt} \frac{dk_i^{(2)}(t)}{dt} = \frac{d}{dt}(I_1 \ln |k^{(1)}(t)|).$$

Therefore,

$$\frac{d}{dt}(I_1 \ln |k^{(1)}(t)|) = \frac{d}{dt}(I_2 \ln |k^{(2)}(t)|)$$

and

$$|k^{(2)}(t)| = C_1 |k^{(1)}(t)|^{I_1/I_2}.$$

The lemma is proven. $\square$

The behavior of solutions of systems (8) and (9) depends on the values of the first integrals I_1, I_2 and the length $|J|$ of the vector. We shall consider different cases.

Case I: $|J| = 0$. In this case the vectors $k^{(1)}(t)$, $k^{(2)}(t)$ are proportional to each other, $k^{(1)}(t) = a(t)k^{(2)}(t)$, $a(t) \neq 0$ and $V^{(1)}(t) = a^{-1}(t)(k^{(1)}(t), v^{(1)}(t)) = 0$. Therefore, $k^{(2)}(t) = k^{(2)}(0)$, $v^{(1)}(t) = v^{(1)}(0)$. In the same way, $V^{(2)}(t) = a(t)(k^{(2)}(t), v^{(2)}(t)) = 0$ and $v^{(2)}(t) = v^{(2)}(0)$, $k^{(1)}(t) = k^{(1)}(0)$.

https://doi.org/10.1017/S0143385703000683 Published online by Cambridge University Press

Case II: $|J| \neq 0$, $I_1 = I_2 = 0$. In this case $V^{(1)}(t) = V^{(2)}(t) = 0$, $v^{(1)}(t) = v^{(1)}(0)$, $v^{(2)}(t) = v^{(2)}(0)$, $k^{(1)}(t) = k^{(1)}(0)$, $k^{(2)}(t) = k^{(2)}(0)$.

Case III: $|J| \neq 0$, $I_1 \neq 0$, $I_2 = 0$. In this case $V^{(2)}(t) = 0$ and, therefore, $v^{(2)}(t) = v^{(2)}(0)$, $k^{(1)}(t) = k^{(1)}(0)$, $V^{(1)}(t) = I_1/|k^{(1)}(0)|^2$. From Lemma 2,

$$\begin{aligned}|k^{(1)}(t)|^2 \cdot |k^{(2)}(t)|^2 &= (k^{(1)}(t), k^{(2)}(t))^2 + |[k^{(1)}(t), k^{(2)}(t)]|^2 \\ &= (I_1 t + C_0)^2 + |J|^2.\end{aligned}$$

Therefore, $|k^{(2)}(t)| = |k^{(1)}(0)|^{-1}((I_1 t + C_0)^2 + |J|^2)$ and $|k^{(2)}(t)| \to \infty$ as $t \to \infty$. Denote by $\varphi(t)$ the angle between $k^{(2)}(t)$ and $k^{(1)}(t)$. We can write $\operatorname{ctg} \varphi(t) = (I_1 t + C_0)/|J|$. This shows that $|k^{(2)}(t)| \to \infty$ and its direction tends to the direction of $k^{(1)}(0)$ if $I_1 > 0$ or $-k^{(1)}(0)$ if $I_1 < 0$.

For the vector $v^{(1)}(t)$, we have the equation

$$\frac{dv^{(1)}(t)}{dt} = v^{(2)}(t)V^{(1)}(t) = v^{(2)}(0)\frac{I_1}{|k^{(1)}(0)|^2}.$$

From the last equation, it follows that $v^{(1)}(t)$ grows linearly in time in the direction of $v^{(2)}(0)$ if $I_1 > 0$ and $-v^{(2)}(0)$ if $I_1 < 0$.

Case IV: $|J| \neq 0$, $I_1 = 0$, $I_2 \neq 0$. This case is reduced to the previous one if we interchange the numbers of the particles.

Case V: $|J| \neq 0$, $I_1 = -I_2 \neq 0$. In this case $(k^{(1)}(t), k^{(2)}(t)) = C_0$, $|k^{(1)}(t)||k^{(2)}(t)| = C_1$. The angle between $k^{(1)}(t)$, $k^{(2)}(t)$ does not depend on t because $\cos \varphi(t) = C_0/C_1$. Also

$$V^{(2)}(t) = \frac{I_2}{|k^{(2)}(t)|^2} = \frac{I_2|k^{(1)}(t)|^2}{C_1^2}.$$

Therefore,

$$\begin{aligned}\frac{d|k^{(1)}(t)|^2}{dt} &= 2(k^{(1)}(t), k^{(2)}(t))V^{(2)}(t) \\ &= \frac{2C_0 I_2}{C_1^2}|k^{(1)}(t)|^2\end{aligned}$$

and

$$|k^{(1)}(t)|^2 = |k^{(1)}(0)|^2 e^{(2C_0 I_2/C_1^2)t}. \tag{10}$$

In the same way,

$$|k^{(2)}(t)|^2 = |k^{(2)}(0)|^2 e^{(2C_0 I_1/C_1^2)t}. \tag{11}$$

Assume that $C_0 I_2 < 0$. Then $C_0 I_1 > 0$ and formulas (10) and (11) show that

$$\begin{aligned}|k^{(1)}(t)|^2 &= |k^{(1)}(0)|^2 e^{-C_2 t}, \\ |k^{(2)}(t)|^2 &= |k^{(2)}(0)|^2 e^{C_2 t},\end{aligned}$$

where $C_2 = -2C_0 I_2/C_1^2$. This gives $\lim_{t\to\infty} |k^{(1)}(t)| = 0$, $\lim_{t\to\infty} |k^{(2)}(t)| = \infty$.

The estimation of $|v^{(i)}(t)|$, $i = 1, 2$ is based on a trick which we shall also use in the other remaining cases. The vectors $k^{(1)}(t)$, $k^{(2)}(t)$, J constitute a non-orthogonal basis

https://doi.org/10.1017/S0143385703000683 Published online by Cambridge University Press

in R^3 because $J \perp k^{(1)}(t)$, $J \perp k^{(2)}(t)$ and $k^{(1)}(t)$, $k^{(2)}(t)$ are linearly independent since $J = [k^{(1)}(t), k^{(2)}(t)] \neq 0$. For each t, we can write

$$v^{(i)}(t) = a_i(t)J + b_i(t)k^{(1)}(t) + c_i(t)k^{(2)}(t), \quad i = 1, 2. \tag{12}$$

Taking the inner products of both sides of (11) with $J, k^{(1)}(t)$, $k^{(2)}$ and using the incompressibility condition we get two systems of three linear equations for $a_1(t)$, $b_1(t)$, $c_1(t)$ and $a_2(t)$, $b_2(t)$, $c_2(t)$. Solving them, we have explicit expressions for all coefficients:

$$\begin{aligned} a_1(t) &= \frac{(v^{(1)}(t), J)}{|J|^2}, \\ b_1(t) &= \frac{-2V^{(1)}(t)(k^{(1)}(t), k^{(2)}(t))}{|J|^2}, \\ c_1(t) &= \frac{2V^{(1)}(t)|k^{(1)}(t)|^2}{|J|^2} \end{aligned} \tag{13′}$$

and

$$\begin{aligned} a_2(t) &= \frac{(v^{(2)}(t), J)}{|J|^2}, \\ b_2(t) &= \frac{2V^{(2)}(t)|k^{(2)}(t)|^2}{|J|^2}, \\ c_2(t) &= \frac{-2V^{(2)}(t)(k^{(1)}(t), k^{(2)}(t))}{|J|^2}. \end{aligned} \tag{13″}$$

In deriving (13′), (13″), we have used the formula $|k^{(1)}(t)|^2|k^{(2)}(t)|^2 - (k^{(1)}(t), k^{(2)}(t))^2 = |J|^2$. From (13′), (13″) we get the following system ODEs:

$$\begin{aligned} \frac{da_1(t)}{dt} &= a_2(t)V^{(1)}(t), \\ \frac{da_2(t)}{dt} &= a_1(t)V^{(2)}(t). \end{aligned} \tag{14}$$

Using Lemmas 1–3, we come to the final expressions for $b_i(t)$, $c_i(t)$, $i = 1, 2$, namely

$$\begin{aligned} b_1(t) &= -\frac{2C_0I_1}{|J|^2 \cdot |k^{(1)}(t)|^2} = -\frac{2C_0I_1}{|J|^2|k^{(1)}(0)|^2}e^{C_2t}, \\ c_1(t) &= \frac{2I_1}{|J|^2}, \\ b_2(t) &= \frac{2I_2}{|J|^2}, \\ c_2(t) &= -\frac{2C_0I_2}{|J|^2|k^{(2)}(t)|^2} = -\frac{2C_0I_2}{|J|^2 \cdot |k^{(2)}(0)|^2}e^{-C_2t}. \end{aligned}$$

System (14) implies

$$\frac{d(a_1^2(t) + a_2^2(t))}{dt} = 2a_1(t)a_2(t)(V^{(1)}(t) + V^{(2)}(t)). \tag{15}$$

From (15),

$$-|V^{(1)}(t)| - |V^{(2)}(t)| \leq \frac{d\ln(a_1^2(t) + a_2^2(t))}{dt} \leq |V^{(1)}(t)| + |V^{(2)}(t)|. \tag{16}$$

https://doi.org/10.1017/S0143385703000683 Published online by Cambridge University Press

If we substitute in (16) the explicit expressions for $V^{(1)}(t)$, $V^{(2)}(t)$, we get the inequalities

$$\exp\left\{-\int_0^t\left(\frac{|I_1|}{|k^{(1)}(s)|^2}+\frac{|I_2|}{|k^{(2)}(s)|^2}\right)ds\right\}\le\frac{a_1^2(t)+a_2^2(t)}{a_1^2(0)+a_2^2(0)}$$
$$\le\exp\left\{\int_0^t\left(\frac{|I_1|}{|k^{(1)}(s)|^2}+\frac{|I_2|}{|k^{(2)}(s)|^2}\right)ds\right\}.$$

We can derive an explicit asymptotic for $v^{(i)}(t)$ as $t\to\infty$.

Case VI: $|J|\neq 0$, $I_1\neq 0$, $I_2\neq 0$, $I_1+I_2\neq 0$.

From Lemmas 1 and 2,

$$|k^{(1)}(t)|^2|k^{(2)}(t)|^2=(k^{(1)}(t),k^{(2)}(t))^2+|[k_1(t),k_2(t)]|^2=((I_1+I_2)t+C_0)^2+|J|^2 \quad (17)$$

and from Lemma 3

$$|k^{(1)}(t)|=[C_1^{-2}F(t)]^{I_2/2(I_1+I_2)} \quad (18)$$
$$|k^{(2)}(t)|=C_1[C_1^{-2}F(t)]^{I_1/2(I_1+I_2)} \quad (19)$$

where $F(t)$ is the right-hand side of (17).

It can be easily seen from (18) and (19) that if I_1 and I_2 have the same sign then $\lim_{t\to\infty}|k^{(1)}(t)|=\lim_{t\to\infty}|k^{(2)}(t)|=\infty$. If their signs are different, then the length of one of the vectors $k^{(i)}(t)$, $i=1,2$, tends to zero while the other one tends to infinity. The angle $\varphi(t)$ between $k^{(1)}(t)$ and $k^{(2)}(t)$ tends to zero or π as $t\to\infty$ depending on the sign of I_1+I_2.

The analysis of the behavior of the vectors $v^{(1)}(t)$, $v^{(2)}(t)$ is based on formulas (13). We have

$$b_1(t)=-\frac{2I_1((I_1+I_2)t+C_0)}{|J|^2[C_1^{-2}((I_1+I_2)\,t+C_0)^2+C_1^{-2}|J|^2]^{I_2/(I_1+I_2)}}$$
$$c_1(t)=\frac{2I_1}{|J|^2},$$
$$b_2(t)=\frac{2I_2}{|J|^2},$$
$$c_2(t)=-\frac{2I_2((I_1+I_2)t+C_0)}{|J|^2[C_1^{-2}((I_1+I_2)t+C_0)^2+C_1^{-2}|J|^2]^{I_1/(I_1+I_2)}}.$$

Thus,

$$\lim_{t\to\infty}\frac{b_1(t)}{t^{(I_1-I_2)/(I_1+I_2)}}=\text{constant},\qquad\lim_{t\to\infty}\frac{c_2(t)}{t^{(I_2-I_1)/(I_1+I_2)}}=\text{constant}$$

and

$$\lim_{t\to\infty}|b_1(t)|=\infty,\quad\lim_{t\to\infty}c_2(t)=0\quad\text{if }\frac{I_1-I_2}{I_1+I_2}>0,$$
$$\lim_{t\to\infty}b_1(t)=0,\quad\lim|c_2(t)|=\infty\quad\text{if }\frac{I_1-I_2}{I_1+I_2}<0.$$

The estimates for $a_1(t)$, $a_2(t)$ follow from (15). In particular, they show that $a_1(t)$, $a_2(t)$ are finite for all $t>0$.

We summarize all possible types of behavior of solutions of (8) and (9) in the formulation of the next theorem which gives the main result of this paper.

https://doi.org/10.1017/S0143385703000683 Published online by Cambridge University Press

THEOREM 2. *For any initial conditions* $k^{(i)}(0) \neq 0$, $v^{(i)}(0)$, $i = 1, 2$ *satisfying the incompressibility condition (9), solutions of system (8) exist for all* $t > 0$.

The functions

$$J = [k^{(1)}(0), k^{(2)}(0)],$$
$$I_1 = \tfrac{1}{2}|k^{(1)}(0)|^2(k^{(2)}(0), v^{(1)}(0)),$$
$$I_2 = \tfrac{1}{2}|k^{(2)}(0)|^2(k^{(1)}(0), v^{(2)}(0))$$

are the first integrals of (8) and (9). Depending on their values, solutions can have the following types of behavior as $t \to \infty$.

(1) $|J| = 0$: $v^{(1)}(t) = v^{(i)}(0)$, $k^{(i)}(t) = k^{(i)}(0)$, $i = 1, 2$.

(2) $|J| \neq 0$, $I_1 = I_2 = 0$: $v^{(i)}(t) = v^{(i)}(0)$, $k^{(i)}(t) = k^{(i)}(0)$, $i = 1, 2$.

(3) $|J| \neq 0$, $I_1 \neq 0$, $I_2 = 0$: $k^{(1)}(t) = k^{(1)}(0)$, $v^{(2)}(t) = v^{(2)}(0)$.
$\lim_{t\to\infty} |k^{(2)}(t)| = \infty$, $v^{(1)}(t) = v^{(2)}(0)I_1 \cdot t/|k^{(1)}(0)|^2 + v^{(1)}(0)$.

(4) $|J| \neq 0$, $I_1 = 0$, $I_2 \neq 0$: *This case is reduced to (3) if we interchange the numbers of the particles.*

(5) $|J| \neq 0$, $I_1 = -I_2 \neq 0$: *In this case* $(k^{(1)}(t), k^{(2)}(t)) = (k^{(1)}(0), k^{(2)}(0) = C_0$. *Assume that* $C_0 \neq 0$. *If* $C_0 I_1 > 0$ *then* $|k^{(1)}(t)|$ *decreases exponentially,* $|k^{(2)}(t)|$ *grows exponentially,* $\lim_{t\to\infty} |v^{(1)}(t)| = \infty$. *If* $C_0 I_1 < 0$ *then* $|k^{(2)}(t)|$ *decreases exponentially,* $|k^{(1)}(t)|$ *grows exponentially,* $\lim_{t\to\infty} |v^{(2)}(t)| = \infty$. *If* $C_0 = 0$ *both norms* $|k^{(1)}(t)|$, $|k^{(2)}(t)|$ *remain constant.*

(6) $|J| \neq 0$, $I_1 \neq 0$, $I_2 \neq 0$, $I_1 + I_2 \neq 0$: *If* $\operatorname{sgn} I_1 = \operatorname{sgn} I_2$ *then* $\lim_{t\to\infty} |k^{(i)}(t)| = \infty$, $i = 1, 2$. *If* $\operatorname{sgn} I_1 = -\operatorname{sgn} I_2$, *then the length of one of the vectors* $k^{(i)}(t)$, $i = 1, 2$ *tends to* ∞ *while the length of the other one tends to* 0. *The angle* $\varphi(t)$ *between* $k^{(1)}(t)$, $k^{(2)}(t)$ *tends to zero if* $I_1 + I_2 > 0$ *and to* π *if* $I_1 + I_2 < 0$ *when* $t \to \infty$.

Theorem 2 shows that the behavior of solutions of (8) and (9) can differ from the behavior of solutions of a similar system with $\nu > 0$.

Acknowledgements. We thank the anonymous referee for many useful remarks, corrections and suggestions. The authors thank Russian Foundation of Basic Research for financial support (grant no. 02-01-00158). The second author is grateful to NSF for the financial support (grant no. DMS-00706689).

REFERENCES

[1] E. I. Dinaburg and Ya G. Sinai. A quasi-linear approximation for the 3D Navier–Stokes system. *Moscow Math. J.* **1**(3) (2001), 381–388.

[2] E. I Dinaburg. New finite-dimensional approximations of the 3D Navier–Stokes system. *Dokl. Russian Acad. Sci.* **353**(2) (2002), 258–262.

https://doi.org/10.1017/S0143385703000683 Published online by Cambridge University Press