Math. Proc. Camb. Phil. Soc. (1988), **103**, 463
Printed in Great Britain

Mal'cev H^*-algebras

BY M. CABRERA-GARCIA, J. MARTINEZ-MORENO AND A. RODRIGUEZ-PALACIOS

Departamento de Análisis Matemático, Universidad de Granada, 18071-Granada, Spain

(*Received* 1 *July* 1986; *revised* 25 *September* 1987)

Abstract

The only finite-dimensional simple non-Lie Mal'cev complex algebra is given the structure of an H^*-algebra and it is proved that this is the only topologically simple non-Lie Mal'cev H^*-algebra.

1. *Introduction*

Since the beginning of the study of associative H^*-algebras in the pioneering paper by Ambrose [**1**], a widespread effort has been made to obtain analogous results in several classes of non-associative algebras. In [**5**] Cuenca and Rodriguez showed that every non-associative H^*-algebra with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals, each of which is a topologically simple H^*-algebra. This result reduces the task of description of the H^*-algebras of any hereditary class to the topologically simple case.

Sagle studied in full detail the finite-dimensional Mal'cev algebras in §3 of [**9**]. He introduced a 7-dimensional simple non-Lie Mal'cev algebra A_7 over any field $\mathbb{K}$, and conjectured that if $\mathbb{K}$ is algebraically closed of characteristic zero then every finite-dimensional simple non-Lie Mal'cev algebra over $\mathbb{K}$ is isomorphic to A_7. In [**10**] Sagle gives an affirmative answer to this conjecture under the additional assumption that the algebra contains an element whose multiplication operator is not nilpotent. In [**7**] Loos proves that this assumption is superfluous, thus completing the description of finite-dimensional simple Mal'cev algebras over algebraically closed fields of characteristic zero.

In this paper the complex Mal'cev algebra A_7 is given the structure of an H^*-algebra in a natural way, and we prove that this algebra is the only topologically simple non-Lie Mal'cev H^*-algebra.

2. *Some previous results on H*-algebras*

An H^*-algebra is a complex non-associative algebra A whose underlying vector space is equipped with an inner product $(x, y) \to (x/y)$ (i.e. a sesqui-linear, conjugate-symmetric, positive-definite function from $A \times A$ to $\mathbb{C}$) under which it is a Hilbert space, together with an algebra involution * (i.e. $(a+b)^* = a^* + b^*$; $(\alpha a)^* = \bar{\alpha} a^*$; $(ab)^* = b^* a^*$; $a^{**} = a$), such that

$$(xy/z) = (x/zy^*) = (y/x^*z) \tag{1}$$

for all x, y, z in A.

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press

The *annihilator ideal* of an algebra A is defined by

$$\mathrm{Ann}(A) := \{x \in A : ax = 0 = xa \quad \text{for all} \quad a \text{ in } A\}.$$

It is shown in proposition 2 of [5] that every H^*-algebra A is a normed algebra (the product is continuous) and that if A has zero annihilator (in particular, if A has an identity) then the algebra involution * is isometric, that is

$$(x/y) = (y^*/x^*) \quad \text{for all} \quad x, y \text{ in } A.$$

Following [11], we recall that a complex algebra A with identity is called a *quadratic algebra* over $\mathbb{C}$ if $A \neq \mathbb{C}1$ and for each x in A we have

$$x^2 - t(x)\,x + n(x)\,1 = 0 \quad \text{for some} \quad t(x), n(x) \text{ in } \mathbb{C}. \tag{2}$$

If x is not in $\mathbb{C}1$, then the complex numbers $t(x), n(x)$ are uniquely determined. Put

$$t(\alpha 1) = 2\alpha \quad \text{and} \quad n(\alpha 1) = \alpha^2 \quad \text{for all} \quad \alpha \text{ in } \mathbb{C},$$

so that the *trace* $t(x)$ and the *norm* $n(x)$ are unique in (2) for all x in A. The trace function t so defined is a linear function: see p. 49 of [11].

Given an H^*-algebra A with isometric involution *, we can construct a new H^*-algebra A' with identity in which A has codimension 1:

DEFINITION. A' is equal to $\mathbb{C} \oplus A$ as a vector space, with product $.$, involution $^\diamond$, and inner product $(/)$ defined as follows:

$$(\alpha, x).(\beta, y) = (\alpha\beta + (x/y^*), \beta x + \alpha y + xy)$$

$$(\alpha, x)^\diamond = (\bar{\alpha}, x^*)$$

and

$$((\alpha, x)/(\beta, y)) = \alpha\bar{\beta} + (x/y)$$

for $\alpha, \beta \in \mathbb{C}$ and $x, y \in A$.

THEOREM 1. *A' is an H^*-algebra with identity $(1,0)$. Moreover A' is a quadratic algebra if and only if either A is a non-zero anti-commutative algebra or A equals $\mathbb{C}$.*

Proof. The proof of the first assertion is rather long but straightforward, so we omit it. Also it is routine that if either A is a non-zero anti-commutative algebra or A equals $\mathbb{C}$, then A' is a quadratic algebra. Conversely, if A' is a quadratic algebra, then we have

$$((x/x^*), x^2) = (0, x)^2 = (-n(0, x), t(0, x)\,x) \quad \text{for all} \quad x \text{ in } A.$$

So

$$x^2 = f(x)\,x \quad \text{for all} \quad x \text{ in } A, \tag{3}$$

where $f(x) = t(0, x)$ is a linear functional in A. If $f = 0$, clearly A is an anti-commutative algebra with dimension at least 1. Suppose that f is non-zero. Since the product in A is continuous and * is an algebra involution it follows that f is continuous and $f(x^*) = \overline{f(x)}$ for all x in A. Linearization of (3) gives

$$xy + yx = f(x)\,y + f(y)\,x \quad \text{for all} \quad x, y \text{ in } A. \tag{4}$$

As A is an H^*-algebra we have

$$(xy + yx/z) = (y/x^*z + zx^*) \quad \text{for all} \quad x, y, z \text{ in } A. \tag{5}$$

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press

Now, when (4) is substituted in (5), we have

$$f(y)(x/z) = \overline{f(z)}(y/x^*) \quad \text{for all} \quad x, y, z \text{ in } A.$$

Since f is a non-zero continuous linear functional on A, the Riesz representation theorem gives us a non-zero element x_0 in A so that $f(x) = (x/x_0)$ for all x in A. So the above equation can be written

$$(y/x_0)(x/z) = (x_0/z)(y/x^*) \quad \text{for all} \quad x, y, z \text{ in } A.$$

Hence

$$(x^*/x_0)x = \|x^*\|^2 x_0 \quad \text{for all} \quad x \text{ in } A.$$

Therefore A is a 1-dimensional algebra with non-zero product, so A is isomorphic to $\mathbb{C}$.

Remark. It may be interesting to note that (up to a positive multiple of the inner product) every H^*-algebra with identity can be obtained as an A' for a suitable H^*-algebra A. Indeed, let B be an H^*-algebra with identity (and so with an isometric algebra involution) such that $\|1\| = 1$. We consider the subspace A orthogonal to the identity in B. Then A is a closed *-invariant subspace of B and so A is a Hilbert space with isometric involution. If we define in A the product

$$x \circ y := xy - (x/y^*)1 \quad \text{for all} \quad x, y \text{ in } A$$

($x \circ y$ is the orthogonal projection of xy in A), it is routine to verify that A is an H^*-algebra with isometric involution and B is isomorphic to A'.

Recall that the (linear) map $x \otimes z$ is defined on A by $(x \otimes z)(y) = (y/z)x$. We will need the following result:

PROPOSITION 1. *Suppose A is an H^*-algebra with isometric involution, and A' is as in Theorem* 1. *Then A' is alternative if and only if A satisfies the identity*

$$L_x^2 = (x/x^*)I - x \otimes x^* + L_{x^2}, \tag{6}$$

where L_x denote the left multiplication operator by x on A.

Proof. This is a long but straightforward computation, and it will be omitted.

3. *The main result*

A *Mal'cev algebra* is, by definition, a non-associative algebra A satisfying the identities

$$xy = -yx$$

and

$$J(x, y, xz) = J(x, y, z)x$$

for all x, y, z in A, where

$$J(x, y, z) = (xy)z + (yz)x + (zx)y.$$

Let $\mathbb{O}^-$ denote the Mal'cev algebra obtained by skew-symmetrization of the product of the algebra $\mathbb{O}$ of complex octonions. Since $\mathbb{O}$ can be endowed with an H^*-algebra structure (see [8]), the algebra $\mathbb{O}^-$ is a non-Lie Mal'cev H^*-algebra. An H^*-algebra structure can be given to the algebra A_7, the quotient of $\mathbb{O}^-$ by its annihilator ideal $\mathbb{C}1$, in a natural way. With this structure A_7 becomes a simple non-Lie Mal'cev H^*-algebra.

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press

Recall that an H^*-algebra A is called *topologically simple* if $A^2 \neq 0$ and A contains no non-zero proper closed ideal. Now we can state our main result.

THEOREM 2. *Let A be a topologically simple non-Lie Mal'cev H^*-algebra. Then A is, up to a positive scalar multiple of the inner product, the simple non-Lie Mal'cev H^*-algebra A_7.*

The strategy of our proof will be to show that the quadratic H^*-algebra A' associated to our Mal'cev algebra A is an alternative algebra. Thus our main purpose will be to prove that the algebra A in our theorem satisfies the equality (6) in Proposition 1 (up to a positive scalar multiple of the inner product, if necessary).

If u is a fixed element in a complex algebra A, the *eigenspace* $\{x \in A : ux = \alpha x\}$ of the operator L_u relative to the complex number α will be denoted A_α. If $\alpha \neq \beta$ then $A_\alpha \cap A_\beta = 0$. If A is an H^*-algebra then A_α is a closed subspace of A. Also if u is a symmetric element in A (i.e. $u = u^*$) then by (1) L_u is a symmetric element of the C^*-algebra $BL(A)$ of bounded linear operators on A. So the eigenvalues of L_u are real numbers, and for $\alpha \neq \beta$, A_α and A_β are orthogonal subspaces. If A is in addition anti-commutative, then clearly $A_\alpha^* = A_{-\alpha}$.

If A is a Lie H^*-algebra with zero annihilator, C a Cartan subalgebra (a maximal self-adjoint abelian subalgebra) of A and f a linear functional on C, we adopt the following definition from [**12**]:

$$A_f = \{x \in A : cx = f(c)\,x \quad \text{for all} \quad c \text{ in } C\}.$$

Then A_f is a closed subspace of A, and f is called a *root* (relative to C) if and only if $A_f \neq 0$. It is shown in [**13**] that A has a Cartan decomposition (relative to C); that is $A = \overline{\bigoplus A_f}$, the sum being taken over the distinct roots relative to C. This allows us to decompose each topologically simple Mal'cev H^*-algebra as the closure of the orthogonal sum of the eigenspaces relative to the multiplication operator of any symmetric element in A. This decomposition will be the cornerstone of our proof and it will play an analogous role to the algebraic decomposition used in [**10**] for the finite-dimensional case.

PROPOSITION 2. *Let A be a Mal'cev H^*-algebra with continuous involution and let u be a symmetric element in A. Then A is the closure of the orthogonal sum of the eigenspaces of the operator L_u.*

Proof. Let $B = \overline{A^2}$ and $E = B^\perp$. Then for $e \in E$ and $x, y \in A$ we have

$$(ex/y) = (e/yx^*) = 0,$$

so by non-degeneracy of $(\,/\,)$, $ex = 0$. Thus we easily see that $E = \mathrm{Ann}\,(A)$, hence $A = B \oplus \mathrm{Ann}\,(A)$. Let $u = a + b$, where $a \in \mathrm{Ann}\,(A)$ and $b \in B$. Since $\mathrm{Ann}\,(A)$ and B are clearly *-invariant, we have $a^* + b^* = u^* = u = a + b$, with $a^* \in \mathrm{Ann}\,(A)$ and $b^* \in B$, so b in particular is symmetric. Let C be a maximal self-adjoint abelian subalgebra of B containing b. Now suppose A is a Lie algebra. Then B is a Lie H^*-algebra with zero annihilator, so B has a Cartan decomposition relative to C, that is $B = \overline{\bigoplus B_f}$. If $x \in B_f$ then

$$ux = (a+b)\,x = bx = f(b)\,x;$$

this proves $B_f \subset A_{f(b)}$ relative to u, for each root f relative to C, so we have

$$B = \overline{\bigoplus B_f} \subset \overline{\bigoplus A_{f(b)}} \subset \overline{\bigoplus A_\alpha}.$$

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press

From $L_u(\mathrm{Ann}\,(A)) = 0$ it follows that $\mathrm{Ann}\,(A) \subset A_0$ and therefore

$$A = \mathrm{Ann}\,(A) \oplus B = \overline{\oplus A_\alpha}.$$

Now, let A be a general Mal'cev H^*-algebra. Let v a symmetric element in A. Then the closed subalgebra D generated by u and v is a Lie subalgebra of A (see corollary 4·4 of [**9**]). Obviously D is an H^*-algebra, so $v \in \overline{\oplus D_\alpha}$, where $D_\alpha = \{d \in D : ud = \alpha d\}$. Since D_α is contained in A_α it follows that $\overline{\oplus A_\alpha}$ contains every symmetric element of A and so equals A.

For a Mal'cev algebra A of characteristic not 2 or 3 it can be seen from theorem 3·5, corollary 5·13 and proposition 5·11 of [**9**] that the *J-nucleus*

$$N = \{x \in A : J(x, A, A) = 0\}$$

and the linear space $J(A, A, A)$ spanned by all elements of the form $J(x, y, z)$ for x, y, z in A are ideals of A such that

$$NJ(A, A, A) = 0. \tag{7}$$

Furthermore if A is a simple non-Lie Mal'cev algebra then by theorem 5·14 of [**9**]

$$A = J(A, A, A), \quad \text{and} \quad N = 0.$$

An analogous result holds in our context, namely if A is a topologically simple non-Lie Mal'cev H^*-algebra then

$$A = \overline{J(A, A, A)}, \quad \text{and} \quad N = 0. \tag{8}$$

In fact, since A is non-Lie, $J(A, A, A)$ is non-zero, and therefore its closure is a non-zero closed ideal of A, so the topological simplicity of A gives $A = \overline{J(A, A, A)}$. Now by (7) we have $N.A = 0$ and therefore $N \subset \mathrm{Ann}\,(A)$, and again by the topological simplicity of A we have $\mathrm{Ann}\,(A) = 0$, which yields the second assertion.

The following result is an adaptation of proposition 3·13 in [**10**]. The author's assumption of finite-dimensionality there is not in fact used, once the algebra has been decomposed into a finite direct sum of subspaces A_α (called weight spaces) whose definition is more complicated than ours. Also we can exploit the fact that A is an H^*-algebra. Nevertheless the proof is long and tedious, and we refer the reader to [**10**] for a large part of it.

PROPOSITION 3. *Let A be a topologically simple non-Lie Mal'cev H^*-algebra and let u be a non-zero symmetric element of A. Then there exists a non-zero real number α such that $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ relative to u, with $A_0 = \overline{A_\alpha A_{-\alpha}}$.*

Proof. The existence of a non-zero eigenvalue α of L_u follows from Proposition 2 and the fact that u is non-zero and A has zero annihilator. Since $A_{-\alpha} = A_\alpha^*$ we deduce that $-\alpha$ is also an eigenvalue. Following the terminology of [**10**], we define

$$X := A_\alpha A_{-\alpha} + A_\alpha + A_{-\alpha}.$$

For two distinct eigenvalues β, γ we have $A_\beta A_\gamma \subset A_{\beta+\gamma}$ (see (2·5) of [**10**]), so the above sum is orthogonal. Since A is topologically simple and X is a non-zero subspace, if we prove that $\bar{X}$ is an ideal then we will have

$$A = \bar{X} = \overline{A_\alpha A_{-\alpha}} \oplus A_\alpha \oplus A_{-\alpha} \subset A_0 \oplus A_\alpha \oplus A_{-\alpha}.$$

Hence

$$A = A_0 \oplus A_\alpha \oplus A_{-\alpha} \quad \text{and} \quad A_0 = \overline{A_\alpha A_{-\alpha}}.$$

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press

In order to prove that $\bar{X}$ is an ideal it is enough by Proposition 2, together with the continuity of the product, to show that $XA_\beta \subset X$ for each eigenvalue β. We see easily that it suffices to prove

$$A_0^2 = 0 \tag{9}$$

$$A_\alpha^2 \subset A_{-\alpha} \tag{10}$$

$$A_\alpha A_\beta = 0 \quad \text{for} \quad \beta \neq 0, \pm\alpha \tag{11}$$

$$(A_\alpha A_{-\alpha}) A_\beta = 0 \quad \text{for} \quad \beta \neq 0, \pm\alpha. \tag{12}$$

For any eigenvalues β, γ we have, in an obvious notation,

$$(x_\beta y_\beta / z_\gamma) = (x_\beta / z_\gamma y_\beta^*) = (x_\beta / z_\gamma y'_{-\beta}) = (x_\beta / w_{\gamma-\beta}) \quad \text{if} \quad \gamma \neq -\beta,$$

and this is zero if $\beta \neq \gamma - \beta$. Thus A_β^2 is orthogonal to every A_γ for which $\gamma \neq -\beta$ or 2β, whence $A_\beta^2 \subset A_{2\beta} + A_{-\beta}$ by Proposition 2.

Taking $\beta = 0$ we find in particular that $A_0^2 \subset A_0$. By (8), Proposition 2, and the continuity of the product, an analogous method to the one used in [**10**] yields that $A_0^2 \subset N = 0$ (see (2·11) of [**10**]).

Taking $\beta = \alpha$ we can deduce (10) by the method of (2·14) of [**10**].

The identity (11) can be established by the method of (3·12) of [**10**].

Now by (3·4) of [**10**] and (11) we have for $x \in A_\alpha$, $y \in A_{-\alpha}$, $z \in A_\beta$ that

$$0 = J(x, y, z) = (xy)z + (yz)x + (zx)y,$$

and so

$$(A_\alpha A_{-\alpha}) A_\beta = 0.$$

COROLLARY 1. *If A is a topologically simple non-Lie Mal'cev H^*-algebra and u is a symmetric element in A then*

$$\mathrm{Sp}\,(BL(A), L_u) = \{0, \|L_u\|, -\|L_u\|\},$$

where $\mathrm{Sp}\,(BL(A), L_u)$ *denotes the spectrum of* L_u *in the algebra* $BL(A)$.

Proof. With the notation of Proposition 3, we clearly have

$$\mathrm{Sp}\,(BL(A), L_u) = \{0, \alpha, -\alpha\}.$$

Because of the symmetry of L_u we have

$$|\alpha| = r(L_u) = \|L_u\|,$$

where $r(L_u)$ denotes the spectral radius of L_u.

LEMMA 1. *Let A be a topologically simple non-Lie Mal'cev H^*-algebra and let u be a non-zero symmetric element in A. Then $A_0 = \mathbb{C}u$.*

Proof. As in §4 of [**10**], $\{L_x : x \in A_0\}$ is a commutative subset of $BL(A)$, thus it is contained in a maximal commutative subset C of $BL(A)$, which is a closed commutative subalgebra of $BL(A)$ containing the identity operator on A (see proposition 15·3 of [**2**]). By theorem 15·4 of [**2**] we have

$$\mathrm{Sp}\,(C, L_u) = \mathrm{Sp}\,(BL(A), L_u).$$

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press

Now, by Corollary 1, $\|L_u\| \in \mathrm{Sp}(C, L_u)$ and by proposition 16·9 of [2] there exists a non-zero homomorphism ψ of C into $\mathbb{C}$ such that

$$\psi(L_u) = \|L_u\|.$$

The function F from $\mathrm{Sym}(A_0)$ into $\mathbb{R}$ defined by $F(x) := \psi(L_x)$ is a linear function. If we define for all elements x in $\mathrm{Sym}(A_0)$

$$|x| := \|L_x\|,$$

then $\mathrm{Sym}(A_0)$ becomes a real normed space. Using the fact that ψ is automatically continuous on C and $\|\psi\| \leqslant 1$ (see proposition 16·3 of [2]) we have

$$|F(x)| = |\psi(L_x)| \leqslant \|L_x\| = |x|$$

and so the function F is $|.|$-continuous. If $x \in \mathrm{Sym}(A_0)$ such that $|x| = 1$, then by proposition 16·9 of [2] and Corollary 1 we have

$$F(x) = \psi(L_x) \in \{-1, 0, 1\}.$$

Hence the function F is $|.|$-continuous in the unit sphere of the normed space $(\mathrm{Sym}(A_0), |.|)$ with values in the set $\{-1, 0, 1\}$. Since $F(u/|u|) = 1$ and $F(-u/|u|) = -1$, it follows that the image of the unit sphere of $(\mathrm{Sym}(A_0), |.|)$ under F is not connected. Therefore the unit sphere of $(\mathrm{Sym}(A_0), |.|)$ is not connected, so we deduce that $\mathrm{Sym}(A_0)$ is 1-dimensional, hence equals $\mathbb{R}u$. Since A_0 is *-invariant, we have $A_0 = \mathrm{Sym}(A_0) + i\,\mathrm{Sym}(A_0)$; thus $A_0 = \mathbb{C}u$.

COROLLARY 2. *If A is a topologically simple non-Lie Mal'cev H*-algebra and u is a symmetric element in A, then we have*

$$\|u\|^2 L_u^2 = \|L_u\|^2(\|u\|^2 I - u \otimes u). \tag{13}$$

Proof. We can assume $u \neq 0$. Let z be any element in A. Then by Proposition 3, Lemma 1 and Corollary 1 we have

$$z = \lambda u + x + y \quad \text{where} \quad \lambda \in \mathbb{C},\ ux = \|L_u\|\, x,\ uy = -\|L_u\|\, y.$$

Therefore
$$L_u^2(z) = \|L_u\|^2(x+y) = \|L_u\|^2(z - \lambda u). \tag{14}$$

Because of the orthogonality of the decomposition we have $(z/u) = \lambda\|u\|^2$ and so

$$\lambda u = \frac{(z/u)}{\|u\|^2}\, u = \frac{u \otimes u}{\|u\|^2}(z).$$

This, together with (14), concludes the proof.

LEMMA 2. *Let A be a topologically simple non-Lie Mal'cev H*-algebra. Then there exists a positive real number k such that $\|L_u\| = k\|u\|$ for every symmetric element u in A.*

Proof. It clearly suffices to show that

$$\|L_u\| = \|L_v\| \tag{15}$$

for all non-zero symmetric elements u, v in A such that $\|u\| = \|v\| = 1$. If u and v are linearly dependent this is trivial. Assume that u and v are linearly independent. Since u, v are symmetric and the involution is isometric we have $(u/v) = (v/u)$. With this

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press

fact in mind, by changing u to $u+\lambda v$ in (13), where λ is a real number, and computing right derivatives at $\lambda = 0$ we obtain

$$L_u L_v + L_v L_u + 2(u/v) L_u^2 = \|L_u\|^2 [2(u/v) I - u \otimes v - v \otimes u] + \mu[I - u \otimes u]$$

for a suitable real number $\mu = \mu(u,v)$. Now, by (13) we have

$$L_u L_v + L_v L_u = \|L_u\|^2 [2(u/v) I - u \otimes v - v \otimes u] + [\mu - 2(u/v) \|L_u\|^2][I - u \otimes u]. \tag{16}$$

Interchanging u and v in this yields

$$L_u L_v + L_v L_u = \|L_v\|^2 [2(u/v) I - u \otimes v - v \otimes u] + [\mu' - 2(u/v) \|L_v\|^2][I - v \otimes v]. \tag{17}$$

On comparing (16) and (17) we see that some linear combination

$$\epsilon I + \alpha u \otimes u + \beta u \otimes v + \gamma v \otimes u + \delta v \otimes v$$

equals the zero operator. Since A is non-Lie, $\dim A > 2$, so there exists $x \in A \backslash 0$ with $(x/u) = (x/v) = 0$. Applying the above operator to x we find $\epsilon = 0$. Then applying it to u and v, and setting $p = (u/v)$, we have (since $\|u\| = \|v\| = 1$) $\alpha + \beta p = 0 = \gamma + \delta p$ and $\alpha p + \beta = 0 = \gamma p + \delta$. If it is false that $\alpha = \beta = \gamma = \delta = 0$, then $p^2 = 1$, so $p = 1$ or -1, whence $\|u - v\|^2 = 0$ (resp. $\|u + v\|^2 = 0$), contradicting linear independence of u and v. Thus in particular $\beta = 0$, so that $\|L_u\| = \|L_v\|$, as required.

COROLLARY 3. *If A is a topologically simple non-Lie Mal'cev H^*-algebra then there exists a positive real number k such that*

$$L_x^2 = k^2[(x/x^*) I - x \otimes x^*] \quad \textit{for all} \quad x \textit{ in } A. \tag{18}$$

Proof. Lemma 2 and Corollary 2 give us the existence of a positive real number k which satisfies

$$L_u^2 = k^2[(u/u) I - u \otimes u] \tag{19}$$

for all symmetric elements u in A. By linearization we have that for u and v symmetric elements in A the following identity is true:

$$L_u L_v + L_v L_u = k^2[2(u/v) I - u \otimes v - v \otimes u]. \tag{20}$$

Finally, each element x in A can be written as $x = u + iv$, where u and v are symmetric elements in A, hence (18) follows from (19) and (20).

Conclusion of the proof of Theorem 2. Let B be the H^*-algebra obtained from A by redefining the inner product as follows:

$$[x/y] := k^2(x/y) \quad \text{for all} \quad x, y \text{ in } A.$$

Then, by Corollary 3, B satisfies

$$L_x^2 = [x/x^*] I - x \otimes x^*,$$

where now $\otimes$ is referred to the new inner product. Now, by Theorem 1 and Proposition 1, B' is a quadratic alternative H^*-algebra. Since B is not 1-dimensional it follows that the dimension of B' is not two and so B' is a central simple algebra (see p. 217 of [3]). Since B is a non-Lie Mal'cev subalgebra of the Mal'cev algebra B'^- it follows that B' is not associative. Hence the main theorem of [6] tells us that B' is

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press

the algebra of complex octonions structured as a H^*-algebra. By the essential uniqueness of the H^*-algebra structure (see corollary 3·5 of [4]) we conclude that $\tilde{B}$, and so A, is essentially the algebra A_7.

The authors wish to thank the referee for suggesting some improvements in the exposition of this paper.

REFERENCES

[1] W. Ambrose. Structure theorems for a special class of Banach algebras. *Trans. Amer. Math. Soc.* **57** (1945), 364–386.
[2] F. F. Bonsall and J. Duncan. *Complete Normed Algebras* (Springer-Verlag, 1973).
[3] H. Braun and M. Koecher. *Jordan-Algebren* (Springer-Verlag, 1966).
[4] J. A. Cuenca and A. Rodriguez. Isomorphisms of H^*-algebras. *Math. Proc. Cambridge Philos. Soc.* **97** (1985), 93–99.
[5] J. A. Cuenca and A. Rodriguez. Structure theory for noncommutative Jordan H^*-algebras. *J. Algebra* **106** (1987), 1–14.
[6] E. Kleinfeld. Simple alternative rings. *Ann. of Math.* (2) **58** (1953), 544–547.
[7] O. Loos. Über eine Beziehung zwischen Mal'cev-Algebren und Lie-Tripelsystemen. *Pacific J. Math.* **18** (1966), 553–562.
[8] I. Perez de Guzman. Structure theorem for alternative H^*-algebras. *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 437–446.
[9] A. A. Sagle. Mal'cev algebras. *Trans. Amer. Math. Soc.* **101** (1961), 426–458.
[10] A. A. Sagle. Simple Mal'cev algebras over fields of characteristic zero. *Pacific J. Math.* **12** (1962), 1057–1078.
[11] R. D. Schafer. *An Introduction to Nonassociative Algebras* (Academic Press, 1966).
[12] J. R. Schue. Hilbert space methods in the theory of Lie algebras. *Trans. Amer. Math. Soc.* **95** (1960), 69–80.
[13] J. R. Schue. Cartan decompositions for L^*-algebras. *Trans. Amer. Math. Soc.* **98** (1961), 334–349.

https://doi.org/10.1017/S0305004100065063 Published online by Cambridge University Press