Since the volume V of a cone of radius x and slant height 1 is given by

$$V = \tfrac{1}{3}\pi x^2\sqrt{1 - x^2},$$

it remains only to show that the function $g(x) = x^2\sqrt{1-x^2}$ is strictly superadditive on a sufficiently large interval $(0, B)$. Write $g(x) = xf(x)$, where

$$f(x) = x\sqrt{1 - x^2}$$
$$= \sqrt{x^2(1 - x^2)}.$$

Since the function $\sqrt{u(1-u)}$ is strictly increasing on $0 < u < \frac{1}{2}$, it follows that g is strictly superadditive on $0 < x < \sqrt{\frac{1}{2}}$.

Finally, let r and s denote the radii of the cones corresponding to the two sectors with angles a and b, respectively. Then

$$a = 2\pi r \qquad \text{and} \qquad b = 2\pi s.$$

Accordingly,

$$2\pi(r + s) = a + b \leqslant \tfrac{4}{3}\pi$$

so that

$$r + s \leqslant \tfrac{2}{3}.$$

The proof is over:

$$\frac{2}{3} < \sqrt{\frac{1}{2}}$$

and (1) follows at once.

Remarks

A simple induction argument shows that the n sector problem never has an optimal solution when $n > 2$. Moreover, in this case the combined volume of the n corresponding cones is always less than the maximal volume obtainable with two cones. As is well known, the optimal solution for the single cone problem need not rely on calculus: it follows almost at once from the inequality between the arithmetic and geometric means. It would be interesting to know if the two cone problem also has a purely algebraic solution.

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On 106.17: Alan Beardon writes: In Note 106.17 the authors seek all polynomials p that satisfy the functional relation

$$p(x)^2 = 1 + p(x+1)p(x-1).$$

First, by elementary manipulation, they reduce this to the relation $p(x)q(x+1) = p(x+1)q(x)$, where $q(x) = p(x+1) + p(x-1)$. Their argument then proceeds by several steps, each of which is based on finding,

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or comparing, the coefficients of certain polynomials. Readers may be interested in the following, more geometric, perspective on their argument.

We can write $p(z) = p_1(z)r(z)$ and $q(z) = q_1(z)r(z)$, where p_1 and q_1 have no common zero; then $p_1(x)q_1(x+1) = p_1(x+1)q_1(x)$. If p_1 is not constant, then it has a zero z_0, and as $p_1(z_0) = 0$ and $q_1(z_0) \neq 0$, we see that $p(z_0+1) = 0$. Since this implies the false assertion that p_1 has zeros $z_0, z_0+1, z_0+2, \ldots$, we see that p_1 must be a constant polynomial. The same holds for q_1 so there is some constant λ such that $\lambda p(z) = p(z+1) + p(z-1)$. By considering the leading term on each side we find that $\lambda = 2$ so that, for all real x, $2p(x) = p(x+1) + p(x-1)$. This shows that $(x, p(x))$ is the mid-point of the segment with endpoints $(x-1, p(x-1))$ and $(x+1, p(x+1))$. However, as p'' has only a finite number of zeros, we see that, for some x_0, either $p'' \geqslant 0$ on $\{x > x_0\}$, or $p'' \leqslant 0$ on $\{x > x_0\}$; thus p is either convex, or concave, on $\{x > x_0\}$. Now if a function f is convex (or concave) on an interval, and if its graph contains three collinear points, then f is linear on the smallest interval which contains these points. Thus p is linear on some interval, and so is a linear polynomial.

We could also argue as follows. Suppose that the graph of a polynomial p has three collinear points, say $(a, p(a))$, $(b, p(b))$ and $(c, p(c))$, where $a < b < c$. Then, by the mean value theorem, there are s and t with $a < s < b < t < c$ and

$$p'(s) = \frac{p(b) - p(a)}{b - a} = \frac{p(c) - p(b)}{c - b} = p'(t).$$

Another application of the mean value theorem shows that for some r with $a < s < r < t < c$ we have $p^n(r) = 0$. It follows that a non-linear polynomial cannot satisfy the collinearity assumption for infinitely many *mutually disjoint* intervals $\{a, c]$.

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On 'How to impress a chemist – again!': Paul Stephenson writes: In [1], Nick Lord makes a radial slit in a filter paper disk and overlaps the edges by some angle to form a cone whose generator is a radius of the disk. A paper clip allows him to alter the angle of overlap and therefore the apical angle. He then asks: "For a given slant height, what apical angle maximises the volume?" He finds the answer is $\cos^{-1}(-\frac{1}{3})$, the bond angle of methane.

Make the following thought experiment. Take the right tetrahedron on an equilateral base tangent to this special cone. It shares an apex with the cone and the lines of tangency are the medians of the faces through the apex. Fit four together so that their apices coincide. The bases of these shallow tetrahedra become the faces of a regular tetrahedron. You have modelled the minimal surface spanning its edges.

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