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INTRODUCTION TO THE SPECIAL ISSUE ON INVERSE PROBLEMS

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A structural econometric model may be defined by three elements: a "parameter" φ (which may be functional) defined by the economic theory, a data generating mechanism characterized, for example, by an i.i.d. sample of a random elements X whose distribution is denoted F, and a link between these two components. This link takes the form of an equation $A(\varphi, F) = 0$ associating the object of interest, φ, with the object that may be estimated, F. In this construction the parameter is only implicitly defined (by contrast with the usual setting, where the parameter is a direct transformation of F). Natural questions are then existence and unicity of the solution or, in more econometric terms, overidentification or identification of the model.

This construction is very common in econometrics, at least in the case where φ is a finite dimensional parameter. Historically the first examples of a such a construction were probably the linear simultaneous equations models where φ denotes the parameters of the structural form and F appears through the linear regression coefficients of the reduced form. The usual GMM method generates a class of examples where $A(\varphi, F)$ takes the form $E^F(h(\varphi, F)) = 0$.

As this model is very often underidentified, the equation $A(\varphi, F) = 0$ has no solution for a general F, and the main way to address that question is to replace the resolution by the minimization of a norm $||A(\varphi, F)||$.

The inverse problem theory considers the extension of this approach to the case where φ is a function and A takes its values in a functional space. The main implication of this extension is that the value of φ that minimizes $||A(\varphi, F)||$ is not a continuous function of F: The problem becomes ill-posed. This implies that the estimation of φ obtained by replacing F by an estimate is not in general consistent. This question is then treated by a regularization of the problem, for example by the addition of a penalization term $\alpha||\varphi||$ to the norm $||A(\varphi, F)||$.

The theory of inverse problems has a long history in numerical analysis and is also relevant in case of finite but high dimensional problems. It has been applied

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to many situations, the image treatment for example. This theory has been seen as a suitable framework for econometric problems essentially for the analysis of the nonparametric estimation of instrumental variables models or in game theoretic models. Many other applications are now considered in the literature.

An important class of inverse problem is the class of linear problem (with respect to φ) when the equation takes the form $T_F\varphi = r_F$, where r_F is an estimable element and T_F is a linear operator, both functions of F. One of the main characteristics of econometric inverse problems is that both r_F and T_F should be estimated using the same sample. Another property of the statistical inverse problem is that the error on this estimation of r_F is not only bounded, but some properties (e.g., its variance) are obtainable.

This special issue of *Econometric Theory* presents a selection of recent developments of the application of the theory of the inverse problem to econometric identification and estimation. This is an exciting area with many future developments to anticipate. For example, the study of problems with singular equations is only in its infancy. The issues raised by nonstationarity and cross-sectional dependence have only recently been addressed.

The analysis of a parameter defined by a functional equation requires one to consider the identification of this parameter or equivalently the unicity of a solution. In a linear case this unicity is obtained if the operator is one-to-one. In many cases this operator is the conditional expectation operator, and its one-to-one property may be viewed as a dependence condition between, for example, the endogenous variables and the instruments. The paper of d'Haultfoeuille addresses this question and gives some conditions based on the characteristic function that imply the injectivity of the conditional expectation operator.

As usual in nonparametric statistics, a central question is the rate of convergence of the estimators and the optimality of this rate. This question is complex in econometric applications because nonparametric estimation is used at two levels: for the estimation of the equation and for the resolution of the equation. Under some conditions the regularity conditions for these two nonparametric estimations may be linked (thanks to the concept of Hilbert scale in particular) and an optimal rate may be derived, as is done in the paper by Chen and Reiss.

The paper by Kim and Linton applies the general theory of inverse problems to the estimation of IGARCH models in a semiparametric form where the dependence of the variance on the lagged value of the process is treated nonparametrically. The process considered is not weakly stationary, which departs from earlier treatments, although the first differences are stationary. The news impact function is the solution to a linear type one integral equation. An estimation method is proposed, and the asymptotic properties are derived.

The paper by Hoderlein and Holzmann presents an empirical application of linear inverse problems to the estimation of Engle curves. The model is in this case severely ill-posed due to assumed normality and leads to a low rate of convergence under very weak assumptions.

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The paper of Horowitz and Mammen considers a nonparametric additive model with an unknown link function. The unknown components are thereby implicitly related through a nonlinear integral equation. They propose an estimator that achieves the optimal rate of convergence regardless of the dimension of the explanatory variable. Thus, the estimator has no curse of dimensionality. Moreover, the asymptotic distribution of the estimator of each additive component is the same as it would be if the link function and the other components were known with certainty. Thus, asymptotically there is no penalty for not knowing the link function or the other components.

Carrasco and Florens propose a new estimator for the density of a random variable observed with an additive measurement error, which is a classic inverse problem. Their estimator is based on the spectral decomposition of the convolution operator, which is compact for an appropriate choice of reference spaces. The density is approximated by a sequence of orthonormal eigenfunctions of the convolution operator. The resulting estimator is shown to be consistent and asymptotically normal. Unlike previous work, they also allow for isolated zeros in the characteristic function of the error. They show that, in the presence of zeros, the problem is identified even though the convolution operator is not one-to-one. They propose two consistent estimators of the density.

The Johannes, Van Bellegem, and Vanhems paper studies the estimation of a nonparametric function φ from the linear inverse problem $r = T\varphi$ given estimates of r and T. The rate of convergence of the estimator is derived under two assumptions expressed in a Hilbert scale. The approach provides a unified framework that allows them to compare various sets of structural assumptions used in the econometrics literature. General upper bounds are derived for the risk of the estimator of the structural function φ as well as of its derivatives.

Florens, Johannes, and Van Bellegem consider nonparametric estimation of a regression function from conditional moment restrictions involving instrumental variables, which leads naturally to the study of an inverse problem. The rate of convergence of penalized estimators is studied in the case where the regression function is not identified from the conditional moment restriction. They also study the gain of modifying the penalty in the estimation, considering derivatives in penalty.

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