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Parametric resonances revisited: comparison with kinetic description

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Abstract. A previous paper, McKenzie and Hagfors (1996 *J. Plasma Phys.* **55**(3), 539–585) henceforth referred to as P1, discussed the excitation of parametric instability in a homogeneous plasma pumped by an external electrostatic field. The discussion was carried out in the fluid approximation, and growth rates and resonance frequencies for several different modes were calculated, some of which had previously been overlooked because of inconsistent cut-off of certain series expansions. In this brief note we show that the fluid treatment gives growth rates and resonance frequencies that agree well with the kinetic treatment.

1. Introduction

The observation of enhanced ion and Langmuir waves by scattering of electromagnetic waves in ionospheric modification experiments was first thought to be fully explained by RF induced parametric instabilities (PDI) and an oscillating two stream instability (OTSI), see Fejer (1979) and Stubbe (1996). Recent work has emphasized nonlinear effects and claims to explain the observed heater-induced phenomena better by treating the problem nonlinearly by the Zakharov equation (Dubois et al. 1993). Although this may be true, we have, nevertheless, found certain interesting features in the parametric instability treatment leading to hitherto overlooked extra growth modes. The problem in our previous work, P1, was treated in the fluid approximation. Here we carry out some numerical calculations in the kinetic approximation allowing for Landau damping and compare some of the results with those of the fluid description.

2. The coupled recursion relations and the dispersion equation for a harmonic pump field

Fourier–Floquet analysis of either the fluid equations or their kinetic (Vlasov) counterpart, describing harmonic pumping at angular frequency ω_0 of Langmuir and ion acoustic waves, lead to the following set of recursion relations for the Fourier

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transform of the density perturbation $N_{\alpha n}$ with $\alpha = \mathrm{e}, \mathrm{i}$ and $n =$ harmonic number:

$$\begin{aligned} d_{en} N_{en} &= \sum_{r=-\infty}^{r=\infty} a_{r,n}^{(i)} N_{\mathrm{e}(r+n)} \\ d_{in} N_{in} &= \sum_{r=-\infty}^{r=\infty} a_{r,n}^{(e)} N_{\mathrm{i}(r+n)}, \end{aligned} \tag{2.1}$$

where

$$a_{r,n}^{(\alpha)} = \sum_{q=-\infty}^{\infty} \frac{J_q(-x) J_{r-q}(x)}{d_{\alpha(q+n)}}$$

and where $J_n(x)$ is the Bessel function of order n, and where the argument x is defined as $x = ku_0/\omega_0$, a number defining the strength of the pump. The fluid and the kinetic equations have identical structure and are of the form

$$d_{\alpha n} = \begin{cases} \left[\Omega_\alpha^2 - (\omega + n\omega_0)^2\right]/\omega_{p\alpha}^2 & \text{(fluid)} \\ 1 + \chi_{\alpha n}^{-1} & \text{(kinetic)} \end{cases} \tag{2.2}$$

where Ω_α is the 'oscillator' frequency defined by

$$\Omega_\alpha^2 = \omega_{p\alpha}^2 + v_\alpha^2 k^2, \tag{2.3}$$

where:

$$\begin{aligned} \omega_{p\alpha}^2 &= \frac{N_\alpha q_\alpha^2}{m_\alpha \epsilon_0}, \\ v_\alpha^2 &= \gamma_\alpha \kappa T_\alpha / m_\alpha, \\ \lambda_\alpha &= \omega_{p\alpha}/v_\alpha = \text{Debye length}, \\ \gamma_\alpha &= \text{adiabatic index}, \\ \kappa &= \text{Boltzmann's constant}, \end{aligned}$$

and where the susceptibility in the kinetic representation is given by

$$\chi_\alpha(\omega) = \frac{\omega_{p\alpha}^2}{k^2 N_{\alpha 0}} \int_{-\infty}^{\infty} \frac{\mathbf{k} \cdot \partial f_{\alpha 0}/\partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \, d\mathbf{v}. \tag{2.4}$$

The formidable infinite array of coupled recursion relations can, for weak pump fields $x \ll 1$ be approximated by a 'resonant' interaction $n-1, n, n+1$, which for the system leads to a truncated dispersion equation of the form

$$D_n = \left(\frac{x}{2}\right)^2 (\omega_{\mathrm{pe}}\omega_{p\mathrm{i}})^2 \left[\frac{(d_{\alpha n+1} - d_{\alpha n})^2/d_{\alpha n+1} d_{\alpha n}}{D_{n+1}} + \frac{(d_{\alpha n-1} - d_{\alpha n})^2/d_{\alpha n-1} d_{\alpha n}}{D_{n-1}}\right], \tag{2.5}$$

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where $D(\omega) = 0$ is the dispersion equation for the unperturbed system and where $D_n = D(\omega + n\omega_0)$. With the fluid description we have:

$$\begin{aligned} D_n &= \left(\omega_n^2 - \omega_{\mathrm{u}}^2\right)\left(\omega_n^2 - \omega_{\mathrm{l}}^2\right) \\ \omega_n &= \omega + n\omega_0 \\ \omega_{\mathrm{u}}^2 &= \Omega_{\mathrm{e}}^2 + (\omega_{\mathrm{pe}}\omega_{pi})^2/\Omega_{\mathrm{e}}^2 \approx \omega_{\mathrm{pe}}^2 + v_{\mathrm{e}}^2 k^2 \\ \omega_{\mathrm{l}}^2 &= \Omega_{\mathrm{i}}^2 - (\omega_{\mathrm{pe}}\omega_{pi})^2/\Omega_{\mathrm{i}}^2 = v_{ia}^2 k^2/\left(1 + k^2\lambda_{\mathrm{e}}^2\right). \end{aligned} \tag{2.6}$$

Here ω_{u} and ω_{l} represent, respectively, the 'high' frequency Langmuir wave and the 'low' frequency ion-acoustic wave. In the kinetic description

$$D_n = (\omega_{\mathrm{pe}}\omega_{pi})^2(d_{en}\, d_{in} - a_{0,n}^{(i)}), \tag{2.7}$$

where the quantity $d_{\alpha,n}$ is now given in terms of the susceptibility according to the prescription (2.2).

The form of the dispersion equation (2.5) shows the nature of the possible resonant interaction which may lead to instability. For example, there are resonances between 'upshifted' $D_{n-1} \approx 0$ and $D_{n+1} \approx 0$ in which D_n plays the role of a passive catalyst. Similarly there are resonances between the $D_n \approx 0$ mode representing either a Langmuir or an ion-acoustic mode, and the downshifted $D_{n\pm1} \approx 0$ modes representing a shifted ion-acoustic or Langmuir wave.

3. Expressions for the oscillating two stream instability (OTSI) and the parametric decay instability (PDI) in the fluid description

In this section we develop simple formulae for the growth rates of the OTSI and the PDI. The OTSI describes the resonance between the upshifted and the downshifted modes, $D_{n\pm1} \approx 0$ with D_n as a catalyst. This occurs when the pump frequency ω_0 is close to but is slightly less than the Langmuir frequency ω_{u} so that D_1 shifted down and D_{-1} shifted up may interact at 'zero' frequency. The dispersion equation (2.5) then approximates to

$$\omega^2 - \delta'^2 = C_{\mathrm{u}}\delta', \tag{3.1}$$

where

$$\begin{aligned} \delta' &= \omega_0 - \omega_{\mathrm{u}} = \omega_0 - \omega_{\mathrm{pe}}\sqrt{1 + k^2\lambda_{\mathrm{e}}^2}, \\ \omega_{\mathrm{u}} &= \omega_{\mathrm{pe}}\sqrt{1 + k^2\lambda_{\mathrm{e}}^2}, \\ \delta &= \frac{\omega_0}{\omega_{\mathrm{pe}}} - 1 \qquad \text{mismatch}, \\ \omega_0 &= (1 + \delta)\omega_{\mathrm{pe}} \qquad \text{mismatch}. \end{aligned}$$

With these quantities substituted we find the following relations between δ and δ'

$$\delta' \approx \omega_{\mathrm{pe}}\left(\delta - \tfrac{1}{2}k^2\lambda_{\mathrm{e}}^2\right) \tag{3.2}$$

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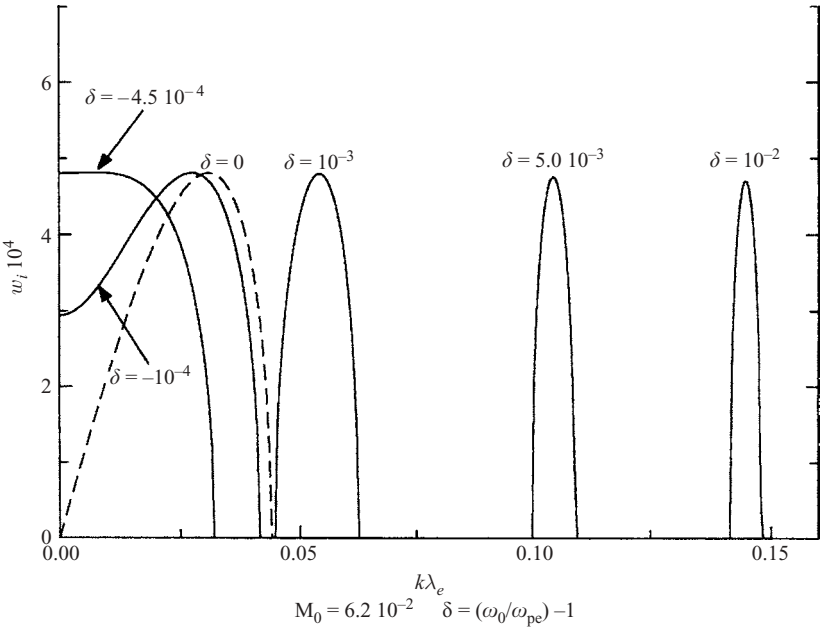


Figure 1. Growth rates of OTSI in the fluid description plotted against $k\lambda_e$ for a number of mismatch parameters δ. No collisional losses.

with C_u given by the expression

$$C_u = \frac{u_0^2 \omega_{pe} \sqrt{1 + k^2 \lambda_e^2}}{4\omega_0^2 \lambda_e^2}. \tag{3.3}$$

With the 'quiver' velocity u_0 expressed in terms of the Mach number $M_0 = u_0/v_e$ and with the assumption that $k^2\lambda_e^2$ is small we obtain the following dispersion relation:

$$\left(\frac{\omega}{\omega_{pe}}\right)^2 = \left(\delta - \frac{1}{2}k^2\lambda_e^2\right)\left[\left(\frac{M_0}{2(1+\delta)}\right)^2 + \delta - \frac{1}{2}k^2\lambda_e^2\right]. \tag{3.4}$$

The Mach number can also be expressed in terms of the pump field, E_0, the pump frequency, f_0, and the electron temperature, T_e, as

$$M_0 = (0.42)\frac{E_0}{f_0\sqrt{T_e}},$$

where the field is measured in V m^{-1}, the pump frequency in MHz and the electron temperature in units of 300 K. Figure 1 shows actual growth rates computed for a Mach number 6.2×10^{-2} for a range of values of δ. This will be compared with the corresponding growth rates derived from kinetic theory in the next section, see Figs 3 and 4.

In the case of the parametric decay instability (PDI), the dispersion relation takes the form, see Equation (45) of P1

$$(\omega - \omega_u)(\omega - (\omega_0 - \omega_l)) = -\gamma^2. \tag{3.5}$$

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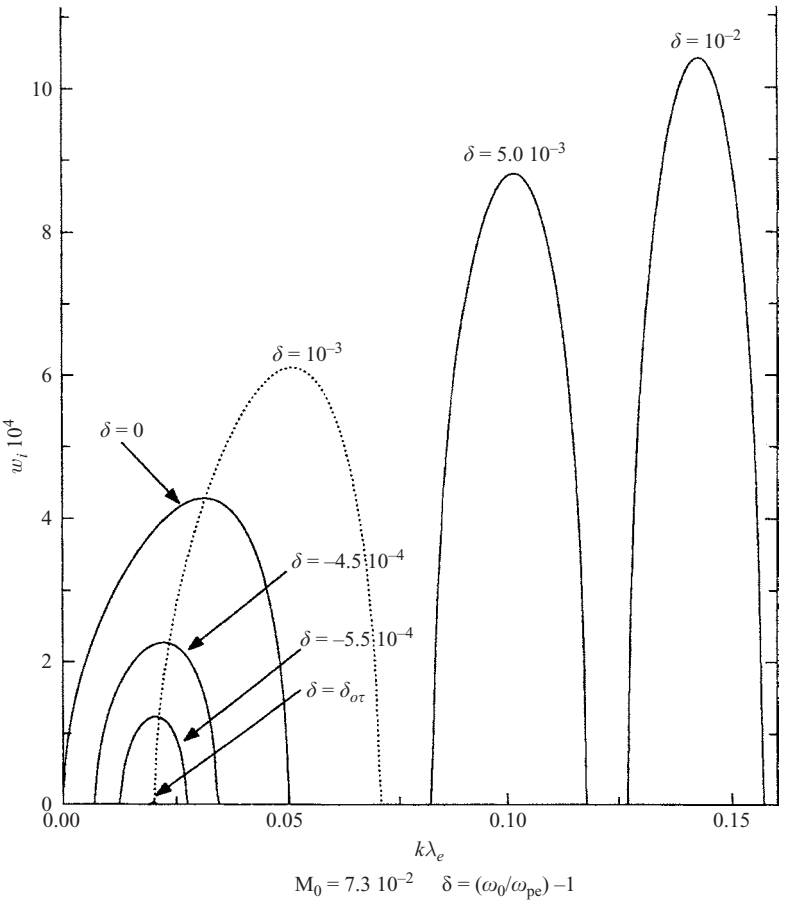


Figure 2. Growth rates of PDI in the fluid description plotted against $k\lambda_e$ for a number of mismatch parameters δ. No collisional losses.

Under the assumption that ω_l is negligible, see Equation (27c) of P1, we obtain for the complex frequency

$$\omega \approx \frac{(\omega_0 + \omega_u)}{2} \pm i\sqrt{\gamma^2 - (\omega_u - \omega_0)^2/4}. \tag{3.6}$$

Growth can only occur when the argument under the integral sign is positive. Substitution for γ in terms of (3.5) yields

$$\gamma^2 = \omega_{pe}^2 k\lambda_e \left(\frac{M_0}{2(1+\delta)}\right)^2 \left(\frac{m_e}{m_i}\right)^{1/2}. \tag{3.7}$$

We can derive the forms shown in Fig. 2 for the PDI in the fluid description. This is to be compared with the kinetic results.

Numerical results for the growth rates as a function of $k\lambda_e$ for one particular value of the pump amplitude, $M_0 = 7.3 \times 10^{-2}$ for different values of the mismatch parameter are shown. In the next section this will be compared with the corresponding growth rates derived from kinetic theory, examples of which are shown in Figs 5 and 6 later.

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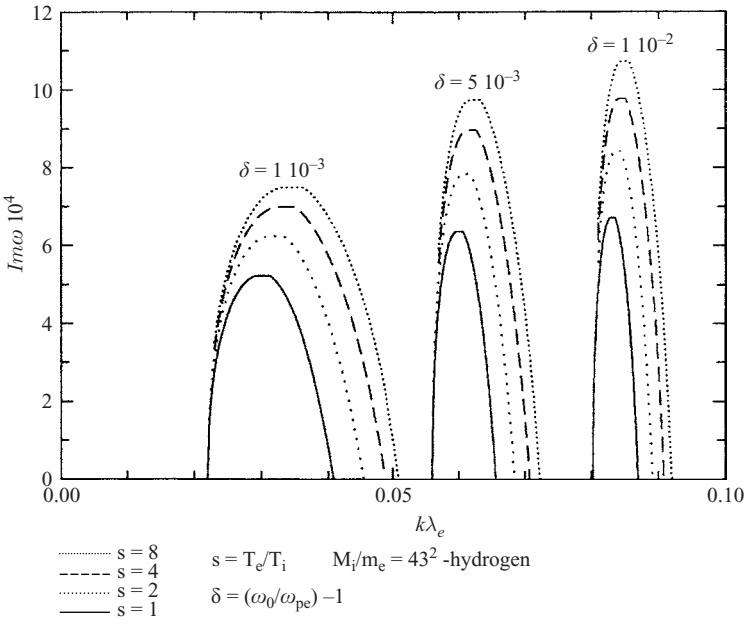


Figure 3. Growth rates of OTSI in the kinetic description plotted against $k\lambda_e$ for a number of mismatch parameters δ and temperature ratios $s = T_e/T_i$. Landau damping but no collisional losses. The plot is for hydrogen ions.

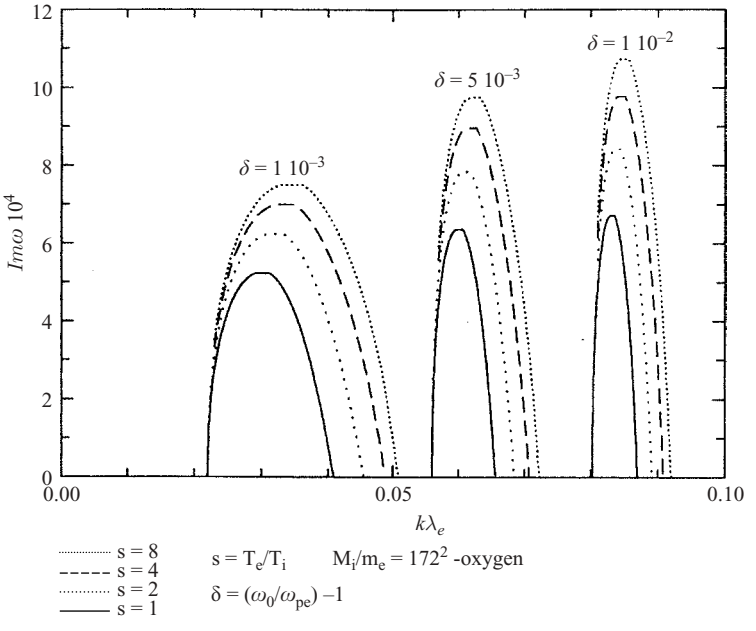


Figure 4. Growth rates of OTSI in the kinetic description plotted against $k\lambda_e$ for a number of mismatch parameters δ and temperature ratios $s = T_e/T_i$. Landau damping but no collisional losses. The plot is for oxygen ions.

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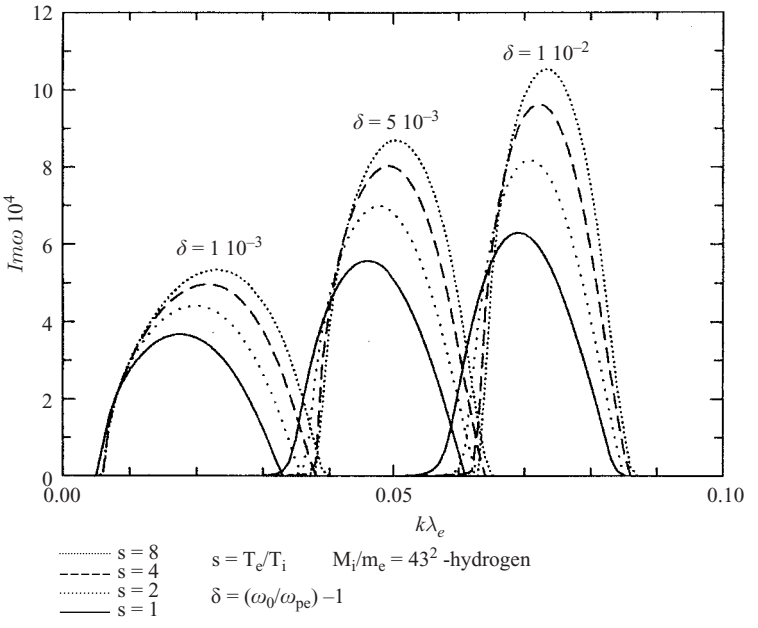


Figure 5. Growth rates of PDI in the kinetic description plotted against $k\lambda_e$ for a number of mismatch parameters δ and temperature ratios $s = T_e/T_i$. Landau damping but no collisional losses. The plot is for hydrogen ions.

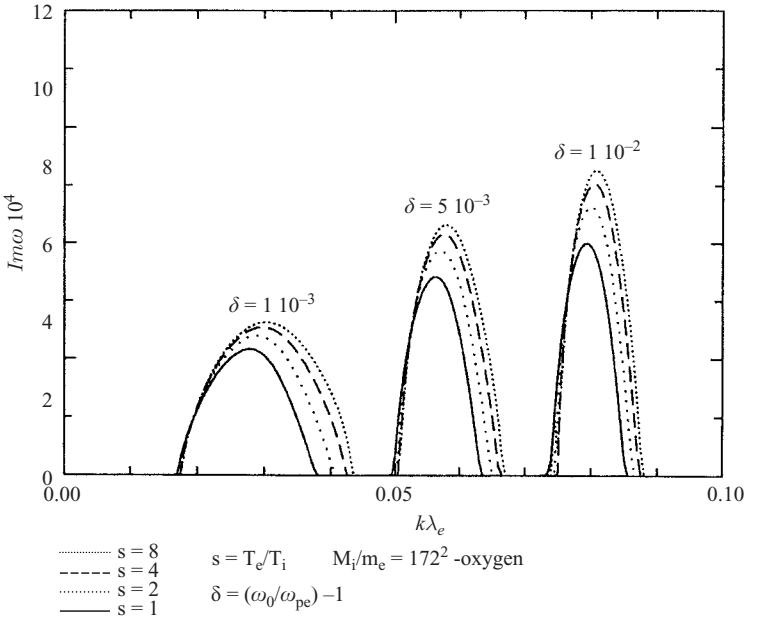


Figure 6. Growth rates of PDI in the kinetic description plotted against $k\lambda_e$ for a number of mismatch parameters δ and temperature ratios $s = T_e/T_i$. Landau damping but no collisional losses. The plot is for oxygen ions.

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4. The governing equation in the kinetic description, comparison and conclusions

The governing equations can be found in earlier papers by Sanmartin (1970), Hagfors and Gieraltowski (1972) and in Appendix A of the paper P1 referred to in the abstract. For the Fourier transform of the density fluctuations in sinusoidally moving reference frames, i.e. frames moving with the plasma particles in the externally impressed RF field, one has

$$N_{\mathrm{e}}(\omega)+\chi_{\mathrm{e}}(\omega)\left[N_{\mathrm{e}}(\omega)-\sum_{m=-\infty}^{m=\infty}J_m(-x)N_{\mathrm{i}}(\omega+m\omega_0)\right]=0 \tag{4.1}$$

$$N_{\mathrm{i}}(\omega)+\chi_{\mathrm{i}}(\omega)\left[N_{\mathrm{i}}(\omega)-\sum_{l=-\infty}^{l=\infty}J_l(-x)N_{\mathrm{e}}(\omega+l\omega_0)\right]=0, \tag{4.2}$$

where the susceptibility was defined in (2.4) and where the other quantities were all defined in Sec. 2.

Truncating these equation to allow for $N_{\mathrm{i}}(\omega)$ in the vicinity of $\omega=0$ and $N_{\mathrm{e}}(\omega)$ for $N_{\mathrm{e}}(\omega\pm l\omega_0)$ for $l=0,\pm1$, and denoting these four quantities as $N_{\mathrm{i}}^0, N_{\mathrm{e}}^0, N_{\mathrm{e}}^1$ and N_{e}^{-1} we obtain the following dispersion equation for the truncated system:

$$\chi_{\mathrm{e}}^0+\chi_{\mathrm{i}}^0+\chi_{\mathrm{e}}^0\chi_{\mathrm{i}}^0\left(1-J_0^2\right)-\chi_{\mathrm{i}}^0\left(1+\chi_{\mathrm{e}}^0\right)\left(J_1^2\frac{\chi_{\mathrm{e}}^1}{1+\chi_{\mathrm{e}}^1}+J_{-1}^2\frac{\chi_{\mathrm{e}}^{-1}}{1+\chi_{\mathrm{e}}^{-1}}\right)=0. \tag{4.3}$$

Using the same values for the mismatch parameter δ and for the relative quiver velocity u_0/v_{e} we computed the growth rate as a function of $k\lambda_{\mathrm{e}}$ as in the fluid treatment. Figure 3 shows the results for the OTSI case for hydrogen ions, and Fig. 4 shows the same for oxygen, with the same parameters as in the fluid case in Fig. 1. In Figs 5 and 6 show the case of the classical PDI for hydrogen and oxygen respectively. This should be compared with the fluid results of Fig. 2.

We see that there is good agreement between the two approaches as far as the structure of the growth rates as a function of $k\lambda_{\mathrm{e}}$ is concerned, and even for the peak growth rates. There is not an exact correspondence between the two methods, however, and this we assume stems from the slightly different definitions of the thermal velocities and the definition of the 'oscillator' frequencies Ω_α and their implicit equivalents in the kinetic description. The thermal velocities and the ion-acoustic velocities depend on $T_{\mathrm{e}}/T_{\mathrm{i}}$ and are determined by a series expansion, which is not always accurate. Hence, on the whole we find the comparison of the results satisfactory, and as good as is to be expected.

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