Math. Proc. Camb. Phil. Soc. (1988), **104**, 317
Printed in Great Britain

A generalized Bernstein approximation theorem

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(*Received* 23 *April* 1987; *revised* 2 *December* 1987)

1. *Introduction*

A celebrated theorem of Weierstrass states that any continuous real-valued function f defined on the closed interval $[0,1] \subset \mathbb{R}$ is the limit of a uniformly convergent sequence of polynomials. One of the most elegant and elementary proofs of this classic result is that which uses the Bernstein polynomials of f

$$(B_n f, x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \quad (x \in [0,1]),$$

one for each integer $n \geqslant 1$. Bernstein's Theorem states that $B_n(f) \to f$ uniformly on $[0,1]$ and, since each $B_n(f)$ is a polynomial of degree at most n, we have as a consequence Weierstrass' theorem. See for example Lorentz [**9**]. The operator B_n defined on the space $C([0,1]; \mathbb{R})$ with values in the vector subspace of all polynomials of degree at most n has the property that $B_n(f) \geqslant 0$ whenever $f \geqslant 0$. Thus Bernstein's Theorem also establishes the fact that each positive continuous real-valued function on $[0,1]$ is the limit of a uniformly convergent sequence of positive polynomials. This raises the following natural question: consider a compact Hausdorff space X and the convex cone $C^+(X) := \{f \in C(X; \mathbb{R}); f \geqslant 0\}$. Now the analogue of Bernstein's Theorem would be a theorem stating when a convex cone contained in $C^+(X)$ is dense in it. More generally, one raises the question of describing the closure of a convex cone contained in $C(X; \mathbb{R})$, and, in particular, the closure of $A^+ := \{f \in A; f \geqslant 0\}$, where A is a subalgebra of $C(X; \mathbb{R})$.

The present paper is concerned with such generalizations of Bernstein's Theorem. Let $A \subset C(X; \mathbb{R})$ and $f \in C(X; \mathbb{R})$. The distance of f from A is defined to be

$$\operatorname{dist}(f; A) := \inf\{\|f-g\|; g \in A\}$$

where, for any $h \in C(X; \mathbb{R})$, $\|h\| = \sup\{|h(x)|; x \in X\}$. Clearly f belongs to the closure of A if, and only if, $\operatorname{dist}(f; A) = 0$. When A is an algebra, a 'strong' version of the Stone–Weierstrass Theorem states that

$$\operatorname{dist}(f; A) = \sup_{x \in X} \operatorname{dist}(f_{[x]}; A_{[x]}) \qquad (*)$$

(see the end of this section for explanation of notation). In fact in this version A need not be an algebra: it can be a vector subspace $A \subset C(X; \mathbb{R})$ which is a module over some subalgebra $M \subset C(X; \mathbb{R})$, i.e. $MA \subset A$, and now $[x]$ in (*) denotes the equivalence class of the point x modulo the equivalence relation defined by M, $x \equiv y \pmod{M}$ if, and only if, $\phi(x) = \phi(y)$ for all $\phi \in M$. It is in this strong form that we present our generalization of Bernstein's Theorem: let A be a non-empty subset

https://doi.org/10.1017/S030500410006549X Published online by Cambridge University Press

of $C(X;\mathbb{R})$ such that $A+A\subset A$; let $M\subset C(X;\mathbb{R})$ be another non-empty subset such that $MA\subset A$. Assume that M has the following properties: (1) each $\phi\in M$ is such that $0\leqslant\phi\leqslant 1$; (2) the function $1-\phi$ belongs to M, for each $\phi\in M$; and (3) the product $\phi\psi\in M$, whenever ϕ and ψ belong to M. Then formula (*) is true for any $f\in C(X;\mathbb{R})$. See Theorem 1 below, where we prove a vector-valued version of this result, and Theorem 2, where A is a convex cone.

In Corollary 5 we describe the closure of $A^+=\{f\in A; f\geqslant 0\}$ where A is some subalgebra of $C(X;\mathbb{R})$, in a way perfectly analogous to the classical Stone–Weierstrass description of the closure of A. Corollary 5 follows from Theorem 3 which gives formula (*) for A^+, in terms of the equivalence classes mod A^+. On the other hand, Bernstein's Theorem follows from Theorem 1 by using Corollary 1 and 2. (See Corollary 8 below.) In fact, even a type of Bohman–Korovkin's theorem on positive operators can be proved by this method. (See §3.)

In §4 we show that our Theorem 1 implies the usual Stone–Weierstrass theorem for modules or algebras. These theorems go back to Glicksberg [**6**], but it was L. Nachbin who established their importance, showing how they provide a framework for the solution of many problems in weighted approximation (i.e. the case in which X is not assumed to the compact). See his article [**11**], where it is clear that the two conditions in the statement of the Stone–Weierstrass theorem, which both refer to the algebra A in the case of algebras, have to be split in the case of modules. One condition refers to equivalence classes modulo the algebra M of multipliers and the other condition refers to the behaviour of the module A itself when restricted to such equivalence classes. (See for example Corollary 1 below.)

In §5 we give an analogous result for convex cones of Machado's generalization of Bishop's Stone–Weierstrass Theorem. (See Machado [**10**] or Burckel [**4**].) It generalizes théorème 8 of [**6**], and gives more information.

A weaker version of some of the present results were presented by us at the International Workshop on Multivariate Approximation, Santiago, Chile, 15–19 December, 1986.

Let us define our notation and terminology. Throughout this paper X denotes a compact Hausdorff topological space, and E denotes a normed space over the real or the complex field. The norm of E will be denoted by $\|\cdot\|$. The vector space of all continuous functions on X with values on E is denoted by $C(X;E)$ and for any $f\in C(X;E)$ we write $\|f\|=\sup\{\|f(x)\|; x\in X\}$. For any non-empty subset $S\subset X$ and $f\in C(X;E)$ we denote by f_S the restriction of f to S. Clearly f_S belongs to $C(S;E)$. If $A\subset C(X;E)$, then $A_S=\{f_S; f\in A\}$. The vector space $C(S;E)$ is equipped with its own supremum norm, which we denote by $\|\cdot\|_S$, and to simplify notation we set $\|f\|_S:=\|f_S\|_S$, whenever $f\in C(X;E)$. If $A\subset C(X;E)$ and $\bar{A}$ denotes its closure in $C(X;E)$, then

$$\operatorname{dist}(f_S;A_S)=\operatorname{dist}(f_S;(\bar{A})_S)$$

if $f\in C(X;E)$. Sometimes this fact will be used without explicit mention. It follows easily from A_S being densely contained in $(\bar{A})_S$.

The set of bounded functions on X is denoted by $l_\infty(X;\mathbb{R})$, and is equipped with the sup-norm. Since X is assumed to be compact, $C(X;\mathbb{R})\subset l_\infty(X;\mathbb{R})$.

https://doi.org/10.1017/S030500410006549X Published online by Cambridge University Press

2. *Convex cones*

In this section we prove our main result establishing the formula

$$\operatorname{dist}(f; A) = \sup_{x \in X} \operatorname{dist}(f_{[x]}; A_{[x]})$$

for convex cones $A \subset C(X; E)$ which are modules over a set $M \subset C(X; [0,1])$ with some property. The consideration of modules is essential for vector valued theorems, but even in the scalar case they are important since they allow one to describe the closure of vector subspaces and of convex cones which are not themselves algebras.

Definition 1. A set $M \subset C(X; [0,1])$ has property V if $0 \in M$ and

(i) $\phi \in M$ implies $1 - \phi \in M$;

(ii) $\phi, \psi \in M$ implies $\phi\psi \in M$.

Property V was defined by Jewett [**8**], who says that the idea of considering such a property comes from von Neumann who states, without proof, in [**13**] that the smallest closed subset of $C([0,1]^m; [0,1])$ having property V, containing the projections and at least one constant $0 < c < 1$, is $C([0,1]^m, [0,1])$ itself. In his paper [**8**], Jewett gives von Neumann's statement as a corollary to his version of the Stone–Weierstrass Theorem characterizing closed, point-separating subsets of $C(X; [0,1])$ having property V. Lemma 2 and theorem 1 of Jewett [**8**] are used in the proof of Lemma 1 below. Notice that the proof of Lemma 1 is inspired by a result of Brosowski and Deutsch [**3**], as modified by Ransford [**15**]. Lemma 2 of Jewett [**8**] was used by S. Machado [**10**] in his strong version of Bishop's generalized Stone–Weierstrass Theorem (see also Prolla [**14**], chapter 1, §8, and Burckel [**4**]).

Throughout this section, for any $x \in X$, $[x]$ will denote its equivalence class modulo M, i.e. $[x] = \{t \in X; \phi(t) = \phi(x) \text{ for all } \phi \in M\}$.

Example 1. Let $W \in C(X; E)$ be a real vector subspace, and let A be the set of its multipliers (see Ellis [**5**]), i.e.

$$A := \{\phi \in C(X; \mathbb{R}); \phi f \in W \quad \text{for all} \quad f \in W\}.$$

It is easy to see that A is a unital subalgebra of $C(X; \mathbb{R})$, which is closed, when W is closed. Let $M := A \cap C(X; [0,1])$. Clearly M has property V and W is an M-module, i.e. $MW \subset W$.

Example 2. Let A be a subset of $C(X; E)$ such that $A + A \subset A$ and $0 \in A$. The set M of the multipliers of A is the collection of all $\phi \in C(X; [0,1])$ such that $\phi f + (1-\phi) g \in A$ whenever $f, g \in A$. It is easy to see that M has property V, and that $MA \subset A$. When A is a convex cone in $C(X; E)$ the definition above is due to Feyel and de La Pradelle [**6**].

Lemma 1. *Let $M \subset C(X; [0,1])$ be a set having property V. Given any $x \in X$ and any open neighbourhood $N \supset [x]$, there is an open neighbourhood $V \supset [x]$, contained in N, and such that for each $0 < \delta < 1$ there is $\phi \in \overline{M}$, such that*

(1) $\phi(t) < \delta$ *for all* $t \notin N$, *and*

(2) $1 - \delta < \phi(t)$ *for all* $t \in V$.

Proof. Let K be the complement of N. For each $y \in K$ there is some $\phi_y \in M$ such that

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$\phi_y(y) < \phi_y(x)$. By lemma 2 of Jewett[8], we can find a polynomial of the form $p_y(t) = (1-t^m)^n$ such that $p_y(\phi_y(x)) < \frac{1}{4}$ and $p_y(\phi_y(y)) > \frac{3}{4}$. Let

$$\mathscr{V}_y = \{t \in X\,;\, p_y(\phi_y(t)) > \tfrac{3}{4}\}.$$

Then $\mathscr{V}_y$ is an open neighbourhood of y, disjoint from $[x]$, because $p_y(\phi_y) \in M$. By compactness, K is contained in a finite union, say $K \subset \mathscr{V}_{y_1} \cup \ldots \cup \mathscr{V}_{y_m}$. For each $i = 1, \ldots, m$, let ϕ_i denote $p_{y_i}(\phi_{y_i})$, and let $\psi = \phi_1 \vee \ldots \vee \phi_m$. By theorem 1 of Jewett[8], $\overline{M}$ is a lattice. Hence $\psi \in \overline{M}$. Notice that $\psi(x) < \frac{1}{4}$, and $\psi(t) > \frac{3}{4}$ for all $t \in K$. Define $V = \{t \in X\,;\, \psi(t) < \frac{1}{4}\}$. Clearly $V \supset [x]$ and $N \supset V$. For each $n \geqslant 1$ define $h_n = (1-\psi^n)^{3^n}$. Then $h_n \in \overline{M}$ and for each $t \in V$, Bernoulli's inequality implies that

$$h_n(t) = (1-\psi^n(t))^{3^n} \geqslant 1-(3\psi(t))^n \geqslant 1-(\tfrac{3}{4})^n.$$

On the other hand, for each $t \in K$ we have $\psi(t) > \frac{1}{2}$, and therefore

$$h_n(t) = (1-\psi^n(t))^{3^n} \leqslant (1+\psi^n(t))^{-3^n} \leqslant (3^n\psi^n(t))^{-1} \leqslant (\tfrac{2}{3})^n.$$

Hence, given $0 < \delta < 1$, $\phi := h_n$, for n sufficiently large, will satisfy (1) and (2). ∎

THEOREM 1. *Let $A \subset C(X\,;E)$ be a non-empty subset such that $A+A \in A$ and $MA \subset A$, where $M \subset C(X\,;[0,1])$ has property V. Then for each $f \in C(X\,;E)$ we have*

$$\operatorname{dist}(f\,;A) = \sup_{x \in X} \operatorname{dist}(f_{[x]}\,;A_{[x]})$$

and, in fact, there is some equivalence class $[x]$ such that

$$\operatorname{dist}(f\,;A) = \operatorname{dist}(f_{[x]}\,;A_{[x]}).$$

Proof. Let $d := \operatorname{dist}(f\,;A)$, $d(x) := \operatorname{dist}(f_{[x]}\,;A_{[x]})$ and $d^* := \sup\{d(x)\,;x \in X\}$.

Clearly $d(x) \leqslant d$ for all $x \in X$, and therefore $d^* \leqslant d$.

Let $\epsilon > 0$ be given. For each $x \in X$, choose $g_x \in A$ such that $\|f-g_x\|_{[x]} < d^*+\epsilon$. Then $N_x = \{t \in X\,;\, \|f(t)-g_x(t)\| < d^*+\epsilon\}$ is an open neighbourhood of $[x]$. Choose another open neighbourhood V_x according to Lemma 1. Since X is compact, there are finitely many points $x_1, x_2, \ldots, x_m$ in X, such that X is contained in the union of the family $\{V_{x_1}, V_{x_2}, \ldots, V_{x_m}\}$. Choose $\delta > 0$ so that $\delta < \min(1, (km)^{-1}\epsilon)$, where $k = \max\{\|f\|, \|f-g_{x_1}\|, \ldots, \|f-g_{x_m}\|\}+1$. For $i = 1, 2, \ldots, m$, choose $\phi_i \in \overline{M}$ such that

$$\phi_i(t) < \delta \quad \text{for all} \quad t \notin N_{x_i}, \tag{1}$$

$$1-\delta < \phi_i(t) \quad \text{for all} \quad t \in V_{x_i}. \tag{2}$$

Define $h_1 = \phi_1$, $h_2 = (1-\phi_1)\phi_2, \ldots, h_m = (1-\phi_1)\ldots(1-\phi_{m-1})\phi_m$. Then

$$h_i \in \overline{M} \ (i = 1, 2, \ldots, m)$$

and

$$h_1+\ldots+h_m = 1-(1-\phi_1)(1-\phi_2)\ldots(1-\phi_m). \tag{3}$$

Given $x \in X$, there is some index $i \leqslant m$ such that $x \in V_{x_i}$. By (2), $\phi_i(x) > 1-\delta$, and then

$$1 \geqslant \sum_{i=1}^{m} h_i(x) = 1-(1-\phi_i(x)) \prod_{j \neq 1} (1-\phi_j(x)) > 1-\delta. \tag{4}$$

On the other hand $h_i(t) \leqslant \phi_i(t)$ and (1) implies that

$$0 \leqslant h_i(t) < \delta \quad \text{if} \quad t \notin N_{x_i}. \tag{5}$$

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Let $g = h_1 g_{x_1} + \ldots + h_m g_{x_m}$. Since $\bar{A}$ is an $\bar{M}$-module, and $\bar{A}+\bar{A} \subset \bar{A}$, we see that $g \in \bar{A}$. For each $x \in X$ we have

$$\|f(x)-g(x)\| \leqslant \left\|f(x)-\sum_{i=1}^{m} h_i(x) f(x)\right\| + \left\|\sum_{i=1}^{m} h_i(x)\,(f(x)-g_i(x))\right\|, \qquad (*)$$

where we have written $g_i = g_{x_i}$ for each $i = 1, 2, \ldots, m$. For the first term on the right-hand side of (*) the following estimate holds:

$$\left\|f(x)-\sum_{i=1}^{m} h_i(x) f(x)\right\| \leqslant \left|1-\sum_{i=1}^{m} h_i(x)\right| \|f\| \leqslant \delta \|f\| \leqslant \epsilon$$

where we have used (4) to obtain $\left|1-\sum_{i=1}^{m} h_i(x)\right| < \delta$.

To estimate the second term, define

$$I_x := \{i \leqslant m;\, x \in N_{x_i}\} \quad \text{and} \quad J_x := \{i \leqslant m;\, x N_{x_i}\}.$$

Then for all $i \in I_x$ we have

$$h_i(x)\,\|f(x)-g_i(x)\| \leqslant h_i(x)\,(d^*+\epsilon) \leqslant d^*+\epsilon$$

and for $i \in J_x$ we get

$$h_i(x)\,\|f(x)-g_i(x)\| \leqslant \delta \|f-g_i\| < \delta k$$

where we have used (5) to obtain $h_i(x) < \delta$.

Hence for the second term on the right-hand side of (*) the following estimate is true:

$$\left\|\sum_{i=1}^{m} h_i(x)\,(f(x)-g_i(x))\right\| < d^*+\epsilon+\delta k m < d^*+2\epsilon. \qquad (7)$$

From (6) and (7), $\|f-g\| \leqslant d^*+3\epsilon$. Hence $\operatorname{dist}(f; \bar{A}) \leqslant d^*+3\epsilon$, and from this $d = \operatorname{dist}(f; A) = \operatorname{dist}(f; \bar{A}) \leqslant d^*$.

Now let Y denote the quotient space of X by the equivalence relation $(\operatorname{mod} M)$, and let $\pi := X \to Y$ be the quotient map. Since Y is a compact Hausdorff space, the map $d_g : Y \to \mathbb{R}_+$ defined by $d_g(y) := \sup\{\|f(t)-g(t)\|;\, t \in \pi^{-1}(y)\}$ is upper semi-continuous, for each $g \in A$. Hence the map $h(y) := \inf\{d_g(y);\, g \in A\}$ for $y \in Y$ is upper semi-continuous too. Therefore it attains its supremum at some point $y_0 \in Y$. Take any $x \in \pi^{-1}(y_0)$. Clearly $d(x) = h(y_0) = \sup_{y \in Y} h(y) = d^*$. ∎

Second proof. By Zorn's Lemma the collection $\mathscr{F}$ of all closed non-empty subsets $T \subset X$ such that $\operatorname{dist}(f; A) = \operatorname{dist}(f_T; A_T)$ has a minimal element S. We claim that $S \subset [x]$ for some $x \in X$. If this is false, there is a pair of points $y, z \in S$ such that $y \not\equiv z \,(\operatorname{mod} M)$. Hence $\psi(y) \neq \psi(z)$ for some $\psi \in M$ and we may assume $\psi(z) < \psi(y)$. By lemma 2 of Jewett[8], we can find a polynomial of the form $p(t) = (1-t^m)^n$ for $t \in [0,1]$, such that $p(\psi(y)) < \frac{1}{3}$ and $p(\psi(z)) > \frac{2}{3}$. Let $\phi = p \circ \psi$. Clearly $\phi \in M$, because M has property V. We now note that $Y = S \cap \phi^{-1}([0, \frac{2}{3}])$ and $Z = S \cap \phi^{-1}([\frac{1}{3}, 1])$ are both non-empty proper closed subsets of S. By minimality of S, there exist $v, w \in A$ such that $\|f-v\|_Y < \operatorname{dist}(f; A)$ and $\|f-w\|_Z < \operatorname{dist}(f; A)$. For each integer $n \geqslant 1$, put $h_n = (1-\phi^n)^{2^n}$ and $g_n = h_n v + (1-h_n) w$. Clearly $h_n \in M$ and $g_n \in A$. Now we can argue exactly as in Ransford[15] to get

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$\|f-g_n\|_S < \operatorname{dist}(f; A)$, for n sufficiently large. This contradicts the fact that $S \in \mathscr{F}$. Hence $S \subset [x]$ for some $x \in X$. It now remains merely to notice that this implies $\operatorname{dist}(f_S; A_S) \leqslant \operatorname{dist}(f_{[x]}; A_{[x]})$. Hence $\operatorname{dist}(f; A) = \operatorname{dist}(f_{[x]}; A_{[x]})$. ∎

THEOREM 2. *Let $A \subset C(X; E)$ be a convex cone and let M be the set of its multipliers. Then, for each $f \in C(X; E)$, there is some $x \in X$ such that* $\operatorname{dist}(f; A) = \operatorname{dist}(f_{[x]}; A_{[x]})$.

Proof. The set M has property V, and A, being a convex cone, satisfies $A + A \subset A$. Since $0 \in A$, it follows that $MA \subset A$. It now suffices to apply Theorem 1. ∎

COROLLARY 1. *Let $A \subset C^+(X)$ be a convex cone satisfying the following two conditions:*

(1) *Given any two distinct points x and y in X, there is some multiplier ϕ of A such that $\phi(x) \neq \phi(y)$;*

(2) *Given any point $x \in X$, there is some $g \in A$ such that $g(x) > 0$.*

Then A is dense in $C^+(X)$.

Proof. Let $f \in C^+(X)$ be given. By (1) and Theorem 2 there is some point $x \in X$ such that $\operatorname{dist}(f; A) = \operatorname{dist}(f(x); A(x))$. By condition (2), we have $A(x) = \mathbb{R}_+$. However $\operatorname{dist}(f(x); \mathbb{R}_+) = 0$, because $f(x) \geqslant 0$. ∎

We can apply Corollary 1 to the problem of uniform approximation by polynomials with positive coefficients. Let $\phi \in C(X; [0, 1])$ be given. Consider the convex cone of all polynomials with positive coefficients in ϕ and $1-\phi$, i.e. $P \in A$ if and only if $P \in C^+(X)$ is of the form

$$P(x) = \Sigma a_{kl} \phi^k (1-\phi)^l \quad (a_{kl} \geqslant 0)$$

where the sum is over all positive integers k and l such that $k+l \leqslant n$, for some $n \in \mathbb{N}$ (depending on P). Clearly ϕ belongs to the set M of multipliers of A. Assume that ϕ is one-to-one. Then condition (1) of Corollary 1 is satisfied. On the other hand, at each $x \in X$ either $\phi(x) > 0$ or $1-\phi(x) > 0$, because they cannot be both zero, and condition (2) is satisfied too. By Corollary 1, the convex cone A is dense in $C^+(X)$.

COROLLARY 2. *Let $A \subset C(X; E)$ be a convex cone which is a $C(X; [0, 1])$-module. Then for each $f \in C(X; E)$ there is some point $x \in X$ such that* $\operatorname{dist}(f; A) = \operatorname{dist}(f(x); A(x))$.

Proof. For any two distinct points $x \neq y$ in X there is some $\phi \in C(X; [0, 1])$ such that $\phi(x) = 0$ and $\phi(y) = 1$. Clearly $C(X; [0, 1])$ has property V. It now suffices to apply Theorem 2. ∎

COROLLARY 3. *Let V be a convex cone in E and let A be either $C(X; [0, 1]) \otimes V$ or $C^+(X) \otimes V$. Then for each $f \in C(X; E)$ there is some point x in X such that* $\operatorname{dist}(f; A) = \operatorname{dist}(f(x); V)$. *Consequently A is dense in $C(X; V)$.*

Proof. A is a $C(X; [0, 1])$-module and $A(x) = V$ for any $x \in X$. ∎

COROLLARY 4. *Let X and Y be two compact Hausdorff spaces. Then $C^+(X) \otimes C^+(Y)$ is dense in $C^+(X \times Y)$.*

Proof. Any function $\phi \in C(X; [0, 1])$ (resp. $\psi \in C(Y; [0, 1])$) is a multiplier of the convex cone $C^+(X) \otimes C^+(Y)$. Hence condition (1) of Corollary 1 is satisfied. Since the positive constants belong to $C^+(X) \otimes C^+(Y)$, condition (2) is satisfied too, and so the convex cone $C^+(X) \otimes C^+(Y)$ is dense in $C^+(X \times Y)$. ∎

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Remark. Corollary 4 can be easily extended to any product $X_1 \times X_2 \times \ldots \times X_m$, for finite m.

Let us finish this section by giving an example of an application of Theorem 1 in a case in which the set A is not a cone. Let $X = [0,1]$ and $A = \{f \in C([0,1]; \mathbb{R}); f(0) \in \mathbb{Z}$ and $f(1) \in \mathbb{Z}\}$. Then A is uniformly closed, $A+A \subset A$, $A-A \subset A$ and $0 \in A$, but A is not a cone. Furthermore $1 \in A$ and $AA \subset A$. Hence the set $M = \{\phi \in A; 0 \leqslant \phi \leqslant 1\}$ has property V and $MA \subset A$. Since the function ϕ defined by $\phi(t) = t$ for all $t \in [0,1]$ belongs to M, the equivalence classes mod M reduce to points. It is easy to see that in this case $A(0) = A(1) = \mathbb{Z}$ and that $A(t) = \mathbb{R}$ for all $0 < t < 1$. It is also easy to see that, given any two distinct points $x, y \in [0,1]$, there is some $\phi \in M$ such that $\phi(x) = 0$ and $\phi(y) = 1$. Hence Theorem 1 implies that for each $f \in C([0,1]; \mathbb{R})$ the following is true:

$$\operatorname{dist}(f; A) = \max(\operatorname{dist}(f(0); \mathbb{Z}), \operatorname{dist}(f(1); \mathbb{Z})).$$

It is known that A is the uniform closure in $C([0,1]; \mathbb{R})$ of the ring $\mathbb{Z}[x]$ of all polynomials with integral coefficients. Hence, for each $f \in C([0,1]; \mathbb{R})$,

$$\operatorname{dist}(f; \mathbb{Z}[x]) = \max(\operatorname{dist}(f(0); \mathbb{Z}), \operatorname{dist}(f(1); \mathbb{Z})).$$

3. *Bernstein's Theorem*

THEOREM 3. *Let $A \subset C(X; \mathbb{R})$ be a subalgebra and let $A^+ = \{f \in A; f \geqslant 0\}$. Then for each $f \in C(X; \mathbb{R})$ there is a point $x \in X$ such that*

$$\operatorname{dist}(f; A^+) = \operatorname{dist}(f_{[x]}; A^+_{[x]}),$$

and consequently

$$\operatorname{dist}(f; A^+) = \sup_{x \in X} \operatorname{dist}(f_{[x]}; A^+_{[x]}),$$

where $[x]$ denotes the equivalence class of x mod A^+.

Proof. Let A_1 be the algebra generated by A and 1. Then $A_1 A \subset A + A \subset A$. Hence $A_1^+ A^+ \subset A^+$. Define $M := \{f \in A_1; 0 \leqslant f \leqslant 1\}$. Clearly $MA^+ \subset A^+$, $A^+ + A^+ \subset A^+$ and M has property V. Hence, for $f \in C(X; \mathbb{R})$, there is $x \in X$ such that $\operatorname{dist}(f; A^+) = \operatorname{dist}(f_{[x]_1}; A^+_{[x]_1})$, where $[x]_1$ is the equivalence class of x mod A_1^+, because A_1^+ and M define the same equivalence class. Now $A^+ \subset A_1^+$ implies that $[x]_1 \subset [x]$, where $[x]$ denotes the equivalence class of x mod A^+. Hence $\operatorname{dist}(f_{[x]_1}; A^+_{[x]_1}) \leqslant \operatorname{dist}(f_{[x]}; A_{[x]})$, and then $\operatorname{dist}(f; A^+) = \operatorname{dist}(f_{[x]}; A^+_{[x]})$. ∎

COROLLARY 5. *Let $A \subset C(X; \mathbb{R})$ be a subalgebra and let $f \in C^+(X)$. Then f belongs to the closure of A^+ if, and only if, the following two conditions are satisfied:*

(1) *for any $x, y \in X$ such that $f(x) \neq f(y)$, there is some $g \in A^+$ such that $g(x) \neq g(y)$;*

(2) *for any $x \in X$ such that $f(x) > 0$, there is some $g \in A^+$ such that $g(x) > 0$.*

Proof. The conditions are easily seen to be necessary. Conversely, assume that (1) and (2) are true. By Theorem 3 there is some point $x \in X$ such that $\operatorname{dist}(f; A^+) = \operatorname{dist}(f_{[x]}; A^+_{[x]})$, where $[x]$ is the equivalence class of x mod A^+. By (1), f is constant on $[x]$. Let $u \geqslant 0$ be its value on $[x]$. If $u = 0$, then clearly $\|f - 0\|_{[x]} = 0$. If $u > 0$, then by (2) there is some $g \in A^+$ such that $g(x) > 0$. Now $h = (u/g(x))g$ belongs to A^+ and $h(t) = u$ for all $t \in [x]$. Hence $\|f - h\|_{[x]} = 0$, and $f_{[x]}$ belongs to $A^+_{[x]}$ i.e. $\operatorname{dist}(f_{[x]}; A^+_{[x]}) = 0$ and so $\operatorname{dist}(f; A^+) = 0$. ∎

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COROLLARY 6. *Let $A \subset C(X;\mathbb{R})$ be a subalgebra such that A^+ separates the points of X and such that, for each $x \in X$, there is some $f \in A^+$ such that $f(x) > 0$. Then A^+ is dense in $C^+(X)$.*

When $X = [a,b] \subset \mathbb{R}$ we can apply Corollary 6 to the algebra A of all algebraic polynomials and conclude that every $f \geqslant 0$ continuous on $[a,b]$ is the uniform limit of a sequence of positive polynomials. Let us now show that Theorem 1 via Corollary 1 implies even more, namely, that on the unit interval $I = [0,1]$, for any $f \in C^+(I)$, $B_n(f) \to f$ as $n \to \infty$ uniformly on I. Thus Theorem 1 is a generalization of the classical Bernstein approximation theorem.

We start proving a version of Bohman–Korovkin's Theorem.

THEOREM 4. *Let H_n $(n = 1,2,3,\ldots)$ be a sequence of monotone linear operators on $C(X;\mathbb{R})$. Assume that for some function $P: X \times X \to \mathbb{R}_+$ the following holds (where $P_t(x) := P(x,t)$ for all $x \in X$):*

(i) *for any open set V containing the diagonal Δ of X^2, $\inf\{P(x,t);(x,t) \notin V\} > 0$;*

(ii) *for each $t \in X$, $P_t \in C(X;\mathbb{R})$ and $H_n(P_t,t) \to 0$ as $n \to \infty$ uniformly on $t \in X$;*

(iii) *there exists a function $g \in C^+(X)$ such that $g > 0$ on X and $H_n(g) \to g$ as $n \to \infty$.*

Then $H_n(f) \to f$ as $n \to \infty$ for all $f \in C(X;\mathbb{R})$.

Proof. Since $f = f^+ - f^-$ for each f, it is enough to prove that $H_n(f) \to f$ as $n \to \infty$ for all $f \in C^+(X)$. Let A be the closed convex cone defined by

$$A = \{f \in C^+(X); H_n(f) \to f \quad \text{as} \quad n \to \infty\}.$$

By (iii), $H_n(g) \to g$ for some $g \in C^+(X)$ such that $g > 0$ on X. Therefore $g \in A$, and condition (2) of Corollary 1 is satisfied. To prove that condition (1) is satisfied too, we show that every $\phi \in C(X;[0,1])$ is a multiplier of A. Since A is a convex cone, it is enough to show that $\phi f \in A$, for every $\phi \in C(X;[0,1])$ and $f \in A$. Let $f \in A$ and $\phi \in C(X;[0,1])$ be given. There exists n_1 such that $\|H_n(f) - f\| < \epsilon$ for all $n \geqslant n_1$. Then

$$|\phi(t) H_n(f,t) - \phi(t) f(t)| < \epsilon \tag{1}$$

for all $t \in X$ and $n \geqslant n_1$. Since ϕ is uniformly continuous on X, there exists some open set $V \subset X^2$, containing the diagonal $\Delta = \{(x,t) \in X^2; t = x\}$ and such that $|\phi(t) - \phi(x)| < \epsilon$ if $(x,t) \in V$. Let K be the complement of V in X^2. By (i), $k = \inf\{P(x,t);(x,t) \in K\} > 0$. Hence

$$|\phi(t) - \phi(x)| < \epsilon + \frac{2}{k} P(x,t)$$

for all $(x,t) \in X^2$. Assuming (without loss of generality) that $\|f\| \leqslant 1$, we get

$$|\phi(x) f(x) - \phi(t) f(x)| < \epsilon e_0(x) + \frac{2}{k} P(x,t)$$

for all $(x,t) \in X^2$ (where $e_0(x) = 1$ for all $x \in X$).

Since H_n is monotone, it follows that

$$|H_n(\phi f) - \phi(t) H_n(f)| \leqslant \epsilon H_n(e_0) + \frac{2}{k} H_n(P_t)$$

and

$$|H_n(\phi f, t) - \phi(t) H_n(f,t)| < \epsilon H_n(e_0,t) + \epsilon$$

for all $x \in X$ and $n \geqslant n_2$, where n_2 is chosen so that $H_n(P_t,t) < k\epsilon/2$ for all $n \geqslant n_2$ and $t \in X$. Let $a = \min\{g(x); x \in X\}$. Then $e_0(x) = 1 \leqslant a^{-1} g(x)$ for all $x \in X$ and

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$H_n(e_0) \leqslant a^{-1}H_n(g)$ for all $n \geqslant 1$. Since $\{H_n(g)\}_{n=1}^{\infty}$ is convergent, the sequence $\{H_n(e_0)\}_{n=1}^{\infty}$ is bounded: there is a constant M_0 such that $\|H_n(e_0)\| \leqslant M_0$ for all $n \geqslant 1$. Hence

$$|H_n(\phi f, t) - \phi(t) H_n(f, t)| \leqslant \epsilon(M_0 + 1) \tag{2}$$

for all $t \in X$ and $n \geqslant n_2$. Let $n_0 = \max(n_1, n_2)$. The inequalities (1) and (2) imply

$$|H_n(\phi f, t) - \phi(t) f(t)| \leqslant \epsilon(M_0 + 2)$$

for all $n \geqslant n_0$ and consequently $\phi f \in A$. ∎

Let us show how Theorem 4 can be used to prove a well-known version of Bohman–Korovkin's Theorem. (See Lorentz [9], pp. 7–8.)

COROLLARY 7. *Let H_n $(n = 1, 2, 3. \ldots)$ be a sequence of monotone linear operators on $C(X; \mathbb{R})$. Assume that for some finite set $\{g_1, g_2, \ldots, g_m\} \subset C(X; \mathbb{R})$ the following is true:*

(i) *$H_n(g_k) \to g_k$ as $n \to \infty$ for all $k = 1, 2, \ldots, m$;*

(ii) *there is a finite set of bounded functions $\{a_1, a_2, \ldots, a_m\} \subset l_\infty(X; \mathbb{R})$ and a continuous function $F: X \times X \to \mathbb{R}_+$ such that $F(x, t) = 0$ if and only if $x = t$; and, if we define*

$$P(x, t) := \sum_{k=1}^{m} a_k(t) g_k(m) \quad \text{for all} \quad (x, t) \in X^2,$$

then $P(x, t) \geqslant F(x, t)$ and $P(x, x) = 0$.

Then $H_n(f) \to f$ as $n \to \infty$ for all $f \in C(X; \mathbb{R})$.

Proof. For any open set V containing the diagonal, the complement K of V in X^2 is compact and $\inf\{P(x, t); (x, t) \in \mathbb{K}\} \geqslant \inf\{F(x, t); (x, t) \in K\} > 0$. Hence (i) of Theorem 4 is satisfied. For each $t \in X$, let $P_t(x) := P(x, t)$ for all $x \in X$. We claim that $H_n(P_t) \to P_t$ as $n \to \infty$, uniformly on $t \in X$. Indeed, let $\epsilon > 0$ be given. Choose $n(\epsilon)$ so that for all $n \geqslant n(\epsilon)$, $\|H_n(g_k) - g_k\| < \epsilon/Mm$, where $M = \max\{\|a_k\|, k = 1, 2, \ldots, m\}$. Then, for any $(x, t) \in X^2$ and $n \geqslant n(\epsilon)$,

$$|H_n(P_t, x) - P_t(x)| = \left| \sum_{k=1}^{m} a_k(t) H_n(g_k, x) - \sum_{k=1}^{m} a_k(t) g_k(x) \right| \tag{*}$$

$$\leqslant M \sum_{k=1}^{m} \|H_n(g_k) - g_k\| < \epsilon.$$

Hence $\|H_n(P_t) - P_t\| \leqslant \epsilon$ for all $t \in X$ and $n \geqslant n(\epsilon)$. By taking $t = x$ in (*) we get $|H_n(P_t, t)| < \epsilon$, for all $n \geqslant n(\epsilon)$ and $t \in X$, i.e. $H_n(P_t, t) \to 0$ as $n \to \infty$, uniformly on X, and (ii) of Theorem 4 is satisfied.

Now take a pair $(u, v) \in X^2$ with $u \neq v$. Then

$$H_n(P_u + P_v) = H_n(P_u) + H_n(P_v) \to P_u + P_v$$

as $n \to \infty$. Hence $g = P_u + P_v$ is such that $g(x) > 0$ for all $x \in X$ and $H_n(g) \to g$ as $n \to \infty$. Hence (iii) of Theorem 4 is satisfied too. ∎

COROLLARY 8 (S. N. Bernstein). *For any $f \in C(I; \mathbb{R})$, $B_n(f) \to f$ as $n \to \infty$, uniformly on $I = [0, 1]$.*

Proof. Let $g_k(x) = x^{k-1}$, for all $x \in I$, and $k = 1, 2, 3$, and $P(x, t) = F(x, t) = (x - t)^2$ for all $(x, t) \in I^2$. It is well known that $B_n(g_k) \to g_k$ for $k = 1, 2, 3$. (See Lorentz [9].) ∎

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4. *Vector subspaces, subalgebras and semi-algebras*

In this section we show how Theorem 1 implies the usual Stone–Weierstrass Theorems for modules or algebras. Let us start with a vector subspace $W \subset C(X; E)$ which is an A-module for some *real* subalgebra $A \subset C(X; \mathbb{R})$, i.e. $AW \subset W$. Now W is a B-module, where B is the unital subalgebra of $C(X; \mathbb{R})$ generated by A and the constants. Hence $MW \subset W$, if M is the set $\{\phi \in B; 0 \leqslant \phi \leqslant 1\}$. Now M has property V and W is a convex cone. Applying Theorem 1 and noticing that M, B and A define the same equivalence relation on X, we get the following strong version of the Stone–Weierstrass Theorem for modules.

THEOREM 5. *Let W be a vector subspace of $C(X; E)$ which is an A-module for some subalgebra $A \subset C(X; \mathbb{R})$. Then for each $f \in C(X; E)$ there is some equivalence class $[x] \bmod A$ such that* $\operatorname{dist}(f; W) = \operatorname{dist}(f_{[x]}; W_{[x]})$.

COROLLARY 9. *Let W be a real vector subspace of $C(X; E)$, and let A be the algebra of its multipliers. Then, for each $f \in C(X; E)$, there is some equivalence class $[x] \bmod A$ such that* $\operatorname{dist}(f; W) = \operatorname{dist}(f_{[x]}; W_{[x]})$.

Proof. The vector subspace W is an A-module, where $A = \{\phi \in C(X; \mathbb{R}); \phi f \in W$ for all $f \in W\}$. It remains to notice that A is a subalgebra. ∎

When $E = \mathbb{R}$, Theorem 5 implies the usual version of the Stone–Weierstrass Theorem.

THEOREM 6. *Let $A \subset C(X; \mathbb{R})$ be a subalgebra and let $f \in C(X; \mathbb{R})$. Then f belongs to the closure of A if and only if the following two conditions hold:*

(1) *for any $x, y \in X$ such that $f(x) \neq f(y)$ there is some $g \in A$ such that $g(x) \neq g(y)$;*

(2) *for any $x \in X$ such that $f(x) \neq 0$ there is some $g \in A$ such that $g(x) \neq 0$.*

Proof. The two conditions are obviously necessary. Conversely, assume that f satisfies conditions (1) and (2). Let $x \in X$ be given by Theorem 5. For any $t \in [x]$, we have by (1) $f(x) = f(t)$. Hence f is constant on $[x]$. Let $u \in \mathbb{R}$ be its constant value. If $u = 0$, then $g = 0 \in A$ is such that $g(t) = 0$ for all $t \in [x]$, and so $\operatorname{dist}(f_{[x]}; A_{[x]}) = 0$. If $u \neq 0$, then by (2) there is some $g \in A$ such that $g(x) \neq 0$. Then $h = (u/g(x))g$ belongs to A and $g(t) = u$ for all $t \in [x]$. Hence

$$\operatorname{dist}(f_{[x]}; A_{[x]}) \leqslant \sup_{t \in [x]} |f(t) - g(t)| = 0.$$

By Theorem 5, $\operatorname{dist}(f; A) = 0$, i.e. f belongs to the closure of A. ∎

The way in which the self-adjoint case of the Stone–Weierstrass Theorem is deduced from the real case is well known, so we do not say anything about this. When the algebra $A \subset C(X; \mathbb{C})$ is not assumed to be self-adjoint, Shilov [16] considered the subalgebra

$$A_{\mathbb{R}} = \{\phi \in A; \phi \text{ is real valued on } X\}.$$

Clearly any vector subspace $W \in C(X; E)$ which is an A-module is, *a fortiori*, an $A_{\mathbb{R}}$-module. Let us consider the closed partition of X defined by the equivalence classes modulo $A_{\mathbb{R}}$, i.e. $x \equiv y (\bmod A_{\mathbb{R}})$. This is the so-called Shilov partition of X. Theorem 5 implies the following vector-valued version of Shilov's Theorem.

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THEOREM 7. *Let $W \subset C(X; E)$ be a vector subspace which is an A-module for some subalgebra $A \subset C(X; \mathbb{C})$. Then, for each $f \in C(X; E)$, there is some Shilov set $S \subset X$ such that*

$$\operatorname{dist}(f; A) = \operatorname{dist}(f_S, W_S).$$

Proof. The set $A_{\mathbf{R}}$ is an algebra, the Shilov sets (with respect to A) are the equivalence classes mod $A_{\mathbf{R}}$, and W is an $A_{\mathbf{R}}$-module. ∎

Recall that a non-empty subset $A \subset C(X; \mathbb{R})$ is called a *semi-algebra*, according to F. F. Bonsall [2], if $f+g$, λf and fg belong to A whenever $f, g \in A$ and $\lambda \geqslant 0$. Clearly if A is an algebra, then A^+ is a semi-algebra. Theorem 3 suggests the following question: for which semi-algebras $A \subset C^+(X)$, given $f \in C(X; \mathbb{R})$ there is some equivalence class $[x]$ mod A such that $\operatorname{dist}(f; A) = \operatorname{dist}(f_{[x]}; A_{[x]})$? Let us recall that a semi-algebra A is said to be of *type* n, if $f^n(1+f)^{-1}$ belongs to A, whenever $f \in A$. Clearly $1 \in A$ whenever A is a semi-algebra of type 0. Let us say that a semi-algebra A is of *type* V if $1-\phi \in A$ whenever $\phi \in A$ and $0 \leqslant \phi \leqslant 1$. By lemma 7 of Bonsall [2], any closed semi-algebra of type 0 is of type V.

THEOREM 8. *Let A be a semi-algebra whose uniform closure is of type V. For each $f \in C(X; \mathbb{R})$, there is some equivalence class $[x]$ mod A such that*

$$\operatorname{dist}(f; A) = \operatorname{dist}(f_{[x]}; A_{[x]}).$$

Proof. Let $M = \{\phi \in \bar{A}; 0 \leqslant \phi \leqslant 1\}$. Clearly M has property V. The result follows from Theorem 1 and the fact that A, $\bar{A}$ and M define the same equivalence relation. ∎

As a consequence of Theorem 8, we can give a very short proof of theorem 4 of Bonsall [2]. Given any non-empty subset $A \subset C(X; \mathbb{R})$, define

$$\Delta_1(A) = \{f \in C(X; \mathbb{R}); x \equiv y \pmod{A} \text{ implies } f(x) = f(y)\}$$

and

$$\Delta_1^+(A) = \Delta_1(A) \cap C^+(X).$$

THEOREM 9 (Bonsall). *Given a non-empty closed subset $A \in C(X; \mathbb{R})$, then $A = \Delta_1^+(A)$ if and only if A is a closed semi-algebra of type* 0.

Proof. Clearly for any non-empty $A \subset C(X; \mathbb{R})$, $\Delta_1^+(A)$ is a closed semi-algebra of type 0.

Conversely, assume that A is a closed semi-algebra of type 0. By lemma 6 of Bonsall [2], we have $A \subset C^+(X)$. Hence $A \subset \Delta_1^+(A)$. Now let $f \in \Delta_1^+(A)$ be given. By Theorem 8, there is some $x \in X$ such that $\operatorname{dist}(f; A) = \operatorname{dist}(f_{[x]}; A_{[x]})$. Since $f_{[x]}$ is a positive constant and A contains all positive constants, $\operatorname{dist}(f_{[x]}; A_{[x]}) = 0$, and therefore $f \in A$, because A is closed. ∎

It is interesting to notice that Bonsall's proof of the above result uses Bernstein polynomials on two variables. Notice also that $\Delta_1(A)$ is a closed subalgebra containing the constants. Hence Theorem 9 tells us that the only closed semi-algebras of type 0 are the positive parts of closed unital subalgebras.

Given a non-empty subset $A \subset C(X; \mathbb{R})$ define $N(A) = \{x \in X; g(x) = 0 \text{ for all } g \in A\}$, $\Delta(A) = \{f \in \Delta_1(A); f \text{ vanishes on } N(A)\}$, and $\Delta^+(A) = \Delta(A) \cap C^+(X)$. It is easily seen that $\Delta(A)$ is a closed subalgebra of $C(X; \mathbb{R})$, and therefore Theorem 3

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applies to the semi-algebra $\Delta^+(A)$: given $f \in C(X;\mathbb{R})$, there is some equivalence class $[x] \bmod \Delta^+(A)$ such that $\operatorname{dist}(f;\Delta^+(A)) = \operatorname{dist}(f_{[x]};\Delta^+(A)_{[x]})$.

Let us show that the only closed semi-algebras contained in $C^+(X)$ for which Theorem 3 is valid are the positive parts of closed subalgebras.

THEOREM 10. *Given a non-empty closed subset $A \subset C^+(X)$, then $A = \Delta^+(A)$ if and only if A is a closed semi-algebra such that, given $f \in C(X;\mathbb{R})$ there is some equivalence class $[x] \bmod A$ satisfying $\operatorname{dist}(f;A) = \operatorname{dist}(f_{[x]};A_{[x]})$.*

Proof. We have just seen that $\Delta^+(A)$ is a closed semi-algebra contained in $C^+(X)$ with the property stated above.

Conversely, assume that A is a closed semi-algebra contained in $C^+(X)$ and with the property stated above. Clearly $A \subset \Delta^+(A)$. Let $f \in \Delta^+(A)$ be given. There is some $x \in X$ such that

$$\operatorname{dist}(f;A) = \operatorname{dist}(f_{[x]};A_{[x]}),$$

where $[x]$ is the equivalence class of $x \bmod A$. Since $f \in \Delta^+(A)$, it follows that $f_{[x]}$ is a positive constant, say $u \geqslant 0$. If $u = 0$, then $0 \in A$ and coincides with u on $[x]$. If $u > 0$, there is some $g \in A$ such $g(x) > 0$, and then $h = (u/g(x))g$ belongs to A and agrees with f on $[x]$. Hence $\operatorname{dist}(f_{[x]};A_{[x]}) = 0$ and $f \in \bar{A} = A$. ∎

5. *Bishop sets and Machado's Theorem*

Let A be a convex cone in $C(X;E)$. The pair (X,A) is called *antisymmetric* (according to [6]) if X is the only Shilov set (relative to A), i.e. if the only multipliers of A are the constants between 0 and 1. A subset $K \in X$ is called A-antisymmetric if the pair (K,A_K) is antisymmetric. Clearly any set K of the form $\{x\}$ with $x \in X$ is A-antisymmetric. The union of all A-antisymmetric sets containing $\{x\}$ is also A-antisymmetric. Hence any $x \in X$ is contained in a maximal A-antisymmetric set, and therefore the collection of all maximal A-antisymmetric sets is a partition of X into closed subsets (it is easily seen that the closure of an A-antisymmetric set is also A-antisymmetric). Following [6], we call every maximal A-antisymmetric set a *Bishop set* (relative to A). Let us give an equivalent description of the Bishop partition of X. Let S be the class of all ordinal numbers whose cardinal numbers are less or equal than the cardinal number of 2^X. For each $s \in S$, we define a partition of X into closed, pairwise disjoint non-empty subsets of X, to be denoted by P_s.

For $s = 1$, we define $P_1 = \{X\}$.

Assume that P_t has been defined for all $t < s$. We consider two cases.

(*a*) If $s = t+1$, for some $t \in S$, we define P_s as follows. Let $T \subset X$ be an element of P_t. Then A_T is a convex cone in $C(T;E)$. We let $M(T)$ be the set of multipliers $\phi \in C(T;[0,1])$ of A_T, and consider the partition of T into equivalence classes $\bmod M(T)$. The partition P_s is then defined as the collection of all such equivalence classes as T ranges over P_t.

The partition P_2 of X into equivalence classes modulo M is called the *Shilov partition of* X, and its elements are called *Shilov sets* (relative to A).

(*b*) If s has no predecessor, i.e., s is a limit ordinal, define $x \equiv y$ for $x,y \in X$ if and only if x and y belong to the same set belonging to P_t for each $t < s$, and define P_s as the set of equivalence classes modulo the above equivalence relation.

This defines P_s for all $s \in S$. Each P_s is a refinement of P_t whenever $s > t$. The

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argument on p. 781 of Bishop[1] is valid here, and there is some ordinal $r \in S$ such that $P_{r+1} = P_r$. We claim that each element $T \in P_r$ is A-antisymmetric. Indeed, every $T \in P_r$ belongs to P_{r+1}, hence it is an equivalence class modulo $M(T)$, the set of multipliers of A_T, and so (T, A_T) is antisymmetric.

Our next result is the analogue for convex cones of Machado's Theorem [**10**].

THEOREM 11. *Let A be a convex cone in $C(X; E)$ and let $f \in C(X; E)$. For each $s \in S$, there is some $T(s) \in P_s$ such that*

(*a*) $T(s) \subset T(t)$, *for all* $t < s$, $t \in S$;

(*b*) $\operatorname{dist}(f; A) = \operatorname{dist}(f_{T(s)}; A_{T(s)})$.

Proof. Let $s \in S$ be given. Assume that a set $T(t) \in P_t$ with (*a*) and (*b*) has been found for all $t < s$.

Case 1. $s = t+1$, for some $t \in S$. By the induction hypothesis there is some $T(t) \in P_t$ such that $T(t) \subset T(u)$ for all $u < t$, $u \in S$, and

$$\operatorname{dist}(f; A) = \operatorname{dist}(f_{T(t)}; A_{T(t)}).$$

Let us consider the convex cone $B = A_{T(t)}$ contained in $C(T(t); E)$. By Theorem 2, there is some equivalence class Y (modulo the set $M(B)$ of multipliers of B in $C(T(t); [0, 1])$) such that $\operatorname{dist}(g; B) = \operatorname{dist}(g_Y; B_Y)$ where $g = f_{T(t)} \in C(T(t); E)$. But, if we define $T(s) = Y$, then $T(s) \in P_s$ and $g_Y = f_{T(s)}$, $B_Y = A_{T(s)}$. Hence (*a*) and (*b*) hold in this case.

Case 2. The ordinal s has no predecessor. Define $T(s) = \cap \{T(t); t \in S, t < s\}$. Then $T(s) \in P_s$ and $T(s) \subset T(t)$ for all $t < s$, $t \in S$. This proves (*a*) for $T(s)$.

Assume that $\operatorname{dist}(f_{T(s)}; A_{T(s)}) < d$, where $d = \operatorname{dist}(f; A)$. (The case $d = 0$ is trivial.) Let $g \in A$ be such that $\|f-g\|_{T(s)} < d$, and define $U = \{x \in X; \|f(x)-g(x)\| < d\}$. Then U is open and contains $T(s)$. By compactness of $X \backslash U$ there are finitely many indices $t_1 < t_2 < \ldots < t_n < s$ such that $X \backslash U \subset (X \backslash T(t_1)) \cup \ldots \cup (X \backslash T(t_n))$. Since $T(t_n) \subset \ldots \subset T(t_1)$, it follows that $T(t_n) \subset U$, a contradiction because $t_n < s$. Hence (*b*) is true for $T(s)$. ∎

COROLLARY 10. *Let $A \subset C(X; E)$ be a convex cone and let $f \in C(X; E)$. There exists a Bishop set K (relative to A) such that*

$$\operatorname{dist}(f; A) = \operatorname{dist}(f_K; A_K).$$

Proof. Let $r \in S$ be such that $P_{r+1} = P_r$. By Theorem 11 there is some $T(r) \in P_r$ such that $\operatorname{dist}(f; A) = \operatorname{dist}(f_{T(r)}; A_{T(r)})$. We have seen that each element of P_r is A-antisymmetric. Hence here is some Bishop set K (relative to A) such that $T(r) \subset K$. Therefore $\operatorname{dist}(f; A) = \operatorname{dist}(f_K; A_K)$. ∎

Remark. Since the Bishop sets form a partition of X, say $B(A)$, it follows from Corollary 10 that

$$\operatorname{dist}(f; A) = \sup_{K \in B(A)} \operatorname{dist}(f_K; A_K)$$

and therefore f belongs to the closure of A if and only if f_K belongs to the closure of A_K in $C(K; E)$, for every Bishop set K (relative to A). Hence Corollary 10 generalizes théorème 8 of Feyel and de La Pradelle[**6**].

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https://doi.org/10.1017/S030500410006549X Published online by Cambridge University Press