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An Eigenvalue problem for holomorphic solutions to a certain class of functional differential equations

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In this paper, we study a class of linear functional differential equations of which the pantograph equation is a prominent member. Specifically, we study the existence of solutions holomorphic at a fixed point of the functional argument. The local theory for equations with attracting fixed points is known [17, 13], but little is known about the case where the fixed point is repelling. We formulate an eigenvalue problem for the repelling fixed point case and show that the corresponding spectrum is discrete. Hence, that holomorphic solutions occur only as special cases. The second order pantograph equation is used to illustrate this result. A key step in this process is to reformulate the problem in terms of a compact operator. Aside from exploiting well known results for the spectra of such operators, we use results such as the Fredholm Alternative to derive existence results for the non-homogeneous problem.

1 Introduction

The existence of non-trivial solutions y holomorphic at $z = 0$ to the pantograph equation

$$y'(z) - by(z) - \lambda y(\alpha z) = 0, \tag{1.1}$$

where α, b and λ are constants and $\lambda \neq 0$, depends crucially upon whether $|\alpha| \leqslant 1$. Suppose that $|\alpha| > 1$ and that there is a non-trivial solution y holomorphic at $z = 0$. Then y can be represented as a power series of the form

$$y(z) = \sum_{n=0}^{\infty} c_n z^n,$$

convergent in some disc centred at $z = 0$. Substituting this expression into (1.1.1) gives

$$c_n = \frac{c_0}{n!} \prod_{k=0}^{n-1} (b + \lambda \alpha^k).$$

Since y is non-trivial, $c_0 \neq 0$. Suppose that $\lambda \neq -b/\alpha^j$ for any non-negative integer j.

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Then $c_k \neq 0$ and

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{|b+\lambda\alpha^n|}{n+1} \to \infty$$

as $n \to \infty$; hence, the series must diverge for any $z \neq 0$. We thus conclude that if a non-trivial holomorphic solution exists then $\lambda = -b/\alpha^j$ for some non-negative integer j. Evidently, these solutions are polynomials. In contrast, suppose that $|\alpha| < 1$. Then it is clear from the above calculation that the above series converges for any choice of $\lambda \in \mathbb{C}$.

Numerous authors have studied (1.1.1), including Kato & McLeod [8], Fredrickson [4], Iserles [7], Morris *et al.* [12] and Derfel & Iserles [1], among others. This equation arises in many applications ranging from current collection in an electric train [14] to cell growth [5]. The second order version of equation (1.1.1), i.e.

$$y''(z) - ay'(z) - by(z) - \lambda y(\alpha z) = 0, \tag{1.2}$$

where $\alpha \in \mathbb{R}$, $\alpha > 1$ has been studied by Wake *et al.* [19] in the context of cell growth, and by Kim [9] and van Brunt *et al.* [18] as a singular Sturm–Liouville problem. The research on the second order problem focused on the behaviour of solutions on the real line and, in particular, on solutions in $L^1(0,\infty)$. Generically, these solutions have Dirichlet series representations with a singularity at $z = 0$, though they are holomorphic in the half plane $\mathrm{Re} z > 0$.

The above example shows that the first order pantograph (1.1.1) defines an eigenvalue problem when $|\alpha| > 1$ and that the spectrum is discrete with $\lambda = 0$ and a limit point. In contrast, if $|\alpha| < 1$, the pantograph equation always admits holomorphic solutions. The existence of holomorphic solutions thus depends in this case on whether the functional argument is advanced, i.e. $|\alpha z| > |z|$, or retarded, $|\alpha z| \leqslant |z|$. This observation motivates questions concerning the rôle the functional argument plays in the availability of holomorphic solutions. For example, consider a generalization of the pantograph equation

$$y'(z) - by(z) - \lambda y(g(z)) = 0, \tag{1.3}$$

where g is a function holomorphic at $z = 0$ and $g(0) = 0$. If the functional argument is retarded near $z = 0$, i.e. if $|g'(0)| < 1$ then it can be shown [17] that a non-trivial solution holomorphic at $z = 0$ exists for any choice of λ. Suppose now that $|g'(0)| > 1$, so that the functional argument is advanced. In this case one would expect that the availability of holomorphic solutions would follow the pantograph paradigm, i.e. holomorphic solutions exist only for certain values of λ.

The pantograph equation (1.1.1) has been studied extensively, but few have studied generalizations such as (1.1.3) and even fewer have studied the existence of holomorphic solutions for the advanced case. A notable exception is Oberg [13], who studied equations of the form

$$y'(z) = \lambda q(z) y(g(z)), \tag{1.4}$$

where q and g are holomorphic at z_0, $g(z_0) = z_0$ and $|g'(z_0)| > 1$. Oberg's analysis shows that (1.1.4) has non-trivial solutions holomorphic at z_0 only for certain values of λ. He also showed that the spectrum must be discrete.

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The key to Oberg's results is that (1.1.4) can be replaced by one that involves a compact operator. Once this reformulation is done, a formidable array of results from the spectral theory of compact operators can be leveled at the problem. Oberg limits his use of these results to the discrete nature of the spectrum and the finite dimensional nature of the corresponding eigenspaces. His analysis, however, can be readily adapted to more general operators. The requirement of analyticity makes available the Cauchy integral formula, and this coupled with the advanced nature of the functional argument provides the mechanism whereby the problem can be reposed as one that involves a compact operator.

In the next section, we pose the eigenvalue problem and note a straightforward extension of Oberg's work to general second order linear differential operators. For the case $|g'(z_0)| > 1$, the operator can be replaced by a compact one; consequently, the associated eigenvalue problem has a discrete spectrum. We illustrate the theory using the second order pantograph equation (1.1.2) in § 3, where we determine the spectrum and the associated eigenfunctions.

Functional differential equations with nonlinear polynomial arguments are studied in § 4. Nonlinear functional arguments strongly influence the nature of the spectrum. We use results from complex dynamics to show that, generically, such equations have at most one non-zero eigenvalue.

The reformulation of advanced functional differential equations in terms of compact operators opens new vistas from which certain problems can be studied with profit. We limit our study here to existence and uniqueness problems for non-homogeneous problems. In § 5 we exploit the Fredholm Alternative for eigenvalue problems involving compact operators to establish an existence and uniqueness result for holomorphic solutions to non-homogeneous problems and apply this theory to the second order pantograph equation.

2 The Eigenvalue problem

Let g be a function holomorphic at z_0 and let z_0 be a fixed point for g. The point z_0 is called *attracting* if there exists a neighbourhood U of z_0 such that

$$|g(z) - z_0| < |z - z_0|$$

for all $z \in U - \{z_0\}$. Similarly, z_0 is called *repelling* if there exists a neighbourhood U of z_0 such that

$$|g(z) - z_0| > |z - z_0|$$

for all $z \in U - \{z_0\}$. If z_0 is neither attracting nor repelling, then z_0 is called *neutral*. Since g is holomorphic at z_0, fixed points can be characterized as follows:

(1) z_0 is attracting if and only if $|g'(z_0)| < 1$.

(2) z_0 is repelling if and only if $|g'(z_0)| > 1$.

The space of functions holomorphic in a set $\Omega \subseteq \mathbb{C}$ is denoted by $H(\Omega)$ and the disc of radius R centred at z_0 is denoted by $D(z_0; R)$. The closed disc will be denoted by $\bar{D}(z_0; R)$.

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Let $\mathscr{L}$ denote the operator defined by

$$\mathscr{L}y(z) = p(z)y''(z) + q(z)y'(z) + r(z)y(z),$$

where p, q and r are functions holomorphic at z_0 with $p(z_0) \neq 0$. In this section, we consider the problem of finding values of $\lambda \in \mathbb{C}$ such that the equation

$$\mathscr{L}y(z) = \lambda y(g(z)) \tag{2.1}$$

has a non-trivial solution holomorphic at z_0. The set of such λ, i.e. the spectrum, will be denoted by S.

The character of S depends crucially upon the nature of the fixed point. Suppose that z_0 is an attracting fixed point. Then it can be shown [11, 17, 13], for any choice of $y(z_0)$, $y'(z_0)$, that there exists a unique solution holomorphic at z_0 for *any* value of $\lambda \in \mathbb{C}$, and hence $S = \mathbb{C}$. In contrast, suppose that z_0 is a repelling fixed point. We show that the spectrum must be discrete, and that the only possible limit point in S is $\lambda = 0$.

Since $g'(z_0) \neq 0$ there is a unique inverse function h holomorphic at $g(z_0) = z_0$. Moreover, since $|g'(z_0)| > 1$ we have $|h'(z_0)| < 1$; therefore, z_0 is an attracting fixed point for h. Let $\mathscr{L}_h$ be the operator defined by

$$\mathscr{L}_h y(z) = p(h(z))y''(h(z)) + q(h(z))y'(h(z)) + r(h(z))y(h(z)).$$

Equation (2.2.1) can be replaced by

$$\mathscr{L}_h y(z) = \lambda y(z), \tag{2.2}$$

and, since y is holomorphic at z_0 if and only if $y \circ h$ is holomorphic at z_0, the problem of finding holomorphic solutions to (2.2.1) is equivalent to that of finding holomorphic solutions to (2.2.2). This seemingly innocuous step has the effect of transforming the original problem into one that involves a compact operator. The proof that $\mathscr{L}_h$ is compact follows *mutatis mutandis* from Oberg's analysis (cf. Oberg [13, Theorem 4-1]).

Theorem 1 *Let $\mathscr{H}^\infty(D(z_0;\rho))$ denote the Banach space of functions holomorphic in $D(z_0;\rho)$ and continuous in $\bar{D}(z_0;\rho)$ equipped with the norm $\|f\|_\infty = \sup_{|z|=\rho} |f(z)|$. There is a $\rho > 0$ such that $\mathscr{L}_h$ is a compact operator from $\mathscr{H}^\infty(D(z_0;\rho))$ into $\mathscr{H}^\infty(D(z_0;\rho))$.*

A basic property of a compact operator is that its spectrum is discrete; moreover, the associated eigenspace must be finite dimensional. Theorem 2.1 thus yields immediately the following result.

Corollary 2 *The set S has no limit points except perhaps at $\lambda = 0$. The eigenspace associated with any $\lambda \in S$ is finite dimensional.*

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3 Spectrum for the pantograph equation

In this section we study the spectrum associated with equations of the form

$$y''(z) - ay'(z) - by(z) = \lambda y(\alpha z), \tag{3.1}$$

where a and b are complex constants and α is a real constant such that $\alpha > 1$.

The functional argument in (3.3.1) has a repelling fixed point at $z_0 = 0$, and Corollary 2.2 shows that the spectrum must be discrete. We determine S along with the associated eigenfunctions in this section. First, however, we prove two lemmas.

Lemma 3 *Suppose that y is a solution to* (3.3.1)*, and that y is holomorphic at $z = 0$. Then y is an entire function.*

Proof Since y is holomorphic at $z = 0$ there is an $\epsilon > 0$ such that $y \in H(D(0;\epsilon))$. Now, y statisfies (3.3.1), and this equation provides a nontrivial analytic continuation of y from $D(0;\epsilon)$ to $D(0;\alpha\epsilon) \supset D(0;\epsilon)$, because $\alpha > 1$ and $y''(z) - ay'(z) - by(z) \in H(D(0;\epsilon))$. The same argument can be used to analytically continue y to $D(0;\alpha^2\epsilon) \supset D(0;\alpha\epsilon)$. Clearly, repeated applications of this process can be used to show that y can be analytically continued into a set of the form $D(0;\alpha^n\epsilon)$ where n is any positive integer. Since $\alpha^n \to \infty$ as $n \to \infty$, we conclude that $y \in H(\mathbb{C})$ □

Lemma 4 *Let y be an entire function and define $M(R)$ by*

$$M(R) = \sup_{|z|=R} |y(z)|,$$

for all $R > 0$. Suppose that there exist numbers $\alpha > 1$ and $\delta > 0$, and a non-negative integer n such that

$$M(\alpha R) \leqslant \alpha^n M(R + \delta), \tag{3.2}$$

for all R sufficiently large. Then y is a polynomial of degree $d \leqslant n$.

Proof Pòlya & Szergö [15] show that, for any fixed number μ, $0 < \mu < 1$, if y is a transcendental entire function then

$$\lim_{R\to\infty} \frac{M(\mu R)}{M(R)} = 0. \tag{3.3}$$

Given inequality (3.3.2), there is a number γ, $1 < \gamma < \alpha$ such that

$$M(\alpha R) \leqslant \alpha^n M(\gamma R),$$

for all R sufficiently large. Therefore

$$\frac{M(\mu\rho)}{M(\rho)} \geqslant \frac{1}{\alpha^n} > 0, \tag{3.4}$$

for all ρ large, where $1/\alpha < \mu = \gamma/\alpha < 1$. Hence, relation (3.3.3) cannot be satisfied and

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therefore y is a polynomial. That y must be of degree $d \leqslant n$ follows immediately from inequality (3.3.4). □

Theorem 5 *Suppose that $\lambda \neq 0$ and y is a non-trivial solution to* (3.3.1) *that is holomorphic at $z = 0$. Then $\lambda \in \bar{D}(0; |b|)$ and y is a polynomial.*

Proof Lemma 3.1 indicates that y must be an entire function. Let $C_R = \{z \in \mathbb{C} : |z| = R\}$. The Cauchy integral formula implies that for any $R > 0$, non-negative integer k and $z \in C_R$

$$y^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{y(\xi)}{(\xi - z)^{k+1}}\, d\xi,$$

where γ is any simple closed contour oriented anticlockwise that encloses z. Thus for any $\delta > 0$, we may use a contour of the form $\gamma_\delta = \{z : |\xi - z| = \delta\}$, and this yields the inequality

$$|y^{(k)}(z)| \leqslant \frac{k!}{\delta^k} \sup_{\xi \in \gamma_\delta} |y(\xi)|.$$

The maximum modulus principle implies that

$$|y^k(z)| \leqslant \frac{k!}{\delta^k} M(R + \delta). \tag{3.5}$$

For any $z \in C_R$ and $R > \delta > 0$ inequality (3.3.5) and the differential equation (3.3.1) thus yield

$$M(\alpha R) \leqslant \frac{1}{|\lambda|} \left(\frac{2}{\delta^2} + \frac{|a|}{\delta} + |b| \right) M(R + \delta). \tag{3.6}$$

Suppose that $|\lambda| > |b|$ Then for δ and R sufficiently large, $0 < \delta < R$, we have

$$\frac{1}{|\lambda|} \left(\frac{2}{\delta^2} + \frac{|a|}{\delta} + |b| \right) < 1.$$

Fix δ and choose R so large that $\alpha R > R + \delta$. Then the above inequality implies

$$M(\alpha R) < M(R + \delta),$$

which contradicts the maximum modulus principle. Hence, $|\lambda| \leqslant |b|$.

Suppose that $\lambda \in \bar{D}(0; |b|)$. Then there is a non-negative integer n such that

$$\frac{|b|}{\alpha^{n+1}} < |\lambda| \leqslant \frac{|b|}{\alpha^n}. \tag{3.7}$$

Inequalities (3.3.6) and (3.3.7) imply

$$M(\alpha R) \leqslant \frac{\alpha^{n+1}}{|b|} \left(\frac{2}{\delta^2} + \frac{|a|}{\delta} + |b| \right) M(R + \delta),$$

and hence for δ and R sufficiently large

$$M(\alpha R) \leqslant \alpha^{n+1} M(R + \delta).$$

Lemma 3.2 thus implies that y is a polynomial of degree $d \leqslant n + 1$. □

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Note that if $b = 0$ the above analysis shows that the set of non-zero eigenvalues is empty.

Corollary 6 *Let $b \neq 0$. The spectrum associated with* (3.3.1) *is $S = \{-b/\alpha^k : k = 0, 1, \ldots\}$. For any non-negative integer n, the eigenfunction associated with $\lambda_k = -b/\alpha^k$ is a polynomial of degree n that is determined uniquely apart from a multiplicative constant.*

Proof Theorem 3.3 shows that any nontrivial solution to equation (3.3.1) holomorphic at 0 must be a polynomial. Therefore, there exists a non-negative integer n and $n + 1$ constants $c_n, \ldots, c_0$ with $c_n \neq 0$ such that

$$y(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0. \tag{3.8}$$

Substituting this expression for y into (3.3.1) and equating coefficients of z^n gives

$$(b + \lambda \alpha^n)\, c_n = 0,$$

and hence

$$\lambda = -\frac{b}{\alpha^n}.$$

We thus conclude that if $\lambda \in S$ then $\lambda = -b/\alpha^n$ for some non-negative integer n.

Suppose that $\lambda = -b/\alpha^n$ for some non-negative integer n. Substituting expression (3.3.8) into (3.3.1) we have that c_n can be an arbitrary non-zero number. Equating coefficients of z^{n-1} we find

$$c_{n-1} = -\frac{an}{b(1 - 1/\alpha)} c_n, \tag{3.9}$$

and the general recurrence relation is

$$k(k-1)c_k - a(k-1)c_{k-1} - b(1 - \alpha^{k-n-2})c_{k-2} = 0, \tag{3.10}$$

for $k = 2, 3, \ldots, n$. Given a value of c_n, we can thus determine the other coefficients uniquely. Hence for each non-zero integer n the choice $\lambda = -b/\alpha^n$ corresponds to a polynomial solution to (3.3.1), and therefore $\lambda \in S$. Moreover, the polynomial solution is determined uniquely up to a multiplicative constant. □

4 Nonlinear polynomial arguments

Having studied the pantograph equation it seems natural to investigate equations of the form

$$y''(z) - ay'(z) - by(z) = \lambda y(g(z)), \tag{4.1}$$

where a and b are complex constants and g is a polynomial of degree $d \geqslant 2$. One might expect that the discrete spectrum is much like that for the pantograph equation, albeit more complicated to determine, but it turns out that, in general, the spectrum is simpler. In fact, we show in this section that there is precisely one non-zero eigenvalue unless g

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is of a special form. This remarkable property is a consequence of the Julia set defined by g. To establish the above claim we thus need to introduce a few concepts and results from complex dynamics. A fuller account of this material is given by Devaney [2].

Let g_n denote the nth composition of g, e.g. $g_2(z) = g(g(z))$. A point $z_0 \in \mathbb{C}$ such that $g_n(z_0) = z_0$, and $g_k(z_0) \neq z_0$ for $k = 1, 2, \ldots n-1$ is called a *periodic point of period n*. Evidently, a fixed point is a periodic point of period 1. Let z_0 be a periodic point of period n. We say that z_0 is *repelling* if there exists a neighbourhood U of z_0 such that

$$|g_n(z) - z_0| > |z - z_0|$$

for all $z \in U - \{z_0\}$.

We define the *Julia set* $J(g)$ of a holomorphic function g as the closure of the set of repelling periodic points of g. This is not the classical definition of the Julia set, which is in terms of normal families. Nonetheless, the definitions are equivalent. The next theorem gives an important property of the Julia sets in connexion with analytic continuation (cf. [2, 11]).

Theorem 7 *Let $z \in J(g)$, U be a neighbourhood of z, and $g_n(U) = \{g_n(z) : z \in \mathbb{C}\}$. Then the set G defined by*

$$G = \bigcup_{n=1}^{\infty} g_n(U)$$

omits at most one point in $\mathbb{C}$.

Theorem 4.1 shows that there is at most one point $\sigma \in \mathbb{C}$ such that $\sigma \notin G$. This point is called an *exceptional point*. It is clear that a repelling fixed point must be in the Julia set. Suppose that (4.4.1) has a solution holomorphic at the repelling fixed point z_0. Then the functional equation can be used to analytically continue the solutions through compositions of g. This observation coupled with the above theorem gives the following result.

Corollary 8 *Let z_0 be a repelling fixed point for g and suppose that there exists a solution y to* (4.4.1) *holomorphic at z_0. Then this solution can be continued to all points in the complex plane with at most one exception.*

Remark 1 If there is an exceptional point then it need not be a pole or isolated essential singularity of y. The exceptional point could be a branch point for y.

Remark 2 Theorem 4.1 and Corollary 4.2 do not depend on the polynomial character of g, i.e. they are valid for any entire function.

Remark 3 The differential operator plays a somewhat passive rôle. We can generalize (4.4.1) substantially, and still retain the above results. We need only ensure that the operator provides the necessary analytic continuation.

The characterization of the exceptional point is particularly simple for polynomials [2].

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Theorem 9 *Let g be a polynomial. Suppose that there is a point $\sigma \in J(g)$ and a neighbourhood U of z such that*

$$\bigcup_{n=1}^{\infty} g_n(U) = \mathbb{C} - \{\sigma\}$$

for some $\sigma \in \mathbb{C}$. Then $g(z) = \sigma + \kappa(z-\sigma)^k$ for some $\kappa \in \mathbb{C} - \{0\}$ and some $k \in \mathbb{N}$.

It is clear from the above result that if there is an exceptional point σ it must be a fixed point for g. If $k \geqslant 2$, then $|g'(\sigma)| = 0$, so that the fixed point must be attracting. Fixed points ζ for which $g'(\zeta) = 0$ are called *superattracting*. The absence of a superattracting fixed point for g thus indicates that solutions to (4.4.1) can be analytically continued to all points in $\mathbb{C}$.

Corollary 10 *Let g be a polynomial with no superattracting fixed points and let z_0 be a repelling fixed point for g. If y is a solution to* (4.4.1) *that is holomorphic at z_0 then y is an entire function.*

We are now in a position to prove the following result concerning the spectrum for (4.4.1).

Theorem 11 *Let g be a nonlinear polynomial with no superattracting fixed points and let z_0 be a repelling fixed point for g. There exists a nontrivial solution to* (4.4.1) *holomorphic at z_0 if and only if $\lambda = -b$. In this case, y is a constant function.*

Proof It is clear that if $\lambda = -b$ then any solution of the form $y = const.$ satisfies (4.4.1). It remains to show that analyticity at z_0 implies that $\lambda = -b$. The proof of this is close to that given by Marshall *et al.* [11] (Theorem 4-2) to establish the existence of a natural boundary for solutions to (4.4.1) near an attracting fixed point.

Note that, by Corollary 4.4, any solution y to (4.4.1) holomorphic at z_0 must be an entire function. The analysis leading to inequality (3.3.6) can be used to establish the inequality

$$|y(g(z))| \leqslant \frac{1}{|\lambda|}\left(\frac{2}{\delta^2} + \frac{|a|}{\delta} + |b|\right) M(R+\delta), \tag{4.2}$$

where δ and R are numbers such that $0 < \delta < R$. For any $\rho > 0$ let $C_\rho = \{z \in \mathbb{C} : |z| = \rho\}$ Since g is a nonlinear polynomial there is are numbers $\hat{R} > 0$ and $\beta > 0$ such that

$$|g(z)| > \beta R^2$$

for all $z \in C_R$ with $R > \hat{R}$. For such a choice of R the curve $\Gamma = \{g(z) : z \in C_{SR}\}$ is closed, and the curve $C_{\beta R^2}$ is enclosed by Γ. The maximum modulus principle thus implies that there is a $z_1 \in C_R$ such that

$$|y(g(z_1))| \geqslant M(\beta R^2). \tag{4.3}$$

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Inequalities (4.4.2) and (4.4.3) thus give

$$M(\beta R^2) \leqslant \frac{1}{|\lambda|}\left(\frac{2}{\delta^2}+\frac{|a|}{\delta}+|b|\right) M(R+\delta). \tag{4.4}$$

Inequality (4.4.4) is valid for all R sufficiently large and all $0 < \delta < R$. We may thus choose an R and a δ so large that

$$\frac{1}{|\lambda|}\left(\frac{2}{\delta^2}+\frac{|a|}{\delta}+|b|\right) < 1+\frac{|b|}{|\lambda|},$$

and thus for all R and δ large

$$M(\beta R^2) \leqslant \left(1+\frac{|b|}{|\lambda|}\right) M(R+\delta). \tag{4.5}$$

Let η be the function defined by

$$\eta(z) = \begin{cases} \frac{y(z)-y(0)}{z}, & \text{if } z \neq 0 \\ y'(0) & \text{if } z = 0. \end{cases}$$

Since y is entire so is η. Let $M_\eta(R) = \sup_{z\in C_R} |\eta(z)|$. We thus have the following inequalities:

$$M_\eta(R) \leqslant \frac{M(R)+|y(0)|}{R} \tag{4.6}$$

$$M(R) \leqslant RM_\eta(R)+|y(0)|. \tag{4.7}$$

Now if y is a constant function $M_\eta(\rho) = 0$ for any $\rho > 0$; otherwise $M_\eta(\rho) > 0$ for all $\rho > 0$. Suppose that y is not a constant function. Then, arguing as in the proof of Lemma 3.2, the above inequalities and inequality (4.4.5) imply

$$M_\eta(\beta R^2) \leqslant \left\{\frac{\left(\frac{|b|}{|\lambda|}+1\right)(R+\delta)}{\alpha R^2}+\frac{2|y(0)|}{\alpha R^2 M_\eta(R+\delta)}\right\} M_\eta(R+\delta). \tag{4.8}$$

For R sufficiently large we have

$$\frac{\left(\frac{|b|}{|\lambda|}+1\right)(R+\delta)}{\beta R^2}+\frac{2|y(0)|}{\beta R^2 M_\eta(R+\delta)} < 1$$

and

$$\beta R^2 > R+\delta.$$

For such R we also have

$$M_\eta(\beta R^2) < M_\eta(R+\delta).$$

The last inequality contradicts the maximum modulus theorem. Hence y must be a constant. Substituting $y = const.$ into (4.4.1) shows that $\lambda = -b$. □

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Theorem 4.5 shows that if g does not have a superattracting fixed point and if $b \neq 0$ then the only non-zero eigenvalue is $\lambda = -b$. If $b = 0$, then the set of non-zero eigenvalues is empty.

If g has a superattracting fixed point the character of the spectrum can be somewhat different. Consider the equation

$$y''(z) = \lambda y(z^n), \tag{4.9}$$

where $n \geqslant 2$. The functional argument has a superattracting fixed point at $z = 0$ and a repelling fixed point at $z = 1$. Certainly any function of the form

$$y_0(z) = C_0 \frac{1}{z^m},$$

where C_0 is an constant and

$$m = \frac{2}{n-1}, \tag{4.10}$$

is a solution to this equation holomorphic at $z = 1$ for the choice $\lambda = m(m+1)$. There are, however, other solutions holomorphic at $z = 1$ for different choices of $\lambda \neq 0$. In fact, (4.4.9) can be transformed into a pantograph equation. Let

$$y(z) = \frac{1}{z^m} Y(z),$$

where m satisfies (4.4.10). Then (4.4.9) transforms to

$$z^2 Y''(z) - 2mz Y'(z) + m(m+1)Y(z) = \lambda Y(z^n).$$

Let $z = e^w$ and $Y(z) = P(w)$. Then the above equation transforms to

$$P''(w) - (2m+1)P(w) + m(m+1)P(w) = \lambda P(nw), \tag{4.11}$$

which is a second order pantograph equation of the form (3.3.1) with $\alpha = n > 1$. The repelling fixed point $z_0 = 1$ for z^n is mapped to the repelling fixed point $w_0 = 0$ for nw, and we thus know that (4.4.11) has eigenvalues of the form $\lambda = m(m+1)/n^j$, where j is any non-negative integer. We can thus deduce that (4.4.9) also has these eigenvalues and that the corresponding eigenfunctions are of the form

$$y(z) = \frac{1}{z^m} L_j(z),$$

where $L_j(z)$ is a polynomial in $\log z$ of degree j.

5 Non-homogeneous equations

Let $\mathscr{L}$ and h be as defined in § 2 and let f be a given function that is holomorphic at z_0. In this section we study equations of the form

$$\mathscr{L} y(z) - \lambda y(g(z)) = f(z), \tag{5.1}$$

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where z_0 is a repelling fixed point for g. In particular, we are concerned with conditions under which (5.5.1) has a solution holomorphic at z_0.

It was shown in § 2 that the homogeneous equation (2.2.1) can be replaced by (2.2.2), which involves a compact operator. Evidently, (5.5.1) can be replaced by

$$\mathscr{L}_h y(z) - \lambda y(z) = F(z), \tag{5.2}$$

where

$$F(z) = f(h(z)).$$

The value of this reformulation is that extensions of Fredholm type theorems are available for compact operators (cf. [6, 10]). For example, the following result is an immediate consequence of the compactness of $\mathscr{L}_h$.

Theorem 12 (Existence and Uniqueness) *Let z_0 be a repelling fixed point for g and suppose that $\lambda \neq 0$. Then* (5.5.1) *has a solution y that is holomorphic at z_0 for any given function f holomorphic at z_0 if and only if the only solution to* (2.2.1) *that is holomorphic at z_0 is the trivial solution. In this case the solution to* (5.5.1) *is unique.*

Indeed, if λ is in the resolvent set of $\mathscr{L}_h$, the the solution y can be represented by the Neumann series

$$y = -\frac{1}{\lambda}\left(F + \frac{1}{\lambda}\mathscr{L}_h F + \frac{1}{\lambda^2}\mathscr{L}_h^2 F + \cdots\right), \tag{5.3}$$

where, for $n \geqslant 2$,

$$\mathscr{L}_h^n F = \mathscr{L}_h\left(\mathscr{L}_h^{n-1} F\right).$$

Theorem 5.1 contrasts sharply with the analogous problem for the case where z_0 is an attracting fixed point. In this case, it can be shown [17] that holomorphic solutions to (5.5.1) exist for any $\lambda \in \mathbb{C}$ and any choice of f that is holomorphic at z_0. Given λ and f, however, there is no uniqueness because it is always possible to prescribe the initial values $y(z_0)$ and $y'(z_0)$. It is thus remarkable that the repelling fixed point case yields a unique solution without any initial data.

In § 3 we determined the spectrum for the second order pantograph equation. Since this spectrum is known explicitly, we can use the above theorem to generate a result specific to the pantograph equation.

Corollary 13 *Let λ and b be non-zero numbers such that $\lambda \neq -b/\alpha^n$ for any non-negative integer n. For any given function f holomorphic at* 0 *there exists a unique solution y to the equation*

$$y''(z) - ay'(z) - by(z) - \lambda y(\alpha z) = f(z), \tag{5.4}$$

that is holomorphic at 0. *Here, a is a complex constant and α is a real number such that $\alpha > 1$.*

As a simple example, consider the equation

$$y''(z) - y(z) - \lambda y(\alpha z) = z, \tag{5.5}$$

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with $\lambda = -\alpha$. This value of λ is not in the spectrum and hence the above corollary asserts that there is a unique solution to (5.5.5) holomorphic at 0. In fact, the solution is

$$y(z) = \frac{z}{\alpha^2 - 1},$$

which can be found by inspection or by use of the Neumann series. Using the latter approach (for illustration) we have $F(z) = z/\alpha$, so that

$$\mathscr{L}_h F = -\frac{z}{\alpha^2},$$
$$\mathscr{L}_h^2 F = \frac{z}{\alpha^3},$$

and in general

$$\mathscr{L}_h^n F = (-1)^n \frac{z}{\alpha^{n+1}}.$$

Hence

$$\begin{aligned} y(z) &= \frac{1}{\alpha}\left(F(z) - \frac{1}{\alpha}\mathscr{L}_h F(z) + \cdots\right) \\ &= \frac{1}{\alpha}\left(\frac{z}{\alpha} + \frac{z}{\alpha^3} + \frac{z}{\alpha^5} + \cdots\right) \\ &= \frac{z}{\alpha^2 - 1}. \end{aligned}$$

In contrast, consider the equation

$$y''(z) - y(z) + y(\alpha z) = 1, \tag{5.6}$$

which corresponds to the choice $\lambda = -1$ and $f(z) = 1$. This value of λ is in the spectrum and the corresponding eigenfunction is a constant. Suppose that there is a holomorphic solution y to equation (5.5.6). Then y can be represented as a power series,

$$y(z) = \sum_{n=1}^{\infty} c_n z^n,$$

with a non-zero radius of convergence. Substituting the power series in (5.5.6) gives

$$c_2 = \frac{1}{2},$$

and, for all $k \geqslant 2$,

$$c_{2k} = \frac{\prod_{j=1}^{k-1}(1 - \alpha^{2j})}{(2k)!}.$$

Consequently,

$$\limsup_{n\to\infty} |c_n|^{1/n} |z| \to \infty$$

for all $z \neq 0$ and therefore the power series must diverge. There is no solution to (5.5.6) holomorphic at 0.

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Theorem 5.1 is part of a more general result for compact operators, the Fredholm Alternative, that addresses the existence problem for solutions to the inhomogeneous equation when λ is an eigenvalue. A version of the Fredholm Alternative is available for compact operators in general Banach spaces (cf. Kreyszig [10]), but for our case the result would involve a Banach space adjoint and the dual space of $\mathscr{H}^\infty$. It is more tractable thus to first recast the problem in some Hilbert space. We are interested primarily in local existence results as opposed to the behaviour of solutions on some disc boundary, and with this in mind we can recast the problem to one that involves a compact operator on the Hardy space $\mathscr{H}^2(D(z_0;\rho))$ for a suitable choice of $\rho > 0$. Recall that $\mathscr{H}^2(D(z_0;\rho))$ is the set of all functions $f \in H(D(z_0;\rho))$ such that

$$\|f\|_2 = \left\{ \int_0^{2\pi} |f(\rho e^{i\phi})|^2 \, d\phi \right\}^{1/2} < \infty,$$

and that $\mathscr{H}^2(D(z_0;\rho))$ is a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{|z|=\rho} f(z)\bar{g}(z)\frac{1}{z}\, dz,$$

for $f, g \in \mathscr{H}^2(D(z_0;\rho))$. The dual of $\mathscr{H}^2(D(z_0;\rho))$ is $\mathscr{H}^2(D(z_0;\rho))$. Let $\mathscr{T}_\lambda : \mathscr{H}^2(D(z_0;\rho)) \to \mathscr{H}^2(D(z_0;\rho))$ be the operator defined by

$$\mathscr{T}_\lambda y(z) = \mathscr{L}_h y(z) - \lambda y(z)$$

and let $\mathscr{T}_\lambda^*$ denote the (Hilbert space) adjoint of $\mathscr{T}_\lambda$. Since $\mathscr{L}_h : \mathscr{H}^2(D(z_0;\rho)) \to \mathscr{H}^2(D(z_0;\rho))$ is compact we have the following version of the Fredholm Alternative for functional differential equations of the form (5.5.1).

Theorem 14 (Fredholm Alternative) *Suppose that $\lambda \neq 0$. Then one of the two following alternatives holds:*

(1) *The homogeneous equation $\mathscr{T}_\lambda y = 0$ has only the trivial solution. In which case the inhomogeneous equation* (5.5.1) *has exactly one solution for each $f \in \mathscr{H}^2(D(z_0;\rho))$.*

(2) *The homogeneous equation $\mathscr{T}_\lambda y = 0$ has a non-trivial solution. In which case, for $f \in \mathscr{H}^2(D(z_0;\rho))$, the inhomogeneous equation* (5.5.1) *has a solution if and only if $\langle F, K \rangle = 0$ for every solution K to the adjoint equation $\mathscr{T}_\lambda^* K = 0$.*

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