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RESEARCH ARTICLE

Discrete-time joint signature and associated properties

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In this work, we consider two coherent systems with shared components in the case when the component lifetimes are independent and identically distributed with a discrete-time distribution instead of a continuous-time distribution. Then, a discrete-time joint signature is proposed for the two systems by generalizing the traditional joint signature for systems with continuous lifetimes. Some stochastic properties of the proposed joint signature are studied in detail, including joint distribution, stochastic ordering, and transformation formula for comparison of pairs of systems of different sizes. Some illustrative examples are also presented.

1. Introduction

Signatures play an important and unique role in reliability theory [6, 18]. Their theory has developed considerably since the first notion called "system signature" was proposed by Samaniego [17] as $\boldsymbol{s} = (s_1, \ldots, s_n)$ for coherent systems with n components which are independent and identically distributed (i.i.d.) from a continuous-time distribution; here, s_i is the probability that the ith component failure causes the failure of the system. Various signature measures have been proposed for describing structural properties of different types of coherent systems. These include joint signature for coherent systems with shared components [15, 16], dynamic signature for used coherent systems with known number of failed components [19], minimal/maximal signature for coherent systems with exchangeable components [14], survival signature for coherent systems with multiple types of components [5], ordered signature for coherent systems in a life-test of systems [2, 25], and their counterparts for multi-state coherent systems [22–24, 26, 27]. A detailed review of these has been provided in [21].

In practical reliability systems, it is common to see components shared by different systems, which leads to the concept of joint signature. Navarro *et al.* [15] originally tried to define joint signature based on joint distribution of the systems with shared components, and Navarro *et al.* [16] argued subsequently that it will be better to define the joint signature to be $\boldsymbol{s} = (s_{i,j},\ 1 \le i, j \le n)$; here, $s_{i,j}$ is the probability that the ith and jth component failures cause the two systems to fail, respectively. More discussions can be found with regard to statistical inference in [3], generalizations to two or more systems in [13] and [29], and some others in [24, 27, 28].

Signatures are quite useful in describing structures of various reliability systems and in studying their lifetime properties. However, one of the main limitations in the associated theory is that it is based on the assumption that component lifetimes follow a continuous-time distribution, while many practical reliability systems actually have their component lifetimes to be discrete [1, 7–12, 20]. For this reason, Balakrishnan *et al.* [4] recently generalized the Samaniego's [17] notion of system signature to discrete-time signature and discussed some of its stochastic properties. In the present work, we further

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develop joint signature when the component lifetimes follow a discrete-time distribution, in analogy to the work of Navarro *et al.* [15, 16] on joint signature for systems with components possessing a continuous lifetime distribution.

The rest of this paper proceeds as follows. In Section 2, we introduce the definition of discrete-time joint signature for two coherent systems with shared components, whose lifetimes are i.i.d. from a discrete-time distribution, and the joint distribution of the system lifetimes is discussed in detail. In Section 3, some series and parallel systems with shared components are studied based on their discrete-time joint signature for a clear understanding of the concept. In Section 4, stochastic ordering results are presented for two pairs of coherent systems with some shared components. In Section 5, a transformation formula is introduced for pairs of systems of different sizes so that they can be stochastically compared. Some concluding remarks are finally made in Section 6.

2. Joint signature and joint distribution of system lifetimes

In the continuous-time case, Navarro *et al.* [16] defined joint signature for two coherent systems ϕ_1, ϕ_2 with n shared components, whose lifetimes $X_1, \ldots, X_n$ are i.i.d. continuous random variables, as $\boldsymbol{s} = (s_{i,j},\ , 1 \le i,j \le n)$ with its (i,j)-th element $s_{i,j} = P\{T_1 = X_{i:n},\ T_2 = X_{j:n}\}$, where T_1 and T_2 are the lifetimes of systems ϕ_1, ϕ_2, and $X_{1:n} \le \cdots \le X_{n:n}$ are the order statistics corresponding to the component lifetimes $X_1, \ldots, X_n$. Now, along the lines of Balakrishnan *et al.* [4], the concept of joint signature in the discrete-time case is introduced as follows.

Definition 2.1. *For two coherent systems ϕ_1, ϕ_2 with n shared components and lifetimes T_1, T_2, suppose the components have i.i.d. discrete lifetimes $X_1, \ldots, X_n$ from a cumulative distribution function (cdf) $F(\cdot)$ and a probability mass function (pmf) $f(\cdot)$. Let us denote the i-th smallest component lifetime by $X_{i:n}$. Then, the joint signature of the two systems can be defined as*

$$\boldsymbol{s} = \left(s_{i_1,i_2}^{j_1,\ldots,j_m},\ j_1 + \cdots + j_m = n,\ 1 \le i_1, i_2 \le m \le n \right),$$

where

$$s_{i_1,i_2}^{j_1,\ldots,j_m} = P\{T_1 = X_{j_{(i_1-1)}+1:n},\ T_2 = X_{j_{(i_2-1)}+1:n} \,|E_{j_1,\ldots,j_m}\}$$

is the conditional probability that failure of System ϕ_1 is caused by the $(j_{(i_1-1)}+1)$-th component failure and failure of System ϕ_2 is caused by the $(j_{(i_2-1)}+1)$-th component failure, given the event

$$E_{j_1,\ldots,j_m} = \{X_{(1,\ldots,j_{(1)}):n} < X_{(j_{(1)}+1,\ldots,j_{(2)}):n} < \cdots < X_{(j_{(m-1)}+1,\ldots,j_{(m)}):n}\};$$

here, $j_1, \ldots, j_m$ are the numbers of order statistics $X_{1:n}, \ldots, X_{n:n}$ that are equal to the first, second, ..., m-th distinct values, respectively. We use $X_{(u,\ldots,v):n}$ as an abbrevation for $X_{u:n} = \cdots = X_{v:n}$ $(1 \le u \le v \le n)$ and $j_{(i)} = j_1 + \cdots + j_i$ for the i-th patial sum, for $i = 1, \ldots, m$. It is evident that $j_{(1)} = j_1$ and $j_{(m)} = \sum_{i=1}^{m} j_i = n$. For convenience, we set $j_{(0)} = 0$ and $j_{(-1)} = -1$.

Remark 2.1. It is useful to observe that the discrete-time joint signature is distribution-free, namely, the elements $s_{i_1,i_2}^{j_1,\ldots,j_m}$ do not depend on the components' lifetime distribution F. Besides,

$$\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} s_{i_1,i_2}^{j_1,\ldots,j_m} = 1$$

holds for all $j_1 + \cdots + j_m = n$ $(1 \le m \le n)$, which is due to the fact that the discrete-time joint signature has been defined based on conditional event $E_{j_1,\ldots,j_m}$. If not, we can redefine the discrete-time joint signature in an unconditional way as

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$$s_{i_1,i_2}^{j_1,\ldots,j_m} = P\{T_1 = X_{j_{(i_1-1)}+1:n},\ T_2 = X_{j_{(i_2-1)}+1:n},\ E_{j_1,\ldots,j_m}\},$$

which would lead to $\sum_{j_1+\cdots+j_m=n} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} s_{i_1,i_2}^{j_1,\ldots,j_m} = 1$ instead.

Remark 2.2. The discrete-time joint signature can be regarded as a matrix of dimension $C_n \times n^2$, where $C_n = \sum_{m=1}^{n} \binom{n-1}{m-1} = 2^{n-1}$ is the total number of compositions (i.e., distinguishable partitions) of an integer n. For example, in the case of $n = 2$, there are $C_2 = 2^{2-1} = 2$ possible compositions, that is, $j_1 = j_2 = 1$ and $j_1 = 2$. The first composition $j_1 = j_2 = 1$ corresponds to the first row of $\boldsymbol{s}$, that is, $s_{i_1,i_2}^{1,1}$, $1 \le i_1, i_2 \le 2$; and the second composition $j_1 = 2$ corresponds to the second row of $\boldsymbol{s}$, that is, $s_{i_1,i_2}^{1,1}$, $1 \le i_1, i_2 \le 1$; see Example 3.1 for pertinent details. Similar discussions can also be provided for $n = 3$; see Example 3.2 for details.

The above definition of joint signature can be used readily to present the following expressions concerning the joint distribution of system lifetimes.

Proposition 2.1. *The joint pmf of the system lifetimes T_1, T_2 can be given as*

$$P\{T_1 = x,\ T_2 = y\} = \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \binom{n}{j_1,\ldots,j_m} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} s_{i_1,i_2}^{j_1,\ldots,j_m} \left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}=x,\ t_{i_2}=y}} \prod_{u=1}^{m} f^{j_u}(t_u) \right],$$

the joint survival function as

$$P\{T_1 > x,\ T_2 > y\} = \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \binom{n}{j_1,\ldots,j_m} \sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-1} \bar{S}_{i_1,i_2}^{j_1,\ldots,j_m} \left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}\le x<t_{i_1+1},\ t_{i_2}\le y<t_{i_2+1}}} \prod_{u=1}^{m} f^{j_u}(t_u) \right],$$

and the joint cdf as

$$P\{T_1 \le x,\ T_2 \le y\} = \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \binom{n}{j_1,\ldots,j_m} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} S_{i_1,i_2}^{j_1,\ldots,j_m} \left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}\le x<t_{i_1+1},\ t_{i_2}\le y<t_{i_2+1}}} \prod_{u=1}^{m} f^{j_u}(t_u) \right],$$

where $t_0 = 0$, $\bar{S}_{i_1,i_2}^{j_1,\ldots,j_m} = \sum_{l_1=i_1+1}^{m} \sum_{l_2=i_2+1}^{m} s_{l_1,l_2}^{j_1,\ldots,j_m}$ and $S_{i_1,i_2}^{j_1,\ldots,j_m} = \sum_{l_1=1}^{i_1} \sum_{l_2=1}^{i_2} s_{l_1,l_2}^{j_1,\ldots,j_m}$.

Proof. According to Definition 2.1, we have

$$\begin{aligned} &P\{T_1 = x,\ T_2 = y\} \\ &= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} P\{T_1 = x,\ T_2 = y \,\big|\, T_1 = X_{j_{(i_1-1)}+1:n},\ T_2 = X_{j_{(i_2-1)}+1:n},\ E_{j_1,\ldots,j_m}\} \\ &\quad \times P\{T_1 = X_{j_{(i_1-1)}+1:n},\ T_2 = X_{j_{(i_2-1)}+1:n},\ E_{j_1,\ldots,j_m}\} \end{aligned}$$

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$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} P\{X_{j_{(i_1-1)}+1:n} = x,\ X_{j_{(i_2-1)}+1:n} = y \,|E_{j_1,\ldots,j_m}\} \cdot s_{i_1,i_2}^{j_1,\ldots,j_m} \cdot P\{E_{j_1,\ldots,j_m}\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} s_{i_1,i_2}^{j_1,\ldots,j_m} \cdot P\{X_{j_{(i_1-1)}+1:n} = x,\ X_{j_{(i_2-1)}+1:n} = y,\ E_{j_1,\ldots,j_m}\}.$$

Similarly, with the convention that $X_{0:n} = 0$, we have

$$P\{T_1 > x,\ T_2 > y\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} P\{T_1 > x,\ T_2 > y \,\big|T_1 = X_{j_{(l_1-1)}+1:n},\ T_2 = X_{j_{(l_2-1)}+1:n},\ E_{j_1,\ldots,j_m}\}$$

$$\times P\{T_1 = X_{j_{(l_1-1)}+1:n},\ T_2 = X_{j_{(l_2-1)}+1:n},\ E_{j_1,\ldots,j_m}\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} P\{X_{j_{(l_1-1)}+1:n} > x,\ X_{j_{(l_2-1)}+1:n} > y \,|E_{j_1,\ldots,j_m}\} \cdot s_{l_1,l_2}^{j_1,\ldots,j_m} \cdot P\{E_{j_1,\ldots,j_m}\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} s_{l_1,l_2}^{j_1,\ldots,j_m} \cdot P\{X_{j_{(l_1-1)}+1:n} > x,\ X_{j_{(l_2-1)}+1:n} > y,\ E_{j_1,\ldots,j_m}\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} s_{l_1,l_2}^{j_1,\ldots,j_m} \cdot \sum_{i_1=0}^{l_1-1} \sum_{i_2=0}^{l_2-1} P\{X_{j_{(i_1-1)}+1:n} \le x < X_{j_{(i_1)}+1:n},\ X_{j_{(i_2-1)}+1:n} \le y <$$

$$X_{j_{(i_2)}+1:n},\ E_{j_1,\ldots,j_m}\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-1} \left(\sum_{l_1=i_1+1}^{m} \sum_{l_2=i_2+1}^{m} s_{l_1,l_2}^{j_1,\ldots,j_m} \right) \cdot P\{X_{j_{(i_1-1)}+1:n} \le x < X_{j_{(i_1)}+1:n},\ X_{j_{(i_2-1)}+1:n}$$

$$\le y < X_{j_{(i_2)}+1:n},\ E_{j_1,\ldots,j_m}\}.$$

Finally, with the convention that $X_{n+1:n} = \infty$, we have

$$P\{T_1 \le x,\ T_2 \le y\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} P\{T_1 \le x,\ T_2 \le y \,\big|T_1 = X_{j_{(l_1-1)}+1:n},\ T_2 = X_{j_{(l_2-1)}+1:n},\ E_{j_1,\ldots,j_m}\}$$

$$\times P\{T_1 = X_{j_{(l_1-1)}+1:n},\ T_2 = X_{j_{(l_2-1)}+1:n},\ E_{j_1,\ldots,j_m}\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} P\{X_{j_{(l_1-1)}+1:n} \le x,\ X_{j_{(l_2-1)}+1:n} \le y \,|E_{j_1,\ldots,j_m}\} \cdot s_{l_1,l_2}^{j_1,\ldots,j_m} \cdot P\{E_{j_1,\ldots,j_m}\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} s_{l_1,l_2}^{j_1,\ldots,j_m} \cdot P\{X_{j_{(l_1-1)}+1:n} \le x,\ X_{j_{(l_2-1)}+1:n} \le y,\ E_{j_1,\ldots,j_m}\}$$

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$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{l_1=1}^{m} \sum_{l_2=1}^{m} s_{l_1,l_2}^{j_1,\ldots,j_m} \cdot \sum_{i_1=l_1}^{m} \sum_{i_2=l_2}^{m} P\{X_{j_{(i_1-1)}+1:n} \le x < X_{j_{(i_1)}+1:n},\ X_{j_{(i_2-1)}+1:n} \le y < X_{j_{(i_2)}+1:n},\ E_{j_1,\ldots,j_m}\}$$

$$= \sum_{m=1}^{n} \sum_{j_1+\cdots+j_m=n} \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \left(\sum_{l_1=1}^{i_1} \sum_{l_2=1}^{i_2} s_{l_1,l_2}^{j_1,\ldots,j_m} \right) \cdot P\{X_{j_{(i_1-1)}+1:n} \le x < X_{j_{(i_1)}+1:n},\ X_{j_{(i_2-1)}+1:n} \le y < X_{j_{(i_2)}+1:n},\ E_{j_1,\ldots,j_m}\}.$$

In the above expressions, we further have

$$P\{X_{j_{(i_1-1)}+1:n} = x,\ X_{j_{(i_2-1)}+1:n} = y,\ E_{j_1,\ldots,j_m}\} = \binom{n}{j_1,\ldots,j_m} \left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}=x,\ t_{i_2}=y}} \prod_{u=1}^{m} f^{j_u}(t_u) \right],$$

$$P\{X_{j_{(i_1-1)}+1:n} \le x < X_{j_{(i_1)}+1:n},\ X_{j_{(i_2-1)}+1:n} \le y < X_{j_{(i_2)}+1:n},\ E_{j_1,\ldots,j_m}\}$$

$$= \binom{n}{j_1,\ldots,j_m} \left[\sum_{1\le t_1<\cdots<t_m<\infty,\ t_{i_1}\le x<t_{i_1+1},\ t_{i_2}\le y<t_{i_2+1}} \prod_{u=1}^{m} f^{j_u}(t_u) \right].$$

□

3. Illustration with series and parallel systems

In this section, we consider series and parallel systems to provide a clearer understanding of the discrete-time joint signature introduced in the last section.

Proposition 3.1. *Two series systems sharing all their n components have their discrete-time joint signature as*

$$\boldsymbol{s}^{(ser)} = \left(s_{(ser),i_1,i_2}^{j_1,\ldots,j_m},\ j_1+\cdots+j_m=n,\ 1 \le i_1, i_2 \le m \le n \right),$$

with $s_{(ser),i_1,i_2}^{j_1,\ldots,j_m} = I_{\{i_1=i_2=1\}}$ *for all* $j_1+\cdots+j_m = n$ $(1 \le m \le n)$*, while two parallel systems sharing all their n components have their discrete-time joint signature as*

$$\boldsymbol{s}^{(par)} = \left(s_{(par),i_1,i_2}^{j_1,\ldots,j_m},\ j_1+\cdots+j_m=n,\ 1 \le i_1, i_2 \le m \le n \right),$$

with $s_{(par),i_1,i_2}^{j_1,\ldots,j_m} = I_{\{i_1=i_2=m\}}$ *for all* $j_1+\cdots+j_m = n$ $(1 \le m \le n)$*. More generally, a* k_1*-out-of-n: F system and a* k_2*-out-of-n: F system sharing all their n components have their discrete-time joint signature as*

$$\boldsymbol{s}^{(k_1,k_2:n)} = \left(s_{(k_1,k_2:n),i_1,i_2}^{j_1,\ldots,j_m},\ j_1+\cdots+j_m=n,\ 1 \le i_1, i_2 \le m \le n \right),$$

with $s_{(k_1,k_2:n),i_1,i_2}^{j_1,\ldots,j_m} = I_{\{j_{(i_1-1)}<k_1\le j_{(i_1)},\ j_{(i_2-1)}<k_2\le j_{(i_2)}\}}$ *for all* $j_1+\cdots+j_m = n$ $(1 \le m \le n)$*.*

Proof. For series systems, $s_{(ser),i_1,i_2}^{j_1,\ldots,j_m} = I_{\{i_1=i_2=1\}}$ since both systems fail at the first component failure. For parallel systems, $s_{(par),i_1,i_2}^{j_1,\ldots,j_m} = I_{\{i_1=i_2=m\}}$ since both systems fail at the last component failure. For the

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k-out-of-n systems, $s^{j_1,\ldots,j_m}_{(k_1,k_2:n),i_1,i_2} = I_{\{j_{(i_1-1)}<k_1\le j_{(i_1)},\, j_{(i_2-1)}<k_2\le j_{(i_2)}\}}$ since a k-out-of-n system fails at the kth component failure.

Two coherent systems with n shared components with a discrete-time joint signature

$$\boldsymbol{s} = \left(s^{j_1,\ldots,j_m}_{i_1,i_2},\ j_1+\cdots+j_m = n,\ 1\le i_1,i_2\le m\le n\right)$$

can be regarded as a mixture of several pairs of k-out-of-n: F systems; that is,

$$\boldsymbol{s} = \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} s^{1,\ldots,n}_{k_1,k_2}\cdot \boldsymbol{s}^{(k_1,k_2:n)},$$

which leads to the fact that

$$s^{j_1,\ldots,j_m}_{i_1,i_2} = \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} s^{1,\ldots,n}_{k_1,k_2}\cdot I_{\{j_{(i_1-1)}<k_1\le j_{(i_1)},\, j_{(i_2-1)}<k_2\le j_{(i_2)}\}} = \sum_{k_1=j_{(i_1-1)}+1}^{j_{(i_1)}}\sum_{k_2=j_{(i_2-1)}+1}^{j_{(i_2)}} s^{1,\ldots,n}_{k_1,k_2}.$$

This means that we can present a discrete-time joint signature by using a continuous-time joint signature of the same coherent systems. □

Proposition 3.2. *For two coherent systems with discrete-time joint signature* $\boldsymbol{s} = \left(s^{j_1,\ldots,j_m}_{i_1,i_2},\ j_1+\cdots+j_m = n,\ 1\le i_1,i_2\le m\le n\right)$*, their corresponding continuous-time joint signature can be given directly from the first row of* $\boldsymbol{s}$ *as* $\tilde{\boldsymbol{s}} = \left(s^{1,\ldots,1}_{i_1,i_2},\ 1\le i_1,i_2\le n\right)$*. For two coherent systems with continuous-time joint signature* $\boldsymbol{s} = \left(s_{i_1,i_2},\ 1\le i_1,i_2\le n\right)$*, their corresponding discrete-time joint signature can be given as* $\tilde{\boldsymbol{s}} = \left(\tilde{s}^{j_1,\ldots,j_m}_{i_1,i_2},\ j_1+\cdots+j_m = n,\ 1\le i_1,i_2\le m\le n\right)$*, where*

$$\tilde{s}^{j_1,\ldots,j_m}_{i_1,i_2} = \sum_{k_1=j_{(i_1-1)}+1}^{j_{(i_1)}}\sum_{k_2=j_{(i_2-1)}+1}^{j_{(i_2)}} s_{k_1,k_2}.$$

Remark 3.1. The following examples illustrate this point.

(1) For two series systems

$$\phi_1 = \min(X_1,\ldots,X_{n_1+n_{1,2}}),\ \phi_2 = \min(X_{n_1+1},\ldots,X_{n_1+n_{1,2}+n_2})$$

with $n = n_1+n_{1,2}+n_2$ shared components, their continuous-time joint signature is given by $\boldsymbol{s} = (s_{i,j},\ 1\le i,j\le n)$ with

$$\begin{aligned} s_{i,j} =& \frac{n_{1,2}}{n} I_{\{i=j=1\}} + \binom{n}{j-1}^{-1}\binom{n_1}{j-1}\frac{n_2+n_{1,2}}{n-j+1} I_{\{i=1,\ 1<j\le n_1+1\}} \\ &+ \binom{n}{i-1}^{-1}\binom{n_2}{i-1}\frac{n_1+n_{1,2}}{n-i+1} I_{\{1<i\le n_2+1,\ j=1\}}, \end{aligned}$$

which leads to their discrete-time joint signature as

$$\tilde{\boldsymbol{s}} = \left(\tilde{s}^{j_1,\ldots,j_m}_{i_1,i_2},\ j_1+\cdots+j_m = n,\ 1\le i_1,i_2\le m\le n\right),$$

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with

$$\begin{aligned}\tilde{s}_{i_1,i_2}^{j_1,\ldots,j_m} &= \sum_{i=j_{(i_1-1)}+1}^{j_{(i_1)}} \sum_{j=j_{(i_2-1)}+1}^{j_{(i_2)}} s_{i,j} \\ &= \frac{n_{1,2}}{n} I_{\{i_1=i_2=1\}} + \sum_{j=(j_{(i_2-1)}+1)\vee 2}^{j_{(i_2)}\wedge(n_1+1)} \binom{n}{j-1}^{-1} \binom{n_1}{j-1} \frac{n_2+n_{1,2}}{n-j+1} I_{\{i_1=1\}} \\ &\quad + \sum_{i=(j_{(i_1-1)}+1)\vee 2}^{j_{(i_1)}\wedge(n_2+1)} \binom{n}{i-1}^{-1} \binom{n_2}{i-1} \frac{n_1+n_{1,2}}{n-i+1} I_{\{i_2=1\}}.\end{aligned}$$

(2) For two parallel systems

$$\phi_1 = \max(X_1, \ldots, X_{n_1+n_{1,2}}),\ \phi_2 = \max(X_{n_1+1}, \ldots, X_{n_1+n_{1,2}+n_2})$$

with $n = n_1 + n_{1,2} + n_2$ shared components, their continuous-time joint signature is given by $\boldsymbol{s} = (s_{i,j},\ 1 \le i,j \le n)$ with

$$\begin{aligned}s_{i,j} &= \frac{n_{1,2}}{n} I_{\{i=j=n\}} + \binom{n}{j-1}^{-1} \binom{n_1}{j-n_2-n_{1,2}} \frac{n_2+n_{1,2}}{n-j+1} I_{\{i=n,\ n_2+n_{1,2}\le j<n\}} \\ &\quad + \binom{n}{i-1}^{-1} \binom{n_2}{i-n_1-n_{1,2}} \frac{n_1+n_{1,2}}{n-i+1} I_{\{n_1+n_{1,2}\le i<n,\ j=n\}},\end{aligned}$$

which leads to their discrete-time joint signature as

$$\tilde{\boldsymbol{s}} = \left(\tilde{s}_{i_1,i_2}^{j_1,\ldots,j_m},\ j_1+\cdots+j_m = n,\ 1 \le i_1, i_2 \le m \le n\right),$$

with

$$\begin{aligned}\tilde{s}_{i_1,i_2}^{j_1,\ldots,j_m} &= \sum_{i=j_{(i_1-1)}+1}^{j_{(i_1)}} \sum_{j=j_{(i_2-1)}+1}^{j_{(i_2)}} s_{i,j} \\ &= \frac{n_{1,2}}{n} I_{\{i_1=i_2=m\}} + \sum_{j=(j_{(i_2-1)}+1)\vee(n_2+n_{1,2})}^{j_{(i_2)}\wedge(n-1)} \binom{n}{j-1}^{-1} \binom{n_1}{j-n_2-n_{1,2}} \frac{n_2+n_{1,2}}{n-j+1} I_{\{i_1=m\}} \\ &\quad + \sum_{i=(j_{(i_1-1)}+1)\vee(n_1+n_{1,2})}^{j_{(i_1)}\wedge(n-1)} \binom{n}{i-1}^{-1} \binom{n_2}{i-n_1-n_{1,2}} \frac{n_1+n_{1,2}}{n-i+1} I_{\{i_2=m\}}.\end{aligned}$$

We now present a few examples to illustrate the introduced notion which, incidentally, also provides a logical motivation for the given definition.

Example 3.1. Consider a single-component system $\phi_1(\boldsymbol{X}) = X_1$ and a parallel system $\phi_2(\boldsymbol{X}) = \max(X_1, X_2)$ with two shared i.i.d. discrete component lifetimes. In this case, there are 3 different possible outcomes for the ordered component lifetimes, as shown in Table 1, and the corresponding system lifetimes are also shown in the table. In this case, the discrete-time joint signature for the two systems is

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$$\boldsymbol{s} = \begin{pmatrix} s_{1,1}^{1,1} & s_{1,2}^{1,1} & s_{2,1}^{1,1} & s_{2,2}^{1,1} \\ s_{1,1}^{2} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Table 1. *Detailed results for systems $\phi_1(\boldsymbol{X})$ and $\phi_2(\boldsymbol{X})$ in Example 3.1.*

m	$j_1, \ldots, j_m$	Component ordering	Lifetime for $\phi_1(\boldsymbol{X})$	Lifetime for $\phi_2(\boldsymbol{X})$
2	1, 1	$X_1 < X_2$	$X_{1:2}$	$X_{2:2}$
2	1, 1	$X_2 < X_1$	$X_{2:2}$	$X_{2:2}$
1	2	$X_1 = X_2$	$X_{1:2}$	$X_{1:2}$

Remark 3.2. According to Remark 3.1, the above result can also be obtained by the fact that continuous-time joint signature of the same two systems is known to be $\tilde{\boldsymbol{s}} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$, which leads to its discrete-time joint signature as

$$\boldsymbol{s} = \frac{1}{2}\boldsymbol{s}_{1,2:3} + \frac{1}{2}\boldsymbol{s}_{2,2:3} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

exactly as determined above.

Example 3.2. Consider a coherent system $\phi_1(\boldsymbol{X}) = \min(\max(X_1, X_2), X_3)$ (a series-parallel system) and another coherent system $\phi_2(\boldsymbol{X}) = \max(\min(X_1, X_2), X_3)$ (a parallel-series system) with shared three i.i.d. discrete component lifetimes. In this case, there are 13 different possible outcomes for the ordered component lifetimes, as shown in Table 2, and the corresponding system lifetimes are also shown in the table. In this case, the discrete-time joint signature for the two systems is

$$\begin{aligned} \boldsymbol{s} &= \begin{pmatrix} s_{1,1}^{1,1,1} & s_{1,2}^{1,1,1} & s_{1,3}^{1,1,1} & s_{2,1}^{1,1,1} & s_{2,2}^{1,1,1} & s_{2,3}^{1,1,1} & s_{3,1}^{1,1,1} & s_{3,2}^{1,1,1} & s_{3,3}^{1,1,1} \\ s_{1,1}^{1,2} & s_{1,2}^{1,2} & s_{2,1}^{1,2} & s_{2,2}^{1,2} & 0 & 0 & 0 & 0 & 0 \\ s_{1,1}^{2,1} & s_{1,2}^{2,1} & s_{2,1}^{2,1} & s_{2,2}^{2,1} & 0 & 0 & 0 & 0 & 0 \\ s_{1,1}^{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

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Table 2. *Detailed results for systems $\phi_1(\boldsymbol{X})$ and $\phi_2(\boldsymbol{X})$ in Example 3.2.*

$m = 1$	$j_1, \ldots, j_m$	Component ordering	Lifetime for $\phi_1(\boldsymbol{X})$	Lifetime for $\phi_2(\boldsymbol{X})$
3	1, 1, 1	$X_1 < X_2 < X_3$	$X_{2:3}$	$X_{3:3}$
	1, 1, 1	$X_1 < X_3 < X_2$	$X_{2:3}$	$X_{2:3}$
	1, 1, 1	$X_2 < X_1 < X_3$	$X_{2:3}$	$X_{3:3}$
	1, 1, 1	$X_2 < X_3 < X_1$	$X_{2:3}$	$X_{2:3}$
	1, 1, 1	$X_3 < X_1 < X_2$	$X_{1:3}$	$X_{2:3}$
	1, 1, 1	$X_3 < X_2 < X_1$	$X_{1:3}$	$X_{2:3}$
2	1, 2	$X_1 < X_2 = X_3$	$X_{2:3}$	$X_{2:3}$
	1, 2	$X_2 < X_1 = X_3$	$X_{2:3}$	$X_{2:3}$
	1, 2	$X_3 < X_1 = X_2$	$X_{1:3}$	$X_{2:3}$
	2, 1	$X_1 = X_2 < X_3$	$X_{1:3}$	$X_{3:3}$
	2, 1	$X_1 = X_3 < X_2$	$X_{1:3}$	$X_{1:3}$
	2, 1	$X_2 = X_3 < X_1$	$X_{1:3}$	$X_{1:3}$
1	3	$X_1 = X_2 = X_3$	$X_{1:3}$	$X_{1:3}$

Remark 3.3. According to Remark 3.1, the above result can also be obtained by the fact that continuous-time joint signature of the same two systems is known to be $\tilde{\boldsymbol{s}} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$, which leads to its discrete-time joint signature as

$$\begin{aligned} \boldsymbol{s} &= \frac{1}{3}\boldsymbol{s}_{1,2:3} + \frac{1}{3}\boldsymbol{s}_{2,2:3} + \frac{1}{3}\boldsymbol{s}_{2,3:3} \\ &= \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &\quad + \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

exactly as determined above.

Example 3.3. Consider a coherent system $\phi_1(\boldsymbol{X}) = \min(X_1, X_2)$ and another coherent system $\phi_2(\boldsymbol{X}) = \min(X_2, X_3)$ with three shared i.i.d. discrete component lifetimes. As in Example 3.2, based on the detailed results in Table 3, the discrete-time joint signature for the two systems in this case is

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$$\boldsymbol{s} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Table 3. *Detailed results for systems $\phi_1(\boldsymbol{X})$ and $\phi_2(\boldsymbol{X})$ in Example 3.3.*

m	$j_1, \ldots, j_m$	Component ordering	Lifetime for $\phi_1(\boldsymbol{X})$	Lifetime for $\phi_2(\boldsymbol{X})$
3	1, 1, 1	$X_1 < X_2 < X_3$	$X_{1:3}$	$X_{2:3}$
	1, 1, 1	$X_1 < X_3 < X_2$	$X_{1:3}$	$X_{2:3}$
	1, 1, 1	$X_2 < X_1 < X_3$	$X_{1:3}$	$X_{1:3}$
	1, 1, 1	$X_2 < X_3 < X_1$	$X_{1:3}$	$X_{1:3}$
	1, 1, 1	$X_3 < X_1 < X_2$	$X_{2:3}$	$X_{1:3}$
	1, 1, 1	$X_3 < X_2 < X_1$	$X_{2:3}$	$X_{1:3}$
2	1, 2	$X_1 < X_2 = X_3$	$X_{1:3}$	$X_{2:3}$
	1, 2	$X_2 < X_1 = X_3$	$X_{1:3}$	$X_{1:3}$
	1, 2	$X_3 < X_1 = X_2$	$X_{2:3}$	$X_{1:3}$
	2, 1	$X_1 = X_2 < X_3$	$X_{1:3}$	$X_{1:3}$
	2, 1	$X_1 = X_3 < X_2$	$X_{1:3}$	$X_{1:3}$
	2, 1	$X_2 = X_3 < X_1$	$X_{1:3}$	$X_{1:3}$
1	3	$X_1 = X_2 = X_3$	$X_{1:3}$	$X_{1:3}$

Remark 3.4. The same results can also be presented by a discussion similar to Remark 3.3. In addition, according to Part (1) of Remark 3.1, with $n = 3$ and $n_1 = n_2 = n_{1,2} = 1$, we also have

$$\begin{aligned}
s^{1,1,1}_{i_1,i_2} &= \frac{1}{3} I_{\{i_1=i_2=1\}} + c_j I_{\{i_1=1,\ i_2=2,\ j=2\}} + c_i I_{\{i_1=2,\ i_2=1,\ i=2\}},\ 1 \le i_1, i_2 \le 3, \\
s^{1,2}_{i_1,i_2} &= \frac{1}{3} I_{\{i_1=i_2=1\}} + c_j I_{\{i_1=1,\ i_2=2,\ j=2\}} + c_i I_{\{i_1=2,\ i_2=1,\ i=2\}},\ 1 \le i_1, i_2 \le 2, \\
s^{2,1}_{i_1,i_2} &= \frac{1}{3} I_{\{i_1=i_2=1\}} + c_j I_{\{i_1=i_2=1,\ j=2\}} + c_i I_{\{i_1=i_2=1,\ i=2\}},\ 1 \le i_1, i_2 \le 2, \\
s^{3}_{i_1,i_2} &= \frac{1}{3} I_{\{i_1=i_2=1\}} + c_j I_{\{i_1=i_2=1,\ j=2\}} + c_i I_{\{i_1=i_2=1,\ i=2\}},\ 1 \le i_1, i_2 \le 1,
\end{aligned}$$

with $c_j = \binom{3}{j-1}^{-1} \binom{1}{j-1} \frac{2}{4-j}$ (specifically, $c_2 = \frac{1}{3}$), which leads to the same joint signature as presented above.

Example 3.4. Consider a coherent system $\phi_1(\boldsymbol{X}) = \max(X_1, X_2)$ and another coherent system $\phi_2(\boldsymbol{X}) = \max(X_2, X_3)$ with three i.i.d. discrete component lifetimes. As in Example 3.2, based on the detailed results in Table 4, the discrete-time joint signature for the two systems in this case is

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$$\boldsymbol{s} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Table 4. *Detailed results for systems $\phi_1(\boldsymbol{X})$ and $\phi_2(\boldsymbol{X})$ in Example 3.4.*

m	$j_1, \ldots, j_m$	Component ordering	Lifetime for $\phi_1(\boldsymbol{X})$	Lifetime for $\phi_2(\boldsymbol{X})$
3	1, 1, 1	$X_1 < X_2 < X_3$	$X_{2:3}$	$X_{3:3}$
	1, 1, 1	$X_1 < X_3 < X_2$	$X_{3:3}$	$X_{3:3}$
	1, 1, 1	$X_2 < X_1 < X_3$	$X_{2:3}$	$X_{3:3}$
	1, 1, 1	$X_2 < X_3 < X_1$	$X_{3:3}$	$X_{2:3}$
	1, 1, 1	$X_3 < X_1 < X_2$	$X_{3:3}$	$X_{3:3}$
	1, 1, 1	$X_3 < X_2 < X_1$	$X_{3:3}$	$X_{2:3}$
2	1, 2	$X_1 < X_2 = X_3$	$X_{2:3}$	$X_{2:3}$
	1, 2	$X_2 < X_1 = X_3$	$X_{2:3}$	$X_{2:3}$
	1, 2	$X_3 < X_1 = X_2$	$X_{2:3}$	$X_{2:3}$
	2, 1	$X_1 = X_2 < X_3$	$X_{1:3}$	$X_{3:3}$
	2, 1	$X_1 = X_3 < X_2$	$X_{3:3}$	$X_{3:3}$
	2, 1	$X_2 = X_3 < X_1$	$X_{3:3}$	$X_{1:3}$
1	3	$X_1 = X_2 = X_3$	$X_{1:3}$	$X_{1:3}$

Remark 3.5. The same result can also be obtained by using Part (2) of Remark 3.1 as follows:

$$\begin{aligned}
s^{1,1,1}_{i_1,i_2} &= \frac{1}{3} I_{\{i_1=i_2=3\}} + c_j I_{\{i_1=3,\ i_2=2,\ j=2\}} + c_i I_{\{i_1=2,\ i_2=3,\ i=2\}},\ 1 \le i_1, i_2 \le 3, \\
s^{1,2}_{i_1,i_2} &= \frac{1}{3} I_{\{i_1=i_2=2\}} + c_j I_{\{i_1=i_2=2,\ j=2\}} + c_i I_{\{i_1=i_2=2,\ i=2\}},\ 1 \le i_1, i_2 \le 2, \\
s^{2,1}_{i_1,i_2} &= \frac{1}{3} I_{\{i_1=i_2=2\}} + c_j I_{\{i_1=2,\ i_2=1,\ j=2\}} + c_i I_{\{i_1=1,\ i_2=2,\ i=2\}},\ 1 \le i_1, i_2 \le 2, \\
s^{3}_{i_1,i_2} &= \frac{1}{3} I_{\{i_1=i_2=1\}} + c_j I_{\{i_1=i_2=1,\ j=2\}} + c_i I_{\{i_1=i_2=1,\ i=2\}},\ 1 \le i_1, i_2 \le 1,
\end{aligned}$$

with $c_j = \binom{3}{j-1}^{-1} \binom{1}{j-2} \frac{2}{4-j}$ (specifically, $c_2 = \frac{1}{3}$), which yield the same joint signature as presented above.

4. Stochastic ordering results

Following naturally from Proposition 2.1, the discrete-time signature introduced earlier in Definition 2.1 can also be readily used to establish some stochastic ordering results, as shown below.

Theorem 4.1. *Let* $\boldsymbol{s}_1 = \left(s^{j_1,\ldots,j_m}_{1,i_1,i_2},\ j_1 + \cdots + j_m = n,\ 1 \le i_1, i_2 \le m \le n \right)$ *and*

$$\boldsymbol{s}_2 = \left(s^{j_1,\ldots,j_m}_{2,i_1,i_2},\ j_1 + \cdots + j_m = n,\ 1 \le i_1, i_2 \le m \le n \right)$$

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be the joint signatures of two pairs of coherent systems of size n with discrete component lifetimes i.i.d. from the same distribution, and let T_1^1, T_2^1 *and* T_1^2, T_2^2 *denote their corresponding lifetimes. If*

$$\sum_{l_1=i_1+1}^{m}\sum_{l_2=i_2+1}^{m} s_{1,l_1,l_2}^{j_1,\ldots,j_m} \le \sum_{l_1=i_1+1}^{m}\sum_{l_2=i_2+1}^{m} s_{2,l_1,l_2}^{j_1,\ldots,j_m},$$

for all $0 \le i_1, i_2 \le m-1$ *and* $j_1+\cdots+j_m = n$ $(1 \le m \le n)$, *then* $P\{T_1^1 > x, T_2^1 > y\} \le P\{T_1^2 > x, T_2^2 > y\}$ *for any* x, y.

Proof. For any $x, y = 1, \ldots, \infty$, we have

$$\begin{aligned}
&P\{T_1^1 > x,\ T_2^1 > y\}\\
&=\sum_{m=1}^{n}\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\ldots,j_m}\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m-1}\left(\sum_{l_1=i_1+1}^{m}\sum_{l_2=i_2+1}^{m} s_{1,l_1,l_2}^{j_1,\ldots,j_m}\right)\left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}\le x<t_{i_1+1},\ t_{i_2}\le y<t_{i_2+1}}}\prod_{u=1}^{m} f^{j_u}(t_u)\right]\\
&\le\sum_{m=1}^{n}\sum_{j_1+\cdots+j_m=n}\binom{n}{j_1,\ldots,j_m}\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m-1}\left(\sum_{l_1=i_1+1}^{m}\sum_{l_2=i_2+1}^{m} s_{2,l_1,l_2}^{j_1,\ldots,j_m}\right)\left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}\le x<t_{i_1+1},\ t_{i_2}\le y<t_{i_2+1}}}\prod_{u=1}^{m} f^{j_u}(t_u)\right]\\
&=P\{T_1^2 > x,\ T_2^2 > y\},
\end{aligned}$$

which completes the proof of the theorem. □

Remark 4.1. The condition

$$\sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} s_{1,i_1,i_2}^{j_1,\ldots,j_m} \le \sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} s_{2,i_1,i_2}^{j_1,\ldots,j_m}, \quad 0 \le l_1, l_2 \le m-1,$$

holds for all $j_1 + \cdots + j_m = n$ $(1 \le m \le n)$ once it holds for the case $j_1 = \cdots = j_n = 1$, that is, the stochastic ordering of discrete-time joint signature can be obtained from the stochastic ordering of continuous-time joint signature. This is so because, according to Remark 3.1, we have

$$\begin{aligned}
\sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} s_{u,i_1,i_2}^{j_1,\ldots,j_m} &= \sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m}\sum_{k_1=j_{(i_1-1)}+1}^{j_{(i_1)}}\sum_{k_2=j_{(i_2-1)}+1}^{j_{(i_2)}} s_{u,k_1,k_2}^{1,\ldots,n}\\
&= \sum_{k_1=j_{(l_1)}+1}^{n}\sum_{k_2=j_{(l_2)}+1}^{n} s_{u,k_1,k_2}^{1,\ldots,n}, \qquad u = 1, 2.
\end{aligned}$$

Remark 4.2. Note that the theorem shows that the stochastic ordering of joint signatures leads to comparison of associated systems in the sense of joint reliability. Besides, with given $j_1, \ldots, j_m$ such that $m \ne n$, the condition for stochastic ordering of $(s_{i_1,i_2}^{j_1,\ldots,j_m}, 1 \le i_1, i_2 \le m)$ (corresponding row of discrete-time joint signature) is easier to get satisfied than the continuous-time joint signature, which leads to comparison of associated systems in the sense of conditional joint reliability.

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Theorem 4.2. *Let $\boldsymbol{s}^{(ser)}$ be the joint signature of two series systems sharing all their n i.i.d. discrete components. Then, their lifetimes $T_1^{(ser)}, T_2^{(ser)}$ are such that*

$$P\{T_1^{(ser)} > x,\ T_2^{(ser)} > y\} \le P\{T_1 > x,\ T_2 > y\},$$

where T_1, T_2 are the lifetimes of any pair of coherent systems with n shared discrete components i.i.d. from the same distribution.

Proof. Denote the discrete-time joint signature of the systems with lifetimes T_1, T_2 by $\boldsymbol{s}$. Then, for all $0 \le l_1, l_2 \le m-1$ and $j_1 + \cdots + j_m = n$ $(1 \le m \le n)$, from Propositon 3.1, we have $s^{j_1,\ldots,j_m}_{(ser),i_1,i_2} = I_{\{i_1=i_2=1\}}$, which means

$$\sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} s^{j_1,\ldots,j_m}_{(ser),i_1,i_2} = \sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} I_{\{i_1=i_2=1\}} = I_{\{l_1=l_2=0\}} \le \sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} s^{j_1,\ldots,j_m}_{i_1,i_2},$$

which leads to the required result by using Theorem 4.1. □

Remark 4.3. Note that the theorem shows that the joint reliability of two series systems sharing all their n i.i.d. discrete components is no more than the joint reliability of any two coherent systems with n shared discrete components i.i.d. from the same distribution.

Theorem 4.3. *Let $\boldsymbol{s}^{(par)}$ be the joint signature of two parallel systems sharing all their n i.i.d. discrete components. Then, their lifetimes $T_1^{(par)}, T_2^{(par)}$ are such that*

$$P\{T_1^{(par)} > x,\ T_2^{(par)} > y\} \ge P\{T_1 > x,\ T_2 > y\},$$

where T_1, T_2 are the lifetimes of any pair of coherent systems with n shared discrete components i.i.d. from the same distribution.

Proof. Denote the discrete-time joint signature of the systems with lifetimes T_1, T_2 by $\boldsymbol{s}$. Then, for all $0 \le l_1, l_2 \le m-1$ and $j_1 + \cdots + j_m = n$ $(1 \le m \le n)$, from Propositon 3.1, we have $s^{j_1,\ldots,j_m}_{(par),i_1,i_2} = I_{\{i_1=i_2=m\}}$, which means

$$\sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} s^{j_1,\ldots,j_m}_{(par),i_1,i_2} = \sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} I_{\{i_1=i_2=m\}} = 1 \ge \sum_{i_1=l_1+1}^{m}\sum_{i_2=l_2+1}^{m} s^{j_1,\ldots,j_m}_{i_1,i_2},$$

which leads to the required result by using Theorem 4.1. □

Remark 4.4. Note that the theorem shows that the joint reliability of two parallel systems sharing all their n i.i.d. discrete components is no less than the joint reliability of any two coherent systems with n shared discrete components i.i.d. from the same distribution.

We now present two examples to illustrate these results.

Example 4.1. Consider the two pairs of coherent systems discussed earlier in Examples 3.2 and 3.3, with their discrete-time joint signatures denoted by $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$, respectively. We notice that their lifetimes satisfy $P\{T_1^1 > x, T_2^1 > y\} \ge P\{T_1^2 > x, T_2^2 > y\}$ for any x, y, because the requirements of Theorem 4.1 are satisfied.

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Example 4.2. Consider the two pairs of coherent systems discussed earlier in Examples 3.2 and 3.4, with their discrete-time joint signatures denoted by $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$, respectively. We notice that their lifetimes satisfy $P\{T_1^1 > x, T_2^1 > y\} \le P\{T_1^2 > x, T_2^2 > y\}$ for any x, y, because the requirements of Theorem 4.1 are satisfied.

5. Transformation formula for discrete-time joint signatures of systems of different sizes

In the previous section, we discussed stochastic orderings of discrete-time joint signatures of systems of the same size. Now, we provide a transformation formula for discrete-time joint signatures of different sizes, which then would facilitate the comparison of systems of different sizes.

Theorem 5.1. *For two coherent systems with n shared i.i.d. components and discrete-time joint signature $\boldsymbol{s} = \left(s_{i_1,i_2}^{j_1,\ldots,j_m},\ j_1+\cdots+j_m=n,\ 1\le i_1,i_2\le m\le n\right)$, the discrete-time joint signature of their equivalent systems with size $n+1$ can be given as*

$$\tilde{\boldsymbol{s}} = \left(\tilde{s}_{i_1,i_2}^{l_1,\ldots,l_m},\ l_1+\cdots+l_m=n+1,\ 1\le i_1,i_2\le m\le n+1\right),$$

where

$$\tilde{s}_{i_1,i_2}^{l_1,\ldots,l_m} = \binom{n+1}{\boldsymbol{l}}^{-1}\sum_{v=1}^{m}\left[s_{i_1,i_2}^{1}\binom{n}{\boldsymbol{l}_{v,m}^1} + s_{i_1-I_{\{v\le i_1-1\}},\ i_2-I_{\{v\le i_2-1\}}}^{2}\binom{n}{\boldsymbol{l}_{v,m}^2}I_{\{l_v=1,\ v\ne i_1,i_2\}}\right],$$
$$\boldsymbol{l}=(l_1,\ldots,l_m),\ \ \boldsymbol{l}_{v,m}^1=(l_1-I_{\{v=1\}},\ldots,l_m-I_{\{v=m\}}),\ \ \boldsymbol{l}_{v,m}^2=(l_1,\ldots,l_{v-1},l_{v+1},\ldots,l_m),$$

for all $l_1+\cdots+l_m=n+1$ and $1\le i_1,i_2\le m\le n$. Note that $s_{i_1,i_2}^{l_1,\ldots,l_m}\ne 0$ only for $l_1+\cdots+l_m=n$ with $1\le i_1,i_2\le m\le n$.

Proof. According to Proposition 2.1, we obtain the following equalities for the joint pmf of the lifetimes of the two systems:

$$P\{T_1=x,\ T_2=y\} = \sum_{m=1}^{n}\sum_{j_1+\cdots+j_m=n}\binom{n}{\boldsymbol{j}}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}s_{i_1,i_2}^{j_1,\ldots,j_m}\left[\sum_{1\le t_1<\cdots<t_m<\infty,\ t_{i_1}=x,\ t_{i_2}=y}\ \prod_{u=1}^{m}f^{j_u}(t_u)\right].$$

With the convention that $\sum_{t=1}^{\infty}f(t)=1$ for discrete-time mass function f, we have

$$\begin{aligned}
&P\{T_1=x,\ T_2=y\}\\
&=\sum_{m=1}^{n}\sum_{j_1+\cdots+j_m=n}\binom{n}{\boldsymbol{j}}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}s_{i_1,i_2}^{j_1,\ldots,j_m}\left[\sum_{1\le t_1<\cdots<t_m<\infty,\ t_{i_1}=x,\ t_{i_2}=y}\ \prod_{u=1}^{m}f^{j_u}(t_u)\right]\sum_{t=1}^{\infty}f(t)\\
&=\sum_{m=1}^{n}\sum_{j_1+\cdots+j_m=n}\binom{n}{\boldsymbol{j}}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}s_{i_1,i_2}^{j_1,\ldots,j_m}\sum_{v=1}^{m}\left[\sum_{1\le t_1<\cdots<t_m<\infty,\ t_{i_1}=x,\ t_{i_2}=y,\ t=t_v}\ \prod_{u=1}^{m}f^{j_u}(t_u)f(t)\right]\\
&\quad+\sum_{m=1}^{n}\sum_{j_1+\cdots+j_m=n}\binom{n}{\boldsymbol{j}}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}s_{i_1,i_2}^{j_1,\ldots,j_m}\sum_{v=1}^{m+1}\left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\ t_{i_1}=x,\ t_{i_2}=y\\ t_{v-1}<t<t_v,\ t_0=0,\ t_{m+1}=\infty}}\ \prod_{u=1}^{m}f^{j_u}(t_u)f(t)\right],
\end{aligned}$$

where the former term comes from the case $t\in\{t_1,\ldots,t_m\}$, and the latter term comes from the case $t\notin\{t_1,\ldots,t_m\}$. Now, by replacing $j_1,\ldots,j_m$ with $l_1-I_{\{v=1\}},\ldots,l_m-I_{\{v=m\}}$ (denoted by $\boldsymbol{l}_{v,m}^1$), the

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former term can be rewritten as

$$\sum_{m=1}^{n}\sum_{l_1+\cdots+l_m=n+1}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\sum_{v=1}^{m} s^{l^1_{v,m}}_{i_1,i_2}\binom{n}{l^1_{v,m}}\left[\sum_{1\le t_1<\cdots<t_m<\infty,\ t_{i_1}=x,\ t_{i_2}=y}\prod_{u=1}^{m} f^{l_u}(t_u)\right].$$

Similarly, by replacing $j_1,\ldots,j_m$ with $l_1,\ldots,l_{v-1},l_{v+1},\ldots,l_{m+1}$ (denoted by $l^2_{v,m}$) and denoting $l_v=1$, the latter term can be rewritten as

$$\sum_{m=1}^{n}\sum_{l_1+\cdots+l_{m+1}=n+1}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\sum_{v=1}^{m+1} s^{l^2_{v,m}}_{i_1,i_2}\binom{n}{l^2_{v,m+1}}\left[\sum_{\substack{1\le t_1<\cdots<t_{m+1}<\infty,\\ t_{i_1+I_{\{v\le i_1\}}}=x,\\ t_{i_2+I_{\{v\le i_2\}}}=y}}\prod_{u=1}^{m+1} f^{l_u}(t_u)\right] I_{\{l_v=1\}},$$

which can be further rewritten as

$$\begin{aligned}
&\sum_{m=2}^{n+1}\sum_{l_1+\cdots+l_m=n+1}\sum_{i_1=1}^{m-1}\sum_{i_2=1}^{m-1}\sum_{v=i_1\vee i_2+1}^{m} s^{l^2_{v,m}}_{i_1,i_2}\binom{n}{l^2_{v,m}}\left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}=x,\ t_{i_2}=y}}\prod_{u=1}^{m} f^{l_u}(t_u)\right] I_{\{l_v=1\}}\\
&+\sum_{m=2}^{n+1}\sum_{l_1+\cdots+l_m=n+1}\sum_{i_1=2}^{m}\sum_{i_2=2}^{m}\sum_{v=1}^{i_1\wedge i_2-1} s^{l^2_{v,m}}_{i_1-1,i_2-1}\binom{n}{l^2_{v,m}}\left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}=x,\ t_{i_2}=y}}\prod_{u=1}^{m} f^{l_u}(t_u)\right] I_{\{l_v=1\}}\\
&+\sum_{m=2}^{n+1}\sum_{l_1+\cdots+l_m=n+1}\sum_{i_1=1}^{m-1}\sum_{i_2=2}^{m}\sum_{v=i_1+1}^{i_2-1} s^{l^2_{v,m}}_{i_1,i_2-1}\binom{n}{l^2_{v,m}}\left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}=x,\ t_{i_2}=y}}\prod_{u=1}^{m} f^{l_u}(t_u)\right] I_{\{l_v=1\}}\\
&+\sum_{m=2}^{n+1}\sum_{l_1+\cdots+l_m=n+1}\sum_{i_1=2}^{m}\sum_{i_2=1}^{m-1}\sum_{v=i_2+1}^{i_1-1} s^{l^2_{v,m}}_{i_1-1,i_2}\binom{n}{l^2_{v,m}}\left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}=x,\ t_{i_2}=y}}\prod_{u=1}^{m} f^{l_u}(t_u)\right] I_{\{l_v=1\}}\\
=&\sum_{m=2}^{n+1}\sum_{l_1+\cdots+l_m=n+1}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\sum_{v=1}^{m} s^{l^2_{v,m}}_{i_1-I_{\{v\le i_1-1\}},i_2-I_{\{v\le i_2-1\}}}\binom{n}{l^2_{v,m}}\left[\sum_{\substack{1\le t_1<\cdots<t_m<\infty,\\ t_{i_1}=x,\ t_{i_2}=y}}\prod_{u=1}^{m} f^{l_u}(t_u)\right] I_{\{l_v=1,\ v\ne i_1,i_2\}}
\end{aligned}$$

by considering four different cases: (1) $v>i_1, v>i_2$, (2) $v\le i_1, v\le i_2$, (3) $v>i_1, v\le i_2$ and (4) $v\le i_1, v>i_2$, and replacing i_1,i_2,m with $i_1-I_{\{v\le i_1\}}, i_2-I_{\{v\le i_2\}}, m-1$, respectively. On the other hand, we also have the expression

$$P\{T_1=x,\ T_2=y\}=\sum_{m=1}^{n+1}\sum_{l_1+\cdots+l_m=n+1}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\tilde{s}^{l_1,\ldots,l_m}_{i_1,i_2}\binom{n+1}{\boldsymbol{l}}\left[\sum_{1\le t_1<\cdots<t_m<\infty,\ t_{i_1}=x,\ t_{i_2}=y}\prod_{u=1}^{m} f^{l_u}(t_u)\right].$$

Upon comparing the above two expressions, we obtain the desired result. □

Remark 5.1. Related discussions can also be generalized to systems of size n and size $n+l$ with $l=2,3,\ldots,$ directly. But, we restrain from presenting it here for the sake of brevity.

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We now apply Theorem 5.1 for comparing the two pairs of systems discussed earlier in Examples 3.1 and 3.2, which have different numbers of components.

Example 5.1. For the coherent systems $\phi_1(\boldsymbol{X}) = X_1$ and $\phi_2(\boldsymbol{X}) = \max(X_1, X_2)$ considered in Example 3.1, we can obtain the discrete-time joint signature of its equivalent systems of size 3 as $\tilde{\boldsymbol{s}} = \left(\tilde{s}_{i_1,i_2}^{l_1,\ldots,l_m},\ l_1+\cdots+l_m=3,\ 1\le i_1,i_2\le m\le 3\right)$, where

$$\tilde{s}_{i_1,i_2}^{l_1,\ldots,l_m} = \binom{3}{\boldsymbol{l}}^{-1} \sum_{v=1}^{m} \left[s_{i_1,i_2}^{\boldsymbol{l}_{v,m}^1} \binom{2}{\boldsymbol{l}_{v,m}^1} + s_{i_1-I_{\{v\le i_1-1\}},\ i_2-I_{\{v\le i_2-1\}}}^{\boldsymbol{l}_{v,m}^2} \binom{2}{\boldsymbol{l}_{v,m}^2} I_{\{l_v=1,\ v\ne i_1,i_2\}} \right],$$
$$\boldsymbol{l} = (l_1,\ldots,l_m),\ \ \boldsymbol{l}_{v,m}^1 = (l_1 - I_{\{v=1\}},\ldots,l_m - I_{\{v=m\}}),\ \ \boldsymbol{l}_{v,m}^2 = (l_1,\ldots,l_{v-1},l_{v+1},\ldots,l_m),$$

namely,

$$\begin{aligned}
\tilde{s}_{i_1,i_2}^{1,1,1} =& 0+0+0+\frac{1}{3}s_{i_1-I_{\{i_1\ge 2\}},i_2-I_{\{i_2\ge 2\}}}^{1,1} I_{\{i_1\ne 1,\ i_2\ne 1\}} + \frac{1}{3}s_{i_1-I_{\{i_1=3\}},i_2-I_{\{i_2=3\}}}^{1,1} \\
& \cdot I_{\{i_1\ne 2,\ i_2\ne 2\}} + \frac{1}{3}s_{i_1,i_2}^{1,1} I_{\{i_1\ne 3,\ i_2\ne 3\}}, \qquad 1\le i_1,i_2\le 3, \\
\tilde{s}_{i_1,i_2}^{1,2} =& 0+\frac{2}{3}s_{i_1,i_2}^{1,1} + \frac{1}{3}s_{i_1-I_{\{i_1\ge 2\}},i_2-I_{\{i_2\ge 2\}}}^{2} I_{\{i_1\ne 1,\ i_2\ne 1\}} + 0, \qquad 1\le i_1,i_2\le 2, \\
\tilde{s}_{i_1,i_2}^{2,1} =& \frac{2}{3}s_{i_1,i_2}^{1,1} + 0+0+\frac{1}{3}s_{i_1,i_2}^{2} I_{\{i_1\ne 2,\ i_2\ne 2\}},\ 1\le i_1,i_2\le 2, \qquad \tilde{s}_{1,1}^{3} = s_{1,1}^{2},
\end{aligned}$$

that is,

$$\begin{aligned}
\tilde{\boldsymbol{s}} =& \begin{pmatrix}
\frac{2}{3}s_{1,1}^{1,1} & \frac{1}{3}s_{1,2}^{1,1} & \frac{1}{3}s_{1,2}^{1,1} & \frac{1}{3}s_{2,1}^{1,1} & \frac{1}{3}s_{1,1}^{1,1}+\frac{1}{3}s_{2,2}^{1,1} & \frac{1}{3}s_{1,2}^{1,1} & \frac{1}{3}s_{2,1}^{1,1} & \frac{1}{3}s_{2,1}^{1,1} & \frac{2}{3}s_{2,2}^{1,1} \\
\frac{2}{3}s_{1,1}^{1,1} & \frac{2}{3}s_{1,2}^{1,1} & \frac{2}{3}s_{2,1}^{1,1} & \frac{2}{3}s_{2,2}^{1,1}+\frac{1}{3}s_{1,1}^{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3}s_{1,1}^{1,1}+\frac{1}{3}s_{1,1}^{2} & \frac{2}{3}s_{1,2}^{1,1} & \frac{2}{3}s_{2,1}^{1,1} & \frac{2}{3}s_{2,2}^{1,1} & 0 & 0 & 0 & 0 & 0 \\
s_{1,1}^{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
=& \begin{pmatrix}
0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{3} \\
0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\end{aligned}$$

Upon comparing this joint signature with the joint signature of the systems in Example 3.2, we observe the former to be better than the latter in the sense of stochastic ordering, according to Theorem 4.1.

Remark 5.2. The above result can also be given by the transformation formula of continuous-time joint signature (see Theorem 2.1 of Yi *et al.* [23]), by writing directly

$$\begin{aligned}
\tilde{\boldsymbol{s}} =& \frac{1}{6}\boldsymbol{s}_{1,2:3} + \frac{1}{6}\boldsymbol{s}_{1,3:3} + \frac{1}{6}\boldsymbol{s}_{2,2:3} + \frac{1}{6}\boldsymbol{s}_{2,3:3} + \frac{1}{3}\boldsymbol{s}_{3,3:3} \\
=& \frac{1}{6}\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} + \frac{1}{6}\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{aligned}$$

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$$+\frac{1}{6}\begin{pmatrix}0&0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0\end{pmatrix}+\frac{1}{6}\begin{pmatrix}0&0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0&0\end{pmatrix}$$
$$+\frac{1}{3}\begin{pmatrix}0&0&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\1&0&0&0&0&0&0&0&0\end{pmatrix}=\begin{pmatrix}0&\frac{1}{6}&\frac{1}{6}&0&\frac{1}{6}&\frac{1}{6}&0&0&\frac{1}{3}\\0&\frac{1}{3}&0&\frac{2}{3}&0&0&0&0&0\\\frac{1}{3}&\frac{1}{3}&0&\frac{1}{3}&0&0&0&0&0\\1&0&0&0&0&0&0&0&0\end{pmatrix}.$$

We notice that the results obtained here are exactly the same as the results in Example 5.1, which not only verifies the correctness of the transformation formula in Theorem 5.1, but also verifies the relationship between continuous-time joint signature and discrete-time joint signature.

6. Concluding remarks

In this work, we discuss joint signatures for the discrete-time case and thus generalizing the traditional (continuous-time) joint signature proposed earlier by Navarro *et. al.* [16]. Some stochastic properties of the new notion have been discussed in detail, including the joint distribution of system lifetimes, stochastic ordering of pairs of systems, and a transformation formula for pairs of systems of different sizes. Related applications can be widely found in practical reliability systems such as communication network systems, smart street light systems, unmanned aerial vehicle systems, hardware load balancer group systems, and so on. The present work can be generalized to different types of coherent systems to study their structural properties under the discrete-time setup; for example, we can propose a discrete-time dynamic signature for used coherent systems with a known number of failed components. Moreover, discrete-time multi-state signature, discrete-time multi-state joint signature, and discrete-time multi-state dynamic signature can also be introduced for the corresponding multi-state systems. We are currently working on these problems and hope to report the findings in a future paper.

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