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The lap-counting function for linear mod one transformations III: the period of a Markov chain

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Abstract. Linear mod one transformations are those maps of the unit interval given by $f_{\beta,\alpha}(x) = \beta x + \alpha \pmod 1$, with $\beta > 1$ and $0 \le \alpha < 1$. The lap-counting function is $L_{\beta,\alpha}(z) = \sum_{n=1}^{\infty} L_n z^n$, where L_n essentially counts the number of monotonic pieces of the nth iterate $f^n_{\beta,\alpha}$. Part I showed that the function $L_{\beta,\alpha}(z)$ is meromorphic in the unit disk $|z| < 1$ and analytic in $|z| < 1/\beta$, and part II showed that the singularities of $L_{\beta,\alpha}(z)$ on the circle $|z| = 1/\beta$ are contained in the set $\{(1/\beta)\exp(2\pi i l/N_{\beta,\alpha}) : 0 \le l < N_{\beta,\alpha}\}$, where $N_{\beta,\alpha}$ is the period of the ergodic part of a Markov chain associated to $f_{\beta,\alpha}$. This paper proves that the set of singularities on $|z| = 1/\beta$ is identical to the set $\{(1/\beta)\exp(2\pi i l/N_{\beta,\alpha}) : 0 \le l < N_{\beta,\alpha}\}$. Part II showed that $N_{\beta,\alpha} = 1$ for $\beta > 2$, and this paper determines $N_{\beta,\alpha}$ in the remaining cases where $1 < \beta \le 2$.

1. *Introduction*

Linear mod one transformations $f_{\beta,\alpha}$ are those maps

$$f_{\beta,\alpha}(x) = \{\beta x + \alpha\}, \quad 0 \le x < 1,$$

where $\beta > 1$ and $0 \le \alpha < 1$. (Here $\{x\} = x - \lfloor x \rfloor$.) The *lap-counting function* is given by

$$L_{\beta,\alpha}(z) = \sum_{n=0}^{\infty} L_n z^n,$$

where L_n counts the number of symbol sequences of length n for the usual symbolic dynamics of iterating $f_{\beta,\alpha}$, i.e. it counts the 'laps' in the kneading theory of Milnor and Thurston [**10**] and Preston [**14**]. Part I [**2**] showed that $L_{\beta,\alpha}(z)$ is analytic in the open disk $|z| < 1/\beta$, and meromorphic in the larger disk $|z| < 1$. Part I also observed that the singularities of $L_{\beta,\alpha}(z)$ in the open disk $|z| < 1$ are identical with those of the Artin–Mazur–Ruelle zeta function $\zeta_{\beta,\alpha}(z)$.

This paper continues the study of the singularities of $L_{\beta,\alpha}(z)$ on the circle $|z| = 1/\beta$ using the auxiliary Markov chain $\Sigma_{\beta,\alpha}$ introduced in part II [**3**]. The main result of

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part II was that all singularities of $L_{\beta,\alpha}(z)$ on $|z| = 1/\beta$ are simple poles, and are located at a subset of $\{(1/\beta)\exp(2\pi il/N) : 0 \le l \le N-1\}$, for some $N \ge 1$. Let $\bar{N}_{\beta,\alpha}$ denote the minimal such N. The main result of part II showed that $\bar{N}_{\beta,\alpha}$ always divides the *period* $N_{\beta,\alpha}$ of the ergodic part $\Sigma_{\beta,\alpha}^{\max}$ of the Markov chain $\Sigma_{\beta,\alpha}$. In addition, for $\beta > 2$, Theorem 2.3 of part I shows that $\bar{N}_{\beta,\alpha} = 1$ and Theorem 5.1 of part II shows that $N_{\beta,\alpha} = 1$.

Our objective in this paper is to determine $\bar{N}_{\beta,\alpha}$ and $N_{\beta,\alpha}$ in the remaining range $1 < \beta \le 2$. In particular, we show that it is always the case that $\bar{N}_{\beta,\alpha} = N_{\beta,\alpha}$, and that $L_{\beta,\alpha}(z)$ has a simple pole at every one of the points $\{(1/\beta)\exp(2\pi il/N_{\beta,\alpha}) : 0 \le l < N_{\beta,\alpha}\}$. Formulae for $N_{\beta,\alpha}$ appear in Theorems 1.1–1.4 below.

The value of $N_{\beta,\alpha}$ is closely related to the topological transitivity properties of $f_{\beta,\alpha}$, which we now review. The values of (β,α) where $f_{\beta,\alpha}$ is topologically transitive were determined by Hofbauer [**7**], and are given in Proposition 4.1 of part I. It is natural to treat the cases $\alpha+\beta > 2$ and $\alpha+\beta \le 2$ separately, as they require different considerations. For $\alpha+\beta > 2$ the map $f_{\beta,\alpha}$ is topologically transitive exactly when (β,α) is not in a set $\tilde{R}$, called the 'big bubble', given by

$$\tilde{R} := \left\{(\beta,\alpha) : \frac{1}{\beta} \le \alpha \le 3 - \beta - \frac{1}{\beta}\right\}, \tag{1.1}$$

except at the 'tip' $(\beta,\alpha) = (2, \frac{1}{2})$ of $\tilde{R}$, in which case $f_{\beta,\alpha}$ is topologically transitive. In particular, $f_{\beta,\alpha}$ is topologically transitive whenever $\beta > 2$. In the remaining case $1 < \alpha+\beta \le 2$, the map $f_{\beta,\alpha}$ is topologically transitive if and only if either it is not in certain regions $R_{N,K}$ of the (β,α)-space called 'bubbles', which are indexed by positive integers (N,K) with $N > K \ge 1$ and $(N,K) = 1$, or else is the 'tip' of some 'bubble' $R_{N,K}$. (See §3.3 for a definition of $R_{N,K}$.) The 'bubble' $R_{N,K}$ is defined to be the region where $f_{\beta,\alpha}$ has a certain kind of periodic orbit, a primary (N,K)-cycle in the sense of Palmer [**11**] and Glendinning [**4**], and these regions coincide with Hofbauer's regions $W(k,k,\ldots,k_m,k)$ (introduced in Hofbauer [**7**]), as is proved in Appendix A.

Our first main result treats the case when $1 < \alpha+\beta \le 2$, and gives necessary and sufficient conditions for $N_{\beta,\alpha} = 1$, as follows.

THEOREM 1.1. *For $1 < \alpha+\beta \le 2$, the following are equivalent:*

(i) *the maximal component of the Markov chain is aperiodic, i.e. $N_{\beta,\alpha} = 1$;*

(ii) *(β,α) is not in any 'bubble' $R_{N,K}$;*

(iii) *the lap-counting function $L_{\beta,\alpha}(z)$ has no singularities on the circle $|z| = 1/\beta$, except for a simple pole at $z = 1/\beta$.*

We prove this result in §3. This proof makes heavy use of results of Hofbauer [**5–7**], and also requires additional arguments of a combinatorial nature. These include Theorem 3.1 and lemmas that relate the notion of a 'bubble' $R_{N,K}$ to the criteria of Hofbauer for non-transitivity.

Let P_n count the number of periodic points of $f_{\beta,\alpha}$ of period dividing n, i.e. the number of fixed points of $f_{\beta,\alpha}^n$. Hofbauer [**8**, Theorem 3] proved that if the maximal component of $\Sigma_{\beta,\alpha}$ is aperiodic, then

$$P_n = \beta^n(1 + o(1)) \quad \text{as } n \to \infty.$$

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Theorem 1.1 above gives sufficient conditions when aperiodicity holds.

Our second main result completely determines $N_{\beta,\alpha}$ in the range $1 < \alpha + \beta \leq 2$. Recall from §4 in part I that to each such (β, α) there is associated a finite sequence of successive 'renormalizations' in 'bubbles' $R_{N_1,K_1}, \ldots, R_{N_s,K_s}$ (see §4 for definitions).

THEOREM 1.2. *Suppose that* $1 < \alpha + \beta \leq 2$, *and that* $f_{\beta,\alpha}$ *has its complete set of renormalizations in the successive 'bubbles'* R_{N_1,K_1}, $R_{N_2,K_2}, \ldots, R_{N_s,K_s}$. *Then*

$$\bar{N}_{\beta,\alpha} = N_{\beta,\alpha} = N_1 N_2 \cdots N_s. \tag{1.2}$$

Furthermore, $L_{\beta,\alpha}(z)$ *has a simple pole at every one of the points* $\{(1/\beta)\exp(2\pi i l/N_{\beta,\alpha}) : 0 \leq l < N_{\beta,\alpha}\}$.

We prove this result in §4. The value of $\bar{N}_{\beta,\alpha}$ is obtained by combining Theorem 4.3 of part I with Theorem 4.2 of part II. Theorem 3.3 of part I guarantees that $L_{\beta,\alpha}(z)$ has certain singularities on $|z| = 1/\beta$, and Theorem 4.2 of part II is used to show that there are no 'extra' singularities. The value of $N_{\beta,\alpha}$ is obtained by study of the behavior of the graph of $\Sigma_{\beta,\alpha}$ under 'renormalization'.

In §5 and §6 we consider the case $2 < \alpha + \beta \leq 3$. (Actually there only remains the subcase where $\alpha + \beta > 2$ and $\beta \leq 2$, but the arguments cover the wider range.) The results differ according to whether (β, α) lies inside or outside the 'big bubble' $\tilde{R}$. In §5 we treat the case where (β, α) is outside the 'big bubble,' and prove the following theorem.

THEOREM 1.3. *If* $2 < \alpha + \beta \leq 3$ *and* (β, α) *is not in the 'big bubble'*

$$\tilde{R} = \left\{(\beta, \alpha) : \frac{1}{\beta} \leq \alpha \leq 3 - \beta - \frac{1}{\beta}\right\},$$

then

$$\bar{N}_{\beta,\alpha} = N_{\beta,\alpha} = 1. \tag{1.3}$$

Finally, in §6 we consider the remaining case where (β, α) is contained in the 'big bubble.' The results involve a 'rescaling map' found by Palmer [**11**], which sends (β, α) to

$$(\bar{\beta}, \bar{\alpha}) := \left(\beta, \frac{\beta\alpha - 1}{\beta - 1}\right).$$

In part I we observed that this 'rescaling map' is a homeomorphism from the 'big bubble' $\tilde{R}$ to the region $T = \{(\bar{\beta}, \bar{\alpha}) : \bar{\beta} > 1 \text{ and } 1 < \bar{\alpha} + \bar{\beta} \leq 2\}$, and that the lap-counting functions of $f_{\beta,\alpha}$ and $f_{\bar{\beta},\bar{\alpha}}$ are related by $(1 - z)L_{\beta,\alpha}(z) = L_{\bar{\beta},\bar{\alpha}}(z)$. We prove the following theorem.

THEOREM 1.4. *Suppose that* (β, α) *lies in the 'big bubble'* $\tilde{R}$. *If*

$$(\bar{\beta}, \bar{\alpha}) = \left(\beta, \frac{\beta\alpha - 1}{\beta - 1}\right),$$

then $1 < \bar{\alpha} + \bar{\beta} \leq 2$ *and*

$$\bar{N}_{\beta,\alpha} = N_{\beta,\alpha} = N_{\bar{\beta},\bar{\alpha}}. \tag{1.4}$$

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From Theorems 1.1 and 1.2 we conclude that, for $1 < \alpha + \beta \le 2$, $N_{\beta,\alpha} = 1$ implies topological transitivity of $f_{\beta,\alpha}$ but not vice-versa (because $f_{\beta,\alpha}$ is topologically transitive at the 'tip' of the bubble $R_{N,K}$, but $N_{\beta,\alpha} = N > 1$). For $2 < \alpha+\beta \le 3$, the implications are in reverse. Indeed, Theorems 1.1 and 1.4 show that there is a substantial region inside the 'big bubble' where $N_{\beta,\alpha} = 1$, hence for $2 < \alpha + \beta \le 3$ the topological transitivity of $f_{\beta,\alpha}$ implies $N_{\beta,\alpha} = 1$ but not vice-versa.

Finally, we remark that the quantity $N := \bar{N}_{\beta,\alpha}$ is alternatively characterized by the fact that $\{\exp(2\pi i l/N) : 1 \le l \le N\}$ is the complete point spectrum for the operator $U_{\beta,\alpha}$ on $L^2([0,1])$ given by $U(g(x)) := g(f_{\beta,\alpha}(x))$. We omit a proof, which can be derived from the formula $\bar{N}_{\beta,\alpha} = N_1 N_2 \cdots N_s$ of Theorem 1.2 along with arguments given in Palmer [**11**, Theorem 6].

2. *Review*

We recall basic facts about $f_{\beta,\alpha}$ and the Markov chain $\Sigma_{\beta,\alpha}$ from part II, specialized to the case where $1 < \alpha + \beta \le 3$. For $1 < \alpha + \beta \le 2$ the graph of f has a unique *turning point* $\gamma = (1-\alpha)/\beta$, while for $2 < \alpha + \beta \le 3$ it has two *turning points* $\gamma_i = (i-\alpha)/\beta$ for $i = 1, 2$. The analysis of the structure of $\Sigma_{\beta,\alpha}$ splits into four cases:

(i) ('generic' case) $\alpha \ne 0$ and $\alpha + \beta \notin \mathbb{Z}$;
(ii) $\alpha = 0$ and $\alpha + \beta \notin \mathbb{Z}$;
(iii) $\alpha \ne 0$ and $\alpha + \beta \in \mathbb{Z}$;
(iv) $\alpha = 0$ and $\alpha + \beta \in \mathbb{Z}$.

We give detailed proofs only for the 'generic' case, because the remaining cases (ii)–(iv) are covered by earlier work.

The *state graph* $S_{\beta,\alpha}$ of $\Sigma_{\beta,\alpha}$ has *vertex set* $D_{\beta,\alpha}$ consisting of a subset of

$$D^*_{\beta,\alpha} := \{(i, j) \in \mathbb{Z}^2 : i, j \ge 0 \text{ and } f^i(0) < f^j(1^-)\}.$$

We make $D^*_{\beta,\alpha}$ into a directed graph $S^*_{\beta,\alpha}$ by adding edges using the following rules.

Rule (a). $(i, j) \to (i+1, j+1)$, if the open interval $(f^i(0), f^j(1^-))$ contains no turning point.

Rule (b). $(i, j) \to (i+1, 0)$ and $(0, j+1)$, if $(f^i(0), f^j(1^-))$ contains exactly one turning point.

Rule (c). $(i, j) \to (i+1, 0)$, $(0, j+1)$ and $(0, 0)$, if $(f^i(0), f^j(1^-))$ contains two turning points.

Note that rule (c) only occurs if $2 < \alpha + \beta \le 3$. The vertex set $D_{\beta,\alpha}$ is the set of all vertices in $S^*_{\beta,\alpha}$ reachable from $(0, 0)$ by a directed path, and $S_{\beta,\alpha}$ is the (directed) subgraph on $D_{\beta,\alpha}$ induced from $S^*_{\beta,\alpha}$.

We associate to $S_{\beta,\alpha}$ two sequences $\{r_l : l \ge 1\}$ and $\{s_l : l \ge 1\}$ of positive integers, defined below, and also their partial sums, given by

$$R_l = \sum_{i=0}^{l} r_i, \quad l \ge 0, \tag{2.1a}$$

$$S_l = \sum_{j=0}^{l} s_j, \quad l \ge 0, \tag{2.1b}$$

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where $r_0 = s_0 = 0$. The sequences $\{r_i\}$ and $\{s_j\}$ are defined inductively. If $\{r_i : 0 \le i \le l\}$ is defined, then set

$$r_{l+1} := \min\{k \ge 1 : (f^{R_l+k}(0),\ f^{k-1}(1^-)) \text{ contains a turning point}\}. \tag{2.2a}$$

Similarly,

$$s_{l+1} = \min\{k \ge 1 : (f^{k-1}(0),\ f^{S_l+k}(1^-)) \text{ contains a turning point}\}. \tag{2.2b}$$

Next, we recall the description of $S_{\beta,\alpha}$ given in Proposition 3.1 of part II, specialized to $1 < \alpha + \beta \le 3$.

PROPOSITION 2.1. (Hofbauer) *Suppose that $\alpha \ne 0$ and $\alpha + \beta \notin \mathbb{Z}$, with $1 < \alpha + \beta \le 3$. Then:*

(i) *The set of vertices of $S_{\beta,\alpha}$ is $\{c_i : i \ge 1\} \cup \{d_j : j \ge 1\} \cup \{(0,0)\}$, with vertex labels*

$$c_i = (i, i - R_l - 1) \quad \text{for } R_l < i \le R_{l+1}, l \ge 0,$$
$$d_j = (j + S_l - 1, j) \quad \text{for } S_l < j \le S_{l+1}, l \ge 0.$$

(ii) *The directed edges of $S_{\beta,\alpha}$ consist of $(0,0) \to c_1$, $(0,0) \to d_1$, the two 'tracks'*

$$c_i \to c_{i+i}, \quad i \ge 0,$$
$$d_i \to d_{i+1}, \quad i \ge 0,$$

plus the 'cross-edges'

$$c_{R_l} \to d_{r_l}, \quad \text{all } l \ge 1,$$
$$d_{S_l} \to c_{s_l}, \quad \text{all } l \ge 1.$$

Finally, if $2 < \alpha + \beta \le 3$, then $S_{\beta,\alpha}$ contains the loop $(0,0) \to (0,0)$ and the additional edges

$$c_{R_l} \to (0,0) \quad \text{if } I_1 \subseteq (f^{R_l}(0),\ f^{r_l-1}(1^-)),$$
$$d_{S_l} \to (0,0) \quad \text{if } I_1 \subseteq (f^{s_l-1}(0),\ f^{S_l}(1^-)).$$

(iii) *For every $l \ge 1$, there exist non-negative integers $P(l)$, $Q(l)$ such that*

$$r_l = 1 + s_1 + \cdots + s_{P(l)} = 1 + S_{P(l)}, \tag{2.3a}$$
$$s_l = 1 + r_1 + \cdots + r_{Q(l)} = 1 + R_{Q(l)}. \tag{2.3b}$$

The graph $S_{\beta,\alpha}$ in the 'generic case' is pictured in Figure 1. It consists of two 'tracks' together with 'stations' $\{c_{R_l} : l \ge 1\}$ and $\{d_{S_l} : l \ge 1\}$, where 'cross-edges' occur between the 'tracks.'

We also recall the structure of the strongly connected components of $S_{\beta,\alpha}$. Proposition 3.2 of part II proved that there is a finite set of strongly connected components of the graph $S_{\beta,\alpha}$, and we denote their vertex sets by $D_1, D_2, \ldots, D_k$. All components are finite except for the set D_k, which is infinite. We call D_k the *maximal component* of $D_{\beta,\alpha}$, and sometimes denote it $D^{\max}_{\beta,\alpha}$. We let $S^{\max}_{\beta,\alpha}$ denote the (induced) subgraph of $S_{\beta,\alpha}$ restricted to the vertex set $D^{\max}_{\beta,\alpha}$, and by analogy we call $S^{\max}_{\beta,\alpha}$ the *maximal component*

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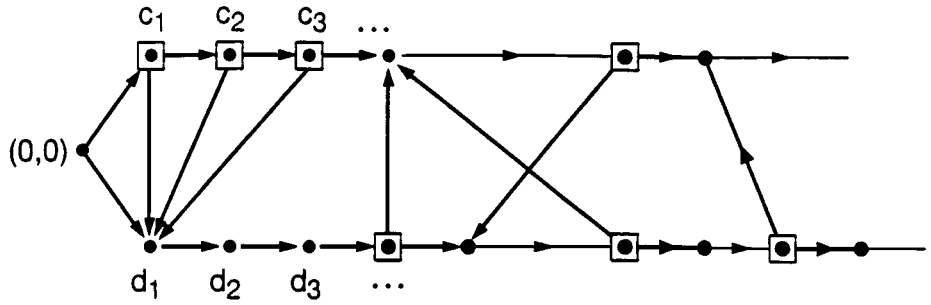


FIGURE 1. Graph $S_{\beta,\alpha}$ in a 'generic' case.

of $S_{\beta,\alpha}$. The maximal component $D^{\max}_{\beta,\alpha}$ either consists of all vertices $\{c_i : i \geq i_0\}$ and $\{d_j : j \geq j_0\}$ for some values $i_0 \geq 1$, $j_0 \geq 1$, or else $D^{\max}_{\beta,\alpha} = D_{\beta,\alpha}$. The latter case never occurs when $1 < \alpha + \beta \leq 2$. A criterion when $D^{\max}_{\beta,\alpha} = D_{\beta,\alpha}$ for $2 < \alpha + \beta \leq 3$ appears as Lemma 5.2 in §5.

Let $\mathbf{M}_i = \mathbf{M}|_{D_i}$ be the incidence matrix of the graph $S_{\beta,\alpha}$ restricted to the vertex set D_i, where $D_1, D_2, \ldots, D_k$ are the components of $D_{\beta,\alpha}$. Let $\rho(\mathbf{M}_i)$ denote the spectral radius of the matrix $\mathbf{M}_i$, when viewed as an operator acting on column vectors in l^∞, or as an operator acting on row vectors in l^1. It has the following interpretation: for any $\mathbf{m} \in D_i$, let $N_n(\mathbf{m})$ denote the number of paths in the graph of $\mathbf{M}_i$ that start at $\mathbf{m}$ and are of length n. Then

$$\rho(\mathbf{M}_i) = \lim_{n\to\infty} N_n(\mathbf{m})^{1/n}, \tag{2.4}$$

with the limit in (2.4) being independent of $\mathbf{m}$.

PROPOSITION 2.2. *All non-maximal components of $S_{\beta,\alpha}$ have spectral radius 1, i.e.*

$$\rho(\mathbf{M}_i) = 1 \quad \text{for } 1 \leq i \leq k-1. \tag{2.5}$$

Proof. The graph $S_{\beta,\alpha}$ coincides with Hofbauer's graph S_M. For S_M this is stated in Hofbauer [**7**] Theorem 1(ii), and in his equation (1.5). He says that it is proved using the proofs of Lemmas 2 and 3 and Theorem 2 in Hofbauer [**6**]. □

3. *Case $1 < \alpha + \beta \leq 2$ outside 'bubbles'*

We suppose that $1 < \alpha + \beta \leq 2$, with the object of proving Theorem 1.1. To prove Theorem 1.1, we observe that (i) $\Rightarrow$ (iii) of that theorem was established by Theorem 4.2 of part II, while (iii) $\Rightarrow$ (ii) follows from Theorem 4.3 of part I. Thus it only remains to prove that (ii) $\Rightarrow$ (i), which is accomplished in Theorem 3.3 below.

We need only consider the 'generic' case (i) of §2. For the remaining 'non-generic' cases (ii)–(iv), the first two parts of Theorem 1.1 are trivially checked. The last part of Theorem 1.1 is proved in the non-generic cases (ii) and (iv) by Flatto [**1**] and by Ito and Takahashi [**9**]. Case (iii) reduces to case (ii) using the relation

$$L_{\beta,\alpha}(z) = L_{\beta,\alpha^*}(z) \quad \text{with } \alpha^* = \lceil \alpha + \beta \rceil - \alpha - \beta,$$

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as observed in the introduction of part I. Note that $\alpha + \beta = 2$ falls in cases (iii) and (iv), so in what follows we suppose that $1 < \alpha + \beta < 2$.

A key part in the analysis is played by the concept of *head cycle*, described below. Theorem 3.1 gives a criterion for $N_{\beta,\alpha} = 1$ in terms of the head cycle. Theorem 3.2 converts this criterion to a condition given in terms of the symbolic itineraries $i(0)$ and $i(1^-)$. We then relate these symbolic itineraries to the structure of 'bubbles' (these are defined in §3.3), and obtain Theorem 3.3 by combinatorial arguments.

3.1. *The head cycle.* The *head cycle* $H_{\beta,\alpha}$ is the subgraph of $S_{\beta,\alpha}$ which consists of the shortest cycle among all those cycles in $S_{\beta,\alpha}$ that contain vertices c_p and d_q that minimizes $p + q$. Proposition 3.1 below asserts that these conditions define $H_{\beta,\alpha}$ uniquely. The concept of head cycle is due to Hofbauer [**6, 7**], who calls it the 'period'. Roughly speaking, the head cycle $H_{\beta,\alpha}$ is the cycle in the graph $S_{\beta,\alpha}$ which is 'closest' to the vertex (0,0), which explains our choice of name.

The structure of the head cycle and various properties of it are detailed in the following four propositions and two lemmas. The propositions are due to Hofbauer.

PROPOSITION 3.1. *Suppose that* $1 < \alpha + \beta < 2$. *There exists a unique head cycle for* $S_{\beta,\alpha}$ *and its vertex set has the form*

$$\{c_p, c_{p+1}, \ldots, c_{p+q-1}\} \cup \{d_q, d_{q+1}, \ldots, d_{p+q-1}\},$$

where (c_{p-1}, c_{p+q-1}) *and* (d_{q-1}, d_{p+q-1}) *are each consecutive pairs of stations, except that if* $p = 1$ *then* c_{p+q-1} *is the first station on the* c_i*-track, and similarly if* $q = 1$ *then* d_{p+q-1} *is the first station on the* d_j*-track. The head cycle has length* $p + q$.

Proof. This is proved in Hofbauer [**7**, pp. 98–99]. □

We say that the head cycle given in Proposition 3.1 is of *type* $[p, q]$. It is pictured in Figure 2.

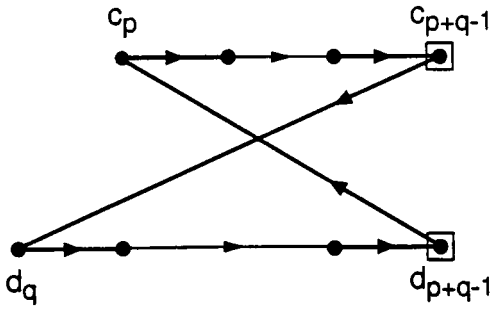


FIGURE 2. The head cycle.

The next proposition describes how the head cycle can be determined† from the

† The definition of period in Hofbauer [**7**] is actually in terms of these itineraries. Note that Hofbauer uses the symbols 1 and 2 where our symbols are 0 and 1.

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symbolic itineraries $i(0)$ and $i(1^-)$. Set

$$i(0) := A_0^* A_1^* A_2^* \dots$$
$$i(1^-) := C_0^* C_1^* C_2^* \dots .$$

Since $1 \le \alpha + \beta < 2$, only the symbols 0 and 1 appear in $i(0)$ and $i(1^-)$.

Given a finite string $P = p_0 p_1 \dots p_n$ of zeros and ones, its *length* $|P| = n + 1$. We say that the finite string P is an *initial segment* of P' if P is the first $|P|$ symbols of P', and we write $P \subseteq P'$ (here P' can be finite or infinite). For example, $A_0^* \dots A_n^*$ and $C_0^* \dots C_n^*$ are initial segments of $i(0)$ and $i(1^-)$, respectively. If $P = p_0 p_1 \dots p_m$ and $Q = q_0 q_1 \dots q_n$ then PQ denotes the concatenated string $p_0 p_1 \dots p_m q_0 q_1 \dots q_n$.

PROPOSITION 3.2. *Suppose that* $1 < \alpha + \beta < 2$.

(i) *There exists an integer* $N \ge 1$ *such that there are* $N + 1$ *initial segments* $\{P_i 0\}$ *of* $i(0)$ *with* $P_0 0 \subseteq P_1 0 \subseteq \dots \subseteq P_N 0$ *and* $N + 1$ *initial segments* $\{Q_i 1\}$ *of* $i(1^-)$ *with* $Q_0 1 \subseteq \dots \subseteq Q_N 1$, *that satisfy the recursion:*

(a) $P_0 = Q_0 = \emptyset$;

(b) *for* $0 \le k \le N - 2$ *precisely one of the following alternatives hold:*

$$P_{k+1} = P_k 0 Q_k \quad \textit{and} \quad Q_{k+1} = Q_k$$

or

$$P_{k+1} = P_k \quad \textit{and} \quad Q_{k+1} = Q_k 1 P_k;$$

(c) $P_N = P_{N-1} 0 Q_{N-1}$ *and* $Q_N = Q_{N-1} 1 P_{N-1}$.

(ii) *The type* $[p, q]$ *for the head cycle of* $S_{\beta,\alpha}$ *is given by*

$$p = |P_{N-1}| + 1, \quad q = |Q_{N-1}| + 1,$$

and

$$|P_N| = |Q_N| = p + q - 1.$$

Proof. This is proved in Hofbauer [**7**, p. 99]. □

The connection of the symbolic itineraries $i(0)$ and $i(1^-)$ and the head cycle in $S_{\beta,\alpha}$ given in Proposition 3.2 can be made more explicit by assigning *labels* to the directed edges of $S_{\beta,\alpha}$, as follows. The directed edge $(i, j) \to (i', j')$ is defined so that $[f^{i'}(0), f^{j'}(1^-))$ is the image under $f_{\beta,\alpha}$ of $[f^i(0), f^j(1^-)) \cap I_k$, where $0 \le k \le \lfloor \alpha + \beta \rfloor$. We assign to this directed edge the label k. For $1 < \alpha + \beta < 3$ edge labels are unique, while for $\alpha + \beta \ge 3$, uniqueness may fail, as $[f^i(0), f^j(1^-))$ may contain several I_k's over which f has full range, in which case $(i, j) \to (0, 0)$ will carry more than one label. As we restrict attention to $1 < \alpha + \beta < 2$, edge labels are automatically unique.

The following fact brings out the significance of edge labels.

LEMMA 3.1. *Suppose there is a directed path in* $S_{\beta,\alpha}$ *between vertices* (i, j) *to* (i', j') *with edges on the path having labels* $a_0, \dots, a_{n-1}$, *each drawn from the label sets. Then* $(a_0, a_1, \dots, a_{n-1})$ *is an admissible sequence, i.e.* $I_{a_0 a_1 \dots a_{n-1}} \ne \emptyset$. *In the case that* $(i, j) = (0, 0)$, *then the restriction of* $f_{\beta,\alpha}^n$ *to* $I_{a_0 \dots a_{n-1}}$ *has the range* $[f^{i'}(0), f^{j'}(1^-))$.

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Proof. In the special case that $(i, j) = (0, 0)$, the two properties above follow from the definition of edge labels. For the general case of (i, j), let $b_0, \ldots, b_m$ be the successive labels of a path from (0,0) to (i, j). $b_0 \ldots b_m a_0 \ldots a_n$ is an admissible sequence by the special case, hence $a_0 \ldots a_n$ is an admissible sequence. □

The edge labels can be simply described in terms of the itineraries $i(0) = \{A_n^*\}$ and $l(1^-) = \{C_n^*\}$ (see Figure 3). First, A_0^* and C_0^* are the respective labels of the edges $(0, 0) \to c_1$ and $(0, 0) \to d_1$. The other A_n^*'s for $n \geq 1$ are the successive labels of the edges on the c_i-track and the C_n^*'s for $n \geq 1$ are the successive labels of the edges on the d_j-track. For any station on the c_i-track, the edge to its immediate right has the label 0 and the edge from it to the d_j-track has the label 1. Similarly, for any station on the d_j-track, the edge to its immediate right has the label 1 and the edge from it to the c_i-track has the label 0.

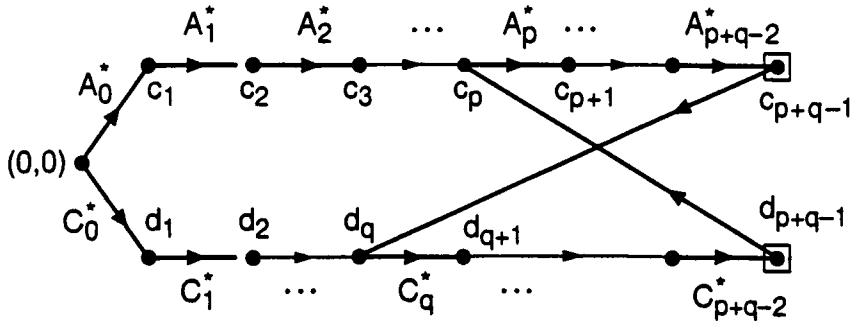


FIGURE 3. Edge labelling of tracks.

Proposition 3.2 allows the following fact to be readily established. If $0 < k < N$ and $P_k \neq \emptyset$, then $c_{|P_k|}$ is a station, and P_k is the sequence of successive labels on edge $(0, 0) \to c_1$ followed by the $\{c_i\}$-track leading to station $c_{|P_N|}$. A similar statement holds for all $Q_k \neq \emptyset$ on the $\{d_j\}$-track. Furthermore, those $c_{|P_k|}$ and $d_{|Q_k|}$ for $1 \leq k \leq N$, with $|P_k|, |Q_k| \geq 1$ account for all stations up to and including c_{p+q-1}, d_{p+q-1}.

The structure of the labelled head cycle is pictured in Figure 4.

Using part (c) of Proposition 3.2(i), we see that Q_{N-1} is the sequence of labels carried by the edges $c_p \to c_{p+1} \to \cdots \to c_{p+q-1}$, and P_{N-1} is the sequence of labels carried by the edges $d_q \to d_{q+1} \to \cdots \to d_{p+q-1}$.

Another consequence of Proposition 3.2 is a simple restriction on the pairs $[p, q]$ that can be the type of a head cycle. Let (p, q) denote the greatest common divisor of p and q.

LEMMA 3.2. *If $[p, q]$ is the type of some head cycle, then the greatest common divisor $(p, q) = 1$.*

Proof. We use Proposition 3.2 to prove inductively on k that

$$(|P_k| + 1, |Q_k| + 1) = 1, \quad 1 \leq k \leq N - 1,$$

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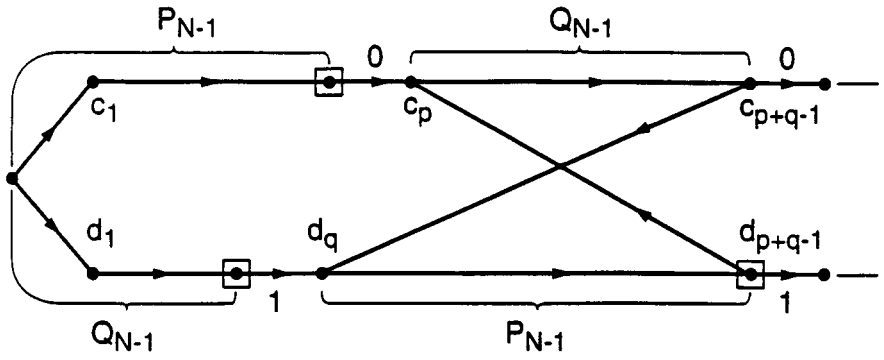


FIGURE 4. Labelled head cycle.

the case $k = N - 1$ giving the lemma. For $k = 0$, we have

$$(|P_0| + 1, |Q_0| + 1) = (1, 1) = 1.$$

To complete the induction step, the recursions for P_{k+1} and Q_{k+1} give either

$$|P_{k+1}| + 1 = (|P_k| + 1) + (|Q_k| + 1) \quad \text{and} \quad |Q_{k+1}| + 1 = |Q_k| + 1,$$

or

$$|P_{k+1}| + 1 = |P_k| + 1 \quad \text{and} \quad |Q_{k+1}| + 1 = (|Q_k| + 1) + (|P_k| + 1).$$

In either case

$$(|P_{k+1}| + 1, |Q_{k+1}| + 1) = (|P_k| + 1, |Q_k| + 1) = 1,$$

completing the induction step. □

The next proposition gives more detailed information relating $i(0)$ and $i(1^-)$ to the head cycle.

PROPOSITION 3.3. *Let the head cycle be of type* $[p, q]$.

(i) *The itineraries* $i(0)$ *and* $i(1^-)$ *have the form*

$$i(0) = P0Q0\ldots, \tag{3.1a}$$

$$i(1^-) = Q1P1\ldots, \tag{3.1b}$$

where $|P| = p - 1$ *and* $|Q| = q - 1$ *and, furthermore,*

$$P0Q = Q1P \text{ except in the first coordinate.} \tag{3.2}$$

(ii) *If both* $p \geq 2$ *and* $q \geq 2$, *then*

$$i(0) = P0Q0\underline{1}\ldots, \tag{3.3a}$$

$$i(1^-) = Q1P1\underline{0}\ldots. \tag{3.3b}$$

We have underlined the last term in (3.3) *to emphasize the improvement over* (3.1).

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Proof. (i) Relations (3.1) follow from Proposition 3.2 on taking $P = P_{N-1}$, $Q = Q_{N-1}$. Relation (3.2) can also be derived from Proposition 3.2. More generally, we prove

$$P_k 0 Q_k = Q_k 1 P_k \text{ except in the first coordinate } 0 \le k \le N-1. \tag{3.4}$$

The base case $k = 0$ has $P_0 0 Q_0 = 0$ and $Q_0 1 P_0 = 1$, so (3.4) holds. For the induction step the recursions for P_k and Q_k in Proposition 3.2 give either

$$P_{k+1} 0 Q_{k+1} = P_k 0 Q_k 0 Q_k \quad \text{and} \quad Q_{k+1} 1 P_{k+1} = Q_k 1 P_k 0 Q_k, \tag{3.5a}$$

or

$$P_{k+1} 0 Q_{k+1} = P_k 0 Q_k 1 P_k \quad \text{and} \quad Q_{k+1} 1 P_{k+1} = Q_k 1 P_k 1 P_k. \tag{3.5b}$$

In either case we conclude from (3.5) that (3.4) holds with k replaced by $k+1$, which completes the induction step.

(ii) Equations (3.3) follow from equation (2.9) in Hofbauer [**7**] by noting that the underlined $\underline{1}$ and $\underline{0}$ necessarily are in location $k + k_1 + \cdots + k_m + k$ of that equation. □

The aperiodicity criteria we develop depend on whether the head cycle $H_{\beta,\alpha}$ is in the maximal component $S^{\max}_{\beta,\alpha}$ or not†. We have the following dichotomy.

LEMMA 3.3. *Suppose that* $1 < \alpha + \beta < 2$, *and that the head cycle* $H_{\beta,\alpha}$ *is of type* $[p, q]$. *Then exactly one of the following conditions holds.*

(i) *There exists some edge* $c_k \to d_l$ *or* $d_k \to c_l$ *in* $S_{\beta,\alpha}$ *with* $k \ge p+q$ *and* $l < p+q$. *In this case* $H_{\beta,\alpha} \subseteq S^{\max}_{\beta,\alpha}$.

(ii) *There exists no edge* $c_k \to d_l$ *or* $d_k \to c_l$ *in* $S_{\beta,\alpha}$ *with* $k \ge p+q$ *and* $l < p+q$. *In this case the vertex set* $D^{\max}_{\beta,\alpha}$ *of* $S^{\max}_{\beta,\alpha}$ *has*

$$D^{\max}_{\beta,\alpha} \subseteq \{c_i : i \ge p+q\} \cup \{d_j : j \ge p+q\} \tag{3.6}$$

and $H_{\beta,\alpha}$ *and* $S^{\max}_{\beta,\alpha}$ *are disjoint.*

Proof. This lemma follows using Proposition 3.1 as follows.

(i) Suppose that such an edge exists, and let it be $d_k \to c_l$. We claim that $l = p$. Suppose not. Then Proposition 3.1 implies that if $l \ne p$, then necessarily $l < p$, because either c_{l-1} is a station or else $l = 1$. However, we can now construct a cycle containing c_l and d_q, namely $c_l \to c_{l+1} \to \cdots \to c_{p+q-1} \to d_q \to d_{q+1} \to \cdots \to d_k \to c_l$, and this cycle has $l + q < p + q$, which contradicts the definition of head cycle. Thus $l = p$. Similar reasoning shows that if the edge is $c_l \to d_l$ then $l = q$.

In either of these cases, the strongly connected component D_l of $S_{\beta,\alpha}$ that contains the head cycle contains two independent connected cycles, e.g. if there is an edge $d_k \to c_p$ then the two cycles in question are the head cycle and $c_p \to c_{p+1} \to c_{p+q-1} \to d_q \to d_{p+1} \to \cdots \to d_k \to c_p$. This implies that

$$\rho(\mathbf{M}_l) > 1$$

using (2.4), and Proposition 2.2 then implies that $D_l = D^{\max}_{\beta,\alpha}$. Thus $H_{\beta,\alpha} \subseteq S^{\max}_{\beta,\alpha}$.

† The condition that $H_{\beta,\alpha} \subseteq S^{\max}_{\beta,\alpha}$ can be shown to be equivalent to $D^{\max}_{\beta,\alpha}$ being the unique strongly connected component of $D_{\beta,\alpha}$. We omit a proof of this fact, as it is not used in what follows.

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(ii) Choose $c_k \in D_{\beta,\alpha}^{\max}$ with $k \geq p+q$. From the hypothesis, it follows that no path starting at c_k can lead to any c_i or d_j for which $i, j < p+q$. As $D_{\beta,\alpha}^{\max}$ is the union of all paths starting at c_k, we conclude that (3.6) holds. This implies that $H_{\beta,\alpha}^{\max}$ and $S_{\beta,\alpha}^{\max}$ are disjoint. □

The next proposition gives a criterion of Hofbauer for the head cycle to be in $S_{\beta,\alpha}^{\max}$, which involves symbolic dynamics. Let σ denote the shift operation on symbol sequences, i.e. $\sigma(S_0 S_1 S_2, \ldots) = S_1 S_2 S_3 \ldots$, and let $\leq$ denote the lexicographic ordering on symbol sequences.

PROPOSITION 3.4. *Suppose that the head cycle $H_{\beta,\alpha}$ is of type $[p, q]$. Then $H_{\beta,\alpha}$ is contained in the maximal component $S_{\beta,\alpha}^{\max}$ if and only if*

$$\sigma(i(0)) \leq \sigma^{p+q+1}(i(0)),$$

and

$$\sigma(i(1^-)) \geq \sigma^{p+1+1}(i(1^-)).$$

Proof. This is Hofbauer [**7**, Lemma 7]. □

3.2. *Aperiodicity and the head cycle.* We next derive two theorems giving necessary and sufficient conditions for aperiodicity of the Markov chain $\Sigma_{\beta,\alpha}^{\max}$. The first of these is stated in terms of the head cycle.

THEOREM 3.1. *Suppose that $1 < \alpha+\beta < 2$, and that the head cycle $H_{\beta,\alpha}$ is of type $[p, q]$.*
(i) *If $H_{\beta,\alpha} \not\subseteq S_{\beta,\alpha}^{\max}$, then $p+q$ divides $N_{\beta,\alpha}$, hence $N_{\beta,\alpha} > 1$.*
(ii) *If $H_{\beta,\alpha} \subseteq S_{\beta,\alpha}^{\max}$, then $N_{\beta,\alpha} = 1$.*

Proof. (i) Proposition 3.1 states that c_{p+q-1} and d_{p+q-1} are both stations, hence there are integers $i_0, j_0 \geq 1$ such that

$$R_{i_0} = S_{j_0} = p+q-1. \tag{3.7}$$

Our immediate objective is to show that the hypothesis $H_{\beta,\alpha} \not\subseteq S_{\beta,\alpha}^{\max}$ implies that all stations beyond R_{i_0} and S_{j_0} have indices congruent to $-1 \pmod{p+q}$, i.e.

$$R_i \equiv -1 \pmod{p+q} \quad \text{for } i \geq i_0, \tag{3.8a}$$

and

$$S_j \equiv -1 \pmod{p+q} \quad \text{for } j \geq j_0. \tag{3.8b}$$

To show this, we consider the set of stations

$$\mathcal{J} := \{R_i : i \geq i_0\} \cup \{S_j : j \geq j_0\},$$

and we prove (3.8) by induction on the size of $t \in \mathcal{J}$. The base case $t = p+q-1$ is established by (3.7). Let $t \geq p+q$ and assume that (3.8) is proved for all $u \in \mathcal{J}$ with $u < t$. We assume that $t = R_i$ for some $i > i_0$, the argument being similar if $t = S_j$ for some $j > j_0$. By Proposition 2.1(iii)

$$R_i = R_{i-1} + S_{P(i)} + 1 \tag{3.9}$$

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for some $P(i)$, so by Proposition 2.1(ii), $c_{R_i} \to d_{S_{P(i)+1}}$ is an edge of $S_{\beta,\alpha}$. Now $H_{\beta,\alpha} \not\subseteq S^{\max}_{\beta,\alpha}$ and Lemma 3.2 tell us that $S^{\max}_{\beta,\alpha}$ contains no edges $c_k \to d_l$ or $d_k \to c_l$ with $k \geq p+q$ and $l < p+q$. This implies that necessarily $P(i) \geq j_0$. The induction hypothesis (3.8) therefore gives $S_{P(i)} \equiv R_{i-1} \equiv -1 \pmod{p+q}$, so (3.9) gives

$$R_i \equiv R_{i-1} \equiv -1 \pmod{p+q},$$

thus proving (3.8) for t. This completes the induction step.

The period $N_{\beta,\alpha}$ of $\Sigma^{\max}_{\beta,\alpha}$ is by definition the greatest common divisor of all cycles contained in $S^{\max}_{\beta,\alpha}$, so it suffices to show that any cycle Γ in $S^{\max}_{\beta,\alpha}$ has length $|\Gamma|$ divisible by $p+q$. Now Lemma 3.2 shows that the vertices of Γ are contained in $\{c_i : i \geq p+q\} \cup \{d_j : j \geq p+q\}$. Next split Γ into segments joining consecutive stations. We prove that each such segment has length divisible by $p+q$. If the two consecutive stations are on the same track, this follows directly from (3.8). If the segment is a path $c_k \to d_l \to d_{l+1} \to \cdots \to d_{l+r}$ then (3.8) implies that both $l \equiv 0 \pmod{p+q}$ and $l+r \equiv -1 \pmod{p+q}$, since $d_l - 1$ is a station in $\mathcal{J}$, so $p+q$ divides $d_{l+r} - d_l + 1$, which is the length of this path. The case $d_k \to c_l \to c_{l+1} \to \cdots \to c_{l+r}$ is handled similarly. Thus $(p+q)$ divides the length of every segment, hence it divides the length of Γ, and (i) is proved.

(ii) It suffices to produce two cycles Γ_1, Γ_2 in $S^{\max}_{\beta,\alpha}$ such that $(|\Gamma_1|, |\Gamma_2|) = 1$. We choose Γ_1 to be the head cycle, so that $|\Gamma_1| = p+q$.

We are given that $H_{\beta,\alpha} \subseteq S^{\max}_{\beta,\alpha}$. The proof of Lemma 3.3(i) shows that there exists some edge $c_k \to d_q$ or $d_k \to c_p$ with $k \geq p+q$. We consider that edge for which k is minimal.

Suppose first that it is $c_k \to d_q$. Now c_k is a station so

$$k = R_i \quad \text{for some } i > i_0,$$

where $R_{i_0} = p+q-1$. By Proposition 2.1(iii) we have

$$q = R_i - R_{i-1}. \tag{3.10}$$

Now consider the set

$$\mathcal{J}' = \{c_{R_l} : p+q-1 \leq R_l < k\} \cup \{d_{S_l} : p+q-1 \leq S_l < k\}.$$

We now have for all R_l, $S_l \in \mathcal{J}'$ that

$$R_l \equiv -1 \pmod{p+q}, \tag{3.11a}$$

and

$$S_l \equiv -1 \pmod{p+q}. \tag{3.11b}$$

This is proved by exactly the same argument used to prove (3.8), using the hypothesis that $k \geq p+q$ was the minimal vertex having an edge going to the head cycle. In particular, we conclude that

$$R_{i-1} \equiv -1 \pmod{p+q}. \tag{3.12}$$

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Now take for Γ_2 the cycle $c_p \to c_{p+1} \to \cdots \to c_k \to d_q \to d_{q+1} \to \cdots \to d_{p+q-1} \to c_p$. Using (3.10) and (3.12) we have

$$k = R_i \equiv q - 1 \pmod{p+q}. \tag{3.13}$$

Thus

$$|\Gamma_2| = (k - p + 1) + p \equiv q \pmod{p+q}.$$

In consequence

$$(|\Gamma_1|, |\Gamma_2|) = (p+q, q) = (p, q) = 1,$$

using Lemma 3.2. Thus (ii) holds in this case.

The remaining case $d_k \to c_p$ is handled by a similar argument. □

By combining Theorem 3.1 with Proposition 3.4 we obtain a criterion for aperiodicity in terms of the symbolic dynamics of $i(0)$ and $i(1^-)$.

THEOREM 3.2. *Suppose that* $1 < \alpha + \beta < 2$ *and that the head cycle* $H_{\beta,\alpha}$ *is of type* $[p, q]$. *Then the period* $N_{\beta,\alpha}$ *of the Markov chain* $\Sigma_{\beta,\alpha}^{\max}$ *is greater than 1 if and only if*

$$\sigma(i(0)) \le \sigma^{p+q-1}(i(0)), \tag{3.14}$$

and

$$\sigma(i(1^-)) \ge \sigma^{p+q-1}(i(1^-)). \tag{3.15}$$

Proof. Immediate. □

3.3. '*Bubbles*'. We now relate the notion of '*bubble*' $R_{N,K}$ in Palmer [**11**] and Glendinning [**4**] to the period $N_{\beta,\alpha}$ of the Markov chain $\Sigma_{\beta,\alpha}^{\max}$. Given a pair of integers (N, K) with $1 \le K < N$ and $(N, K) = 1$, a *primary* (N, K)*-cycle* for $f_{\beta,\alpha}$ is a set of points $\{z_1, z_2, \ldots, z_N\} \subseteq (0, 1)$ such that:

(a) if $\gamma = (1 - \alpha)/\beta$ is the (unique) turning point of $f_{\beta,\alpha}$, then

$$0 < z_1 < z_2 < \cdots < z_{N-K} < \gamma < z_{N-K+1} < \cdots < z_N < 1;$$

(b) for $1 \le i \le N$,

$$f_{\beta,\alpha}(z_i) = z_{i+K},$$

where subscripts are interpreted (mod N);

(c) the values $f_{\beta,\alpha}(0)$ and $f_{\beta,\alpha}(1)$ satisfy

$$z_K \le f_{\beta,\alpha}(0) < f_{\beta,\alpha}(1) \le z_{K+1}.$$

The '*bubble*' $R_{N,K}$ consists of all pairs (β, α) with $1 < \alpha + \beta < 2$ such that $f_{\beta,\alpha}$ has a primary (N, K)-cycle.

The object of the rest of §3 is to prove the following result.

THEOREM 3.3. *For* $1 < \alpha + \beta < 2$, *the following two conditions are equivalent:*

(S1) *The period* $N_{\beta,\alpha}$ *of* $\Sigma_{\beta,\alpha}^{\max}$ *is greater than 1.*

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(S2) (β, α) *lies in some 'bubble'* $R_{N,K}$. *In this case the type* $[p, q]$ *of the head cycle* $H_{\beta,\alpha}$ *is related to* (N, K) *by*

$$N = p + q \quad \text{and} \quad pK \equiv -1 \pmod{N}. \tag{3.16}$$

This result implies (ii) $\Rightarrow$ (i) of Theorem 1.1, which completes the proof of Theorem 1.1, as explained at the beginning of §3.

The main work in the proof of Theorem 3.3 is contained in three preliminary lemmas. They use the symbolic dynamics criterion of Theorem 3.2 for $N_{\beta,\alpha} > 1$ to construct a symbolic primary (N, K)-cycle with (N, K) given by (3.16). Let Σ_f^+ denote the closure of the one-sided subshift Σ_f of $\{0, 1\}^N$ consisting of all itineraries $[i(x) : 0 \le x < 1\}$. A symbolic primary (N, K)-cycle consists of $Z_1, \ldots, Z_N \in \Sigma_f^+$ which obey analogues of conditions (a)–(c) of a primary (N, K)-cycle, with $\le$ denoting the lexicographic ordering on symbol sequences, and the symbols $0, \gamma, 1$ being replaced by $i(0), i(\gamma), i(1^-)$, respectively.

To begin, for any integer m, define $r(m)$ by $r(m) \equiv m \bmod p + q$ with $1 \le r(m) \le p + q$.

LEMMA 3.4. *Suppose that the head cycle* $H_{\beta,\alpha}$ *has type* $[p, q]$. *Write* $i(0) = P0Q \ldots$ *where* $|P| = p - 1$ *and* $|Q| = q - 1$. *Then the number of 0's in* $P0Q$ *equals* l *where*

$$1 \le l \le p + q - 1 \quad \text{and} \quad pl \equiv 1 \pmod{p + q}. \tag{3.17}$$

These 0's are located exactly in the positions

$$r(1 - jp), \quad 0 \le j \le l - 1, \tag{3.18}$$

in $P0Q$, *hence* $P0Q$ *is uniquely determined by* $[p, q]$.

Proof. Suppose first that $p = 1$, so that $P = \emptyset$. Now (3.2) of Proposition 3.3 states that $0Q = Q1$ except in the first coordinate, which with $|Q| = q - 1$ gives $Q = 1^{q-1}$. (For $q > 1$, 1^{q-1} is defined to be the string of $q - 1$ consecutive 1's; for $q = 1$, 1^{q-1} is defined to be $\emptyset$.) Thus $P0Q = 01^{q-1}$, so we have $l = 1$. Now (4.17) and (4.18) are easily checked, so the lemma holds in this case.

The case that $q = 1$ is handled by a similar argument which we omit.

Suppose now that $p \ge 2$, $q \ge 2$, so that $P \ne \emptyset$ and $Q \ne \emptyset$. Since $i(0)$ begins with a 0 and $i(1^-)$ begins with a 1, these are the leading terms in P and Q, respectively, so we may write

$$P = 0\tilde{P}, \quad Q = 1\tilde{Q},$$

where $|\tilde{P}| = p - 2$ and $|\tilde{Q}| = q - 2$. Now the criterion (6.2) asserts that

$$\tilde{P}01\tilde{Q} = \tilde{Q}10\tilde{P}. \tag{3.19}$$

We reformulate this as

$$\tilde{P}01\tilde{Q}01 = \tilde{Q}10\tilde{P}01, \tag{3.20}$$

and note that these are zero-one strings of length $p + q$.

We view (3.20) as expressing the statement that $\tilde{P}01\tilde{Q}01$ is a fixed point of a certain permutation Φ of zero-one strings $(S(1), \ldots, S(p+q))$ of length $p+q$. The permutation

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Φ first exchanges $S(p+q-1)$ and $S(p+q)$ and then cyclically permutes this vector p coordinates to the left. The right-hand side of (3.20) just expresses the action of Φ on the left-hand side of (3.20). Setting $(S(1), \ldots, S(p+q)) = \tilde{P}01\tilde{Q}01$, (3.20) says that

$$S(r(m-p)) = \begin{cases} 1 - S(m) & \text{if } m \equiv 0, -1 \pmod{p+q}, \\ S(m) & \text{otherwise.} \end{cases}$$

Taking $1 \le l \le p+q$ with $lp \equiv 1 \pmod{p+q}$, these relations imply that $S(r(-(j+1)p)) = S(r(-jp))$ unless $j = 0$ or l, when $S(r(-(j+1)p)) = 1 - S(r(-jp))$. Since $S(r(0)) = S(p+q) = 1$ and $S(r(-lp)) = S(p+q-1) = 0$, we conclude that

$$\begin{cases} S(r(-jp)) = 0 & 1 \le j \le l, \\ S(r(-jp)) = 1 & l+1 \le j \le p+q. \end{cases} \tag{3.21}$$

Now (3.21) shows that there are exactly l values of 0 in $\tilde{P}01\tilde{Q}01$ which occur in positions $\{r(-jp), 1 \le j \le l\}$. Since $P0Q = 0\tilde{P}01\tilde{Q}$, we conclude that there are exactly l zeros in $P0Q$. The zeros in $P0Q$ are obtained from those of $\tilde{P}01\tilde{Q}01$ by removing the zero in position $p+q-1$ and inserting it in front of $\tilde{P}$. Thus the zeros of $P0Q$ are in positions $1 + r(-jp) = r(1-jp)$ for $0 \le j \le l-1$. □

LEMMA 3.5. *Suppose that the head cycle $H_{\beta,\alpha}$ is of type $[p, q]$ and is not contained in $S^{\max}_{\beta,\alpha}$. Then necessarily*

$$i(0) = P0Q0\underline{1}\ldots \tag{3.22a}$$

$$i(1^-) = Q1P1\underline{0}\ldots, \tag{3.22b}$$

where $|P| = p-1$ and $|Q| = q-1$.

When $p \ge 2$ and $q \ge 2$ the conclusion of this lemma already follows from Proposition 3.3(ii). The point of this lemma is that the conditions (3.22) still hold in certain cases when $p = 1$ or $q = 1$.

Proof. We need only treat the cases $p = 1$ and $q = 1$.

Suppose that $p = 1$. Then Proposition 3.3 implies that $P = \emptyset$, $Q = 1^{q-1}$, as indicated in the proof of Lemma 3.3, and (3.1) becomes

$$i(0) = 01^{q-1}0\ldots, \tag{3.23a}$$

$$i(1^-) = 1^{q+1}\ldots. \tag{3.23b}$$

In this case the vertices in the head cycle $H_{\beta,\alpha}$ are given by $\{c_1, c_2, \ldots, c_q\} \cup \{d_q\}$. The symbolic itineraries (3.23) imply that $d_1, d_2, \ldots, d_q$ are stations and $d_1 \to c_1, \ldots, d_q \to c_1$ are edges and, furthermore, that c_q is the first station on the $\{c_i\}$-track, and that there is an edge $c_q \to d_q$. If $i(1^-) = 1^{q+2}\ldots$, then d_{q+1} would also be a station with $d_{q+1} \to c_1$. According to Lemma 3.2 this implies that $H_{\beta,\alpha} \subseteq S^{\max}_{\beta,\alpha}$, contradicting our hypothesis. Thus

$$i(1^-) = 1^{q+1}\underline{0}\ldots,$$

verifying (3.22b).

Similarly, if $q = 1$, then the above argument with p and q interchanged gives

$$i(0) = 0^{p+1}1\ldots,$$

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verifying (3.22a). Hence (3.22) holds for $p = q = 1$. If $p = 1$ and $q \geq 2$, then (3.23a) gives

$$i(0) = 01\ldots \quad \text{and} \quad \sigma^q(i(0)) = 0\ldots. \tag{3.24}$$

Since $i(0) \leq \sigma^q(i(0))$, we conclude from (3.24) that

$$\sigma^q(i(0)) = 01\ldots$$

and combining this with (3.23a) gives

$$i(0) = 01^{q-1}0\underline{1}\ldots,$$

so (3.22a) is verified. This completes the case $p = 1$.

Finally, the case $q = 1$ is treated analogously to $p = 1$. □

The next lemma assumes the criterion of Proposition 3.4 and establishes the existence of a symbolic primary (N, K)-cycle. To explain this, let $\overline{P0Q1}$ denote the periodic symbol sequence

$$\overline{P0Q1} := P0Q1\ P0Q1\ P0Q1\ldots,$$

where $|P| = p - 1$ and $|Q| = q - 1$. Next, for all integers i, set

$$\Omega_i = \sigma^{i-1}(\overline{P0Q1})$$

and define Z_i by

$$Z_i = \Omega_{1-(i-1)p}.$$

Then $\Omega_{i+p+q} = \Omega_i$ and $Z_{i+p+q} = Z_i$ for all i. The set $\{Z_i : 1 \leq i \leq p+q\}$ is called the *symbolic $(p+q)$-cycle associated to (P, Q)*. In the following lemma $\leq$ denotes lexicographic ordering on symbol sequences.

LEMMA 3.6. *Suppose that the head cycle $H_{\beta\alpha}$ is of type $[p, q]$ and is not contained in $S^{\max}_{\beta,\alpha}$. Then $i(0) = P0Q\ldots$ with $|P| = p-1$, $|Q| = q-1$, and let $\{Z_i : 1 \leq i \leq p+q\}$ be the symbolic $(p+q)$-cycle associated to (P, Q). Define (N, K) with $1 \leq K < N$ by*

$$N = p + q \quad \textit{and} \quad pK \equiv -1 \pmod{N}.$$

Then each $Z_i \in \Sigma_f^+$ and the Z_i's satisfy:

(a) *the Z_i's are ordered, and if $\gamma = (1-\alpha)/\beta$, then,*

$$i(0) < Z_1 < \cdots < Z_{N-K} < i(\gamma) < Z_{N-K+1} < \cdots < Z_N < i(1^-);$$

(b) *for $1 \leq i \leq N$,*

$$\sigma(Z_i) = Z_{i+K};$$

(c) *there holds*

$$Z_K \leq \sigma(i(0)) < \sigma(i(1^-)) \leq Z_{K+1}.$$

The properties (a)–(c) above, together with all $Z_i \in \Sigma_f^+$, define a *symbolic primary* (N, K)-cycle. Lemma 3.6 states that a symbolic $(p+q)$-cycle can also be identified as a symbolic primary (N, K)-cycle, with (p, q) and (N, K) related by (3.16).

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Proof. We prove the lemma in six steps. To simplify some notation, we set

$$a := i(0), \quad b := i(1^-), \quad \text{and} \quad \hat{\gamma} := i(\gamma).$$

Step 1. $Z_1 < Z_2 < \cdots < Z_N$. To show this, we have $\Omega_1 = \overline{P0Q1}$. Lemma 3.4 shows that the zeros in Ω_1 among the first $p+q$ symbols are located at positions $r(1-jp)$ for $0 \le j \le l-1$. Hence the zeros of Ω_k among the first $p+q$ symbols are at positions $r(2-k-jp)$, $0 \le j \le l-1$, all k. Replacing k by $k+p$, we conclude that the zeros of Ω_{k+p} among the first $p+q$ symbols are in positions $r(2-k-jp)$, $1 \le j \le l$. Comparing Ω_k and Ω_{k+p}, we see that the first $p+q$ symbols are identical except for those in positions $r(1-k) = r(2-k-lp)$ and $r(2-k)$. At these respective positions Ω_k has symbols 1 and 0, while Ω_{k+p} has 0 and 1. Now

$$r(1-k) < r(2-k) \quad \text{if } k \not\equiv 1 \pmod{p+q}, \tag{3.25}$$

while if $k = 1$, then $r(1-k) = p+q > r(2-k) = 1$. Consequently,

$$\Omega_{k+p} < \Omega_k \quad \text{for } k \not\equiv 1 \pmod{p+q}, \tag{3.26a}$$

$$\Omega_{k+p} > \Omega_k \quad \text{for } k \equiv 1 \pmod{p+q}. \tag{3.26b}$$

This yields

$$\Omega_1 < \Omega_{1-p} < \cdots < \Omega_{1-(p+q-1)p} \equiv \Omega_{1+p},$$

which is just $Z_1 < Z_2 < \cdots < Z_N$, where $N = p+q$.

Step 2. $i(0) < Z_1$ *and* $Z_N < i(1^-)$. To show this, using Proposition 3.3 we obtain

$$i(0) = P0Q0\ldots < P0Q1\ldots = \Omega_1 = Z_1.$$

Similarly, we obtain

$$i(1^-) = Q1P1\ldots > Q1P0\ldots = \Omega_{1+p} = Z_N.$$

Step 3. $Z_{N-K} < \hat{\gamma} < Z_{N-K+1}$. To show this, from $pK \equiv -1 \pmod{N}$ we get

$$Z_{N-K} := \Omega_{1-(N-K-1)p} = \Omega_p = \overline{0Q1P},$$
$$Z_{N-K+1} := \Omega_{1-(N-K)p} = \Omega_0 = \overline{1P0Q}.$$

Since $f(\gamma) = 0$, we have $\hat{\gamma} = 1a = 1P0Q0\ldots$, hence

$$Z_{N-K} = 0\ldots < 1\ldots = \hat{\gamma},$$

and

$$Z_{N-K+1} = 1P0Q1\ldots > 1P0Q0\ldots = \hat{\gamma}.$$

Step 4. $\sigma(Z_i) = Z_{i+K}$ *for* $1 \le i \le N$. Since $pK \equiv -1 \pmod{p+q}$, this follows from

$$\sigma(Z_i) = \Omega_{1-(i-1)p+1} = \Omega_{1-(i+K-1)p} = Z_{i+K}.$$

Step 5. $Z_K \le \sigma(i(0))$ *and* $\sigma(i(1^-)) \le Z_{K+1}$. By assumption, $H_{\beta,\alpha} \not\subseteq S_{\beta,\alpha}$. Hence by Lemma 3.5,

$$a := i(0) = P0Q01\sigma^{p+q+1}(a), \tag{3.27a}$$

$$b := i(1^-) = Q1P10\sigma^{p+q+1}(b), \tag{3.27b}$$

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and by Proposition 3.4,

$$\sigma(a) \le \sigma^{p+q+1}(a), \tag{3.28a}$$

$$\sigma(b) \ge \sigma^{p+q+1}(b). \tag{3.28b}$$

Suppose first that $p \ge 2$ and $q \ge 2$. We may thus write $P = 0\tilde{P} = 1\tilde{Q}$, and, using (3.23), (3.27a) and (3.28a), we get

$$\begin{aligned} \sigma(a) &= \tilde{P}01\tilde{Q}01\sigma^{p+q+1}(a) = \tilde{Q}10\tilde{P}01\sigma^{p+q+1}(a) \\ &\ge \tilde{Q}10\tilde{P}01\sigma(a). \end{aligned}$$

Iterating this inequality *ad infinitum*, we obtain

$$\sigma(a) \ge \overline{\tilde{Q}10\tilde{P}01} = \Omega_{p+2} = \Omega_{1-(K-1)p} = Z_{K+1}.$$

Similarly, from (3.2), (3.27b) and (3.28b), we get

$$\begin{aligned} \sigma(b) &= \tilde{Q}10\tilde{P}10\sigma^{p+q+1}(b) = \tilde{P}01\tilde{Q}10\sigma^{p+q+1}(b) \\ &\le \tilde{P}01\tilde{Q}10\sigma(b). \end{aligned}$$

Repeating this inequality *ad infinitum*, we obtain

$$\sigma(b) \le \overline{\tilde{P}01\tilde{Q}10} = \Omega_2 = \Omega_{1-Kp} = Z_{K+1}.$$

There remain the cases $p = 1$ and $q = 1$. We describe only the case $p = 1$, the case $q = 1$ being handled analogously. For $p = 1$ we have $P = \emptyset$ and $Q = 1^{q-1}$. From (3.27a) and (3.28a) we get

$$\begin{aligned} \sigma(a) &= 1^{q-1}01\sigma^{p+q+1}(a) \\ &\ge 1^{q-1}01\sigma(a). \end{aligned}$$

Repeating this inequality *ad infinitum* gives

$$\sigma(a) \ge \overline{1^{q-1}01} = \Omega_{p+2} = Z_K.$$

Similarly, from (3.27b) and (3.28b) we obtain

$$\begin{aligned} \sigma(b) &= 1^q0\sigma^{p+q+1}(b) \\ &\le 1^q0\sigma(b). \end{aligned}$$

Repeating this inequality *ad infinitum*, we obtain

$$\sigma(b) \le \overline{1^q0} = \Omega_2 = Z_{K+1}.$$

This completes the case $p = 1$.

Step 6. Each $Z_i \in \Sigma_f^+$. To show this, we use the criterion of Hofbauer [**5**, Theorem 2]: $Z \in \Sigma_f^+$ if and only if

$$i(0) \le \sigma^i(Z) \le i(1^-) \quad \text{for all } i \ge 0.$$

Since the $\{Z_i\}$ have period N, we need only show that

$$i(0) < Z_i < i(1^-) \quad \text{for } 1 \le i \le N,$$

which follows from Steps 1 and 2. □

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Proof of Theorem 3.3. $(S1) \Rightarrow (S2)$. Suppose that $N_{\beta,\alpha} > 1$. Theorem 3.2 and Lemma 3.6 together imply that $f_{\beta,\alpha}$ has a symbolic primary (N, K)-cycle, where (N, K) have the relations (3.16) to $[p, q]$. We now show that $f_{\beta,\alpha}$ has a primary (N, K)-cycle.

We claim that there exist $\{z_k : 1 \le k \le N\}$ in $[0, 1)$ whose symbolic itineraries are

$$i(z_k) = Z_k, \quad 1 \le i \le N, \tag{3.29}$$

i.e. that each $Z_i \in \Sigma_f$. To prove (3.29), we use the following sharper form of the nested intervals theorem. If $I_n = [a_n, b_n)$ is a sequence of nested half-open intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, such that the b_n's are not identical from some point onwards, then $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$. We observe that $Z_i = \Omega_1 = \overline{P0Q1}$ is a sequence of edge labels of a path $\gamma(Z_i)$ in $S_{\beta,\alpha}$ beginning at vertex d_q and wrapping infinitely around the head cycle, see Figure 4, noting that $P_{N-1} = P$ and $Q_{N-1} = Q$. Set $Z_1 = a_0a_1a_2\ldots$, and note that by Lemma 3.1 the sequences $a_0a_1\ldots a_n$, $n \ge 0$ are admissible, i.e. $I_n := I_{a_0a_1\ldots a_n} \neq \emptyset$. The sequence $I_1 \supseteq I_2 \supseteq \cdots$ is a nested sequence of half-open intervals. The path $\gamma(Z_1)$ passes infinitely often through d_{p+q-1}, thus $a_n = 0$ for infinitely many n. Furthermore, for these n, both $a_0a_1\ldots a_{n-1}0$ and $a_0a_1\ldots a_{n-1}1$ are admissible by Lemma 3.1. Now

$$I_{a_0\ldots a_{n-1}} = I_{a_0\ldots a_{n-1}0} \cup I_{a_0a_1\ldots a_{n-1}1},$$

hence the right endpoint of $I_{a_0\ldots a_{n-1}}$ does not equal the right endpoint of $I_{a_0\ldots a_n}$. The nested intervals theorem quoted above applies to yield a point $z_1 \in \bigcap_{n=0}^{\infty} I_n$. We have $i(z_1) = a_0a_1\ldots a_n\ldots$ as $z \in I_n$, hence

$$i(z_1) = \overline{P0Q1} = Z_1.$$

This proves (3.29) for Z_1. A similar argument works for Z_i, $1 \le i \le n$. Alternatively, it suffices to define

$$z_i := f^{j-1}(z_1), \quad 2 \le j \le p + q = N,$$

with $j \equiv 1-(i-1)p \pmod{p+q}$, and then use $\sigma(i(f^{j-1}(z))) = \sigma^{j-1}(i(z))$ to conclude (3.29).

Next we claim that $\{z_i : 1 \le i \le N\}$ is a primary (N, K)-cycle. We use the fact that the itinerary map $i(1)$ is order-preserving:

$$x_1 \le x_2 \Leftrightarrow i(x_1) \le i(x_2), \tag{3.30}$$

together with Lemma 3.6. The properties

$$0 < z_1 < z_2 < \cdots < z_{N-K} < \gamma < z_{N-K-1} < \cdots < z_N < 1 \tag{3.31}$$

and

$$f_{\beta,\alpha}(z_i) = z_{i+K}, \quad 1 \le i \le N,$$

immediately follow from Lemma 3.6(a) and (b). We consider Lemma 3.6(c), which states

$$Z_K \le \sigma(i(0)) < \sigma(i(1^-)) \le Z_{K+1}.$$

We now claim that

$$Z_K \le \sigma(i(0)) < \sigma(i(1)) \le Z_{K+1}. \tag{3.32}$$

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This is certainly true if $\sigma(i(1^-)) = \sigma(i(1))$. The remaining possibility is $\sigma(i(1^-)) < \sigma(i(1))$, which only occurs if some $f^M(1) = 0$. Choosing the minimal such M, the itineraries are

$$i(1) = B_0^* B_1^* \dots B_{M-2} 1(i(0)), \tag{3.33a}$$

$$i(1^-) = B_0^* B_1^* \dots B_{M-2}^* 0(i(1^-)), \tag{3.33b}$$

according to Lemma 2.3 of Part I. Set $Z_{K+1} = b_1 b_2 b_3 \dots$. It remains to show that $\sigma(i(1)) \le z_{K+1}$. From (3.33b) and $\sigma(i(1^-)) \le z_{K+1}$, we obtain

$$b_1 \dots b_{M-1} \ge B_1^* \dots B_{M-2}^* 0. \tag{3.34}$$

We now treat three cases.

Case 1. $b_1 \dots b_{M-1} > B_1^* \dots B_{M-2}^* 1$. We conclude from (3.33a) that $\sigma(i(1)) \le z_{K+1}$.

Case 2. $b_1 \dots b_{M-1} = B_1^* \dots B_{M-2}^* 1$. We know that $\sigma(Z_j) = Z_{j+K}$ and that $i(z_j) = Z_j$, hence $\sigma^{M-1}(Z_{K+1}) = i(z_{K+1+K(M-1)}) \ge i(0)$. Combining this with (3.33a) gives

$$\sigma(i(1)) = B_1^* \dots B_{M-2}^* 1 i(0) \le B_1^* \dots B_{M-2}^* 1 \sigma^{M-1}(Z_{K+1}) = Z_{K+1}.$$

Case 3. $b_1 \dots b_{M-1} < B_1^* \dots B_{M-2}^* 1$. This inequality, together with (3.34), implies that $b_1 \dots b_{M-1} = B_1^* \dots B_{M-2}^* 0$. Hence

$$Z_{K+1} = B_1^* \dots B_{M-2}^* \circ \sigma^{M-2}(Z_{K+1}).$$

Similar to Case 2, this converts to

$$Z_{K+1} = B_1^* \dots B_{M-2}^* 0 i(z_{KM+1}). \tag{3.35}$$

Comparing this with (3.33b) and using $\sigma(i(1^-)) \le Z_{K+1}$ yields

$$i(z_{KM+1}) \ge i(1^-).$$

However, the only point $z \in [0, 1]$ with $i(z) \ge i(1^-)$ is $z = 1$. Thus $z_{KM+1} = 1$, which contradicts (3.31). This completes Case 3.

Now (3.32) is proved, and (3.29) and (3.30) then give

$$z_k \le f_{\beta,\alpha}(0) < f_{\beta,\alpha}(1) \le z_{K+1}.$$

Hence $\{z_i : 1 \le i \le N\}$ is a primary (N, K)-cycle, i.e. $(\beta, \alpha) \in R_{N,K}$.

$(S2) \Rightarrow (S1)$. We prove the contrapositive. Suppose $N_{\beta,\alpha} = 1$. Then Theorem 3.2 states that the lap-counting function $L_{\beta,\alpha}(z)$ has no singularities on the circle $|z| = 1/\beta$ except at $z = 1/\beta$. Now Corollary 3.3 of part I shows that (β, α) cannot lie in any 'bubble' $R_{N,K}$. □

4. *Case $1 < \alpha + \beta \le 2$ inside 'bubbles'*

In this section we prove Theorem 1.2, which determines $\bar{N}_{\beta,\alpha}$ and $N_{\beta,\alpha}$ for $1 < \alpha + \beta \le 2$.

The notion of 'renormalization' traces back to Palmer [**11**] and Parry [**12**], and is made explicit in Glendinning [**4**]. Various equivalent definitions of it are given in §4 of

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part I. We take here a definition of 'renormalization' implied by Theorem 3.2(i) of part I. If $(\beta, \alpha) \in R_{N,K}$ then there exists a 'renormalized' pair $(\bar{\beta}, \bar{\alpha})$, given by

$$\bar{\beta} = \beta^N \quad \text{and} \quad \bar{\alpha} = 1 = \beta^N \left[\frac{\frac{1-\alpha}{\beta} - f_{\beta,\alpha}^{N-1}(0)}{f_{\beta,\alpha}^{N-1}(1) - f_{\beta,\alpha}^{N-1}(0)} \right], \tag{4.1}$$

with $1 < \bar{\alpha} + \bar{\beta} \leq 2$. We call $(\bar{\beta}, \bar{\alpha})$ the *renormalization* of (β, α). Theorem 4.2 of part I states that the lap-counting functions of (β, α) and $(\bar{\beta}, \bar{\alpha})$ are related by

$$L_{\beta,\alpha}(z) = \frac{1 - z^N}{(1-z)^2} L_{\bar{\beta},\bar{\alpha}}(z^N). \tag{4.2}$$

For any pair $(\beta, \alpha) := (\beta_1, \alpha_1) \in R_{N_1,K_1}$, there exists a finite sequence of pairs $(\beta_i, \alpha_i) \in R_{N_i,K_i}$ for $2 \leq i \leq s$, plus $(\beta_{s+1}, \alpha_{s+1})$, such that each (β_i, α_i) is a renormalization of $(\beta_{i-1}, \alpha_{i-1}), 2 \leq i \leq s+1$, and the final pair $(\beta_{s+1}, \alpha_{s+1})$ has $1 < \alpha_{s+1} + \beta_{s+1} \leq 2$ with the property that $(\beta_{s+1}, \alpha_{s+1})$ does not lie in any 'bubble'. In this case we say that $f_{\beta,\alpha}$ has a *complete set of renormalizations* lying in the successive 'bubbles' $R_{N_1,K_1}, \ldots, R_{N_s,K_s}$. Theorem 1.1 implies that a set of renormalizations is complete if and only if $N_{\beta_{s+1},\alpha_{s+1}} = 1$.

LEMMA 4.1. *If (β, α) has $1 < \alpha + \beta \leq 2$, with $f_{\beta,\alpha}$ having a complete set of renormalizations lying in successive 'bubbles' $R_{N_1,K_1}, \ldots, R_{N_s,K_s}$, then*

$$\bar{N}_{\beta,\alpha} = N_1 N_2 \ldots N_s. \tag{4.3}$$

Proof. According to Theorem 4.3 of part I, after setting $N_0 = 1$, we have

$$L_{\beta,\alpha}(z) = \left(\frac{1 - z^{N_1 \ldots N_s}}{1 - z} \right) \prod_{i=0}^{s-1} (1 - z^{N_0 N_1 \ldots N_i})^{-1} L_{\tilde{\beta},\tilde{\alpha}}(z^{N_1 \ldots N_s}), \tag{4.4}$$

where $\tilde{\beta} = \beta^{N_1 N_2 \ldots N_s}$. Also, $N_{\tilde{\beta},\tilde{\alpha}} = 1$ since this set of renormalizations is complete. Thus $L_{\tilde{\beta},\tilde{\alpha}}(z)$ has no singularities on the circle $|z| = 1/\tilde{\beta}$ except for a simple pole at $z = 1/\tilde{\beta}$, according to Theorem 4.2 of part II. Now (4.4) says that $L_{\beta,\alpha}(z)$ has poles on $|z| = 1/\beta$ exactly at the set $\{(1/\beta)\exp(2\pi i l/N_1 N_2 \ldots N_s) : 0 \leq l < N_1 N_2 \ldots N_s\}$, because all other factors on the right-hand side of (4.4) are non-vanishing on the open disk $|z| < 1$. Thus $\bar{N}_{\beta,\alpha} = N_1 N_2 \ldots N_s$. □

We proceed to determine $N_{\beta,\alpha}$. Given $(\beta, \alpha) \in R_{N,K}$ we determine the relation of the state graphs $S_{\beta,\alpha}$ and $S_{\bar{\beta},\bar{\alpha}}$ of the Markov chains $\Sigma_{\beta,\alpha}$ and $\Sigma_{\bar{\beta},\bar{\alpha}}$. Lemmas 4.2 and 4.3 below formulate relations between $S_{\beta,\alpha}$ and $S_{\bar{\beta},\bar{\alpha}}$ and between $N_{\beta,\alpha}$ and $N_{\bar{\beta},\bar{\alpha}}$, from which Theorem 1.2 is an easy consequence.

Let $D_{\beta,\alpha}$ and $S_{\beta,\alpha}$ denote the state space and the state graph of the Markov chain $\Sigma_{\beta,\alpha}$. Assume $(\beta, \alpha) \in R_{N,K}$. We construct a new graph $S'_{\beta,\alpha}$ from $S_{\beta,\alpha}$ by 'contracting by the factor N'. (This notion is explained below in the comments preceding Lemma 4.2.) The vertices $D'_{\beta,\alpha}$ of $S_{\beta,\alpha}$ consist of

$$\{(0,0)\} \cup \{c'_i : i \geq 1\} \cup \{d'_j : j \geq 1\},$$

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where we identify c'_i and d'_j with certain vertices of $D_{\beta,\alpha}$ by the correspondence

$$c'_i = c_{(i+1)N-1} \quad \text{and} \quad d'_j = d_{(j+1)N-1}, \quad \text{for } i \geq 1.$$

To describe the edges of $S'_{\beta,\alpha}$, recall from Theorem 3.3 that the head cycle $H_{\beta,\alpha}$ of $S_{\beta,\alpha}$ is of type $[p, q]$, where $p+q = N$. Thus c_{N-1} and d_{N-1} are the rightmost vertices of $H_{\beta,\alpha}$ and are both stations, and all vertices c'_i and d'_j are to the right of $H_{\beta,\alpha}$. The proof of Theorem 3.1 showed that every station of $S_{\beta,\alpha}$ to the right of $H_{\beta,\alpha}$ is either a c'_i or a d'_j. By Proposition 2.1, in $S_{\beta,\alpha}$ any $c_{(i+1)N-1}$ with $i \geq 1$ that is a station has a cross-edge to d_{jN} for some $j \geq 1$, and similarly any $d_{(j+1)N-1}$ with $j \geq 1$ that is a station has a cross-edge to some c_{iN} for some $i \geq 1$. In what follows, we denote directed edges in the graph $S'_{\beta,\alpha}$ by $\Rightarrow$ and in $S_{\beta,\alpha}$ by $\rightarrow$. The directed edges in $S'_{\beta,\alpha}$ are given by $(0,0) \Rightarrow c_1$, $(0,0) \Rightarrow d_1$, the 'tracks'

$$\begin{cases} c'_i \Rightarrow c'_{i+1}, & i \geq 1, \\ d'_j \Rightarrow d'_{j+1}, & j \geq 1, \end{cases} \tag{4.5a}$$

together with the 'cross-edges'

$$\begin{cases} c'_i \Rightarrow d'_j, & \text{if } c'_i \rightarrow d_{jN}, \\ d'_j \Rightarrow c'_i, & \text{if } d'_j \rightarrow c_{iN}. \end{cases} \tag{4.5b}$$

We call the vertices c'_i and d'_j which satisfy (4.5b) the *stations* of $S'_{\beta,\alpha}$. The vertices c'_i (respectively d'_j) are stations in $S'_{\beta,\alpha}$ if and only if they are stations in $S_{\beta,\alpha}$; if they are stations then their destination vertices on the other track depend on whether they are in $S'_{\beta,\alpha}$ or in $S_{\beta,\alpha}$.

The above edge rules not involving $(0,0)$ can be alternatively described as follows. If $x, y \in D'_{\beta,\alpha} - \{0,0\}$, then $x \Rightarrow y$ if and only if there are N distinct vertices $x = x_1, x_2, \ldots, x_N = y$ with $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_N$, such that $x_2, \ldots, x_{N-1}$ are not in $D'_{\beta,\alpha}$ but are in $D_{\beta,\alpha}$. This explains the phrase 'contracting by the factor N'.

LEMMA 4.2. *If $(\beta,\alpha) \in R_{N,K}$, let $(\bar{\beta},\bar{\alpha})$ be its renormalization. Then the graph $S'_{\beta,\alpha}$ is isomorphic to the state graph $S_{\bar{\beta},\bar{\alpha}}$ of the Markov chain $\Sigma_{\bar{\beta},\bar{\alpha}}$.*

Proof. Let $\{r_l\}$ and $\{s_l\}$ denote the sequences of gaps between stations of $S_{\beta,\alpha}$ given by Proposition 2.1, and let $\{\bar{r}_l\}$, $\{\bar{s}_l\}$ be the corresponding sequences for $S_{\bar{\beta},\bar{\alpha}}$. We now construct analogous sequences $\{r'_l\}$ and $\{s'_l\}$ for the graph $S'_{\beta,\alpha}$ as follows. Let $\{c_{R_l} : l \geq 1\}$ and $\{d_{S_l} : l \geq 1\}$ be the successive stations on the $\{c_i\}$- and $\{d_j\}$-tracks of $f_{\beta,\alpha}$. Since c_{N-1} and d_{N-1} are stations, we have $N-1 = R_m = S_n$ for some m and n. Hence $\{c_{R_{l+m}} : l \geq 1\}$ and $\{d_{S_{l+n}} : l \geq 1\}$ are the stations to the right of the head cycle $H_{\beta,\alpha}$. Note that these satisfy

$$R_{l+m} \equiv S_{l+n} \equiv -1 \pmod{N}.$$

The stations of $S'_{\beta,\alpha}$ are $\{c'_{R'_l} : l \geq 1\}$ and $\{d_{S'_l} : l \geq 1\}$, where

$$R_{l+m} = (R'_l + 1)N - 1, \quad l \geq 1, \tag{4.6a}$$

$$S_{l+n} = (S'_l + 1)N - 1, \quad l \geq 1. \tag{4.6b}$$

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We have $1 \le R'_1 < R'_2 < \cdots$ and $1 \le S'_1 < S'_2 < \cdots$, and we set $R'_0 = S'_0 = 0$. Next we define the sequences $\{r'_n\}_{l \ge 0}$ and $\{s'_n\}_{n \ge 0}$ by

$$R'_l = \sum_{n=0}^{l} r'_n, \quad S'_l = \sum_{n=0}^{l} s'_n.$$

By Proposition 2.1,

$$c'_{R'_l} \to d_{r'_l N} \quad \text{and} \quad d'_{S'_l} \to c_{s'_l N}.$$

Thus (4.5b) can be rewritten as

$$\begin{cases} c'_{R'_l} \Rightarrow d'_{r'_l} \\ d'_{S'_l} \Rightarrow c'_{s'_l}. \end{cases} \tag{4.7}$$

The directed edges of $S'_{\beta,\alpha}$ are given by $(0,0) \Rightarrow c_1$, $(0,0) \Rightarrow d_1$, (4.5a) and (4.7). Hence, to prove that $S'_{\beta,\alpha}$ is isomorphic to $S_{\bar\beta,\bar\alpha}$, denoted by $S'_{\beta,\alpha} \approx S_{\bar\beta,\bar\alpha}$, we must show that the sequences $\{r'_l\}$ and $\{s'_l\}$ are identical to the sequences $\{\bar r_l\}$ and $\{\bar s_l\}$ associated to $S_{\bar\beta,\bar\alpha}$ by Proposition 2.1. Here $\{\bar r_l\}$ and $\{\bar s_l\}$ are defined by (2.2), after f, r_l, s_l are replaced by $f_{\bar\beta,\bar\alpha}$, $\bar r_l$, $\bar s_l$, respectively. It therefore suffices to show that $\{r'_l\}$ and $\{s'_l\}$ also satisfy (2.2) for $f = f_{\bar\beta,\bar\alpha}$. Since $\bar\gamma := (1-\bar\alpha)/\bar\beta$ is the unique critical point of $f_{\bar\beta,\bar\alpha}$, this amounts to proving the following claim.

CLAIM. *For each integer $l \ge 0$, r'_{l+1} is the smallest positive integer $k \ge 0$ for which*

$$\bar\gamma = \frac{1-\bar\alpha}{\bar\beta} \in (f^{R'_l+k}_{\bar\beta,\bar\alpha}(0),\ f^{k-1}_{\bar\beta,\bar\alpha}(1^-)). \tag{4.8}$$

A similar statement holds for s'_{l+1}, with S'_l replacing R'_l in (4.8).

We prove (4.8) for r'_{l+1}, the proof for s'_{l+1} being similar. We first derive a relation (4.10) between $f_{\bar\beta,\bar\alpha}$ and $f_{\beta,\alpha}$, valid for $(\beta,\alpha) \in R_{N,K}$, and use it to transform (4.8) into an equivalent statement (4.15) formulated in terms of

$$f := f_{\beta,\alpha} \quad \text{and} \quad \gamma := \frac{1-\alpha}{\beta}, \tag{4.9}$$

and then we prove (4.15).

We recall several facts proved in part I. (See there the proof of Theorem 4.2.) For $(\beta,\alpha) \in R_{N,K}$, we have:

(i) $f^{N-1}(0) < \gamma < f^{N-1}(1^-)$;

(ii) $f^N|_I$ maps $I := [f^{N-1}(0), f^{N-1}(1^-))$ to itself;

(iii) the graph of $f^N|_I$ consists of two linear segments of slope $\bar\beta := \beta^N$, with a turning point at γ;

(iv) if

$$\phi(x) = \frac{x - f^{N-1}(0)}{f^{N-1}(1^-) - f^{N-1}(0)}$$

is the linear map sending I to $[0,1]$, then

$$\phi \circ f^N \circ \phi^{-1} = f_{\bar\beta,\bar\alpha}, \tag{4.10}$$

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To use these facts, we iterate (4.10) j times to obtain

$$\phi^{-1} \circ f_{\bar{\beta},\bar{\alpha}}^{j} = f^{N_j} \circ \phi^{-1}. \tag{4.11}$$

Also, from (4.10) and (4.11),

$$\phi^{-1}(0) = f^{N-1}(0), \quad \phi^{-1}(1) = f^{N-1}(1), \quad \phi^{-1}(\bar{\gamma}) = \gamma. \tag{4.12}$$

Applying ϕ^{-1} to both sides of (4.8) and using the formulae above, we obtain

$$\gamma = \frac{1-\alpha}{\beta} \in (f^{N(R_l'+k)}(f^{N-1}(0)), f^{N(k-1)}(f^{N-1}(1)^-)). \tag{4.13}$$

The proof of Theorem 4.2 of part I shows that $f^{N-1}(t)$ is continuous at $t = 1$. Hence $f^{N-1}(t) \uparrow f^{N-1}(1)$ as $t \uparrow 1$, so that

$$f^{N(k-1)}((f^{N-1}(1))^-) = \lim_{t\uparrow 1} f^{N(k-1)}(f^{N-1}(t)) = f^{Nk-1}(1^-).$$

Thus (4.13) is equivalent to

$$\gamma \in (f^{N(R_l'+1)+Nk-1}(0), f^{Nk-1}(1^-)). \tag{4.14}$$

Using (4.6a), we rewrite this as

$$\gamma \in (f^{R_{l+m}+Nk}(0), f^{Nk-1}(1^-)). \tag{4.15}$$

We must show that r_{l+1}' is the smallest positive integer k for which (4.15) holds. But r_{l+m+1} is by definition the smallest positive integer j for which

$$\gamma \in (f^{R_{l+m}+j}(0), f^{j}(1^-)). \tag{4.16}$$

Subtracting R_{l+m} from R_{l+m+1} in (4.6a), we obtain

$$r_{l+m+1} = Nr_{l+1}', \quad l \geq 1. \tag{4.17}$$

Combining (4.16) and (4.17) gives the desired conclusion for r_{l+1}', proving the claim and the lemma. □

LEMMA 4.3. *Let $(\beta, \alpha) \in R_{N,K}$ have the 'renormalization' $(\bar{\beta}, \bar{\alpha})$. Then*

$$N_{\beta,\alpha} = N \cdot N_{\bar{\beta},\bar{\alpha}}. \tag{4.18}$$

Proof. We have established in Lemma 4.2 that $S_{\beta,\alpha}' \approx S_{\bar{\beta},\bar{\alpha}}$. Let $(S_{\beta,\alpha}')^{\max}$ be the image of $S_{\beta,\alpha}^{\max}$ under this isomorphism. Every path in $(S_{\beta,\alpha}')^{\max}$ of length k lifts to a unique path of length Nk in $S_{\beta,\alpha}^{\max}$, as described just before Lemma 4.2. Conversely, every cycle of $S_{\beta,\alpha}^{\max}$ has length divisible by N, call it Nk, and lifts to a cycle of $(S_{\beta,\alpha}')^{\max}$ of length k. To show this, note that every cycle in $S_{\beta,\alpha}^{\max}$ decomposes into a set of paths between stations, and each such path further decomposes into subpaths between vertices of $S_{\beta,\alpha}^{\max}$ that are in $(S_{\beta,\alpha}')^{\max}$. Finally, each subpath has length exactly N, and corresponds to a single edge of $(S_{\beta,\alpha}')^{\max}$ (see the discussion before Lemma 4.2). Hence $N_{\beta,\alpha} = N \cdot N_{\beta,\alpha}'$, where $N_{\beta,\alpha}'$ is the period of $(S_{\beta,\alpha}')^{\max}$. As $(S_{\beta,\alpha}')^{\max} \approx (S_{\bar{\beta},\bar{\alpha}})^{\max}$, we conclude that $N_{\beta,\alpha}' = N_{\bar{\beta},\bar{\alpha}}$, which yields (4.18). □

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Proof of Theorem 1.2. Repeated application of Lemma 4.3 gives

$$N_{\beta,\alpha} = N_1 N_2 \dots N_s N_{\tilde{\beta},\tilde{\alpha}}.$$

But $N_{\tilde{\beta},\tilde{\alpha}} = 1$ by Theorem 1.1, since it is not in any 'bubble'. Thus $N_{\beta,\alpha} = N_1 N_2 \dots N_s$, hence Corollary 3.3 of part I guarantees that $L_{\beta,\alpha}(z)$ has poles at every point of $\{(1/\beta)\exp(2\pi i l/N_{\beta,\alpha}) : 0 \le l < N_{\beta,\alpha}\}$. Applying Lemma 4.1 completes the proof. □

Remarks. (1) Lemmas 4.2 and 4.3 determine the complete structure of the strongly connected components of $S_{\beta,\alpha}$ in the general case, as follows. We omit a proof.

THEOREM 4.1. *Suppose that* $1 < \beta_1 + \alpha_1 \le 2$ *and that* (β_1, α_1) *has the complete set of 'renormalizations'* $(\beta_i, \alpha_i) \in R_{N_i,K_i}$ *for* $1 \le i \le s$. *Then* $S_{\beta,\alpha}$ *has exactly* $s+1$ *strongly connected components* $D_1, D_2, \dots, D_{s+1}$ *with* $D_{s+1} = D^{\max}_{\beta,\alpha}$. *For* $1 \le i \le s$ *the component* D_i *consists of the vertices in a cycle of order* $N_1 N_2 \dots N_i$, *and the maximal component* D_{s+1} *has period* $N_1 N_2 \dots N_s$.

(2) Theorem 4.1 of part II states that for $N = N_{\beta,\alpha}$ the following limits exist

$$r_l = \lim_{k\to\infty} \frac{L_{l+Nk}}{\beta^{l+kN}}, \quad 0 \le l < N.$$

It is of some interest to know how many distinct values are assumed by the numbers r_l. This seems a difficult question. For $s = 1$, we can use the formula for $L_{\beta,\alpha}(z)$, derived in part I, to show that r_l is unimodal in l. It follows that, in this case, the r_l's assume at least $\lceil N_{\beta,\alpha}/2 \rceil$ distinct values.

5. *Case $2 < \alpha + \beta \le 3$ outside 'big bubble'*

In this section we treat the case $2 < \alpha + \beta \le 3$, and prove Theorem 1.3. We treat in detail the 'generic' case (i) of §2, that is $\alpha > 0$ and $\alpha + \beta \notin \mathbb{Z}$, so that $2 < \alpha + \beta < 3$.

For the remaining non-generic cases $N_{\beta,\alpha} = 1$ is trivially checked. For cases (ii) and (iv) the remaining statement in Theorem 1.3 concerning singularities of $L_{\beta,\alpha}(z)$ is covered by Flatto [**1**] and Ito and Takahashi [**9**]. For case (iii), we again reduce to case (ii) using the fact that $L_{\beta,\alpha}(z) = L_{\beta,\alpha^*}(z)$ for $\alpha^* = \lceil \alpha + \beta \rceil - \alpha - \beta$.

For $\alpha + \beta > 2$ the graph $S_{\beta,\alpha}$ always contains the loop $(0,0) \to (0,0)$. We begin with the following simple sufficient condition† for aperiodicity.

LEMMA 5.1. *For* $\alpha + \beta > 2$, *the Markov chain* $\Sigma^{\max}_{\beta,\alpha}$ *is aperiodic if* $(0,0)$ *is in* $S^{\max}_{\beta,\alpha}$.

Proof. By definition, the period $N_{\beta,\alpha}$ of $\Sigma^{\max}_{\beta,\alpha}$ is the greatest common divisor of the lengths of cycles in $S^{\max}_{\beta,\alpha}$. The loop $(0,0) \to (0,0)$ has length 1, so if $(0,0)$ is contained in $S_{\beta,\alpha}$, then $N_{\beta,\alpha} = 1$. □

LEMMA 5.2. *If* $\alpha + \beta > 2$, *then* $(0,0)$ *is in* $S^{\max}_{\beta,\alpha}$ *if and only if* $S_{\beta,\alpha}$ *contains either some edge* $c_i \to (0,0)$ *or some edge* $d_j \to (0,0)$.

† The set of (β, α) in the region $\alpha + \beta > 2$ that do not satisfy this sufficient condition is in fact the 'big bubble' $\tilde{R}$.

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Proof. If $(0,0) \in S^{\max}_{\beta,\alpha}$, then such an edge clearly exists. Conversely, suppose such an edge exists, and suppose it is $c_i \to (0,0)$. Then there is a loop $(0,0) \to c_1 \to c_2 \to \cdots \to c_i \to (0,0)$ in $S_{\beta,\alpha}$. Thus the component D_l containing $(0,0)$ also contains $\{c_1, \ldots, c_i\}$, and it contains two cycles of length $i+1$, namely the one above, and the cycle which traverses the loop $(0,0) \to (0,0)$ exactly $i+1$ times. Thus the number of different paths in D_l of length $m(i+1)$ which start from $(0,0)$ is at least 2^m. Now formula (2.4) gives

$$\rho(\mathbf{M}_l) \geq 2^{1/(i+1)}.$$

Proposition 2.2 forces $D_l = D_K$, so $(0,0) \in S^{\max}_{\beta,\alpha}$. Finally, a similar argument applies if the edge is $d_j \to (0,0)$. □

Proof of Theorem 1.3. If suffices to establish that $N_{\beta,\alpha} = 1$, since Theorem 4.2 of part II then implies that $L_{\beta,\alpha}(z)$ has no singularities on $|z| = 1/\beta$ except at $z = 1/\beta$. The hypothesis $2 < \alpha + \beta < 3$ means that the graph of $f := f_{\beta,\alpha}$ consists of three line segments, and it has a unique fixed point x_f which is located in the middle interval $I_1 = [(1-\alpha)/\beta, (2-\alpha)/\beta]$. Now

$$x_f = \alpha + \beta x_f - 1 \tag{5.1}$$

so $x_f = (1-\alpha)/(\beta-1)$. Also, $f(1^-) = \alpha + \beta - 2 \leq \alpha = f(0)$, and elementary algebra shows that

$$f(1^-) \leq x_f \leq f(0) \Leftrightarrow (\beta, \alpha) \in \tilde{R}. \tag{5.2}$$

Thus $(\beta, \alpha) \notin \tilde{R}$ means that either $f(0) < x_f$ or $x_f < f(1^-)$. Suppose that $f(0) < x_f$. If $f(0), f^2(0), \ldots, f^n(0)$ are all in the middle interval I_1, then the relations

$$f^{j+1}(0) = \alpha + \beta f^j(0) - 1, \quad \text{for } 0 \leq j < n,$$

together with (5.1) yield

$$f^{j+1}(0) - x_f = \beta(f^j(0) - x_f),$$

which in turn gives

$$f^n(0) - x_f = \beta^{n-1}(f(0) - x_f). \tag{5.3}$$

This guarantees that some $f^n(0) \notin I_1$. If $n \geq 1$ is minimal with this property, then $f(0) < x_f$ and (5.3) gives

$$f^n(0) < \frac{1-\alpha}{\beta} \leq f^{n-1}(0) < f^{n-2}(0) < \cdots < f(0) < x_f.$$

The rules of constructing $S_{\beta,\alpha}$ in §2 and Proposition 2.1 now show that $c_i = (i, 0)$ for $1 \leq i \leq n$ and that $c_n \to (0,0)$ in $S_{\beta,\alpha}$. In the remaining case that $x_f < f(1^-)$, we conclude similarly that $d_n \to (0,0)$ for some $n \geq 1$.

In either case Lemma 5.2 applies, and Lemma 5.1 then yields $N_{\beta,\alpha} = 1$. □

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6. *Case $2 < \alpha + \beta \leq 3$ inside 'big bubble'*

We suppose throughout that (β, α) lies in the 'big bubble'

$$\tilde{R} = \left\{ (\beta, \alpha) : \beta > 1 \text{ and } \frac{1}{\beta} \leq \alpha \leq 3 - \beta - \frac{1}{\beta} \right\},$$

and our objective is to prove Theorem 1.4. Recall that Theorem 5.1 of part I showed that if $(\bar{\beta}, \bar{\alpha}) = (\beta, (\alpha\beta - 1)/(\beta - 1))$, then $1 < \bar{\alpha} + \bar{\beta} \leq 2$ and $L_{\beta,\alpha}(z) = (1/(1-z))L_{\bar{\beta},\bar{\alpha}}(z)$, which implies that

$$\bar{N}_{\beta,\alpha} = \bar{N}_{\bar{\beta},\bar{\alpha}}. \tag{6.1}$$

We prove the following theorem.

THEOREM 6.1. *For $(\beta, \alpha) \in \tilde{R}$, the state graphs $S_{\beta,\alpha}$ and $S_{\bar{\beta},\bar{\alpha}}$ of the Markov chains $\Sigma_{\beta,\alpha}$ and $\Sigma_{\bar{\beta},\bar{\alpha}}$ are identical, except for the loop $(0,0) \to (0,0)$, which is present in $S_{\beta,\alpha}$ and absent in $S_{\bar{\beta},\bar{\alpha}}$. Consequently,*

$$N_{\beta,\alpha} = N_{\bar{\beta},\bar{\alpha}}. \tag{6.2}$$

This theorem immediately yields the following.

Proof of Theorem 1.4. This follows by combining (6.1), (6.2) with Theorem 1.2. □

To establish Theorem 6.1 we first relate the itineraries $i_{\beta,\alpha}(0)$ and $i_{\beta,\alpha}(1^-)$ to $i_{\bar{\beta},\bar{\alpha}}(0)$ and $i_{\bar{\beta},\bar{\alpha}}(1^-)$. For convenience we abbreviate these to $i(0)$, $i(1^-)$, $\bar{i}(0)$ and $\bar{i}(1^-)$, respectively. These relations depend on whether (β, α) lies in the interior of $\tilde{R}$ or on its boundary, and these two alternatives are treated below in Lemmas 6.2 and 6.3, respectively.

The map $f_{\beta,\alpha}$ is pictured in Figure 5(a).

As an intermediate step we consider $f_{\beta,\alpha}$ restricted to the domain

$$\tilde{J} = \tilde{J}_{\beta,\alpha} := \left[0, \frac{\beta - 1}{\beta}\right) \cup \left[\frac{1}{\beta}, 1\right) := \tilde{J}_0 \cup \tilde{J}_2.$$

For $\beta \leq 2$, the intervals $\tilde{J}_0$ and $\tilde{J}_2$ are disjoint, with $\tilde{J}_0$ lying to the left of $\tilde{J}_2$. We prove below that if $(\beta, \alpha) \in \tilde{R}$, then

$$f_{\beta,\alpha}(\tilde{J}) \subseteq \tilde{J}.$$

Figure 5(b) pictures the graph of $f_{\beta,\alpha}|_{\tilde{J}}$.

LEMMA 6.1. *Let $(\beta, \alpha) \in \tilde{R}$. Then $\tilde{J}$ is invariant under $f_{\beta,\alpha}$, i.e. $f_{\beta,\alpha}(\tilde{J}) \subseteq \tilde{J}$. The piecewise linear map*

$$\tilde{h}(x) = \begin{cases} \dfrac{\beta x}{\beta - 1}, & \text{if } x \in \tilde{J}_0, \\ \dfrac{\beta x - 1}{\beta - 1}, & \text{if } x \in \tilde{J}_2, \end{cases} \tag{6.3}$$

is a bijection from $\tilde{J}_0$ to $[0,1)$ and also from $\tilde{J}_2$ to $[0,1)$. The restricted map $f_{\beta,\alpha}|_{\tilde{J}}$ satisfies

$$\tilde{h} \circ f_{\beta,\alpha}|_{\tilde{J}} = f_{\bar{\beta},\bar{\alpha}} \circ \tilde{h}. \tag{6.4}$$

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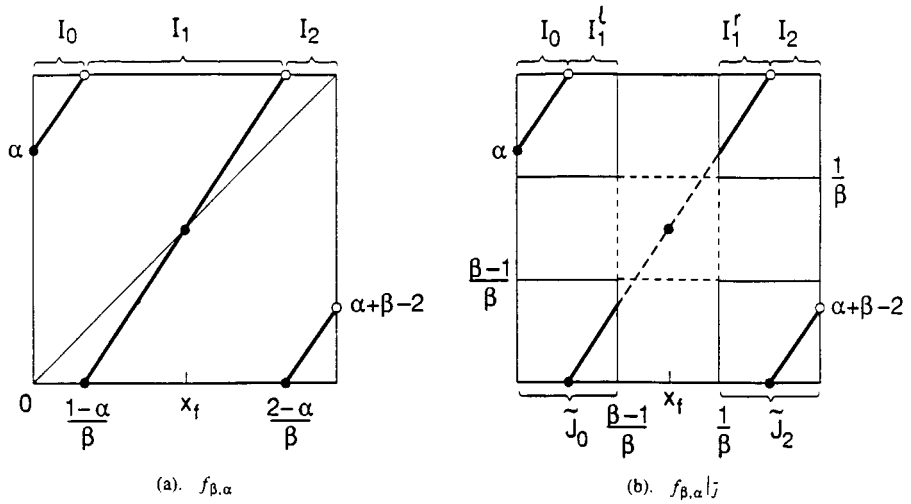


FIGURE 5. Graphs of $f_{\beta,\alpha}$ and $f_{\beta,\alpha}|_{\tilde{J}}$, where $\tilde{J} = \tilde{J}_0 \cup \tilde{J}_2$.

Remark. This lemma should be contrasted with Lemma 5.1 of part I which produced an invariant set $J \subseteq [0, 1]$ and a bijection $h : J \to [0, 1)$ such that $h \circ f_{\beta,\alpha}|_J = f_{\bar{\beta},\bar{\alpha}} \circ h$, whence $f_{\beta,\alpha}|_J$ is conjugate to $f_{\beta,\alpha}$. Here the map $\tilde{h}$ is two-to-one, whence $f_{\bar{\beta},\bar{\alpha}}$ is a factor of $f_{\beta,\alpha}|_{\tilde{J}}$. Use of the two-to-one factor map facilitates keeping track of symbolic dynamics between $f_{\beta,\alpha}$ and $f_{\bar{\beta},\bar{\alpha}}$. We note that $J \subseteq \tilde{J}$.

Proof. Let $f := f_{\beta,\alpha}$. The condition $1/\beta \leq \alpha \leq 3 - \beta - (1/\beta)$ for $(\beta, \alpha) \in \tilde{R}$ yields

$$\alpha + \beta - 2 \leq \frac{\beta - 1}{\beta} \leq x_f = \frac{1 - \alpha}{\beta - 1} \leq \frac{1}{\beta} \leq \alpha. \tag{6.5}$$

Also, since $(\beta - 1)/\beta$ and $1/\beta$ then lie in the interior of I_1,

$$f\left(\frac{\beta - 1}{\beta}\right) = \alpha + \beta - 2, \quad f\left(\frac{1}{\beta}\right) = \alpha. \tag{6.6}$$

Now define the intervals

$$I_1^l := \left[\frac{1 - \alpha}{\beta}, \frac{\beta - 1}{\beta}\right), \quad I_1^r := \left[\frac{1}{\beta}, \frac{2 - \alpha}{\beta}\right). \tag{6.7}$$

which satisfy

$$I_1^r, I_1^l \subseteq I_1.$$

Then I_0 and I_l^r are contiguous, and so are I_1^r and I_2. Also,

$$\tilde{J}_0 = I_0 \cup I_1^l, \quad \tilde{J}_2 = I_1^r \cup I_2 \tag{6.8}$$

and

$$\lambda(I_0) = \lambda(I_1^r) = \frac{1 - \alpha}{\beta} \quad \text{and} \quad \lambda(I_1^l) = \lambda(I_2) = \frac{\alpha + \beta - 2}{\beta}.$$

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where λ denotes Lebesgue measure. These facts are indicated in Figure 5(b). Next, (6.5) and (6.6) yield

$$f(I_0) = f(I_1^r) = [\alpha, 1) \subseteq \tilde{J}_2 \tag{6.9a}$$

$$f(I_2) = f(I_1^l) = [0, \alpha + \beta - 2) \subseteq \tilde{J}_0. \tag{6.9b}$$

These facts imply $f(\tilde{J}) \subseteq \tilde{J}$.

The bijective property of $\tilde{h}$ follows directly from definition (6.3), and relation (6.4) can then be verified by a direct calculation.

We now describe an alternative 'cut-and-paste' construction leading from $f_{\beta,\alpha}$ to $f_{\bar{\beta},\bar{\alpha}}$ that makes transparent the appearance of the map $\tilde{h}(x)$ and relation (6.4). The map $f_{\beta,\alpha}|_{\tilde{J}}$ in Figure 5(b) consists of four line segments contained in the four 'corner' squares with side length equal to $(\beta-1)/\beta$. The graphs in the two upper 'corner' squares are identical, as are those in the two lower 'corner' squares. The four squares are glued together and then scaled by a factor $\beta/(\beta-1)$, to obtain the graph of a function $\tilde{f} : [0, 2) \to [0, 2)$ depicted in Figure 6(a).

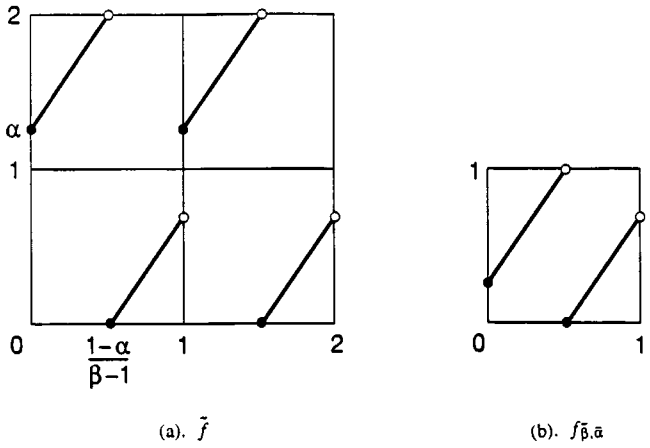


FIGURE 6. Graphs of $\tilde{f}$ and $\tilde{g} = f_{\bar{\beta},\bar{\alpha}}$.

Now $\tilde{f}(x+1) = \tilde{f}(x)$, hence $\tilde{f}(x)$ has the factor map $\tilde{g}$ derived from it by identifying x and $x+1$, for $x \in [0, 1)$, depicted in Figure 6(b). We claim that $\tilde{g} = f_{\bar{\beta},\bar{\alpha}}$. The graph of $\tilde{g}$ is obtained by superposing the left upper corner of Figure 6(a) on top of the left lower corner, so that $\tilde{g}$ is a linear mod one transformation consisting of two line segments of $\tilde{g} = f_{\bar{\beta},\bar{\alpha}}$, where $1 < \bar{\beta} + \bar{\alpha} \leq 2$. It remains to determine $\bar{\beta}$ and $\bar{\alpha}$. Clearly $\bar{\beta} = \beta$, and $\bar{\alpha}$ is determined by computing the turning point $\bar{\gamma}$ in two ways. First, by definition $\bar{\gamma} = (1 - \bar{\alpha})/\bar{\beta}$, and tracing it through the factor map gives $\bar{\gamma} = (1 - \alpha)/(\beta - 1)$. Equating these expressions and using $\bar{\beta} = \beta$, yields $\bar{\alpha} = (\beta\alpha - 1)/(\beta - 1)$. Finally, tracing through the variable changes in the construction gives (6.4), where $\tilde{h}(x)$ is given by (6.3). □

We now use Lemma 6.1 to relate the symbolic dynamics of $f_{\beta,\alpha}$ and $f_{\bar{\beta},\bar{\alpha}}$. Note first that $f_{\beta,\alpha}(\tilde{J}) \subseteq \tilde{J}$ implies that the iterates $f_{\beta,\alpha}^n(0)$ and $f_{\beta,\alpha}^n(1^-)$ remain inside $\tilde{J}$. Let

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$0, 1, 2$ denote the symbols used in the symbolic dynamics for $f_{\beta,\alpha}$, and let $0, 1$ denote the symbols for $f_{\bar\beta,\bar\alpha}$. We first treat the case that $(\beta, \alpha) \in \mathrm{Int}(\tilde R)$.

LEMMA 6.2. *If $(\beta, \alpha) \in \mathrm{Int}(\tilde R)$, then $f_{\beta,\alpha}$ has the itineraries*

$$i(0) = 0\ 1^{j_1}\ 2\ 1^{j_2}\ 0\ 1^{j_3} \ldots, \tag{6.10a}$$

$$i(1^-) = 2\ 1^{k_1}\ 0\ 1^{k_2}\ 2\ 1^{k_3} \ldots, \tag{6.10b}$$

where $\{j_i\}$ and $\{k_i\}$ are infinite sequences of integers ≥ 0. The corresponding itineraries for $(\bar\beta, \bar\alpha)$ are

$$\bar i(0) = 0^{1+j_1}\ 1^{1+j_2}\ 0^{1+j_3} \ldots, \tag{6.11a}$$

$$\bar i(0) = 1^{1+k_1}\ 0^{1+k_2}\ 1^{1+k_3} \ldots. \tag{6.11b}$$

Remark. The symbols in $i(0)$, $i(1^-)$ consist of an alternating pattern of 0's and 2's with blocks of 1's interlaced between them. Also, $\bar i(0)$ is obtained from $i(0)$ by replacing each block $0\ 1^{j_n}$ with 0^{1+j_n}, and each block $2\ 1^{j_n}$ with 1^{1+j_n}; $\bar i(1^-)$ is obtained from $i(1^-)$ similarly.

Proof. For $x \in \tilde J$, let $\bar i(x)$ be the itinerary of x with respect to $f|_{\tilde J}$ for the partition $\mathcal{P} = \{I_0, I_1^l, I_1^r, I_2\}$, where $\mathcal{A} = \{0, 1_l, 1_r, 2\}$ is the symbol alphabet for $\mathcal{P}$. Thus $\tilde i(x)$ is a sequence of symbols from $\mathcal{A}$. For $a, b \in \mathcal{A}$, let $a \to b$ indicate that $f(I_a) \cap I_b \neq \emptyset$, and let $a \nrightarrow b$ indicate that $f(I_a) \cap I_b = \emptyset$. Now (6.8) and (6.9) yield

$$0,\ 1_r \nrightarrow 0,\ 1_l \quad \text{and} \quad 2,\ 1_l \nrightarrow 2,\ 1_r. \tag{6.12}$$

Now, as long as $x, f(x), \ldots, f^n(x) \in I_1$, we have

$$f^{n+1}(x) - x_f = \beta^n (x - x_f). \tag{6.13}$$

If $(\beta, \alpha) \in \mathrm{Int}(\tilde R)$, then $x_f \notin \tilde J$, hence from (6.13) we conclude that the f-iterates of any $x \in \tilde J$ must exit from I_1^l after finitely many iterations, and similarly for I_1^r. We conclude from this and (6.12) that

$$\tilde i(0) = 0\ 1_r^{j_1} 2\ 1_l^{j_2} 0\ 1_r^{j_3} \ldots, \tag{6.14a}$$

$$\tilde i(1^-) = 2\ 1_l^{k_1} 0\ 1_r^{k_2}\ 0\ 1_l^{k_3} \ldots. \tag{6.14b}$$

Now $i(0)$ and $i(1^-)$ are obtained from $\tilde i(0)$ and $\tilde i(1^-)$ simply by replacing 1_l and 1_r by 1, hence (6.10) follows.

To continue, let

$$\bar\gamma := \frac{1 - \bar\alpha}{\bar\beta} = \frac{1 - \alpha}{\beta - 1},$$

and define the partition

$$\bar I_0 := [0, \bar\gamma), \quad \bar I_1 := [\bar\gamma, 1)$$

of $[0, 1)$. From (6.3) we get

$$\tilde h(I_0) = \tilde h(I_1^r) = \bar I_0; \quad \tilde h(I_2) = \tilde h(I_1^l) = \bar I_1. \tag{6.15}$$

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Given $x \in \tilde{J}$, its image $y := \tilde{h}(x)$ lies in $[0, 1)$. Let $\bar{i}(y)$ denote the $f_{\bar{\beta},\bar{\alpha}}$-itinerary of y with respect to the partition $\bar{\mathcal{P}} := \{\bar{I}_0, \bar{I}_1\}$. From (6.4) and (6.15) it follows that $\bar{i}(y)$ is obtained from $\tilde{i}(x)$ using the symbol substitutions

$$0,\ 1_r \to 0 \quad \text{and} \quad 2,\ 1_l \to 1. \tag{6.16}$$

Since $\tilde{h}(0) = 0$ and $\tilde{h}(1) = 1$, we obtain the itineraries (6.11) from (6.14) via this substitution. □

There remains the case when $(\beta, \alpha) \in \partial\tilde{R}$. We partition the boundary $\partial\tilde{R}$ as

$$\partial\tilde{R} := \partial\tilde{R}_l \cup \partial\tilde{R}_r \cup \{(2, \tfrac{1}{2})\},$$

in which

$$\partial\tilde{R}_l := \left\{(\beta, \alpha) : \alpha = \frac{1}{\beta}, 1 < \beta < 2\right\},$$

is the left boundary of $\tilde{R}$, and

$$\partial\tilde{R}_r := \left\{(\beta, \alpha) : \alpha = 3 - \beta - \frac{1}{\beta},\ 1 < \beta < 2\right\},$$

is the right boundary of $\tilde{R}$, and $(2, \frac{1}{2})$ is the 'tip' of $\tilde{R}$. In all these cases $x_f \in \tilde{J}$, with

$$x_f = \frac{1}{\beta} = \alpha \quad \text{for } (\beta, \alpha) \in \partial\tilde{R}_l \cup \{(2, \tfrac{1}{2})\},$$

$$x_f = \frac{\beta - 1}{\beta} = \alpha + \beta - 2 \quad \text{for } (\beta, \alpha) \in \partial\tilde{R}_r \cup \{(2, \tfrac{1}{2})\}.$$

Consequently, one of the itineraries $i(0)$ or $i(1^-)$ has only finitely many symbols 0 or 2, and this necessitates a separate treatment.

LEMMA 6.3.

(a) *If* $(\beta, \alpha) \in \partial R_1$, *then*

$$i(0) = 0\ 1^\infty \tag{6.17a}$$

$$i(1^-) = 2\ 1^{k_1}0\ 1^{k_2}2\ 1^{k_3} \ldots. \tag{6.17b}$$

The k_i*'s are either an infinite sequence of non-negative integers, or a finite sequence of odd length in which case* $i(1^-)$ *ends in* $0\ 1^\infty$. *The corresponding itineraries for* $(\bar{\beta}, \bar{\alpha})$ *are*

$$\bar{i}(0) = 0^\infty \tag{6.18a}$$

$$\bar{i}(1^-) = 1^{1+k_1}0^{1+k_2}1^{1+k_3} \ldots. \tag{6.18b}$$

(b) *If* $(\beta, \alpha) \in \partial R_2$, *then*

$$i(0) = 0\ 1^{j_1}2\ 1^{j_2}0\ 1^{j_3} \ldots \tag{6.19a}$$

$$i(1^-) = 2\ 1^\infty. \tag{6.19b}$$

The j_i*'s are either an infinite sequence of non-negative integers, or a finite sequence of odd length in which case* $i(0)$ *ends in* $2\ 1^\infty$. *The corresponding itineraries for* $(\bar{\beta}, \bar{\alpha})$ *are*

$$\bar{i}(0) = 0^{1+j_1}1^{1+j_2}0^{1+j_3} \ldots \tag{6.20a}$$

$$\bar{i}(1^-) = 1^\infty. \tag{6.20b}$$

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(c) *If* $(\beta, \alpha) = (2, \frac{1}{2})$, *then*

$$i(0) = 0\ 1^{\infty}; \quad \bar{i}(0) = 0^{\infty},$$
$$i(1^-) = 2\ 1^{\infty}; \quad \bar{i}(1^-) = 1^{\infty}.$$

Proof. Similar to Lemma 6.2. Note in (a) that $i(1^-) < 2\ 1^{\infty}$. A similar remark applies in (b). □

Proof of Theorem 6.1. Let $\{r_l : l \geq 0\}$ and $\{s_l : l \geq 0\}$ denote the integers defined by (2.2), for the map $f_{\beta,\alpha}$, and let $\{\bar{r}_l : l \geq 0\}$ and $\{\bar{s}_l : l \geq 0\}$ denote the corresponding integers for $f_{\bar{\beta},\bar{\alpha}}$. To show that the state graphs $S_{\beta,\alpha}$ and $S_{\bar{\beta},\bar{\alpha}}$ are identical except for the loop at $(0, 0)$, is suffices to prove that

$$r_l = \bar{r}_l \quad \text{and} \quad s_l = \bar{s}_l,\ l \geq 1,$$

because, by Proposition 2.1, these determine all 'cross-edges' of $S_{\beta,\alpha}$, respectively $S_{\bar{\beta},\bar{\alpha}}$. Since $2 < \alpha + \beta \leq 3$ while $1 < \bar{\alpha} + \bar{\beta} \leq 2$, we conclude that $S_{\beta,\alpha}$ contains the loop $(0, 0) \to (0, 0)$, while $S_{\bar{\beta},\bar{\alpha}}$ does not.

Let $i(0) = \{A_n^*\}$, $i(1^-) = \{C_n^*\}$, $\bar{i}(0) = \{\bar{A}_n^*\}$, and $\bar{i}(0) = \{\bar{A}_n^*\}$, $\bar{i}(1^-) = \{\bar{C}_n^*\}$. We use the prescription of Hofbauer [**7**, p. 93] for computing $\{r_l : l \geq 1\}$. We partition $\{A_n^* : n \geq 1\}$ into blocks $\{A_{R_{l-1}+1}^*, A_{R_{l-1}+2}^* \ldots, A_{R_l}^*\}$ for $l \geq 1$, and the sequence $\{r_l : l \geq 1\}$ is determined by the properties

$$\begin{cases} A_{R_{l-1}+i}^* = C_{i-1}^* & \text{for } 1 \leq i \leq r_l - 1, \\ A_{R_l}^* = A_{R_{l-1}+r_l}^* \neq C_{r_l-1}^*. \end{cases} \tag{6.21}$$

We now prove that $\bar{r}_l = r_l$ holds for $l \geq 0$ by induction on l. By definition $\bar{r}_0 = r_0 = 1$. Suppose the equality holds up to r_{l-1}, and consider r_l. Since $\{A_{n+k}^*\} < \{C_n^*\}$ lexicographically (see Hofbauer [**5**, Theorem 2]) for any $k \geq 0$, we have

$$\{A_{R_{l-1}+1}^*, \ldots, A_{R_l}^*\} \leq \{C_0^*, C_1^*, \ldots, C_{r_l-1}^*\}. \tag{6.22}$$

From (6.21) and (6.22) we get

$$A_{R_l}^* < C_{r_l-1}^*,$$

which implies $A_{R_l}^* = 0$ or 1. We proceed by cases.

Case (i). $r_l \geq 2$ *and* $A_{R_l}^* = 1$. Then $C_{r_l-1}^* = 2$. Lemmas 6.2 and 6.3 give

$$\{A_{R_{l-1}+1}^*, \ldots, A_{R_l}^*\} = 2\ 1^{k_1} \ldots 0\ 1^{k_n+1}, \tag{6.23a}$$
$$\{C_0^*, \ldots, C_{r_l-1}^*\} = 2\ 1^{k_1} \ldots 0\ 1^{k_n} 2, \tag{6.23b}$$

for some $n \geq 2$, with $k_1, \ldots, k_n \geq 0$. From (6.23), combined with Lemmas 6.2 and 6.3, we conclude that (6.21) holds when all symbols A and C are replaced by $\bar{A}$ and $\bar{C}$, respectively. Thus $\bar{r}_l = r_l$ in this case.

Case (ii). $r_l \geq 2$ *and* $A_{R_l}^* = 0$. Then $C_{r_l-1}^* = 1$ or 2. Let i be the largest index satisfying $R_{l-1} \leq i < R_l$ such that $A_i^* = 2$. From (6.21), we obtain $C_i^* = 2$ and $C_{l+1}^* = \cdots = C_{r_l-2}^* = 1$. Since the 0's and 2's form an alternating pattern in $\{C_n^*\}$, we also conclude that $C_{r_l-1}^* = 1$. Again by Lemmas 6.2 and 6.3, we conclude that

$$\{A_{R_{l-1}+1}^*, \ldots, A_{R_l}^*\} = 2\ 1^{k_1} \ldots 2\ 1^{k_n} 0, \tag{6.24a}$$
$$\{C_0^*, \ldots, C_{r_l-1}^*\} = 2\ 1^{k_1} \ldots 2\ 1^{k_n+1}, \tag{6.24b}$$

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for some $n \geq 1$ and all $k_i \geq 0$. From (6.24), Lemmas 6.2 and 6.3, we conclude that (6.21) holds with all symbols A, C replaced by $\bar{A}$, $\bar{C}$, respectively. Thus $\bar{r}_l = r_l$ in this case.

Case (iii). $r_l = 1$. If $A^*_{R_l} = 0$, then Lemmas 6.2 and 6.3 give $\bar{A}^*_{R_l} = 0$ and $\bar{C}^*_0 = 1$, hence $\bar{r}_l = 1$. If $A^*_{R_l} = 1$, let i be the largest index with $i < R_l$ for which $A^*_i \neq 1$. We claim that $A^*_i = 0$. If $i = 0$ this is clear. If $i > 0$ and A^*_i is in a block with more than one element this follows from either (6.23a) or (6.24a). If $i > 0$ and $\{A^*_i\}$ is a block of one element, then $i = R_k$ for some k, and we conclude from $A^*_{R_k} < C^*_{r_{k-1}}$ that $A^*_i = A^*_{R_k} = 0$. Thus $A^*_i = 0$, and Lemmas 6.2 and 6.3 then yield $\bar{A}^*_{R_l} = 0$ and $\bar{C}^*_0 = 1$, hence $\bar{r}_l = 1$, completing this case.

This completes the induction step that $\bar{r}_l = r_l$ for $l \geq 0$.

We prove $s_l = \bar{s}_l$ for all $l \geq 0$ by a similar induction, using Hofbauer's prescription for computing $\{s_l : l \geq 1\}$.

Finally, to show that $N_{\beta,\alpha} = N_{\bar{\beta},\bar{\alpha}}$, it suffices to show that $S^{\max}_{\beta,\alpha}$ and $S^{\max}_{\bar{\beta},\bar{\alpha}}$ do not contain the vertex $(0, 0)$. This is trivially true for $S^{\max}_{\bar{\beta},\bar{\alpha}}$, and the condition $(0, 0) \notin S^{\max}_{\beta,\alpha}$ is satisfied if and only if (β, α) is in the 'big bubble', see† Hofbauer [**7**, p. 111]. □

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Appendix A. Relations between 'bubbles' and Hofbauer regions $W(k, k_1, \ldots, k_m, k)$.

In studying the support of the maximal measure of $f_{\beta,\alpha}$, Hofbauer [**7**, p. 107] defines sets $W(k, k_1, \ldots, k_m, k)$ consisting of all pairs (β, α) such that:

(i) $f_{\beta,\alpha}$ has itineraries‡

$$i(0) = 0^{k-1}10^{k_2-1}1 \ldots 0^{k_m-1}10^k1 \ldots$$
$$i(1^-) = 10^{k-2}10^{k_2-1}1 \ldots 0^{k_m-1}10^{k-1}1 \ldots$$

with each $k_i = k$ or $k - 1$ for $2 \leq i \leq m$;

(ii) $\tilde{S}_{\beta,\alpha}$ has a head cycle of type $[p, q]$ with $p + q = k + k_2 + \cdots + k_m + k - 1$ such that $P0Q0 = 0^{k-1}10^{k_2-1}1 \ldots 0^{k_m-1}10^{k-1}$;

(iii) this head cycle is *not* contained in the maximal component of $\tilde{S}_{\beta,\alpha}$;

(iv) $1 < \beta < 2^{1/(p+q)}$.

The condition (i) above actually determines the type $[p, q]$ of the head cycle uniquely (if one exists), because (ii) shows that the number of zeros in $P0Q$ is $l = k + k_2 + \cdots + k_m + k - m - 2$, whence Lemma 4.4 now gives

$$p \equiv 1/(k + k_2 + \cdots + k_m + k - m - 2) \pmod{p + q}. \tag{A.1}$$

We have the following lemma.

† Also compare equation (3.8) of Hofbauer [**7**] with Lemma 5.1, after noting that Hofbauer's symbols $1, 2, 3$ correspond to our symbols $0, 1, 2$ respectively.

‡ See Hofbauer [**7**, equation (2.9)].

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LEMMA A.1. *The region $W(k, k_1, \ldots, k_m, k)$ is either the empty set or else is exactly the bubble $R_{N,K}$ with its tip removed, where $N = p + q$ and*

$$K \equiv -1/(k + k_1 + \cdots + k_m + k - m - 2) \pmod{p + q}.$$

All 'bubbles' $R_{N,K}$ arise in this way, from a unique $W(k, k_1, \ldots, k_m, k)$.

Proof. The parameters (N, K) for the 'bubble' $R_{N,K}$ are deduced from Theorem 4.3 from those of (p, q) given above. Each $W(k, k_1, \ldots, k_m)$ is contained in that 'bubble' $R_{N,K}$ determined from $[p, q]$ in (ii) above. The union of the $W(k, k_1, \ldots, k_m, k)$ exhausts the union of all $R_{N,K}$ excluding their tip; see Proposition 3.1 of part (i). (The tip of $R_{N,K}$ has $\beta = 2^{1/N} = 2^{1/p+q}$, and is excluded by condition (iv).) Thus all pairs (N, K) with $(N, K) = 1$ and $1 \leq K < N$ occur associated to the $W(k, k_1, \ldots, k_n)$. However, each pair (N, K) occurs exactly once, because (N, K) determines $[p, q]$, and $[p, q]$ determines the itinerary $i(0)$, by Lemma 3.4, sufficiently to uniquely specify $(k, k_1, \ldots, k_m, k)$. Thus that region $W(k, k_1, \ldots, k_m, k)$ must be exactly $R_{N,K}$ minus its tip. □

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