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A VARIATION ON SELBERG'S APPROXIMATION PROBLEM

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Abstract. Let $\alpha \in \mathbb{C}$ in the upper half-plane and let I be an interval. We construct an analogue of Selberg's majorant of the characteristic function of I that vanishes at the point α. The construction is based on the solution to an extremal problem with positivity and interpolation constraints. Moreover, the passage from the auxiliary extremal problem to the construction of Selberg's function with vanishing is easily adapted to provide analogous "majorants with vanishing" for any Beurling–Selberg majorant.

§1. *Introduction.* In the 1970s, Selberg [**31**, **34**] introduced a useful tool for proving inequalities at the interface of Fourier analysis and number theory. Given a real number $\delta > 0$ and an interval $I \subset \mathbb{R}$, he constructed an integrable function $C : \mathbb{R} \to \mathbb{R}$ that satisfies:

(1) $C(x) \geqslant \chi_I(x)$;
(2) $\hat{C}(\xi) = 0$ whenever $|\xi| > \delta$; and
(3) $\int_{-\infty}^{\infty} C(x)\,dx = \text{Length}(I) + \delta^{-1}$;

where $\hat{C}(\xi)$ is the Fourier transform of $C(x)$ (see §2). It is not difficult to show, and we will see this below, that (2) implies $C(x)$ is the restriction to $\mathbb{R}$ of an entire of *exponential type*. The last property (3) demonstrates that $C(x)$ is a good approximation to $\chi_I(x)$ in L^1-norm. In fact, among all integrable functions satisfying conditions (1) and (2) above, the integral appearing in (3) is minimal if, and only if, $\text{Length}(I)\delta \in \mathbb{Z}$. In unpublished work, Logan [**27**] found the extremal majorant in the case when $\text{Length}(I)\delta \notin \mathbb{Z}$ and has shown that the corresponding extremal majorant is unique. This result has been realized again in the work [**25**] of Littmann, presumably using different methods.

In this paper we study the following variation of this problem suggested to us by E. Bombieri:

> *Let $\alpha \in \mathbb{C}$ be a point in the upper half-plane. Construct an analogue of Selberg's majorant that vanishes at α.*

The motivation for this problem comes primarily from the study of L-functions. To illustrate, suppose that $L(s)$ is an L-function that fails the Riemann hypothesis. This means the function $z \mapsto L(1/2 + \mathrm{i}z)$ has a zero at $z = \alpha$, where $0 < \text{Im}(\alpha) < 1/2$. Suppose further that we wish to use Selberg's majorant in the explicit formula for $L(s)$. We would rather not have the terms in the

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explicit formula that involve α, so it would be desirable to have an analogue of Selberg's function that vanishes at α. Applying such a function in the explicit formulas would cause the terms involving α to vanish, but the information is not lost as α is encoded in the function itself.

We do not determine the extremal majorant with the extra vanishing condition in this paper. Rather, we solve a different extremal problem that allows us to produce a class of majorants that are *good* approximations *and* satisfy vanishing conditions. Our method allows us to obtain non-trivial bounds on the quantity

$$\rho(\alpha, I, \delta) = \inf \int_{-\infty}^{\infty} \{G(t) - \chi_I(t)\}\, dt$$

where the infimum is taken over entire functions $G(z)$ that satisfy:

(i) $G(t) \geqslant \chi_I(t)$ for all $t \in \mathbb{R}$;
(ii) $\hat{G}(\xi) = 0$ whenever $|\xi| > \delta$; and
(iii) $G(\alpha) = 0$.

Our first result concerns non-trivial bounds on $\rho(\alpha, I, \delta)$.

THEOREM 1. *Let $\delta > 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Im}(\alpha) > 0$, and $I \subset \mathbb{R}$ be an interval. Then*

$$\delta^{-2} \ll \rho(\alpha, I, \delta) \ll \delta^{-3} \tag{1.1}$$

as $\delta \to 0$, and

$$\rho(\alpha, I, \delta) \approx \delta^{-1} \tag{1.2}$$

as $\delta \to \infty$. The implied constants are effective and depend only on α and I.

The upper bounds in this theorem are obtained by constructing an entire function $G(z)$ satisfying the conditions (i)–(iii) above. This construction is based on the solution to the following extremal problem.

Problem. Let $\delta > 0$, $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Im}(\alpha) > 0$. Determine the value of

$$\kappa(\alpha, \beta, \delta) = \inf \int_{-\infty}^{\infty} F(x)\, dx \tag{1.3}$$

where the infimum is taken over continuous functions $F : \mathbb{C} \to \mathbb{C}$ with the following properties:

(i) $F(x)$ is real valued and integrable on $\mathbb{R}$;
(ii) $F(x) \geqslant 0$ for each x in $\mathbb{R}$;
(iii) $F(\alpha) = \beta$;
(iv) $\hat{F}(\xi)$ is supported in the interval $[-\delta, \delta]$.

In addition to finding the minimal integral, find explicit extremal functions for which the minimal integral is obtained.

We say a function $F(z)$ is admissible for κ if it satisfies the conditions in the above problem. Our main result is the solution to this problem.

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THEOREM 2. *Let $\delta > 0$, $\alpha \in \mathbb{C}$ be a point in the upper half-plane, and $\beta \in \mathbb{C}$. Then:*

(i) $\kappa(\alpha, \beta, \delta) = |\beta|\kappa(\alpha, \beta/|\beta|, \delta)$;
(ii) $\kappa(\alpha, \beta, \delta) = r^{-1}\kappa(r\alpha, \beta, r\delta)$ *for each* $r > 0$;
(iii) $\kappa(\alpha, \beta, \delta) = \kappa(\alpha + t, \beta, \delta)$ *for each* $t \in \mathbb{R}$; *and*
(iv)

$$\frac{\kappa(\alpha, \beta, \delta)}{2} = \frac{|\beta|K(\alpha, \alpha) - \delta\mathrm{Re}(\beta)}{K(\alpha, \alpha)^2 - \delta^2},$$

where $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is given by

$$K(\omega, z) = \frac{\sin \pi\delta(z - \overline{\omega})}{\pi(z - \overline{\omega})}.$$

The infimum in (1.3) is achieved for a unique admissible function $F(z) = U(z)\overline{U(z)}$ where

$$U(z) = \lambda_1 K(\alpha, z) + \lambda_2 K(\overline{\alpha}, z),$$

and λ_1 and λ_2 are given by

$$\lambda_1 = \frac{\beta K(\alpha, \alpha) - \delta}{K(\alpha, \alpha)^2 - \delta^2} \quad \text{and} \quad \lambda_2 = \frac{K(\alpha, \alpha) - \beta\delta}{K(\alpha, \alpha)^2 - \delta^2}. \tag{1.4}$$

Remark. If $\alpha = x + \mathrm{i}y$ and $\beta = b$ is real, then the formula in (iv) becomes

$$\kappa(x + \mathrm{i}y, b, \delta) = \frac{4\pi y|b|}{\sinh(2\pi y\delta) + \mathrm{sgn}(b)2\pi y\delta}.$$

We will be primarily concerned with the case when $b = -1$.

Let $F(z; \alpha, \beta)$ be the unique extremal function from Theorem 2, note that by using this notation we suppress the dependence of F on δ. Consider the following modification of Selberg's function

$$z \mapsto G_\alpha(z) = C(z) + F(z; \alpha, -C(\alpha)). \tag{1.5}$$

Seeing that $F(x; \alpha, \beta) \geqslant 0$ for real x, it follows that $C(x) \leqslant G_\alpha(x)$ for each real x. In addition, $G_\alpha(\alpha) = 0$, and $\hat{G}_\alpha(\xi) = 0$ whenever $|\xi| > \delta$. So $G_\alpha(z)$ is an admissible function for $\rho(\alpha, I, \delta)$.

There is a philosophical problem with $G_\alpha(z)$, namely that you require knowledge of the value $C(\alpha)$ to define it, and this is a value we may be trying to avoid. Here is a way to get around this issue. Define a new function $C_\alpha(z)$ by

$$z \mapsto C_\alpha(z) = C(z)(1 + F(z; \alpha, -1)). \tag{1.6}$$

This function is admissible for $\rho(\alpha, I, 2\delta)$ and has the virtue that it does not depend on $C(\alpha)$. Moreover, we can produce a majorant that vanishes at many distinct points by repeating this procedure many times. Indeed, for any $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ in the upper half-plane the function

$$z \mapsto C(z) \prod_{n=1}^{N} (1 + F(z; \alpha_n, -1)) \tag{1.7}$$

is admissible for $\rho(\alpha_n, I, N\delta)$ for $n = 1, \ldots, N$.

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The reader may note that the function $C(z)$ plays a rather passive role in this game. If $M(z)$ is an entire function such that $M(x) \geqslant m(x)$ for each real x, then

$$z \mapsto M(z) + F(z; \alpha, -M(\alpha))$$

vanishes at α while remaining a majorant of $m(x)$. The problem of determining the extremal majorants $M(z)$ of a function $m(x)$ is well-studied (see [**1**, **5**–**9**, **12**, **13**, **18**, **28**, **34**]). The above modification for prescribed vanishing applies to the class of functions studied in this collection of papers. So the virtue of this construction is in its adaptability. We know of two cases where extremal majorants with a vanishing constraint have been determined: (1) Vaaler (J. Vaaler, personal communication, 2011) has determined the best majorant of the Kronecker delta function on $\mathbb{R}$ that vanishes at a point α in the upper half-plane; and (2) Littmann–Spanier [**26**] have determined the best majorant of the signum function that vanishes at a prescribed point on the imaginary axis. It would be interesting to determine extremal majorants with several vanishing constraints, even for the Kronecker delta function. Such a majorant would have applications in the theory of L-functions and their related objects.

1.1. *Organization of paper.* Before we jump into the proofs of our main results, we review some basic information about entire functions of exponential type. Then in §3 we prove Theorem 2 and compute the Fourier transform of the extremal function $F(z; \alpha, \beta)$. Section 4 contains an analysis of the extremal function from Theorem 2 in the special case when α is purely imaginary and $\beta = -1$. In §5, we determine a zero free region for Selberg's majorant and prove Theorem 1. Finally, in the last section we discuss minorants, and generalizations of Theorem 2 in de Branges spaces. Here we also discuss Theorem 16, a result in de Branges space that may be of independent interest. Namely, if $E(z)$ is a de Branges function of bounded type, then we give necessary and sufficient conditions for a de Branges reproducing kernel $K_E(\alpha, z)$ to be linearly independent with $K_E(\overline{\alpha}, z)$.

§2. *Background.* Throughout this paper x will be an element of the real numbers $\mathbb{R}$, z will be an element of the complex numbers $\mathbb{C}$, and $\overline{z}$ is the complex conjugate of z. Here $\mathscr{U} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ will denote the upper half-plane of $\mathbb{C}$, where $\mathrm{Im}(z)$ is the imaginary part of z. If $F(z)$ is an entire function, the complex conjugate $F^*(z)$ of $F(z)$ defined by $F^*(z) = \overline{F(\overline{z})}$ is also an entire function. If $I \subset \mathbb{R}$ is an interval, then $\chi_I(t)$ will denote the characteristic function of I. If $F(z) = F^*(z)$, then we say that $F(z)$ is *real entire*. An integrable function $F(x)$ will be called *admissible* if $F(x)$ satisfies the conditions of the extremal problem under consideration (not necessarily extremality). Here $F(x)$ will be called *extremal* if it is admissible and achieves the extreme value defined by the extremal problem. If $f(t)$ is square integrable and continuous function on $\mathbb{R}$, then the Fourier transform $\hat{f}(\xi)$ is defined by

$$\hat{f}(\xi) = \lim_{T \to \infty} \int_{-T}^{T} \mathrm{e}^{-2\pi \mathrm{i} t \xi} f(t)\, dt.$$

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We extend the definition of the Fourier transform to all suitably nice functions in the usual way.

Suppose that $F : \mathbb{R} \to \mathbb{C}$ is a integrable and $\hat{F}(\xi) = 0$ whenever $|\xi| > \delta$

$$z \mapsto \int_{-\delta}^{\delta} e(z \cdot \xi)\hat{F}(\xi)\, d\xi \tag{2.1}$$

is equal to $F(x)$ for almost every x in $\mathbb{R}^N$ and for each closed curve γ in $\mathbb{C}$ we have

$$\int_{\gamma}\int_{-\delta}^{\delta} e(z\xi)\hat{F}(\xi)\, d\xi\, dz = \int_{-\delta}^{\delta}\int_{\gamma} e(z\xi)\, dz\hat{F}(\xi)\, d\xi = 0$$

where we have used Fubini's theorem to interchange the order of integration. Since the curve γ was arbitrary, equation (2.1) defines an entire function by Morera's theorem. Consequentially, $F(x)$ is almost everywhere equal to the restriction to $\mathbb{R}$ of an entire function. We will always identify admissible functions with their extensions to entire functions. The representation (2.1) shows that admissible functions satisfy the following growth estimate in $\mathbb{C}$

$$|F(z)| \ll_{\epsilon} \mathrm{e}^{2\pi(1+\epsilon)\delta\ \mathrm{Im}(z)} \tag{2.2}$$

for each $\epsilon > 0$. Entire functions that satisfy an estimate of the form (2.2) are called *entire functions of exponential type*. An entire function $F(z)$ is said to be of exponential type $2\pi\sigma > 0$ if

$$|F(z)| \ll_{\epsilon} \mathrm{e}^{2\pi\sigma(1+\epsilon)|z|}$$

for each $\epsilon > 0$. From the definition it is immediate that an entire function of exponential type is an entire function of order one, however, the converse does not hold. The Riemann Xi function is an example of an entire function of order one that is not of exponential type.

In this paper we are primarily concerned with entire functions of exponential type which are bounded on the real axis. Although this subject is well studied†, we will collect some relevant material here. Throughout the paper we will use the notation of Stein [**32**, **33**]. Here E_{σ} will denote the space of entire functions with exponential type at most σ, and B_{σ} will be the subspace of E_{σ} consisting of functions which are bounded on the real axis. For $1 \leqslant p \leqslant \infty$ we let L^p be the space of entire functions $F(z)$ such that

$$\|F\|_p = \begin{cases} \left\{\displaystyle\int_{-\infty}^{\infty} |F(x)|^p\right\}^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \displaystyle\sup_{-\infty<x<\infty} |F(x)| & \text{if } p = \infty, \end{cases}$$

is finite. The following classical theorem of Paley and Wiener gives two equivalent ways of looking at $L^2 \cap E_{\pi\sigma}$.

† We refer the interested reader to [**2**] for further information.

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THEOREM 3 (Paley–Wiener). *If $F \in L^2 \cap E_{\pi\sigma}$, then $\hat{F}(\xi) = 0$ if $|\xi| > \sigma/2$. Conversely, any function $F(x)$ that is square integrable on $\mathbb{R}$ and that satisfies $\hat{F}(\xi) = 0$ for $|\xi| > \sigma/2$ is almost everywhere equal to the restriction to the real axis of a function in $F \in L^2 \cap E_{\pi\sigma}$.*

The space of functions characterized in the Paley–Wiener theorem forms a Hilbert space, with respect to the $L^2(\mathbb{R})$ inner product $\langle \cdot, \cdot \rangle$, called the *Paley–Wiener space of type $\sigma/2$*. We will use the notation

$$\mathbf{H}_\sigma = L^2 \cap E_{2\pi\sigma} \cong \{ f \in L^2(\mathbb{R}) \mid \hat{f}(\xi) = 0 \text{ if } |\xi| > \sigma \}$$

when referring to this space. Note that $\mathbf{H}_r$ is the image of $L^2([-r, r])$ through the Fourier transform. This implies that the image of the standard basis through the Fourier transform

$$\{ e(n\xi/\sigma) \mid n \in \mathbb{Z} \} \to \left\{ \frac{\sin 2\pi\sigma(z - n/\sigma)}{\pi\sigma(z - n/\sigma)} \;\middle|\; n \in \mathbb{Z} \right\}$$

is an orthogonal basis for $\mathbf{H}_\sigma$. From this simple observation we obtain the following interpolation result: *for each $F \in H_\sigma$*,

$$F(z) = \sum_{n \in \mathbb{Z}} F(n/\sigma) \frac{\sin 2\pi\sigma(z - n/\sigma)}{\pi\sigma(z - n/\sigma)}$$

where the sum converges in $\mathbf{H}_\sigma$. It follows that an element of $\mathbf{H}_\sigma$ is completely determined by the values it takes on the lattice $\sigma^{-1}\mathbb{Z}$ or its translates. Another important property of $\mathbf{H}_\sigma$, that we will use extensively, is the reproducing kernel property: for every $F \in \mathbf{H}_\sigma$ and $\omega \in \mathbb{C}$,

$$|F(\omega)| \leqslant 2\sigma \|F\|_2. \tag{2.3}$$

This shows that evaluation is a continuous linear functional. It then follows from the Riesz representation theorem that evaluation can be realized as the inner product with an element of $\mathbf{H}_\sigma$. This element is called the reproducing kernel, and for $\mathbf{H}_\sigma$ it is given by

$$K(\omega, z) = \frac{\sin 2\pi\sigma(z - \overline{\omega})}{\pi(z - \overline{\omega})}. \tag{2.4}$$

So (2.3) implies that for each $F \in \mathbf{H}_\sigma$ and $\omega \in \mathbb{C}$,

$$F(\omega) = \langle F, K(\omega, \cdot) \rangle.$$

We close this section with several inequalities that we will use in what follows.

LEMMA 4 (Bernstein). *If $F \in B_\sigma$, then $\|F'\|_\infty \leqslant \sigma \|F\|_\infty$.*

Let L^p be the space of entire functions $F(z)$ for which $\|F\|_p$ is finite. It is known (see [2]) that if $F \in E_\sigma \cap L^p$ for $p \geqslant 1$, then $\|F'\|_p \leqslant \sigma \|F\|_p$. The following can be found in [2, p. 83].

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PROPOSITION 5. *If $f \in B_\sigma$, then*

$$|f(x+\mathrm{i}y)| \leqslant \|f\|_\infty \cosh \sigma y \tag{2.5}$$

for all real numbers x and y.

LEMMA 6 (Stein [32]). *If $f \in E_\sigma \cap L^p$, $1 \leqslant p < \infty$, then for a universal $c > 0$,*

$$\|f\|_\infty \leqslant c\sigma^{1/p}\|f\|_p.$$

§3. *Interpolation in the upper half-plane: Proof of Theorem 2.* In this section we prove Theorem 2. The proof has four steps.

(1) Reduce the L^1 variational problem to an L^2 variational problem by showing that each admissible function $F(z)$ can be factored as $F(z) = U(z)\overline{U(\overline{z})} = U(z)U^*(z)$.
(2) Realize that $U(z)$ is an element of the Paley–Wiener space $\mathbf{H}_{\delta/2}$, and that $U(z) = \langle U, K(z,\cdot)\rangle$ for any $z \in \mathbb{C}$.
(3) Reformulate the problem as minimizing $\|U\|_2^2$ subject to the condition

$$\langle U, K(\alpha,\cdot)\rangle\overline{\langle U, K(\overline{\alpha},\cdot)\rangle} = \beta.$$

This is just another way of writing $F(\alpha) = \beta$.
(4) Solve the problem in the previous step in a generic inner product space.

Steps (1) and (2) are performed in Proposition 7 and step (4) is performed in Lemma 8.

PROPOSITION 7. *Suppose that $F(z) \in \mathbf{H}_\delta$ is real valued and non-negative on the real axis and that $F(z)$ is not identically zero. Then there exists an entire function $U(z) \in \mathbf{H}_{\delta/2}$ such that $U(z)$ is zero free in $\mathscr{U}$ and $F(z) = U(z)U^*(z)$.*

Proof. Let $\{\omega_n \mid n = 1, 2, \ldots\}$ be the zeros of $F(z)$, listed with appropriate multiplicity, in the upper half-plane and let

$$B_N(z) = \prod_{n=1}^{N} \frac{1 - z/\overline{\omega_n}}{1 - z/\omega_n}.$$

We define a sequence of entire functions $F_N(z)$ by $F_N(z) = B_N(z)F(z)$. Each of the functions $F_N(z)$ is in $\mathbf{H}_\delta$ by the Paley–Wiener theorem. Note that $\|F\| = \|F_N\|$ for each N because $|B_N(x)| = 1$ for each $x \in \mathbb{R}$. Thus, by the Banach–Alaoglu theorem, the sequence $\{F_N(z)\}$ is weakly compact in H_δ. Since this space is a reproducing kernel space, it follows that $F_N(z) \to G(z)$ pointwise for a subsequence. Observe that $|B_N(z)| \geqslant 1$ if $z \in \mathscr{U}$. It follows that $G(z)$ is zero free in $\mathscr{U}$ and that $|G(t)| = |F(t)|$ for real t. This shows that $F(z)^2 = F(z)F^*(z) = G(z)G^*(z)$. In particular, the non-real zeros of $G(z)$ occur with even multiplicity.

Since $F(z)$ is real valued and non-negative on $\mathbb{R}$, the zeros of $G(z)$ occur with even multiplicity and so there is an entire function $U(z)$ for which $G(z) = U(z)^2$. Then $F(z)^2 = \{U(z)U^*(z)\}^2$ and since $F(z)$ is real valued and non-negative on $\mathbb{R}$ it follows that $F(z) = U(z)U^*(z)$. □

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LEMMA 8. *Let* $\mathbf{H}$ *be a complex vector space with inner product* $\langle \cdot, \cdot \rangle$, $\beta \in \mathbb{C}$. *Let* $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}$ *be linearly independent,* $\eta = \|\boldsymbol{u}\|\|\boldsymbol{v}\|$ *and* $\nu = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. *If*

$$\langle \mathbf{h}, \boldsymbol{u} \rangle \overline{\langle \mathbf{h}, \boldsymbol{v} \rangle} = \beta \tag{3.1}$$

then

$$\|\mathbf{h}\|^2 \geqslant 2\frac{|\beta|\eta - \mathrm{Re}(\beta\nu)}{\eta^2 - |\nu|^2}. \tag{3.2}$$

If $\mathbf{h}$ *satisfies (3.1), then* $\mathbf{h}$ *achieves equality in (3.2) if and only if*

$$\omega\mathbf{h} = \lambda_1 \boldsymbol{u} + \lambda_2 \boldsymbol{v} \tag{3.3}$$

for some $|\omega| = 1$, *where*

$$\lambda_1 = \frac{\gamma\beta\eta - |\nu|}{\eta^2 - |\nu|^2} \quad \textit{and} \quad \lambda_2 = \frac{\eta - \beta\nu}{\eta^2 - |\nu|^2}$$

and $\gamma = \nu/|\nu|$.

Proof. By scaling considerations, it suffices to prove the claim when $|\beta| = \|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$ and $\nu = \overline{\nu}$.

For any c_1 and c_2

$$\|\mathbf{h} - (c_1\boldsymbol{u} + c_2\boldsymbol{v})\|^2 \geqslant 0. \tag{3.4}$$

Expanding this out gives

$$2\mathrm{Re}\{\langle \mathbf{h}, c_1\boldsymbol{u} + c_2\boldsymbol{v} \rangle\} - \|c_1\boldsymbol{u} + c_2\boldsymbol{v}\|^2 \leqslant \|\mathbf{h}\|^2. \tag{3.5}$$

Equality occurs in (3.5) if and only if $\mathbf{h} = c_1\boldsymbol{u} + c_2\boldsymbol{v}$. We let $\mathbf{h}$ satisfy $\langle \mathbf{h}, \boldsymbol{u} \rangle \overline{\langle \mathbf{h}, \boldsymbol{v} \rangle} = \beta$ and set

$$c_1 = \frac{\beta - \nu}{1 - \nu^2}$$

and

$$c_2 = \frac{1 - \beta\nu}{1 - \nu^2}.$$

It can be checked that

$$\|c_1\boldsymbol{u} + c_2\boldsymbol{v}\|^2 = \frac{2 - 2\mathrm{Re}(\beta)\nu}{1 - \nu^2} = 2\mathrm{Re}(c_2). \tag{3.6}$$

Scaling $\mathbf{h}$ by a suitable constant of absolute value 1 we find that

$$\langle \mathbf{h}, \boldsymbol{u} \rangle = \beta/r \quad \text{and} \quad \langle \mathbf{h}, \boldsymbol{v} \rangle = r$$

for some $r > 0$. Thus,

$$2\mathrm{Re}\{\langle \mathbf{h}, c_1\boldsymbol{u} + c_2\boldsymbol{v} \rangle\} = 2\left\{\frac{\mathrm{Re}(c_1\overline{\beta})}{r} + \mathrm{Re}(c_2)r\right\}.$$

Seeing that $\mathrm{Re}(c_1\overline{\beta}) = \mathrm{Re}(c_2)$, we have

$$\left(\frac{1}{r} + r - 1\right)2\mathrm{Re}(c_2) \leqslant \|\mathbf{h}\|^2. \tag{3.7}$$

This completes the proof. □

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Proof of Theorem 2. If $F(z)$ satisfies

$$\int_{-\infty}^{\infty} F(x)\,dx < \infty,$$

then $F \in L^1 \cap E_{2\pi\delta}$ and, by Lemma 6, $F \in B_{2\pi\delta}$. It follows that $F \in L^1 \cap L^\infty \cap E_{2\pi\delta}$ which in turn implies that $F \in \mathbf{H}_\delta$. It follows from Proposition 7 that there exists a function $U \in \mathbf{H}_{\delta/2}$ such that $F(z) = U(z)U^*(z)$. Using this factorization we can write

$$\kappa(\alpha, \beta, \delta) = \inf \|U\|_2^2$$

where the infimum is taken over functions $U \in \mathbf{H}_{\delta/2}$ such that

$$\langle U, K(\alpha, \cdot)\rangle\overline{\langle U, K(\overline{\alpha}, \cdot)\rangle} = \beta.$$

(By the reproducing property of $\mathbf{H}_{\delta/2}$, this is just another way of writing $F(\alpha) = \beta$.) The functions $K(\alpha, z)$ and $K(\overline{\alpha}, z)$ are linearly independent if, and only if,

$$K(\alpha, \alpha)^2 - |K(\alpha, \overline{\alpha})|^2 = \left\{\frac{\sinh 2\pi\delta\ \mathrm{Im}(\alpha)}{2\pi\ \mathrm{Im}(\alpha)}\right\}^2 - \delta^2 \neq 0.$$

This holds if, and only if, $\mathrm{Im}(\alpha) \neq 0$. The result now follows from Lemma 8 with $\boldsymbol{u} = K(\alpha, z)$ and $\boldsymbol{v} = K(\overline{\alpha}, z)$. □

In the following lemma we compute the Fourier transform of $F(z; \alpha, \beta)$.

LEMMA 9. *Let* $K(\omega, z) = \sin(\pi\delta(z - \overline{\omega}))/\pi(z - \overline{\omega})$, *and* $\mathscr{F}$ *denote the Fourier transform, and*

$$G_{\alpha,\omega}(t) = \mathscr{F}(K(\alpha, \cdot)K(\omega, \cdot))(t), \tag{3.8}$$

then

$$G_{\alpha,\omega}(t) = \mathrm{e}^{-\pi i t(\overline{\omega}+\overline{\alpha})}\frac{\sin\{\pi(\overline{\omega} - \overline{\alpha})(\delta - |t|)_+\}}{\pi(\overline{\omega} - \overline{\alpha})}.$$

Let $F(z) = F(z; \alpha, \beta)$ *be the extremal function identified in Theorem 2, then*

$$\mathscr{F}(F)(t) = (|\lambda_1|^2+|\lambda_2|^2)G_{\alpha,\overline{\alpha}}(t)+(\overline{\lambda_1}\lambda_2 e(-\alpha)+\lambda_1\overline{\lambda_2}e(-\overline{\alpha}))(\delta-|t|)_+, \tag{3.9}$$

where λ_1 *and* λ_2 *are given by (1.4), and* $(x)_+ = \max\{0, x\}$.

Proof. If $a \leqslant b$ and $\xi \in \mathbb{C}$, then

$$\int_a^b \mathrm{e}^{2\pi i s\xi}\,ds = \frac{\sin(\pi\xi(b - a))}{\pi\xi}\mathrm{e}^{\pi i(b+a)\xi}.$$

Now, if $\xi \in \mathbb{C}$ and $\delta > 0$, then

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi i s\xi}\chi_{[-\delta/2,\delta/2]}(s)\chi_{[-\delta/2,\delta/2]}(t - s)\,ds = \frac{\sin\pi\xi(\delta - |t|)_+}{\pi\xi}\mathrm{e}^{\pi i t\xi}.$$

This is seen by observing that

$$\chi_{[-\delta/2,\delta/2]}(s)\chi_{[-\delta/2,\delta/2]}(t - s) = \chi_{[-\delta/2,\delta/2]\cap[t-\delta/2,t+\delta/2]}(s).$$

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Hence

$$\int_{-\infty}^{\infty} e^{2\pi i s\xi}\chi_{[-\delta/2,\delta/2]}(s)\chi_{[-\delta/2,\delta/2]}(t-s)\,ds = \begin{cases} \displaystyle\int_{t-\delta/2}^{\delta/2} e^{2\pi i s\xi}\,ds & \text{if } t \geqslant 0, \\ \displaystyle\int_{-\delta/2}^{t+\delta/2} e^{2\pi i s\xi}\,ds & \text{if } t < 0, \end{cases}$$

when $t \geqslant 0$

$$\int_{t-\delta/2}^{\delta/2} e^{2\pi i s\xi}\,ds = \frac{\sin \pi\xi(\delta - t)}{\pi\xi} e^{\pi i t\xi}$$

and when $t < 0$

$$\int_{-\delta/2}^{t+\delta/2} e^{2\pi i s\xi}\,ds = \frac{\sin \pi\xi(\delta - |t|)}{\pi\xi} e^{\pi i t\xi}.$$

Finally if $|t| < \delta$, then

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-2\pi i s t} K_\alpha(s) K_\omega(s)\,ds \\ &\quad = e^{-2\pi i \overline{\omega} t} \int_{-\infty}^{\infty} e^{2\pi i(\overline{\omega}-\overline{\alpha})s}\chi_{[-\delta/2,\delta/2]}(s)\chi_{[-\delta/2,\delta/2]}(t-s)\,ds \\ &\quad = e^{-2\pi i \overline{\omega} t} e^{\pi i t(\overline{\omega}-\overline{\alpha})} \frac{\sin \pi(\overline{\omega}-\overline{\alpha})(\delta - |t|)}{\pi(\overline{\omega}-\overline{\alpha})} \\ &\quad = e^{-\pi i t(\overline{\omega}+\overline{\alpha})} \frac{\sin \pi(\overline{\omega}-\overline{\alpha})(\delta - |t|)}{\pi(\overline{\omega}-\overline{\alpha})}. \end{aligned}$$

The rest of the proof is straightforward. □

§4. *Vanishing on the imaginary axis.* As before, we let $F(z; \alpha, \beta)$ be the extremal function coming from Theorem 2. In this section we analyze $F(z; \alpha, \beta)$ in the special case $\alpha = ir$ for some $r > 0$ and $\beta = -1$. The estimates we obtain extend to $\alpha \in \mathscr{U}$ by the identity

$$F(z; \alpha, \beta) = F(z - \mathrm{Re}(\alpha); i\,\mathrm{Im}(\alpha), \beta),$$

which follows from item (iii) from Theorem 2.

In the case when $\alpha = ir$ and $\beta = -1$, the function $U(z)$ given in Theorem 2 is given by†

$$U(z) = \frac{K(ir, z) - K(-ir, z)}{K(ir, ir) - K(-ir, ir)}, \tag{4.1}$$

where

$$K(\omega, z) = \int_{-\delta/2}^{\delta/2} e^{-2\pi i(z-\overline{\omega})\xi}\,d\xi = \frac{\sin \pi\delta(z - \overline{\omega})}{\pi(z - \overline{\omega})}.$$

We record some basic estimates for $U(z)$ in the following lemma.

† Note that $K(ir, ir) = K(-ir, -ir) = (2\pi r)^{-1}\sinh(2\pi r\delta)$ and $K(ir, -ir) = \delta$.

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LEMMA 10. *For each $r > 0$, the function $U(z)$ defined by (4.1) has the following properties:*

(i) $\hat{U}(\xi) = 0$ *if* $|\xi| > \delta/2$;
(ii) $U(\mathrm{i}r)U^*(\mathrm{i}r) = -1$;
(iii) $\|U\|_\infty^2 \leqslant 4\pi r\delta(\sinh(2\pi r\delta) - 2\pi r\delta)^{-1}$; *and*
(iv) $\|U\|_2^2 = \kappa(\mathrm{i}r, -1, \delta) = 4\pi r(\sinh(2\pi r\delta) - 2\pi r\delta)^{-1}$
(v)
$$|\hat{U} * \overline{\hat{U}}(\xi)| \leqslant \|U\|_2^2;$$
(vi)
$$|U(z)| \leqslant \frac{\|U\|^2}{|z - \mathrm{i}r|} \exp(\pi\delta \operatorname{Im}(z \pm \mathrm{i}r))$$
where $\pm$ *is chosen according to whether* $\operatorname{Im}(z) > 0$ *or not.*

Proof. Item (i) follows from the definition of $K(\omega, z)$ and linearity of the Fourier transform. Item (ii) is easy to verify by using the definition of $U(z)$ and noting that $U(\mathrm{i}r) = -U^*(\mathrm{i}r) = 1$. For the remaining items, we use the following facts: (a) the function $t \mapsto K(\omega, t)$ is square integrable on $\mathbb{R}$; and (b)

$$K(\omega, \lambda) = \langle K(\omega, \cdot), K(\lambda, \cdot)\rangle = \overline{K(\bar{\lambda}, \bar{\omega})}$$

where $\langle \cdot, \cdot \rangle$ is the usual L^2 inner product on $\mathbb{R}$. To compute $\|U\|_2^2$ we write $\|U\|_2^2 = |\langle U, U\rangle|^2$ and use (4.1):

$$\begin{aligned}\|U\|_2^2 &= 2\frac{K(\mathrm{i}r, \mathrm{i}r) - K(\mathrm{i}r, -\mathrm{i}r)}{(K(\mathrm{i}r, \mathrm{i}r) - K(\mathrm{i}r, -\mathrm{i}r))^2} \\ &= \frac{4\pi r}{\sinh(2\pi r\delta) - 2\pi r\delta}.\end{aligned}$$

By the reproducing property we see that $U(\omega) = \langle U, K(\omega, \cdot)\rangle$. It follows from the Cauchy–Schwarz inequality that

$$|U(t)|^2 \leqslant \delta\|U\|_2^2.$$

To finish off the proof of the lemma, note that by the Plancherel theorem and Young's inequality we have

$$|\hat{U} * \overline{\hat{U}}(\xi)| \leqslant \|U\|_2^2. \tag{4.2}$$

A trivial estimation of $\sin(z)$ yields

$$|U(z)| \leqslant \frac{\|U\|_2^2}{|z - \mathrm{i}r|} \exp(\pi\delta \operatorname{Im}(z \pm \mathrm{i}r)) \tag{4.3}$$

where $\pm$ is chosen according to whether $\operatorname{Im}(z) > 0$ or not. □

§5. *Selberg's majorant with vanishing Proof of Theorem 1.* In this section we will prove Theorem 1. We begin by determining a zero-free region for Selberg's function $C(z)$. This will serve as a model for how to bound $\rho(I, \alpha, \delta)$ from below. Using this model we will prove the lower bounds appearing in Theorem 1.

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This result is a consequence of the fact that admissible functions are Lipschitz on horizontal strips in $\mathbb{C}$ together with a straightforward analysis of the Lipschitz constant. Then we will construct an admissible function for $\rho(I, \alpha, 4\delta)$ and estimate its integral.

PROPOSITION 11. *Suppose* $\delta\text{Length}(I) \ll 1$. *If* $C(\omega) = 0$, *then* $|\omega| = \Omega(\delta^{-1})$.

We will use the following corollary of Lemma 4, Proposition 5 and Lemma 6.

COROLLARY 12. *Let* $F \in E_\sigma \cap L^1$ *such that* $F(0) \geqslant 1$ *and* $\|F\|_1 = L$. *Then if* ω *is a zero of* $F(z)$, *then*

$$1 \ll L\sigma^2|\omega|\cosh(\sigma\ \mathrm{Im}(\omega))$$

where the implied constant is absolute.

Proof. Suppose $F(\omega) = 0$. By the mean value theorem

$$|F'(u\omega)| = \left|\frac{F(\omega) - F(0)}{\omega}\right| \geqslant \frac{1}{|\omega|} \tag{5.1}$$

for some $0 \leqslant u \leqslant 1$. Combining Proposition 5, Lemma 4 and then Lemma 6 we have

$$|F'(u\omega)| \leqslant \cosh(\sigma\ \mathrm{Im}(\omega))\|F'\|_\infty \leqslant c\sigma^2 L\cosh(\sigma\ \mathrm{Im}(\omega)). \tag{5.2}$$

Combining (5.1) with (5.2), we are lead to the inequality

$$1 \leqslant c|\omega|\sigma^2 L\cosh(\sigma\ \mathrm{Im}(\omega))$$

where $c > 0$ is the constant appearing in Lemma 6. □

Proof of Proposition 11. Suppose $C(\omega) = 0$ and that $0 \in I$. By Corollary 12

$$1 \ll (\text{Length}(I) + \delta^{-1})(2\pi\delta)^2|\omega|\cosh 2\pi\delta\ \mathrm{Im}(\omega). \tag{5.3}$$

Combining the fact that $\delta\text{Length}(I) \ll 1$ and (5.3) yields

$$1 \ll 2\pi\delta|\omega|\cosh 2\pi\delta|\omega|,$$

it follows that $\delta|\omega| = \Omega(1)$. □

This demonstrates that vanishing does not come for free with extremal majorants of $\chi_I(t)$ and that forcing vanishing will come at some cost for small values of δ. We will now consider how well Selberg-type functions, which have the property that they vanish at a prescribed point in the upper half-plane, can approximate $\chi_I(t)$. If δ is small, then we know Selberg's function is not admissible and we can use the modifications suggested in the introduction to obtain the upper bounds on $\rho(I; \alpha, \delta)$. We are now in a position to prove the first part of Theorem 1.

Proof of Theorem 1 (lower bound). Following the proof of Proposition 11 we find that

$$1 \ll \rho(I; \alpha, \delta)(2\pi\delta)^2|\alpha|\cosh 2\pi\delta\ \mathrm{Im}(\alpha). \tag{5.4}$$

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Because $\cosh(2\pi\delta\ \mathrm{Im}(\alpha)) \sim 1$ when $\delta \ll 1$ we have

$$\delta^{-2} \ll_\alpha \rho(I;\alpha,\delta) \tag{5.5}$$

when $\delta \ll 1$.

Now, when $\delta \gg 1$, we know that $\rho(I;\alpha,\delta) \gg \delta^{-1}$ because Selberg's function (while not admissible) satisfies weaker assumptions than admissible functions and has L^1 distance δ^{-1} from the characteristic function of I. □

As before, we let $F(z;\alpha,\beta)$ denote the extremal function described in Theorem 2. To produce the upper bound in the theorem, we will use the function

$$C_\alpha(z) = C(z)(1 + F(z;\alpha,-1)),$$

which is admissible for $\rho(I;\alpha,\delta)$.

Proof of Theorem 1 (upper bound). We begin with the observation that

$$\rho(I;\alpha,2\delta) \leqslant \int_{-\infty}^{\infty} C_\alpha(t)\,dt - \mathrm{Length}(I). \tag{5.6}$$

Now when $\delta \ll 1$, we estimate the right-hand side of (5.6) by

$$\int_{-\infty}^{\infty} C_\alpha(t)\,dt \leqslant \|C\|_1(1 + \|F\|_\infty).$$

This inequality, combined with (5.6), and Lemma 10 yields

$$\rho(I;\alpha,2\delta) \leqslant (\mathrm{Length}(I) + \delta^{-1})\left(1 + \frac{8\pi^2 c\delta\ \mathrm{Im}(\alpha)}{\sinh(2\pi\ \mathrm{Im}(\alpha)\delta) - 2\pi\ \mathrm{Im}(\alpha)\delta}\right)$$

and as $\delta \to 0$ this reduces to

$$\rho(I;\alpha,2\delta) \leqslant O(1) + 3(2\pi c + o(1))\frac{4\pi\ \mathrm{Im}(\alpha)}{(2\pi\ \mathrm{Im}(\alpha)\delta)^3} \ll_\alpha \delta^{-3}.$$

In the case when $\delta \gg 1$, we use the uniform estimate

$$F(t;\alpha,-1) \leqslant \frac{4\pi\ \mathrm{Im}(\alpha)\delta}{\sinh(2\pi\ \mathrm{Im}(\alpha)\delta) - 2\pi\ \mathrm{Im}(\alpha)\delta} \ll_\alpha \mathrm{e}^{-2\pi\ \mathrm{Im}(\alpha)\delta}$$

from Lemma 10. Combining this with (5.6) we have

$$\begin{aligned}\rho(I;\alpha,2\delta) &\ll_\alpha (1 + \mathrm{e}^{-2\pi\ \mathrm{Im}(\alpha)\delta})(\mathrm{Length}(I) + \delta^{-1}) - \mathrm{Length}(I)\\ &\ll_\alpha \mathrm{Length}(I)\mathrm{e}^{-2\pi\ \mathrm{Im}(\alpha)\delta} + \delta^{-1}\mathrm{e}^{-2\pi\ \mathrm{Im}(\alpha)\delta} + \delta^{-1}\\ &\ll_{\alpha,I} \delta^{-1}.\end{aligned}$$

□

§6. *Remarks and generalizations.*

6.1. *A word on minorants.* In the literature, when speaking of Beurling–Selberg majorants, it is customary to also speak of Beurling–Selberg *minorants*. We have decided to say just a few words in this direction, owing to the fact that we have not even determined the extremal majorant of the characteristic function of the interval with a vanishing constraint. That being said, we can construct minorants in the same way that we constructed majorants by modifying Selberg's functions.

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Besides the majorant $C(z)$, Selberg constructed an analogous minorant $c(z)$. That is, for each real number $\delta > 0$ and an interval $I \subset \mathbb{R}$, he constructed an integrable function $c : \mathbb{R} \to \mathbb{R}$ that satisfies:

(1) $c(x) \leqslant \chi_I(x)$;
(2) $\hat{c}(\xi) = 0$ whenever $|\xi| > \delta$; and
(3) $\int_{-\infty}^{\infty} c(x)\,dx = \text{Length}(I) - \delta^{-1}$.

The minorant $c(x)$ extends to an entire function of exponential type $2\pi\delta$, like the majorant $C(z)$. A notable difference between $C(z)$ and $c(z)$ is that $c(z)$ becomes a worse L^1-approximation than the constant function $f(x) = 0$ when $\delta < \text{Length}(I)^{-1}$. So the reader may think of δ as being large when we are discussing Selberg's minorant.

Observe that the function

$$z \mapsto c(z) - F(z; \alpha, c(\alpha))$$

is an entire function of exponential type $2\pi\delta$ that minorizes the characteristic function of I on the real line, and vanishes at α. Similarly, if $F(z; \alpha, 1) \leqslant 1$, then

$$z \mapsto c(z)(1 - F(z; \alpha, 1))$$

is an entire function of exponential type $4\pi\delta$ that minorizes the characteristic function of I on the real line, and that vanishes at α. By Lemma 10, the condition that $F(z; \alpha, 1) \leqslant 1$ is satisfied when $\delta \gg 1$. It is not difficult, however, to show that $F(z; \alpha, 1) > 1$ for small values of δ. The exact value of δ where the change occurs can be computed in the following way. First write $F(z; \alpha, 1) = V(z)V^*(z)$ where

$$V(z) = \frac{K(\alpha, z) + K(-\alpha, z)}{K(\alpha, \alpha) + K(-\alpha, \alpha)}.$$

The expression $|V(t)| \leqslant 1$ for each $t \in \mathbb{R}$ is equivalent to

$$4 \sinh \pi\delta \operatorname{Im}(\alpha) \leqslant \sinh 2\pi\delta \operatorname{Im}(\alpha) + 2\pi\delta \operatorname{Im}(\alpha)$$

for $\delta, \operatorname{Im}(\alpha) > 0$. Equality is obtained when $\pi\delta \operatorname{Im}(\alpha) \approx 1.0295$.

6.2. *Vanishing at many points.* In the introduction we mentioned that our method allows for the construction for majorants what vanish at many distinct points. In this section we elaborate on this and explore some of the analytic properties of (1.7).

Given N points $\alpha_1, \ldots, \alpha_N \in \mathscr{U}$ represented by $\boldsymbol{\alpha}$, we define the following analogue of $\rho(\alpha, I, \delta)$:

$$\rho(\boldsymbol{\alpha}, I, \delta) = \inf \int_{-\infty}^{\infty} \{G(t) - \chi_I(t)\}\,dt$$

where the infimum is taken over entire functions $G(z)$ that satisfy:

(i) $G(t) \geqslant \chi_I(t)$ for all $t \in \mathbb{R}$;
(ii) $\hat{G}(\xi) = 0$ whenever $|\xi| > \delta$; and
(iii) $G(\alpha_n) = 0$ for $n = 1, 2, \ldots, N$.

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By using the function defined in (1.7), we can obtain the following bounds on $\rho(\boldsymbol{\alpha}, I, \delta)$.

THEOREM 13. *Let $\delta > 0$, $N > 0$, $\alpha_1, \dots, \alpha_N \in \mathscr{U}$ be represented by $\boldsymbol{\alpha}$, and $I \subset \mathbb{R}$ be an interval. Then*

$$\delta^{-2} \ll \rho(\boldsymbol{\alpha}, I, \delta) \ll \delta^{-2N-1} \tag{6.1}$$

as $\delta \to 0$, and

$$\rho(\boldsymbol{\alpha}, I, \delta) \approx \delta^{-1} \tag{6.2}$$

as $\delta \to \infty$. The implied constants are effective and depend only on N, $\boldsymbol{\alpha}$ and I:

$$G_{\boldsymbol{\alpha}}(z) = \prod_{n=1}^{N} (1 + F(z/N; \alpha_n/N, -1)). \tag{6.3}$$

It is easy to see that $G_{\boldsymbol{\alpha}}(z)$ is an entire function of exponential type $2\pi\delta$ such that $G_{\boldsymbol{\alpha}}(x) \geqslant 1$ on the real axis and $G_{\boldsymbol{\alpha}}(\alpha_n) = 0$ for each $n = 1, \dots, N$. Now observe that the following modification of Selberg's majorant

$$C_{\boldsymbol{\alpha}}(z) = C(z) G_{\boldsymbol{\alpha}}(z) \tag{6.4}$$

has exponential type $4\pi\delta$, $C_{\boldsymbol{\alpha}}(x) \geqslant C(x)$ for all real x, and $C_{\boldsymbol{\alpha}}(\alpha_n) = 0$ for each $n = 1, \dots, N$.

Remark. Instead of giving weight $1/N$ to each function depending on α_ℓ, we could instead take a different convex combination using the functions $F(\lambda_n z; \lambda_n \alpha_n, -1)$ where $\lambda_1, \dots, \lambda_N > 0$ and $\lambda_1 + \cdots + \lambda_N = 1$.

LEMMA 14. *Let $\varphi \in L^1 \cap E_\delta$ and define $\varphi_{\boldsymbol{\alpha}}(x) = \varphi(x) G^+_{\boldsymbol{\alpha}}(x)$. There exists a constant $c = c(\boldsymbol{\alpha}, N) > 0$ such that*

$$|\hat{\varphi_{\boldsymbol{\alpha}}}(\xi)| \leqslant |\hat{\varphi}(\xi)| \Big(1 + c(\boldsymbol{\alpha}, N) \Big(\exp\Big\{-\pi \min_{\ell} \{\mathrm{Im}(\alpha_\ell)\} \delta / N^2\Big\}\Big)\Big)$$

as $\delta \to \infty$ and

$$|\hat{\varphi_{\alpha}}(0)| \ll \|\varphi\|_1 \delta^{-2N}$$

as $\delta \to 0$. The implied constants depend on a, b, and $\boldsymbol{\alpha}$.

Proof. By Lemma 10(v), we have

$$|1 - G_{\boldsymbol{\alpha}}(x)| \ll_{N,\boldsymbol{\alpha}} \exp\Big\{-\pi \min_{\ell} \{\mathrm{Im}(\alpha_\ell)\} \delta / N^2\Big\}$$

and using the fact that $|\hat{\varphi}(\xi)| \leqslant |\hat{\varphi}(0)|$ we have

$$\hat{\varphi_{\boldsymbol{\alpha}}}(\xi) = \hat{\varphi}(\xi) \Big(1 + O_{N,\boldsymbol{\alpha}} \Big(\exp\{-\pi \min_{\ell} \{\mathrm{Im}(\alpha_\ell)\} \delta / N^2\Big\}\Big)\Big).$$

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Now when $\delta \to 0$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_{\boldsymbol{\alpha}}(x)\,dx &= \int_{-\infty}^{\infty} \varphi(x)\,dx \\ &\quad + \sum_{i_1<\cdots<i_k} \int_{-\infty}^{\infty} \varphi(x) F(x/N;\alpha_{i_1}/N,-1)\cdots F(x/N;\alpha_{i_k}/N,-1)\,dx \\ &\leqslant \hat{\varphi}(0) + \|\varphi\|_1 \sum_{i_1<\cdots<i_k} \prod_{\ell=1}^{k} \|F(x/N;\alpha_{i_\ell}/N,-1)\|_\infty. \end{aligned} \tag{6.5}$$

Lemma 10 gives

$$\begin{aligned}\|F(x/N;\alpha_{i_\ell}/N,-1)\|_\infty &\leqslant \|F(x;\alpha_{i_\ell},-1)\|_\infty \\ &\leqslant \frac{4\pi\delta\ \mathrm{Im}(\alpha_{i_\ell})}{K(\alpha_\ell,\alpha_\ell)-\delta},\end{aligned}$$

and by writing

$$K(\alpha_\ell,\alpha_\ell) = \delta + \sum_{n=2}^{\infty} \frac{(2\pi\ \mathrm{Im}(\alpha_\ell))^{2n-2}}{(2n-2)!}\delta^{2n-1}$$

we find that

$$\|F(x/N;\alpha_{i_\ell}/N,-1)\|_\infty \leqslant \frac{2}{2\pi\ \mathrm{Im}(\alpha_{i_\ell})\delta^2},$$

which combined with (6.6) gives

$$\int_{-\infty}^{\infty} \varphi_{\boldsymbol{\alpha}}(x)\,dx \leqslant \hat{\varphi}(0) + N\|\varphi\|_1 \sum_{i_1<\cdots<i_k} \prod_{\ell=1}^{k} \frac{2}{2\pi\ \mathrm{Im}(\alpha_{i_\ell})\delta^2}. \tag{6.6}$$

It follows that

$$\int_{-\infty}^{\infty} \varphi_{\boldsymbol{\alpha}}(x)\,dx \leqslant \hat{\varphi}(0) + \|\varphi\|_1 O(\delta^{-2N}) \quad \text{as } \delta \to 0$$

where the implied constant depends on N and $\boldsymbol{\alpha}$. □

6.3. *de Branges spaces.* In this section we show how Theorem 2 can be generalized so that the minimization occurs in a fairly general *de Branges space*. A Hilbert space H, which is non-trivial and whose elements are *entire functions*, is called a *de Branges space* if:

(i) $F \in H$ and ω is a non-real zero of F, then $(z-\overline{\omega})F(z)/(z-\omega) \in H$ and has the same norm as F;
(ii) $F \in H$ implies $F^* \in H$ and has the same norm as F; and
(iii) for every $\omega \in \mathbb{C}$, then functional $F \mapsto F(\omega)$ is continuous.

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It is a fundamental theorem of de Branges [11] that to each space H there exists an entire function $E(z)$ satisfying the elementary inequality

$$|E(\overline{z})| < |E(z)| \quad \text{for each } z \in \mathscr{U} \tag{6.7}$$

such that the Hilbert space whose elements come from H, but whose inner product is given by

$$\langle F, G\rangle_E = \int_{-\infty}^{\infty} F(t)\overline{G(t)}\frac{dt}{|E(t)|^2}$$

with induced norm $\|\cdot\|_E$, is isometric to H. Following [15], we will call this function a *de Branges function* and we will say that it is *strict* if it has no zeros on the real axis. Condition (iii) implies that a de Branges space is a reproducing kernel Hilbert space. We will let $K_E(\omega, z)$ denote the corresponding reproducing kernel, which is given by the formula [11, Theorem 19]

$$K_E(\omega, z) = \frac{B(z)\overline{A(\omega)} - A(z)\overline{B(\omega)}}{\pi(z - \overline{\omega})}, \tag{6.8}$$

where $A(z) = (1/2)(E(z) + E^*(z))$ and $B(z) = (\mathrm{i}/2)(E(z) - E^*(z))$.

Conversely, given an entire function satisfying (6.7), there exists a de Branges space H_E whose elements $F(z)$ satisfy:

(i) $\|F\|_E < \infty$; and
(ii) $F(z)/E(z)$ and $F^*(z)/E(z)$ are of *bounded type* and non-positive *mean type* in $\mathscr{U}$.

A function $g(z)$ which is analytic in $\mathscr{U}$ is said to be of *bounded type* in $\mathscr{U}$ if it can be expressed as the quotient of bounded analytic functions in $\mathscr{U}$. The *mean type* of a function $g(z)$ of bounded type in $\mathscr{U}$ is the number

$$\nu(g) = \limsup_{y\to\infty} y^{-1}\log|g(\mathrm{i}y)|$$

if $g(z)$ is not identically zero and $-\infty$ if $g \equiv 0$. We can now formulate a generalization of Theorem 2 for de Branges spaces.

THEOREM 15. *Let $\alpha \in \mathscr{U}$, $\beta \in \mathbb{C}$, and $E(z)$ be a de Branges function that is of bounded type in $\mathscr{U}$. Assume in addition that $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly independent. If $F(z)$ is an entire function of exponential type at most $2\nu(E)$ satisfying:*

(1) *$F(x) \geqslant 0$ for real x; and*
(2) *$F(\alpha) = \beta$;*

then

$$\frac{|\beta|K_E(\alpha,\alpha) - \mathrm{Re}(\beta K_E(\alpha,\overline{\alpha}))}{K_E(\alpha,\alpha)^2 - |K_E(\alpha,\overline{\alpha})|^2} \leqslant \frac{1}{2}\int_{-\infty}^{\infty} F(x)|E(x)|^{-2}\,dx. \tag{6.9}$$

Equality occurs in (6.16) if and only if $F(z) = U(z)U^*(z)$, where

$$U(z) = \lambda_1 K_E(\alpha, z) + \lambda_2 K_E(\overline{\alpha}, z). \tag{6.10}$$

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The coefficients λ_1 and λ_2 are given by

$$\lambda_1 = \frac{\gamma\beta K_E(\alpha,\alpha) - |K_E(\alpha,\overline{\alpha})|}{K_E(\alpha,\alpha)^2 - |K_E(\alpha,\overline{\alpha})|^2} \quad \textit{and} \quad \lambda_2 = \frac{K_E(\alpha,\alpha) - \beta K_E(\alpha,\overline{\alpha})}{K_E(\alpha,\alpha)^2 - |K_E(\alpha,\overline{\alpha})|^2},$$

where $\gamma = K_E(\alpha,\overline{\alpha})|K_E(\alpha,\overline{\alpha})|^{-1}$.

Proof. The proof of Theorem 15 of [**13**] shows the existence of a $U(z) \in H_E$ such that $F(z) = U(z)U^*(z)$. The result follows from Lemma 8 by taking $\boldsymbol{u} = K_E(\alpha, z)$ and $\boldsymbol{v} = K_E(\overline{\alpha}, z)$. □

The condition that $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly independent is necessary because of examples such as $E(z) = z + \mathrm{i}$. Using (6.8), we let $A(z) = (1/2)(E(z) + E^*(z)) = z$, $B(z) = (\mathrm{i}/2)(E(z) - E^*(z)) = -1$, and

$$K_E(\alpha, z) = \frac{B(z)\overline{A(\alpha)} - A(z)\overline{B(\alpha)}}{\pi(z - \overline{\alpha})} = \frac{z - \overline{\alpha}}{\pi(z - \overline{\alpha})} = \pi^{-1}.$$

It follows that $K_E(\alpha, z) = K_E(\beta, z)$ for every $\alpha, \beta \in \mathbb{C}$. In fact, $H_E \cong \mathbb{C}$ as a Hilbert space.

The following theorem shows that the only obstruction to $K_E(\alpha, z)$ being linearly independent to $K_E(\overline{\alpha}, z)$ is realized in $E(z) = z + \mathrm{i}$.

THEOREM 16. *Let $E(z)$ be a de Branges function that is of bounded type in $\mathscr{U}$ and $\alpha \in \mathscr{U}$. The functions $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly independent in H_E if, and only if, either of the following conditions hold:*

(1) *$E(z)$ has positive mean type, $\nu(E) > 0$;*
(2) *$E(z)$ has more than one non-real zero.*

To prove Theorem 16, we will need the following lemmas.

LEMMA 17. *Suppose $E(z)$ is a de Branges function, $S(z)$ is a real entire function with only real zeros, and $\alpha \in \mathscr{U}$. The function $\tilde{E}(z) = E(z)S(z)$ is a de Branges structure function. There exists a constant $c \neq 0$ such that*

$$K_{\tilde{E}}(\alpha, z) = cK_{\tilde{E}}(\overline{\alpha}, z)$$

if, and only if,

$$K_E(\alpha, z) = c\frac{S(\alpha)}{S(\overline{\alpha})}K_E(\overline{\alpha}, z).$$

Proof. First we compute $K_{\tilde{E}}(\omega, z)$ in terms of $E(z)$. Note that

$$\tilde{A}(z) = \frac{\tilde{E}(z) + \tilde{E}^*(z)}{2} = S(z)A(z),$$
$$\tilde{B}(z) = i\frac{\tilde{E}(z) - \tilde{E}^*(z)}{2} = S(z)B(z).$$

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By (6.8) it follows that

$$K_{\tilde{E}}(\omega, z) = S(z)S(\overline{\omega})K_{\tilde{E}}(\omega, z).$$

From this identity we see that there exists a non-zero constant c such that $K_{\tilde{E}}(\alpha, z) = cK_{\tilde{E}}(\overline{\alpha}, z)$ if, and only if,

$$K_E(\alpha, z) = c\frac{S(\alpha)}{S(\overline{\alpha})}K_E(\overline{\alpha}, z). \qquad \square$$

LEMMA 18 (Pólya). *Suppose $E(z)$ satisfies the conditions of Theorem 16. Then there are numbers $b, c \in \mathbb{C}$ satisfying $\mathrm{Re}(b) \geqslant 0$ and $a \neq 0$, such that*

$$E(z) = az^k \mathrm{e}^{2\pi \mathrm{i}bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \mathrm{e}^{h_n z} \tag{6.11}$$

where $z_1, z_2, \ldots$ are the non-zero zeros of $E(z)$ (listed with appropriate multiplicity), $k \geqslant 0$ is the order of the zero at 0, and $h_n = \mathrm{Re}(z_n)|z_n|^{-2}$.

Proof. The assumption that $E(z)$ is of bounded type in $\mathscr{U}$ implies [**11**, Problem 34] that $E(z)$ is an entire function of Pólya class. The theorem then follows from [**11**, Theorem 7]. $\square$

LEMMA 19. *Let $E(z)$ be a strict de Branges function and let $\alpha \in \mathscr{U}$. If there is a constant $c \neq 0$ such that*

$$K_E(\alpha, z) = cK_E(\overline{\alpha}, z),$$

then there are numbers $b \geqslant 0$, and $a \neq 0$, such that

$$E(z) = \begin{cases} a\mathrm{e}^{2\pi \mathrm{i}bz}(z - \omega) & \text{if } E(\omega) = 0 \text{ and } \mathrm{Im}(\omega) < 0, \\ a\mathrm{e}^{2\pi \mathrm{i}bz} & \text{if } E(z) \text{ has no zeros.} \end{cases}$$

Proof. If $K_E(\alpha, z) = cK_E(\overline{\alpha}, z)$ for some constant $c \neq 0$, then (6.8) gives

$$\frac{B(z)\overline{A(\alpha)} - A(z)\overline{B(\alpha)}}{\pi(z - \overline{\alpha})} = c\frac{B(z)A(\alpha) - A(z)B(\alpha)}{\pi(z - \alpha)}. \tag{6.12}$$

From this expression, it follows that $A(\alpha) = 0$ if, and only if, $B(\alpha) = 0$. If $A(\alpha) = B(\alpha) = 0$, then $E(\alpha) = 0$ which is impossible in view of (6.7). Thus, we have both $B(\alpha) \neq 0$ and $A(\alpha) \neq 0$. This implies that $A(z)$, $zA(z)$, $B(z)$, and $zB(z)$ are linearly *dependent*, i.e. that there are constants $c_1, \ldots, c_4$ (not all zero) such that

$$c_1A(z) + c_2zA(z) + c_3B(z) + c_4zB(z) = 0. \tag{6.13}$$

To show that the constants are not all zero we will show that c_1 and c_2 cannot both be zero. Observe

$$c_1 = B(\overline{\alpha})\alpha - cB(\alpha)\overline{\alpha} \quad \text{and} \quad c_2 = B(\overline{\alpha}) - cB(\alpha).$$

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If $c_1 = c_2 = 0$, then $c = B(\overline{\alpha})\alpha/(B(\alpha)\overline{\alpha}) = B(\overline{\alpha})/B(\alpha)$. But this is impossible because $\alpha \in \mathscr{U}$.

Rearranging (6.13) yields

$$A(z) = \left(\frac{c_4 z - c_3}{c_2 z - c_1}\right) B(z).$$

Moreover, $E(z) = A(z) - \mathrm{i}B(z)$, which leads us to

$$E(z) = A(z)\left(1 - \mathrm{i}\left(\frac{c_4 z - c_3}{c_2 z - c_1}\right)\right).$$

This factorization for $E(z)$, along with (6.7), implies that $E(z)$ has at most one non-real zero. Therefore, by Lemma 18

$$E(z) = \begin{cases} a\mathrm{e}^{2\pi \mathrm{i} b z}(z - \omega) & \text{if } E(\omega) = 0 \text{ and } \mathrm{Im}(\omega) < 0, \\ a\mathrm{e}^{2\pi \mathrm{i} b z} & \text{if } E(z) \text{ has no zeros} \end{cases}$$

for some $a \neq 0$, and $b \in \mathbb{C}$ with $\mathrm{Re}(b) \geqslant 0$. By Lemma 17 we may assume that b is real. □

Proof of Theorem 16. If $E(z)$ is not a strict de Branges function, then $E(z)$ can be written as $E(z) = E_0(z)S(z)$ where $S(z) = S^*(z)$ and $E_0(z)$ has no real zeros. From (6.7), it follows that $S(z)$ has only real zeros and that $E_0(z)$ is a strict de Branges function. By Lemma 17, it suffices to prove the lemma when $E(z)$ is a strict de Branges function, and we assume that $E(z)$ is strict throughout the remainder of the proof.

By Lemma 19, if $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly dependent, then there exists a $b \geqslant 0$, and $a \neq 0$, such that

$$E(z) = \begin{cases} a\mathrm{e}^{2\pi \mathrm{i} b z}(z - \omega) & \text{if } E(\omega) = 0 \text{ and } \mathrm{Im}(\omega) < 0, \\ a\mathrm{e}^{2\pi \mathrm{i} b z} & \text{if } E(z) \text{ has no non-real zeros.} \end{cases}$$

In light of this structure, it suffices to show that $K_E(\alpha, z)$ and $K_E(\overline{\alpha}, z)$ are linearly dependent if, and only if, $b = 0$.

If $b = 0$, then H_E is one dimensional and $K_E(\alpha, z)$ and $K_E(\alpha, z)$ must be linearly dependent.

Suppose $b > 0$ and ω is the single non-real zero of $E(z)$. The function $G(z) = 1$ is then in H_E since $\nu(1/E) = 0 - \nu(E) \leqslant 0$ and $|t - \omega|^{-2}$ is integrable. Similarly, the function $H(z) = \mathrm{e}^{-2\pi \mathrm{i} b z}$ is in H_E. Now suppose, by way of contradiction, that there is a non-zero constant $c \in \mathbb{C}$ such that $F(\alpha) = cF(\overline{\alpha})$ for all $F \in H_E$. The constant c must be equal to one, because

$$1 = G(\alpha) = cG(\overline{\alpha}) = c.$$

The function $H(z)$ is in H_E but $H(\alpha) \neq cH(\overline{\alpha}) = H(\overline{\alpha})$, a contradiction.

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If $E(z)$ has no zeros, then we must have $b > 0$, by (6.7). In this case, the reproducing kernel for H_E is given by

$$K(\omega, z) = \frac{\sin 2\pi b(z - \overline{\omega})}{\pi(z - \overline{\omega})}. \tag{6.14}$$

Here $K(\alpha, z)$ and $K(\overline{\alpha}, z)$ are linearly independent in H_E if, and only if,

$$K(\alpha, \alpha)^2 - |K(\alpha, \overline{\alpha})|^2 = \left\{ \frac{\sinh 4\pi b \operatorname{Im}(\alpha)}{2\pi \operatorname{Im}(\alpha)} \right\}^2 - 4b^2 \neq 0.$$

This holds if, and only if, $\operatorname{Im}(\alpha) > 0$. □

6.4. *Trigonometric polynomials.* In this final section we prove an analogue of Theorem 2 for trigonometric polynomials.

Let $N \geqslant 1$ and $\mathscr{P}_N \subset \mathbb{C}[z]$ be the complex vector space of polynomials of degree at most N. Define an inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{P}_N$ by

$$\langle p, q \rangle = \int_{S^1} p(\theta)\overline{q(\theta)}\, d\sigma(\theta)$$

where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and σ is the Haar probability measure on S^1. Let $\| \cdot \|$ be the norm induced by $\langle \cdot, \cdot \rangle$ and $e(t) = \mathrm{e}^{2\pi i t}$. To simplify notation, we will define a function $L : \mathbb{C} \times \mathbb{Z}^+ \to \mathbb{R}$ by

$$L(\omega, M) = \sum_{m=0}^{M} |\omega|^m = \begin{cases} \dfrac{1 - |\omega|^M}{1 - |\omega|} & \text{if } |\omega| \neq 1, \\ M + 1 & \text{if } |\omega| = 1. \end{cases}$$

LEMMA 20. *The space $(\mathscr{P}_N, \langle \cdot, \cdot \rangle)$ is a reproducing kernel Hilbert space with reproducing kernel $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by*

$$K(\omega, z) = \sum_{n=0}^{N} z^n \overline{\omega}^n.$$

Furthermore, if $\alpha \neq 0$, the functions $z \mapsto K(\alpha, z)$ and $z \mapsto K(1/\overline{\alpha}, z)$ are linearly independent if, and only if, $\alpha \notin S^1$.

Proof. The form of the reproducing kernel is a standard fact, but we note that it does not depend on the choice of orthonormal basis. We need only verify the latter statement. The functions in question are linearly independent if and only if

$$\|K(\cdot, \alpha)\|^2 \|K(\cdot, 1/\overline{\alpha})\|^2 - |\langle K(\cdot, \alpha), K(\cdot, 1/\overline{\alpha}) \rangle|^2 \neq 0. \tag{6.15}$$

Explicitly, these terms are given by

$$K(\alpha, \alpha) = L(|\alpha|, N), \quad K(\alpha, 1/\overline{\alpha}) = N + 1,$$
$$K(1/\overline{\alpha}, 1/\overline{\alpha}) = L(|\alpha|^{-1}, N).$$

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Now (6.15) can be rewritten as

$$L(|\alpha|, N)L(|\alpha|^{-1}, N) - (N+1)^2 \neq 0.$$

But by the arithmetic–geometric mean inequality we have

$$(N+1)^{-2}\left\{\sum_{n=0}^{N}|\alpha|^n\right\}\left\{\sum_{n=0}^{N}|\alpha|^{-n}\right\} \geqslant \left\{\prod_{n=0}^{N}|\alpha|^n\right\}^{1/(N+1)}\left\{\prod_{m=0}^{N}|\alpha|^{-m}\right\}^{1/(N+1)} = 1.$$

Equality holds if, and only if, $|\alpha|^m = |\alpha|^n$ for $n, m = 1, 2, \ldots, N$ which implies $|\alpha| = 1$. □

COROLLARY 21. *Suppose $\alpha \neq 0$, $\alpha \notin S^1$, $\beta \in \mathbb{C}$, and $N \geqslant 1$. If $F(z)$ is a Laurent polynomial of degree at most N:*

(1) *$F(\theta) \geqslant 0$ for $\theta \in S^1$; and*
(2) *$F(\alpha) = \beta$;*

then

$$\frac{|\beta|L(\alpha, N)^{1/2}L(\alpha^{-1}, N)^{1/2} - (N+1)\mathrm{Re}(\beta)}{L(\alpha, N)L(\alpha^{-1}, N) - (N+1)^2} \leqslant \frac{1}{2}\int_{S^1} F(\theta)\, d\sigma(\theta). \quad (6.16)$$

Equality occurs in (6.16) if, and only if, $F(z) = p(z)p^*(z)$, where

$$p(z) = \lambda_1 K(\alpha, z) + \lambda_2 K(1/\overline{\alpha}, z). \quad (6.17)$$

The coefficients λ_1 and λ_2 can be explicitly computed in terms of K, α, β and N.

Proof. By Fejér's theorem, there exists a $p \in \mathscr{P}_N$ such that

$$F(z) = p(z)p^*(z),$$

where $p^*(z) = \overline{p(1/\overline{z})}$. The result follows from Lemmas 20 and 8. □

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