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THE DE VYLDER–GOOVAERTS CONJECTURE HOLDS WITHIN THE DIFFUSION LIMIT

STEFAN ANKIRCHNER,* *University of Jena*
CHRISTOPHETTE BLANCHET-SCALLIET** AND
NABIL KAZI-TANI,*** *Université de Lyon*

Abstract

The De Vylder and Goovaerts conjecture is an open problem in risk theory, stating that the finite-time ruin probability in a standard risk model is greater than or equal to the corresponding ruin probability evaluated in an associated model with equalized claim amounts. Equalized means here that the jump sizes of the associated model are equal to the average jump in the initial model between 0 and a terminal time T. In this paper, we consider the diffusion approximations of both the standard risk model and its associated risk model. We prove that the associated model, when conveniently renormalized, converges in distribution to a Gaussian process satisfying a simple SDE. We then compute the probability that this diffusion hits the level 0 before time T and compare it with the same probability for the diffusion approximation for the standard risk model. We conclude that the De Vylder and Goovaerts conjecture holds for the diffusion limits.

Keywords: Risk theory; ruin probability; equalized claim; diffusion approximation

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1. Introduction

Ruin theory is concerned with the study of level crossing probabilities of certain stochastic processes with jumps. This is of clear interest in insurance, for example, where the focus is on the probability that the wealth process of a given insurance company reaches a lower level, in particular the level 0: in this case, the jumps represent claim sizes. There are also applications in storage, for example of water in a dam: in that context, the jumps represent water arrival events, such as rainfalls. The probability that the water level reaches critical levels for the dam is important for energy management, for instance. We refer the reader to the monograph [3] for further details on applications of ruin theory to other problems, in particular its duality with queuing theory.

In this paper, we use the terminology and interpretations from insurance. Let $\tau_1, \tau_2, \ldots$ be a sequence of positive random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $T_n := \tau_n - \tau_{n-1}$ ($\tau_0 := 0$). Now define

$$N_t := \#\{n \geq 1 \colon \tau_n \leq t\}, \qquad t \geq 0,$$

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* Postal address: Institute for Mathematics, University of Jena, Ernst-Abbe-Platz 2, 07743 Jena, Germany.
** Postal address: Institut Camille Jordan – Ecole Centrale de Lyon, CNRS UMR 5208, Université de Lyon, 36 Avenue Guy de Collongue, 69134 Ecully Cedex, France.
*** Postal address: Laboratoire SAF, ISFA, Université de Lyon, 50 Avenue Tony Garnier, 69366 Lyon Cedex 07, France.

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the counting process associated with the sequence $(\tau_n)_{n\geq 0}$. This last sequence represents the arrival epochs of the claims, and N_t is the number of claims in the time interval $[0, t]$.

Let $(\xi_i)_{i\geq 0}$ be an independent and identically distributed (i.i.d.) sequence of nonnegative random variables, defined on the same probability space, with finite second moments, i.e. satisfying $\mathbb{E}(\xi_i^2) < \infty$. Moreover suppose that the sequence $(\xi_i)_{i\geq 0}$ is independent of the process N. The aggregate claim size up to time t is given by $\sum_{i=1}^{N_t} \xi_i$. We assume that the insurer receives premia from customers at a rate $c \in \mathbb{R}^+$. The claim surplus process is defined by $S_t = \sum_{i=1}^{N_t} \xi_i - ct$ and the risk reserve process by

$$R_t = x - S_t = x + ct - \sum_{i=1}^{N_t} \xi_i, \qquad t \in [0, \infty), \tag{1.1}$$

where $x \in [0, \infty)$ is the initial reserve. Within the ruin model the insurer's vulnerability to financial distress is measured in terms of the ruin probability up to a given time horizon $T \in (0, \infty)$:

$$\rho(x, T) := \mathbb{P}\Big(\inf_{t\in[0,T]} R_t < 0 \Big). \tag{1.2}$$

De Vylder and Goovaerts [14] suggested the following modification of the risk reserve process:

$$\widetilde{R}_t := x - \widetilde{S}_t := x + ct - \frac{N_t}{N_T} \sum_{i=1}^{N_T} \xi_i,$$

with the convention that if $N_T = 0$, then $(N_t/N_T) \sum_{i=1}^{N_T} \xi_i = 0$, too. The ruin probability is given in that case by

$$\widetilde{\rho}(x, T) := \mathbb{P}\Big(\inf_{t\in[0,T]} \widetilde{R}_t < 0 \Big).$$

The processes R and $\widetilde{R}$ have the same jump times and the same values at times 0 and T, but the jump sizes of $\widetilde{R}$ are *equalized*, in the sense that each jump of $\widetilde{R}$ is equal to the average jump of R between times 0 and T.

The De Vylder and Goovaerts conjecture can then be stated as follows.

Open problem 1.1. *For all $x \geq 0$ and all $T > 0$,*

$$\widetilde{\rho}(x, T) \leq \rho(x, T). \tag{1.3}$$

Define

$$\rho_n(x, T) := \mathbb{P}\Big(\inf_{t\in[0,T]} R_t < 0 \mid N_T = n \Big), \qquad n \geq 0.$$

As explained in [14], (1.3) is equivalent to the fact that $\widetilde{\rho}_n(x, T) \leq \rho_n(x, T)$, for every $n \geq 0$, where $\widetilde{\rho}_n$ is defined as ρ_n, with $\widetilde{R}$ instead of R. Obviously, $\widetilde{\rho}_0(x, T) = \rho_0(x, T)$. It is proved in [14] that $\widetilde{\rho}_1(x, T) = \rho_1(x, T)$, $\widetilde{\rho}_2(x, T) \leq \rho_2(x, T)$, and that the probabilities coincide in the case of an initial reserve equal to zero, $\widetilde{\rho}(0, T) = \rho(0, T)$, but the general case has yet to be established. The results in [14] are derived analytically, using polynomial representations for the ruin probabilities.

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Progress towards a proof of (1.3) was made in [12], where it is proved, using stop-loss order techniques, that for each $u \geq 0$,

$$\int_u^{+\infty} \tilde{\rho}(x, T)\, \mathrm{d}x \leq \int_u^{+\infty} \rho(x, T)\, \mathrm{d}x.$$

We are not aware of any further articles providing results on Problem 1.1.

Our idea is to approximate both processes $(S_t)_{t\in[0,T]}$ and $(\widetilde{S}_t)_{t\in[0,T]}$ with continuous diffusions. Standard references for these type of approximations, using functional limit theorems, include in particular [2], [7], [8], and [9]. The idea is to rescale both time and space, time in order to let the number of claims increase and space in order to make the claim sizes smaller so that the risk reserve process converges in distribution to a continuous process.

It is well known that the surplus process $(S_t)_{t\in[0,T]}$ can be approximated with a Brownian motion with drift (see e.g. Chapter V in [3]). We show also that $(\widetilde{S}_t)_{t\in[0,T]}$ has a diffusion approximation. Conditional to the terminal value at time T, both diffusion approximations are Brownian bridges. The diffusion rate of the Brownian bridge approximating $(S_t)_{t\in[0,T]}$ is greater than or equal to the diffusion rate of the Brownian bridge approximating $(\widetilde{S}_t)_{t\in[0,T]}$. Since the probability for a Brownian bridge to hit zero is increasing in the diffusion rate, the ruin probability for the first diffusion approximation is larger than for the second. The De Vylder–Goovaerts conjecture is therefore true for the diffusion limits.

We consider two cases for computing the diffusion limits. In the first case we assume that $c = \lambda\mathbb{E}(\xi_1)$. In this case the accumulated premia coincide with the expected aggregate claims. In Section 2, we derive the diffusion limits by scaling time with a factor m while scaling the aggregate claim size with $1/\sqrt{m}$, and we also characterize the distribution of the limit as the solution of a stochastic differential equation. In the second case, which is dealt with in Section 3, we assume that c exceeds $\lambda\mathbb{E}(\xi_1)$. The difference $\eta := c - \lambda\mathbb{E}(\xi_1)$ is usually referred to as the safety loading. We compute the diffusion limits by scaling time with $1/\eta^2$ and letting $\eta \downarrow 0$, which is referred to as the *heavy traffic approximation* in risk theory ([3]).

2. Zero safety loading

To deal with diffusion approximations, we introduce $D[0, T]$, the space of real càdlàg functions on [0,T], i.e. the real functions that are right-continuous and have left-hand limits. We next recall the definition of weak convergence on this space.

We say that a sequence of càdlàg processes (X_n) converges in distribution to the càdlàg process X if, for every bounded function $f\colon D[0, T] \to \mathbb{R}$ that is continuous with respect to the Skorokhod topology (see e.g. [5]), we have

$$\mathbb{E}(f(X_n)) \to \mathbb{E}(f(X)).$$

We make the following assumption on the inter-arrival claim times.

Assumption 2.1. *There exist positive constants ν and κ such that the process*

$$\left(\frac{1}{\kappa\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} (T_j - \nu)\right)_{t\in[0,T]}$$

converges in distribution in $D[0, T]$ to a standard Brownian motion as $n \to \infty$.

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Using the central limit theorem, we know that Assumption 2.1 is satisfied if the random variables $(T_n)_{n\geq 1}$ are independent and identically distributed with mean ν and variance κ^2. This is the so-called *renewal theory* framework. In the particular case where the common distribution of the variables $(T_n)_{n\geq 1}$ is exponential, N is a *Poisson process*, and the model defined in (1.1) is the so-called *Cramér–Lundberg* model. By analogy with the Poisson case, we write $\lambda := 1/\nu$.

Assumption 2.1 implies the convergence (Theorem 14.6 in [5])

$$\left(\frac{N_{tn} - \lambda tn}{\kappa \lambda^{3/2}\sqrt{n}}\right)_{t\in[0,T]} \Rightarrow (B_t)_{t\in[0,T]}, \tag{2.1}$$

where here, and for the remainder of the paper, $\Rightarrow$ denotes convergence in distribution on $D := D([0, T)]$ and B is a Brownian motion.

Throughout this section we assume that $c = \lambda\mathbb{E}(\xi_1)$. Note that this implies that in the compound Poisson case, the process $(S_t)_{t\in[0,T]}$ is a martingale.

We first characterize the laws of the diffusion limits of $(S_t)_{t\in[0,T]}$ and $(\widetilde{S}_t)_{t\in[0,T]}$ with the help of a Brownian motion. Let $V(\,\cdot\,)$ denote the variance operator. We introduce the constants

$$\gamma := \sqrt{\lambda V(\xi_1) + \lambda^3\kappa^2\mathbb{E}[\xi_1]^2}, \tag{2.2}$$

$$\sigma := \kappa\lambda^{3/2}\,\mathbb{E}(\xi_1). \tag{2.3}$$

Note that $\gamma \geq \sigma$, and that equality is satisfied if and only if the claims ξ_i are deterministic.

2.1. Limiting diffusions

Proposition 2.1. *The process* $(m^{-1/2}\,S_{tm})_{t\in[0,T]}$ *converges in law on* $D[0, T]$ *to*

$$U_t := \gamma B_t, \qquad t \in [0, T],$$

as $m \to \infty$.

In the proof of Proposition 2.1 we make use of the continuous mapping theorem, which, for the ease of the reader, we recall here.

Proposition 2.2. (For example Theorem 2.7 in [5].) *Let* M *and* M' *be two metric spaces,* $h : M \to M'$ *a measurable mapping and let* D_h *denote the set of discontinuities of* h. *Let* (X_n) *be an* M*-valued sequence of random variables converging to a random variable* X *in distribution as* $n \to \infty$. *If* $\mathbb{P}(X \in D_h) = 0$, *then* $h(X_n)$ *converges to* $h(X)$ *in distribution as* $n \to \infty$.

Proof of Proposition 2.1. Let us write

$$\frac{1}{\sqrt{m}}S_{tm} = \frac{1}{\sqrt{m}}\sum_{i=1}^{N_{tm}}(\xi_i - \mathbb{E}[\xi_1]) + \frac{1}{\sqrt{m}}\mathbb{E}[\xi_1](N_{tm} - \lambda tm).$$

Define

$$Z_m(t) := \frac{1}{\sqrt{m}}\sum_{j=1}^{tm}(\xi_j - \mathbb{E}[\xi_1]).$$

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By Donsker's theorem,

$$\left(\frac{1}{\sqrt{V(\xi_1)}}Z_m(t)\right)_{t\in[0,T]}$$

converges in distribution on D, as m goes to $+\infty$, to a Brownian motion. On the other hand, (2.1) entails that

$$\left(\frac{1}{\sqrt{m}}\mathbb{E}[\xi_1](N_{tm}-\lambda tm)\right)_{t\in[0,T]} \Rightarrow (\sigma B_t)_{t\in[0,T]}.$$

Let W be a Brownian motion that is independent of B. By independence of N and the sequence $(\xi_m)_m \geq 1$, the pair of scaled processes

$$\left(Z_m(t),\ \frac{1}{\sqrt{m}}\mathbb{E}[\xi_1](N_{tm}-\lambda tm)\right)_{t\in[0,T]}$$

converges in distribution to $(\sqrt{V(\xi_1)}W_t, \sigma B_t)_{t\in[0,T]}$.

Since N_{tm}/m converges in distribution on D to the function $\Phi\colon t\mapsto \lambda t$, this implies, using Theorem 3.9 in [5], that

$$\left(\frac{N_{tm}}{m}, Z_m(t),\ \frac{1}{\sqrt{m}}\mathbb{E}[\xi_1](N_{tm}-\lambda tm)\right)_{t\in[0,T]} \Rightarrow (\lambda t, \sqrt{V(\xi_1)}W_t, \sigma B_t)_{t\in[0,T]}.$$

Let D_0 be the space of nondecreasing functions in D. As the function Φ is continuous and strictly increasing, we can use Theorem 3.1 in [15] to obtain that the mapping defined on $D_0 \times D$ by $(y, x)\mapsto x\circ y$ is continuous at (Φ, x') for arbitrary $x'\in D$. Consequently, $\Psi\colon D_0 \times D^2 \to D^2$, given by $\Psi(y, x, z) = (x\circ y, z)$, is continuous at (Φ, x', z') for arbitrary $x', z'\in D$. The continuous mapping theorem now implies

$$\left(Z_m\left(\frac{N_{tm}}{m}\right),\ \frac{1}{\sqrt{m}}\mathbb{E}[\xi_1](N_{tm}-\lambda tm)\right)_{t\in[0,T]} \Rightarrow (\sqrt{\lambda V(\xi_1)}W_t, \sigma B_t))_{t\in[0,T]}. \tag{2.4}$$

By Theorem 4.1 in [15], the addition of two elements in D is continuous at every (x, y), where x and y are continuous functions. By applying the continuous mapping theorem to the sum of the components in (2.4) we arrive at

$$\frac{1}{\sqrt{m}}S_{tm} \Rightarrow (\sigma B_t + \sqrt{\lambda V(\xi_1)}W_t)_{t\in[0,T]}.$$

This last limit process is centred, Gaussian, and has the same covariance structure as the centred Gaussian process γB, which ends the proof. □

We now approximate $(\widetilde{S}_t)_{t\geq 0}$. Note that

$$\mathbb{E}\left[\frac{N_t}{N_T}\sum_{i=1}^{N_T}\xi_i\right] = \lambda\mathbb{E}(\xi_1)t.$$

Hence $\mathbb{E}(\widetilde{S}_t) = 0$ for all $t\in[0, T]$.

Theorem 2.1. *The scaled process* $(m^{-1/2}\widetilde{S}_{tm})_{t\in[0,T]}$ *converges in law on* $D[0, T]$ *to*

$$\widetilde{U}_t := \sigma B_t + (\gamma-\sigma)\frac{t}{T}B_T, \qquad t\in[0, T], \tag{2.5}$$

as $m\to\infty$.

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Proof. First we note that the process $\widetilde{U}$ is a centred Gaussian process with covariance function given by

$$K(s,t) = (\gamma^2 - \sigma^2)\frac{st}{T} + \sigma^2(s \wedge t), \qquad s,t \in [0,T]. \tag{2.6}$$

Consider the decomposition

$$\frac{1}{\sqrt{m}}\widetilde{S}_{tm} = \frac{1}{\sqrt{m}}\frac{N_{tm}}{N_{Tm}}\sum_{i=1}^{N_{Tm}}(\xi_i - \mathbb{E}[\xi_1]) + \frac{1}{\sqrt{m}}\mathbb{E}[\xi_1](N_{tm} - \lambda tm). \tag{2.7}$$

Define

$$Z_m(t) := \frac{1}{\sqrt{m}}\sum_{j=1}^{tm}(\xi_j - \mathbb{E}[\xi_1]).$$

Now, notice that the decomposition (2.7) can be written as

$$\frac{1}{\sqrt{m}}\widetilde{S}_{tm} = \frac{N_{tm}}{N_{Tm}}Z_m\left(\frac{N_{Tm}}{m}\right) + \frac{1}{\sqrt{m}}\mathbb{E}[\xi_1](N_{tm} - \lambda tm).$$

The quantity N_{tm}/N_{Tm} (resp. N_{Tm}/m) converges in distribution on D to $t \mapsto t/T$ (resp. $t \mapsto \lambda T$).

Theorem 3.9 of Billingsley [5] implies that

$$\left(\frac{N_{Tm}}{m}, \frac{N_{tm}}{N_{Tm}}, Z_m(t), \frac{1}{\sqrt{m}}\mathbb{E}[\xi_1](N_{tm} - \lambda tm)\right)_{t\in[0,T]} \Rightarrow \left(\lambda T, \frac{t}{T}, \sqrt{V(\xi_1)}W_t, \sigma B_t\right)_{t\in[0,T]}. \tag{2.8}$$

By Theorems 3.1, 4.2, and 4.1 in [15] the mapping $h : D \times D_0 \times D^2 \to D$, $(x', y', z', v') \mapsto x'(z' \circ y') + v'$, is continuous at every quadruple of continuous functions (x, y, z, v) with y being strictly increasing. The continuous mapping theorem, applied to h and the convergent sequence (2.8), yields

$$\frac{1}{\sqrt{m}}\widetilde{S}_{tm} \Rightarrow \frac{t}{T}\sqrt{\lambda V(\xi_1)}W_T + \sigma B_t =: V_t.$$

Here V and $\widetilde{U}$ are centred Gaussian processes with the same covariance function given in (2.6), so they have the same distribution, and the proof is complete. □

We briefly comment on the method we use for proving Proposition 2.1 and Theorem 2.1. Recall that a more commonly used method for proving convergence in distribution on D is, first, to prove weak convergence of finite-dimensional distributions and, second, to show tightness of the considered sequence of probability measures on D (see e.g. Section 13 in [5]). A convenient method for verifying tightness is to show that the second moment, or another appropriately chosen moment, of the product of two neighbouring process increments can be nicely controlled.

This commonly used approach can also be applied to the two scaled process families of Proposition 2.1 and Theorem 2.1, respectively. A sufficient moment criterion for tightness of the distributions of $(m^{-1/2}\,S_{\cdot m})$, $m \geq 1$, is that there exists a constant $C \in [0,\infty)$ such that, for all $r, s, t \in [0,T]$ with $r \leq s \leq t$, we have

$$\mathbb{E}[(S_s - S_r)^2(S_t - S_s)^2] \leq C(t-r)^2$$

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(see e.g. Section 13 in [5]). The same moment criterion applies to the scaled versions of $\tilde{S}$. Note that the increments of $(S_t)_{t\in[0,T]}$ and $(\tilde{S}_t)_{t\in[0,T]}$ are not independent. Even in the Poisson case where $(S_t)_{t\in[0,T]}$ has independent increments, the increments of $\tilde{S}$ are not independent. The dependence of increments implies that the moment criterion for verifying tightness requires a stronger integrability condition on the claim sizes $(\xi_i)_{i\geq 1}$ than L^2-integrability.

The approach that we follow for proving Proposition 2.1 and Theorem 2.1 requires only L^2-integrability of the claim sizes. The reason is that it falls back on limit theorems for simpler processes and then applies the continuous mapping theorem for the Skorokhod topology. This approach relies on results of [15], and it has been used in particular in [7] in the context of diffusion approximations of risk processes.

Remark 2.1. Let $z \in D([0, T])$ be a given path with finite jump activity, i.e. such that z admits a finite number of jumps in any finite interval (notice that S and $\widetilde{S}$ have finite jump activity, almost surely). Then define $n_t(z)$ as the number of jumps of z during the time interval $[0, t]$. The path z can always be written as

$$z(t) = z_c(t) + \sum_{0<t\leq T} (z(t) - z(t^-)),$$

where z_c is a continuous trajectory and $z(t^-)$ is the left-hand side limit of z at t. The transformation suggested by De Vylder and Goovaerts is the application $R\colon D \to D$,

$$R(z)(t) := z_c(t) + n_t(z)\left(\frac{1}{n_T(z)} \sum_{0<t\leq T} (z(t) - z(t^-))\right),$$

that replaces each jump of z by the average jump of z between 0 and T.

A simple consequence of our results is that the application R is not continuous for the Skorokhod topology. If it were, then Proposition 2.1 would imply that

$$R\left(\frac{1}{\sqrt{m}} S_{\cdot m}\right) = \frac{1}{\sqrt{m}} \widetilde{S}_{\cdot m}$$

converges in distribution on D for the Skorokhod topology to $R(\gamma B) = \gamma B$, which is not the case, by Theorem 2.1.

2.2. Comparison of the ruin probabilities

Let $X_t := x - U_t$ and $\widetilde{X}_t := x - \widetilde{U}_t$. The ruin probability ρ can be approximated by $\chi := \mathbb{P}(\inf_{t\in[0,T]} X_t < 0)$ and the ruin probability $\widetilde{\rho}$ by $\widetilde{\chi} := \mathbb{P}(\inf_{t\in[0,T]} \widetilde{X}_t < 0)$. In this section we show that $\widetilde{\chi} \leq \chi$.

First note that $X_T = \widetilde{X}_T$. Conditional to $X_T = y$, the process $(X_t)_{t\in[0,T]}$ is a Brownian bridge from x to y with diffusion rate γ. We can therefore derive an explicit formula for the conditional probability that $(X_t)_{t\in[0,T]}$ goes below zero at some time before T.

To make this precise, let $\mathcal{B}$ be the collection of Borel sets on the space of continuous functions $\mathcal{C}([0, T])$ endowed with the supremum norm. Moreover, let $\mathcal{B} \times \mathbb{R} \ni (A, y) \mapsto \mathbb{P}(A|X_T = y)$ be a regular conditional distribution of $(X_t)_{t\in[0,T]}$ with respect to X_T.

Lemma 2.1. *For all $y \geq 0$ we have*

$$\mathbb{P}\Big(\inf_{t\in[0,T]} X_t < 0 \mid X_T = y\Big) = \exp\Big(-2\frac{xy}{T\gamma^2}\Big).$$

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Proof. One can show ([11, Chapter I]) that the law of $(X_t)_{t\in[0,T]}$ under $\mathbb{P}(\,\cdot\,|X_T = y)$ coincides with the law for a Brownian bridge from x to y with diffusion rate γ. Therefore,

$$\mathbb{P}\Big(\inf_{t\in[0,T]} X_t < 0 \mid X_T = y\Big) = \mathbb{P}\Big(\inf_{t\in[0,\gamma^2 T]} B_t < -x \mid B_{\gamma^2 T} = y - x\Big).$$

Well-known results on hitting probabilities for Brownian bridges (see e.g. Proposition 3 in [13]) imply the result. □

We next derive an explicit formula for the conditional probability that $\widetilde{X}$ goes below zero at some time before T. To this end, recall that the process

$$Y_t := \left(B_t - \frac{t}{T}B_T\right), \qquad t \in [0, T], \tag{2.9}$$

is a Brownian bridge on $[0, T]$ from zero to zero. Observe that

$$\widetilde{U}_t = \sigma Y_t + \gamma \frac{t}{T}B_T = \sigma Y_t + \frac{t}{T}(x - X_T),$$

and that B_T is independent of Y_t. Conditional to $X_T = y$, the process $(\widetilde{X}_t)_{t\in[0,T]}$ is therefore a Brownian bridge from zero to y with diffusion rate σ. From well-known results on hitting probabilities of Brownian bridges, we arrive at the following result.

Lemma 2.2. *For all $y \geq 0$ we have*

$$\mathbb{P}\Big(\inf_{t\in[0,T]} \widetilde{X}_t < 0 \mid X_T = y\Big) = \exp\Big(-2\frac{xy}{T\sigma^2}\Big).$$

Lemmas 2.1 and 2.2 imply that the difference of ruin probabilities χ and $\widetilde{\chi}$ satisfies

$$\begin{aligned}\chi - \widetilde{\chi} &= \int_0^\infty \Big[\mathbb{P}\Big(\inf_{t\in[0,T]} X_t < 0 \mid X_T = y\Big) - \mathbb{P}\Big(\inf_{t\in[0,T]} \widetilde{X}_t < 0 \mid X_T = y\Big)\Big]\mathbb{P}(X_T \in \mathrm{d}y) \\ &= \int_0^\infty \left[\exp\Big(-2\frac{xy}{T\gamma^2}\Big) - \exp\Big(-2\frac{xy}{T\sigma^2}\Big)\right]\frac{1}{\sqrt{2\pi\gamma^2 T}}\exp\Big(-\frac{(y-x)^2}{2\gamma^2 T}\Big)\,\mathrm{d}y \\ &\geq 0.\end{aligned}$$

Note that the difference is greater than zero if $\gamma > \sigma$. Thus we have obtained the following result.

Theorem 2.2. *It holds that*

$$\mathbb{P}\Big(\inf_{t\in[0,T]} \widetilde{X}_t < 0\Big) \leq \mathbb{P}\Big(\inf_{t\in[0,T]} X_t < 0\Big),$$

where we have equality if and only if the claim sizes $(\xi_i)_{i\geq 0}$ are deterministic.

Remark 2.2. The assumption made on the counting process N in the formulation of De Vylder and Goovaerts [14] differs from our Assumption 2.1. In [14], it is assumed that N is a *homogeneous risk process*, meaning that conditionally on $(N_T = n)$, the vector of jump times $(\tau_1, \ldots, \tau_n)$ has the same distribution as the increasing order statistics of n independent uniform random variables on $[0, T]$. In particular, this property is satisfied when N is a Poisson process. More details and properties of the homogeneous process are given in [10]. If we

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assume that $(N_t)_{t\in\mathbb{R}^+}$ is a homogeneous process, then according to Theorem A2 in [14], it is a mixed Poisson process. This entails, using the same lines of reasoning as in Chapter 4 of [6] (pages 65–67), that our limit results hold, with λ replaced by a structure random variable Λ. Then, by conditioning on this variable, we can still compare the ruin probabilities in the diffusion limits, as in the previous theorem.

Remark 2.3. All the results presented so far remain true if we omit the assumption that each random variable ξ_i is nonnegative but simply assume that $\mathbb{E}(\xi_i) \geq 0$.

2.3. The dynamics of the approximating diffusion

We close this section by characterizing $(\widetilde{U}_t)_{t\in[0,T]}$, the diffusion limit of the claim surplus process within the model with equalized claims, as a solution of a stochastic differential equation (SDE). First observe that the diffusion limit $(\widetilde{U}_t)_{t\in[0,T]}$ is a Gaussian process. Recall that the law of a one-dimensional Gaussian process $(Z_t)_{t\in[0,T]}$ is uniquely determined by its expectation function $m(t) := \mathbb{E}(Z_t)$ and covariance function $K(t, s) := \mathrm{Cov}(Z_t, Z_s)$. Hence, we can characterize the limiting law of $(m^{-1/2}\,\widetilde{S}_{tm})_{t\in[0,T]}$ as that of a Gaussian process with $m(t) = 0$ and

$$K(s, t) = (\gamma^2 - \sigma^2)\frac{st}{T} + \sigma^2(s \wedge t) \quad \text{for } s, t \in [0, T],$$

where the constants γ and σ are defined in (2.2) and (2.3). This also allows us to characterize the limiting law in terms of an SDE.

Proposition 2.3. *There exists a Brownian motion $(\widetilde{W}_t)_{t\in[0,T]}$ such that the process $(\widetilde{U}_t)$ satisfies the SDE*

$$\mathrm{d}\widetilde{U}_t = \widetilde{U}_t \frac{\gamma^2 - \sigma^2}{\sigma^2(T - t) + \gamma^2 t}\,\mathrm{d}t + \sigma\,\mathrm{d}\widetilde{W}_t \tag{2.10}$$

on $[0, T]$.

Proof. We first solve explicitly the SDE (2.10) for a given Brownian motion $(W_t)_{t\in[0,T]}$. To this end, let

$$f(t) = \frac{\gamma^2 - \sigma^2}{\sigma^2(T - t) + \gamma^2 t} \quad \text{for all } t \in [0, T].$$

Note that the SDE $\mathrm{d}Z_t = Z_t f(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$ with initial condition $Z_0 = 0$ has a unique solution given by

$$Z_t = \int_0^t \exp\left(\int_s^t f(u)\,\mathrm{d}u\right)\sigma\,\mathrm{d}W_s, \qquad t \in [0, T].$$

The solution process $(Z_t)_{t\in[0,T]}$ is a Gaussian process. The covariance structure of $(Z_t)_{t\in[0,T]}$ coincides with that of $(\widetilde{U}_t)_{t\in[0,T]}$. Indeed, for any $s, t \in [0, T]$, the Itô isometry implies

$$\mathrm{Cov}(Z_s, Z_t) = \int_0^{s\wedge t} \exp\left(\int_r^s f(u)\,\mathrm{d}u\right)\exp\left(\int_r^t f(u)\,\mathrm{d}u\right)\sigma^2\,\mathrm{d}r.$$

Note that

$$\exp\left(\int_r^k f(u)\,\mathrm{d}u\right) = \frac{\sigma^2 T + (\gamma^2 - \sigma^2)k}{\sigma^2 T + (\gamma^2 - \sigma^2)r} \quad \text{for any } k \in [r, T].$$

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Now a straightforward computation shows that

$$\mathrm{Cov}(Z_s, Z_t) = \sigma^2(s \wedge t) + (\gamma^2 - \sigma^2)\frac{st}{T}.$$

As a consequence, the process $(Z_t)_{t\in[0,T]}$ has the same law as the process $(\widetilde{U}_t)_{t\in[0,T]}$. Now define

$$\widetilde{W}_t := \frac{1}{\sigma}\left(\widetilde{U}_t - \int_0^t \widetilde{U}_s f(s)\,\mathrm{d}s\right).$$

Then the pair $(\widetilde{U}_t, \widetilde{W}_t)_{t\in[0,T]}$ has the same process distribution as $(Z_t, W_t)_{t\in[0,T]}$. In particular, $(\widetilde{W}_t)_{t\in[0,T]}$ is a Brownian motion and $(\widetilde{U}_t)_{t\in[0,T]}$ satisfies (2.10). □

Remark 2.4. The dynamics (2.10) can be derived from (2.5) via a projection argument. To explain this, let $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$ be the smallest filtration such that, first, the usual conditions of right-continuity and completeness are satisfied and, second, $(\widetilde{U}_t)_{t\in[0,T]}$ is adapted. Denote by $\widetilde{U}_t = \widetilde{M}_t + \widetilde{A}_t$ the Doob–Meyer decomposition of $(\widetilde{U}_t)_{t\in[0,T]}$ with respect to $(\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$. Assume that there exists an integrable $(\widetilde{\mathcal{F}}_t)$-predictable process α such that $\widetilde{A}_t = \int_0^t \alpha_s\,\mathrm{d}s$, $t \in [0, T]$. Since $\langle\widetilde{M}, \widetilde{M}\rangle_t = \langle\widetilde{U}, \widetilde{U}\rangle_t = \sigma^2 t$ for all $t \in [0, T]$, we conclude from Levy's theorem that $\widetilde{M}/\sigma$ is a Brownian motion with respect to $(\widetilde{\mathcal{F}}_t)$.

For the Brownian bridge Y, defined in (2.9), there exists a Brownian motion W^Y such that

$$\mathrm{d}Y_t = \mathrm{d}W_t^Y - \frac{Y_t}{T-t}\,\mathrm{d}t.$$

Since B_T is independent of Y_t, the process W^Y is also a Brownian motion with respect to the initially enlarged filtration $(\mathcal{G}_t)$, where $\mathcal{G}_t := \cap_{s>t}(\mathcal{F}_s^Y \vee \sigma(B_T))$. In particular, $(\widetilde{X}_t)_{t\in[0,T]}$ is a $(\mathcal{G}_t)$-Itô process with dynamics

$$\mathrm{d}\widetilde{U}_t = \gamma\frac{1}{T}B_T\,\mathrm{d}t - \frac{\sigma Y_t}{T-t}\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^Y = \frac{\gamma B_T - \widetilde{U}_t}{T-t}\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^Y.$$

Note that $W_t^Y = \widetilde{W}_t - \int_0^t \mu_s\,\mathrm{d}s$, where

$$\mu_t = -\frac{1}{\sigma}\left(\alpha_t - \frac{\gamma B_T - \widetilde{U}_t}{T-t}\right).$$

The process μ is the so-called information drift of $\widetilde{W}$ with respect to the enlarged filtration $(\mathcal{G}_t)$. It is known that the L^2-projection of μ onto the space of square-integrable and $(\widetilde{\mathcal{F}}_t)$-progressively measurable processes vanishes (see e.g. Section 1 in [1]). This entails that

$$\alpha_t = \widetilde{U}_t\frac{\gamma^2 - \sigma^2}{\sigma^2(T-t) + \gamma^2 t}, \qquad t \in [0, T],$$

and hence (2.10).

3. The conjecture in the heavy traffic approximation

We now assume that $\eta = c - \lambda\mathbb{E}(\xi_1) > 0$. By scaling time with $1/\eta^2$ and letting $\eta \downarrow 0$, the laws of the diffusion limits of $(S_t)_{t\in[0,T]}$ and $(\widetilde{S}_t)_{t\in[0,T]}$ are given by the following proposition.

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Proposition 3.1. *The process* $(\eta S_{t/\eta^2})_{t\in[0,T]}$ *converges in law on* $D[0,T]$ *to*

$$V_t := \gamma B_t - t, \qquad t \in [0,T],$$

and the process $(\eta\widetilde{S}_{t/\eta^2})_{t\in[0,T]}$ *converges in law on* $D[0,T]$ *to*

$$\widetilde{V}_t := \sigma B_t + (\gamma-\sigma)\frac{t}{T}B_T - t, \qquad t \in [0,T],$$

as $\eta \to 0$.

Proof. The proof is a straightforward adaptation of [3, Theorem 5.1] using the results from Proposition 2.1 and Theorem 2.1. □

In the heavy traffic approximation, the ruin probability ρ can be approximated by

$$\mathbb{P}\Big(\sup_{t\in[0,T]} V_t > x\Big)$$

and the ruin probability $\widetilde{\rho}$ by

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \widetilde{V}_t > x\Big).$$

Following arguments similar to those of Section 2.2, we have $V_T = \widetilde{V}_T$, and the conditional nonruin probability is equal to

$$\mathbb{P}\Big(\sup_{t\in[0,T]} V_t \le x \mid V_T = y\Big) = \mathbb{P}\Big(B_t \le \frac{x+t}{\gamma},\, 0\le t\le T | B_T = \frac{y+T}{\gamma}\Big), \qquad y < x.$$

In other words, the nonruin probability conditional to $V_T = y$ is equal to the probability that a Brownian bridge starting form 0 and going to $y/\gamma + T$ with length equal to T stays under the line $t \to (x+t)/\gamma$. Using the results of [13] and [4], we obtain

$$\mathbb{P}\Big(\sup_{t\in[0,T]} V_t \le x \mid V_T = y\Big) = 1 - \exp\Big(-2\frac{x(x-y)}{T\gamma^2}\Big).$$

Using arguments similar to those of Section 2.2, it follows that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \widetilde{V}_t \le x \mid V_T = y\Big) = 1 - \exp\Big(-2\frac{x(x-y)}{T\sigma^2}\Big),$$

and hence we obtain that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} V_t \le x \mid V_T = y\Big) - \mathbb{P}\Big(\sup_{t\in[0,T]} \widetilde{V}_t \le x \mid V_T = y\Big) \le 0.$$

This implies that the De Vylder–Goovaerts conjecture holds in the heavy traffic approximation.

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