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Rigidity for periodic magnetic fields

M. L. BIALY
School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel
(e-mail: bialy@math.tau.ac.il)

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Abstract. We study the motion of a charge on a conformally flat Riemannian torus in the presence of a magnetic field. We prove that for any non-zero magnetic field there always exist orbits of this motion which have conjugate points. We conjecture that the restriction of conformal flatness of the metric is not essential for this result. This would provide a 'twisted' version of the recent generalization of Hopf's rigidity result obtained by Burago and Ivanov.

1. *Introduction*

It was proved by Hopf [**9**] that for a Riemannian 2-torus with non-zero curvature there always exist geodesics with conjugate points. In [**5**] this result was finally generalized to higher dimensions. We refer the reader to [**6, 7, 14**] and [**10**] for a non-complete list of previous important contributions. It was discovered recently that the nature of Hopf rigidity is not entirely Riemannian. The reader should consult [**1, 2**] and also [**4**] where the rigidity results of this type are proved for some other classes of variational problems. In each case this requires some new integral geometric tool adapted for the system under consideration.

In this paper we give a proof of such a rigidity result for the motion of a charged particle on a torus in the presence of the magnetic field, provided the Riemannian metric is conformally flat. Notice that the nonlinearity of this problem comes through the magnetic field as well as the Riemannian metric. We show that for any non-zero magnetic field on the n-dimensional torus there are always orbits of the magnetic geodesic flow which have conjugate points (in a special case of dimension two, for constant magnetic field this result was obtained in [**12**]). This result gives the twisted version of a theorem by Knauf and Croke–Fathi [**6, 14**]. We formulate the conjecture that the restriction of conformal flatness is not essential for the result. This, if true, would provide a twisted version of the Burago–Ivanov theorem.

Our approach follows the original one by Hopf. The success of this approach relies heavily on a good choice of coordinates or of the field of horizontal Lagrangian subspaces

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of the cotangent bundle. It is not clear, however, if such a successful choice of coordinates is possible for the general, not necessarily conformally flat, case.

In the situation we consider, it would be reasonable to conjecture a more general fact. Namely, that there always exists a periodic orbit with conjugate points. This fact is, however, much more difficult to approach. It is not proved to the best of our knowledge even in the usual untwisted case. Moreover, it seems that the question of existence of the periodic orbit is still not completely answered for arbitrary magnetic fields on the torus. We refer the reader to a survey paper by Ginzburg [**13**] for known results and techniques.

2. *Main results*

A magnetic field on a torus $\mathbb{T}^n$ is a closed 2-form which we will denote by β. Consider the cotangent bundle $T^*\mathbb{T}^n$ and define the symplectic structure twisted by β

$$\omega = \omega_0 + \pi^*\beta,$$

where ω_0 is a standard structure and π is the canonical projection.

For a Riemannian metric g on $\mathbb{T}^n$, consider the Hamiltonian flow g^t of the function H

$$H = \tfrac{1}{2}\langle p, p\rangle_g$$

computed with the help of the symplectic structure ω. Recall that the no conjugate points condition for the orbit ϑ of g^t means that for any two points $x, y \in \vartheta$

$$g^t_* V(x) \pitchfork V(y),$$

where t is the time difference between y and x and $V(x)$, $V(y)$ are the vertical subspaces at x, y.

THEOREM 1. *Suppose that the Riemannian metric g is conformally flat. Then there always exist orbits of g^t on the level of $\{H = 1/2\}$ with conjugate points unless the 2-form β vanishes identically.*

Remark 1. In other words, the no conjugate points condition implies $\beta \equiv 0$ and then it follows from [**6**] and [**14**] that the metric is flat too. In fact, we will see it once more by our computations later.

The proof which is suggested in this paper follows the original scheme by Hopf. The first important ingredient of the proof is to construct a measurable field of Lagrangian subspaces $l(x) \subseteq T_x(T^*\mathbb{T}^n)$ for any $x \in \{H = 1/2\}$. This field can be constructed by the following limit procedure:

$$l(x) = \lim_{t \to +\infty} g^t_* V(g^{-t}(x)).$$

This construction was first used by Hopf [**9**] and Green [**11**] in the Riemannian case. We refer the reader to a recent paper [**8**] for the proofs in a general optical case.

It follows from the very construction of l that the field l is Lagrangian with respect to the symplectic structure ω. It is invariant under the flow g^t and is transversal to the vertical field V everywhere.

With the construction of l, Theorem 1 becomes a corollary of the following.

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THEOREM 2. *Let l be a measurable field of Lagrangian subspaces invariant under the flow g^t. If l is transversal to V everywhere then β vanishes identically, and the metric g is flat.*

This last theorem has the following dynamical interpretation.

THEOREM 3. *Suppose that the energy shell $\{H = 1/2\}$ is smoothly foliated by Lagrangian tori homologous to the zero section of $T^*\mathbb{T}^n$. Then the 2-form β vanishes identically and the metric is flat.*

Remark 2. In the paper [**14**], the situation described in Theorem 3 is called total integrability and simple examples of totally integrable magnetic geodesic flows are given. Let us remark that the Lagrangian tori in all these examples cannot be homologous to the zero section as it follows from Theorem 3. It is an interesting, completely open problem to characterize totally integrable magnetic geodesic flows.

3. *Proofs*

Let us explain first how Theorem 3 follows from Theorem 2. Since the tori are Lagrangian and homologous to the zero section, the 2-form β must be exact. Denote by α the primitive 1-form. Then one can easily see that the flow g^t is equivalent to the Hamiltonian flow $\tilde{g}^t$ of the function $\tilde{H} = \frac{1}{2}\langle p - \alpha, p - \alpha\rangle_g$ with respect to the standard structure ω_0. This equivalence is given by the diffeomorphism $(q, p) \to (q, p+\alpha)$ which is fibre preserving. Note that the function $\tilde{H}$ is strictly convex with respect to p. It follows from the generalized Birkhoff theorem (see [**3**] for its most general form and for the survey and discussions) that all the Lagrangian tori are the sections of the cotangent bundle. But then the distribution of their tangent spaces meets the conditions of Theorem 2.

Proof of Theorem 2. We shall work in standard coordinates (q, p) on $T^*\mathbb{T}^n$, such that the conformally flat Riemannian metric g is given by

$$ds^2 = \frac{1}{2\lambda}(dq_1^2 + \cdots + dq_n^2).$$

Then

$$H(p, q) = \frac{\lambda}{2}(p_1^2 + \cdots + p_n^2). \tag{3.1}$$

Write the 2-form of the magnetic field as $\beta = d\alpha + \gamma$, where α is a 1-form and γ is a 2-form, having constant coefficients in the coordinates $(q_1, \ldots, q_n)$, $\gamma = \sum_{i<j} \gamma_{ij}\, dq_i \wedge dq_j$. Consider the fibre-preserving diffeomorphism Φ of $T^*\mathbb{T}^n$,

$$\Phi(q, p) = (q, p + \alpha).$$

Then Φ transforms the symplectic form ω to another symplectic form $\tilde{\omega} = (\Phi^{-1})^*\omega$, where

$$\tilde{\omega} = \omega_0 + \gamma = \sum_{i=1}^{n} dp_i \wedge dq_i + \sum_{i<j} \gamma_{ij}\, dq_i \wedge dq_j.$$

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The transformed flow $\tilde{g}^t = \Phi g^t \Phi^{-1}$ is a Hamiltonian flow with respect to $\tilde{\omega}$ of the function $\tilde{H} = H \circ \Phi^{-1}$,

$$\tilde{H}(p,q) = H(p-\alpha, q) = \tfrac{1}{2}\lambda((p_1-\alpha_1)^2 + \cdots + (p_n-\alpha_n)^2).$$

Denote by $\tilde{l}$ the distribution induced by Φ, $\tilde{l} = \Phi_*(l)$. Then $\tilde{l}$ is, obviously, Lagrangian with respect to $\tilde{\omega}$ and invariant under the flow $\tilde{g}^t$. Since l, and hence also $\tilde{l}$, are transversal to V, then at every point $x = (q, p) \in \{\tilde{H} = 1/2\}$ the subspace $\tilde{l}(x)$ is given by a matrix $A(x)$:

$$dp = A(q, p)\, dq.$$

The condition to be Lagrangian with respect to $\tilde{\omega}$ is equivalent to

$$A^{\mathrm{T}} = A - \Gamma, \tag{3.2}$$

where Γ is the skew symmetric matrix of the form γ. Then A is a measurable matrix function. Moreover, the condition on $\tilde{l}$ to be invariant under the flow $\tilde{g}^t$ implies that A has to satisfy the following Riccati equation along the flow $\tilde{g}^t$:

$$\dot{A} + (A-\Gamma)\tilde{H}_{pp}A + (A-\Gamma)\tilde{H}_{pq} + \tilde{H}_{qp}A + \tilde{H}_{qq} = 0,$$

where $\tilde{H}_{pp}$, $\tilde{H}_{pq}$, $\tilde{H}_{qq}$ are the $n \times n$ matrices of second derivatives of $\tilde{H}$.

In order to perform this computation (cf., for example, [**15**]) one has to differentiate along the Hamiltonian equations of $\tilde{H}$ with respect to the symplectic form $\tilde{\omega}$, which have the form

$$\dot{q} = \tilde{H}_p, \quad \dot{p} = -\tilde{H}_q + \Gamma \tilde{H}_p.$$

This Riccati equation can be written in the form

$$\dot{A} + ((A-\Gamma)\tilde{H}_{pp}^{1/2} + \tilde{H}_{qp}\tilde{H}_{pp}^{-1/2})(\tilde{H}_{pp}^{1/2}A + \tilde{H}_{pp}^{-1/2}\tilde{H}_{pq}) + (\tilde{H}_{qq} - \tilde{H}_{qp}\tilde{H}_{pp}^{-1}\tilde{H}_{pq}) = 0.$$

Introduce the function $a(q, p) = \operatorname{tr} A(q, p)$. Since

$$(A-\Gamma)\tilde{H}_{pp}^{1/2} + \tilde{H}_{qp}\tilde{H}_{pp}^{-1/2} = (\tilde{H}_{pp}^{1/2}A + \tilde{H}_{pp}^{-1/2}\tilde{H}_{pq})^{\mathrm{T}},$$

then

$$\operatorname{tr}((A-\Gamma)\tilde{H}_{pp}^{1/2} + \tilde{H}_{qp}\tilde{H}_{pp}^{-1/2})(\tilde{H}_{pp}^{1/2}A + \tilde{H}_{pp}^{-1/2}\tilde{H}_{pq}) \geq 0.$$

Therefore, this implies that

$$\dot{a} + \operatorname{tr}(\tilde{H}_{qq} - \tilde{H}_{qp}\tilde{H}_{pp}^{-1}\tilde{H}_{pq}) \leq 0,$$

with equality possible only when $A = -\tilde{H}_{pp}^{-1}\tilde{H}_{pq}$. Integrate now the last inequality by t, $t \in [0, 1]$, and with respect to the invariant measure $\tilde{\mu}$ on the energy level $\{\tilde{H} = 1/2\}$. The integral of $\dot{a}$ vanishes, since the derivative is along the flow $\tilde{g}^t$ and the measure $\tilde{\mu}$ is invariant under this flow. Thus we obtain the following:

$$\sigma(\tilde{H}) = \int \operatorname{tr}(\tilde{H}_{qq} - \tilde{H}_{qp}\tilde{H}_{pp}^{-1}\tilde{H}_{pq})\, d\tilde{\mu} \leq 0, \tag{3.3}$$

with equality possible only if $A = -\tilde{H}_{pp}^{-1}\tilde{H}_{pq}$ almost everywhere on the level $\{\tilde{H} = 1/2\}$. On the other hand, we shall compute $\sigma(\tilde{H})$ by the following lemmas.

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LEMMA 1. *Let $H(p,q)$ be any Hamiltonian function which is convex and symmetric with respect to p, $H(p,q) = H(-p,q)$, and let $\tilde{H}(p,q) = H(p-\alpha(q),q)$ for any 1-form α. Then it follows that $\sigma(\tilde{H}) = \sigma(H)$. Here $\sigma(H)$ is computed by the integral (3.2) with respect to the Hamiltonian H and the invariant measure μ of g^t.*

LEMMA 2. *Let H be the Hamiltonian of a conformally flat metric given by (3.1). Then $\sigma(H) \geq 0$, if the dimension $n \geq 3$, and $\sigma(H) = 0$ in the two-dimensional case. Moreover, the equality in the case $n \geq 3$ is achieved only when λ is a constant.*

Remark 3. The inequality of the last lemma is, in fact, analogous to one proved in [**14**], but here it appears in the Hamiltonian form. The equality $\sigma(H) = 0$ in the 2-dimensional case is the reflection of the Gauss–Bonnet formula.

Let us complete the proof of the theorem, postponing those of the lemmas. Comparing (3.3) with Lemmas 1 and 2, we obtain that $\sigma(H) = \sigma(\tilde{H}) = 0$, and therefore

$$A = -\tilde{H}_{pp}^{-1}\tilde{H}_{pq} \tag{3.4}$$

holds almost everywhere on the energy level. In addition, the conformal factor λ has to be constant. Indeed, in the case $n \geq 3$ this is just the consequence of Lemma 2, while for $n = 2$ one can argue in the following way. Substitute A from (3.4) into equation (3.2). This gives the following equation:

$$-\tilde{H}_{pq}^{\mathrm{T}} + \tilde{H}_{pq} + \lambda\Gamma = 0.$$

Taking into account the explicit expression $\tilde{H} = \frac{1}{2}\lambda((p_1-\alpha_1)^2+(p_2-\alpha_2)^2)$, one computes immediately that the following equation is satisfied almost everywhere

$$\frac{1}{\lambda}\left(\frac{\partial}{\partial q_2}(\lambda\cdot(p_1-\alpha_1)) - \frac{\partial}{\partial q_1}(\lambda\cdot(p_2-\alpha_2))\right) = \text{constant}.$$

But the right-hand side does not depend on p at all, therefore λ must be a constant. We shall assume, without loss of generality, that λ is normalized to be 1.

Having $\lambda = 1$, one completes the proof of the theorem in both cases in the following way. For $\tilde{H} = \frac{1}{2}(\sum_{i=1}^{n}(p_i-\alpha_i)^2)$, equation (3.4) gives

$$A = (A_{ij}) = \left(\frac{\partial\alpha_i}{\partial q_j}\right), \quad i,j = 1,\ldots,n.$$

Thus equation (3.2) means precisely that

$$d\alpha + \gamma = 0.$$

Therefore, $\beta = 0$ and Theorem 2 follows. □

Proof of Lemma 1. We have to show that σ is the same for H and $\tilde{H} = H(p-\alpha(q),q)$. Direct computation gives the following expressions for the matrices of second derivatives:

$$\begin{gathered}\tilde{H}_{pp} = H_{pp},\\ \tilde{H}_{pq} = -H_{pp}D\alpha + H_{pq}, \quad \tilde{H}_{qp} = -(D\alpha)^{\mathrm{T}}H_{pp} + H_{qp},\\ \tilde{H}_{qq} = (D\alpha)^{\mathrm{T}}H_{pp}(D\alpha) - H_{qp}D\alpha - (D\alpha)^{\mathrm{T}}H_{pq} + H_{qq} + M.\end{gathered}$$

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Here

$$M = (M_{ij}), \quad M_{ij} = -\sum_{k=1}^{n} H_{p_k} \frac{\partial^2 \alpha_k}{\partial q_i \partial q_j}$$

and $D\alpha$ is the matrix of the derivatives of α, $(D\alpha)_{ij} = \partial \alpha_i / \partial q_j$. Then we obtain

$$(\tilde{H}_{qq} - \tilde{H}_{qp}\tilde{H}_{pp}^{-1}\tilde{H}_{pq}) = (H_{qq} - H_{qp}H_{pp}^{-1}H_{pq}) + M.$$

Here the left- and right-hand sides are evaluated at the points (q, p) and $(q, p - \alpha)$ respectively. Therefore, performing the change of variables $(q, p) \to (q, p - \alpha)$, one concludes

$$\sigma(\tilde{H}) = \sigma(H) + \int_{\{H=\frac{1}{2}\}} \sum_{k,i=1}^{n} H_{p_k} \frac{\partial^2 \alpha_k}{\partial q_i^2} \, d\mu.$$

By the symmetry assumption, the additional integral in the last formula vanishes because the integrand is an odd function while the level $\{H = \frac{1}{2}\}$ is even by the symmetry assumption on H. Thus $\sigma(\tilde{H}) = \sigma(H)$. An important fact which was used in this computation is that Φ transforms μ to $\tilde{\mu}$. Indeed the measures on the energy levels are defined by the rules

$$d\mu \wedge dH = \omega^n \quad \text{and} \quad d\tilde{\mu} \wedge d\tilde{H} = \tilde{\omega}^n.$$

However, it is easy to see that the volume forms of the symplectic structures $\omega, \tilde{\omega}, \omega_0$ are all equal. This completes the proof of the lemma. □

Proof of Lemma 2. The second derivatives for the Hamiltonian $H = \lambda/2(p_1^2 + \cdots + p_n^2)$ are given by the formulas

$$H_{p_i p_j} = \lambda \delta_{ij}, \quad H_{q_i q_j} = \tfrac{1}{2}(\lambda)_{q_i q_j}(p_1^2 + \cdots + p_n^2), \quad H_{p_i q_j} = \lambda_{q_j} p_i.$$

Thus for the points of the energy level $\{H = \frac{1}{2}\}$

$$\operatorname{tr}(H_{qq} - H_{qp}H_{pp}^{-1}H_{pq}) = \frac{1}{2\lambda}\Delta\lambda - \frac{1}{\lambda^2}(\operatorname{grad}\lambda)^2.$$

The invariant measure μ on $\{H = \frac{1}{2}\}$ is calculated in accordance with the condition

$$d\mu \wedge dH = \omega_0^n.$$

Then it can be easily computed and equals

$$d\mu = \left(\frac{1}{\sqrt{\lambda}}\right)^n d\sigma \, dq,$$

where $d\sigma$ is a standard measure on the unit sphere in p space. Then we have

$$\sigma(H) = \operatorname{Vol}(S^{n-1}) \int \left(\frac{1}{2\lambda}\Delta\lambda - \frac{1}{\lambda^2}(\operatorname{grad}\lambda)^2\right)\left(\frac{1}{\sqrt{\lambda}}\right)^n dq.$$

Integrating by parts we obtain

$$\sigma(H) = \operatorname{Vol}(S^{n-1})\left(\frac{n-2}{4}\right) \int \lambda^{-2-n/2}(\operatorname{grad}\lambda)^2 \, dq.$$

From this expression the assertion of Lemma 2 follows. □

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4. *Some open problems*

Let us discuss here some natural open problems.

In the proof of Theorem 3 we noticed first that the form β must be exact and then showed that it vanishes by the integral geometry argument. It might be that the weaker condition of existence of an invariant distribution of Lagrangian non-vertical subspaces also implies that the form β must be exact. We were not able, however, to find a direct proof of this fact.

It is an important problem to study those magnetic fields having integrable twisted geodesic flow. The simplest possible topology of the phase portrait is one for the totally integrable flows. It is not clear how to characterize those flows even for dimension two. One natural class of examples is for an S^1 symmetric magnetic field $\beta = \beta(q_1)\, dq_1 \wedge dq_2$. We do not know if there are other examples. Note that the Lagrangian tori of the foliation cannot be homologous to the zero section (Theorem 3).

One would like to generalize the results to the case of any Riemannian metric on $\mathbb{T}^n$. However, it may happen that the twisted version of the Burago–Ivanov proof should be found. It should be mentioned that our approach works each time when it is related to a good choice of coordinates in the phase space. It is not clear, however, if the coordinates of [**5**] are good in this sense.

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