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COACTIONS OF COMPACT GROUPS ON M_n

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Communicated by Aidan Sims

We dedicate this paper to the memory of our friend and colleague Iain Raeburn

Abstract

We prove that every coaction of a compact group on a finite-dimensional C^*-algebra is associated with a Fell bundle. Every coaction of a compact group on a matrix algebra is implemented by a unitary operator. A coaction of a compact group on M_n is inner if and only if its fixed-point algebra has an abelian C^*-subalgebra of dimension n. Investigating the existence of effective ergodic coactions on M_n reveals that SO(3) has them, while SU(2) does not. We give explicit examples of the two smallest finite nonabelian groups having effective ergodic coactions on M_n.

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1. Introduction

Coactions of locally compact groups on C^*-algebras were invented to generalize crossed-product duality to nonabelian groups. Coactions of abelian groups correspond to actions of the dual groups, via Fourier transforms, and the theories correspond remarkably closely. In particular, coactions of discrete groups correspond to actions of compact groups, and coactions of compact groups correspond to actions of discrete groups. Since actions of compact groups are the easiest, it is not surprising that coactions of discrete groups are also easy; in fact, they are essentially the same as Fell bundles (over discrete groups). However, actions of discrete groups can be quite hard and, correspondingly, coactions of compact groups can be expected to be hard too.

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So, we feel that it is a good time to look closely at coactions of compact groups. As a first step, we compensate by restricting the C^*-algebra to be finite-dimensional—in fact, as our title indicates, we concentrate on M_n. A long time ago, Iain Raeburn asked one of us exactly this question, which we now can answer.

As a reward, we show that coactions of compact groups on finite-dimensional C^*-algebras are given by Fell bundles (see Theorem 3.1). In fact, we prove this in the more general context of amenable G since it is no harder, and might be useful elsewhere. By the way, a Fell bundle over a locally compact group is required to have a compatible topology. However, every Fell bundle we encounter over our compact group G has only finitely many nonzero fibers, so the topology of G becomes irrelevant.

Next, since actions of discrete groups on M_n are unitarily implemented, it is natural to expect something along these lines for coactions of compact groups, and we prove this in Theorem 4.1 (see also [Was88, Theorem 3]). The key here is to note that the passage from actions of an abelian group G to coactions of its dual group $\widehat{G}$ involves passing from $C_0(G)$ to $C^*(G)$, and that the individual automorphisms of G can be recovered from $C_0(G)$ by composing with irreducible representations.

Inner coactions of G are determined by homomorphisms of $C(G)$, and we use this to characterize the inner coactions on M_n as those whose fixed-point algebras contain a maximal abelian subalgebra of M_n (see Theorem 5.1). We illustrate our techniques by considering inner coactions on M_2 and M_3 (Examples 5.3 and 5.4).

In some sense, at the opposite extreme from inner coactions are the ergodic ones, and we might as well assume that our ergodic coactions are effective (that is, every spectral subspace A_x is nonzero), and this forces us down to finite groups—indeed, those of order n^2 since our coactions are on M_n. In fact, the existence of an effective ergodic coaction of G on M_n is characterized by a result of Kleppner [Kle62] regarding 'regular elements'. The case of M_p with p prime is unsurprisingly quite special, forcing $G = \mathbb{Z}_p \times \mathbb{Z}_p$ (see Theorem 6.4). As an application of our techniques, we show that SU(2) has no ergodic coaction on any M_n (with $n > 1$), while SO(3) does (see Example 6.5). In Example 6.7, we give an explicit example of two finite nonabelian groups having effective ergodic coactions on M_n. The groups have order 16, and this is minimal.

In Section 7, we examine coactions on M_n of the smallest nonabelian (discrete) group S_3 and, in particular, we characterize the effective inner coactions that require $n \geq 4$ (see Proposition 7.1). Finally, in Section 8, we briefly indicate further possible research (and we comment 'Clearly, we have only scratched the surface').

2. Preliminaries

Throughout, G is a compact group with left Haar measure and $A, B, \ldots$ typically denote C^*-algebras. We refer to [EKQR06, Appendix A] for actions and coactions. If $\delta : A \to M(A \otimes C^*(G))$ is a coaction of G on A, we say that the pair (A, δ) is a coaction.

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Since G is compact, all coactions are normal (and maximal). Furthermore, the Fourier algebra $A(G)$ coincides with the Fourier–Stieltjes algebra $B(G)$, so it is a unital Banach *-algebra (with norm from the dual space $C^*(G)^*$, pointwise multiplication, and involution given by $f^*(x) = \overline{f(x^{-1})}$) that is dense in $C(G)$. Most of the background on $A(G)$ is contained in [Eym64]. Most importantly, [Eym64, Théorème 3.34] says that the dual (that is, the maximal ideal space) of $A(G)$ can be identified with G via

$$\chi_x(f) = f(x) \quad \text{for } f \in A(G), x \in G.$$

We use the general fact that for any C^*-algebra B, the dual space B^* can be regarded as the dual space of the multiplier algebra $M(B)$ when the latter is given the strict topology. The *zero set* of an ideal I of $A(G)$ is $\{x \in G : f(x) = 0 \text{ for all } f \in I\}$, and a *spectral set* in G is a closed subset E for which $\{f \in A(G) : f(x) = 0 \text{ for all } x \in E\}$ is the only closed ideal of $A(G)$ with zero set E.

Every coaction (A, δ) gives rise to a module structure on A over the Fourier algebra $A(G)$ via slicing:

$$f \cdot a := \langle \delta(a), \mathrm{id} \otimes f \rangle \quad \text{for } f \in A(G), a \in A.$$

The same formula also makes A a module over the Fourier–Stieltjes algebra $B(G)$.

If H is a closed subgroup of G, then any coaction of H on A can be *inflated* to a coaction of G on A, using the canonical homomorphism of $C^*(H)$ into $M(C^*(G))$ (see [EKQR06, Example A.29]).

We refer to [FD88b] for Fell bundles (called *C^*-algebraic bundles* in [FD88b]) $\mathcal{A}$ over G, and to [LPRS87] for the associated coaction $\delta_{\mathcal{A}}$—often called the *dual coaction*—on the full cross-sectional algebra $C^*(\mathcal{A})$. If (A, δ) is a coaction of G, then for $x \in G$, the *spectral subspace* is $A_x = \{a \in A : \delta(a) = a \otimes x\}$ and the *spectrum* of δ is $\operatorname{sp}\delta = \{x \in G : A_x \neq \{0\}\}$. If $\operatorname{sp}\delta = G$, we say the coaction is *effective*. Dangerous bend: the spectrum is not always a subgroup of G (see [Qui96, Example 5.3] for a counterexample with $G = \mathbb{Z}_4$; however, note that the Fell bundle described in [Qui96] could alternatively be regarded as coming from an effective coaction of $\mathbb{Z}_3$, as indicated in Example 5.3). When G is finite (or, more generally, discrete, or when the fibers A_x are only finitely nonzero), the family $\{A_x\}_{x \in G}$ forms a Fell bundle over G, and we say that δ is *associated with* $\mathcal{A}$ if $(A, \delta) \simeq (C^*(\mathcal{A}), \delta_{\mathcal{A}})$. Every coaction of a discrete group has an associated Fell bundle, but this is not automatic for arbitrary groups (see [LPRS87, Example 2.3(6)] for a counterexample with $G = \mathbb{T}$). When G is finite, the isomorphism classes of Fell bundles $\mathcal{A}$ and their associated coactions $\delta_{\mathcal{A}}$ are in bijective correspondence (see [Qui96, Theorem 3.8 and Corollary 3.9]).

The spectral subspace A_e is also denoted by A^δ and called the *fixed-point subalgebra*. A coaction δ is called *ergodic* if $A^\delta = \mathbb{C}1$. In this case, for every $x \in \operatorname{sp}\delta$, the spectral subspace A_x is spanned by a (nonunique) unitary U_x, and the spectrum $\operatorname{sp}\delta$ is a subgroup of G.

Suppose that δ is an effective ergodic coaction of G on A. Interestingly, for each $x \in G$, we get an automorphism $\operatorname{Ad} U_x$ of A and, in fact, $x \mapsto \operatorname{Ad} U_x$ is an action of G

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on A. When $A = M_n$ and $|G| = n^2$, this process can be reversed, and we get a bijection between ergodic coactions and ergodic actions of G on A.

We write w_G for the unitary element of

$$M(C(G) \otimes C^*(G)) = C(G, M^\beta(C^*(G)))$$

(where the 'β' signifies that we mean continuity with respect to the strict topology on the multiplier algebra) given by the canonical embedding $G \hookrightarrow M(C^*(G))$. If $\mu : C(G) \to M(A)$ is a unital homomorphism, then the unitary element

$$w_\mu := (\mu \otimes \mathrm{id})(w_G) \in M(A \otimes C^*(G))$$

satisfies

$$(\mathrm{id} \otimes \delta_G)(w_\mu) = (w_\mu \otimes 1)(\mathrm{id} \otimes \Sigma)(w_\mu),$$

where Σ is the 'flip automorphism' of $C^*(G) \otimes C^*(G)$ given on elementary tensors by $\Sigma(x \otimes y) = y \otimes x$, and where δ_G is the canonical coaction on $C^*(G)$ given on group elements by $\delta_G(x) = x \otimes x$. Unitaries satisfying this identity are called *corepresentations*. In the opposite direction, [NT79, Theorem A.1] proves that every corepresentation $w \in M(A \otimes C^*(G))$ is of the form w_μ for a unique homomorphism μ given by slicing:

$$\mu(f) = \langle w, \mathrm{id} \otimes f \rangle \quad \text{for } f \in A(G).$$

The relevance for us is that for any μ, there is an *inner coaction* δ of G on A given by

$$\delta(a) = \mathrm{Ad}\, w_\mu(a \otimes 1) \quad \text{for } a \in A. \tag{2-1}$$

Such coactions are called *unitary* in [LPRS87, Examples 2.3(3)] (and [Qui94, Lemma 1.11] shows that conditions (b) and (c) in [LPRS87] are redundant). More generally, if (A, δ) is any coaction, we say that a unitary $w \in M(A \otimes C^*(G))$ *implements* δ and that δ is *unitarily implemented*, if (2-1) holds. Of course, not all unitaries will implement coactions and, moreover, a moment's thought reveals that a coaction that is implemented by a unitary w might not be inner, since slicing w might not give a *-homomorphism of $A(G)$ (and hence of $C(G)$). For example, when G is finite abelian, we have ergodic actions that are implemented by unitaries twisted by 2-cocycles.

We need the fact that if $c \in M(C^*(G))$, then $\delta_G(c) = c \otimes c$ implies that $c \in G$. This is a folklore result concerning 'group-like elements', and is easy to verify by slicing and using

$$\langle \delta_G(c), f \otimes g \rangle = \langle c, fg \rangle \quad \text{for } f, g \in A(G)$$

together with the identification of G with the spectrum of $A(G)$.

We frequently need to work with unitary multipliers of $M(C^*(G))$ and, for convenience, we write

$$UM(C^*(G)) = \{u \in M(C^*(G)) : u \text{ is unitary}\}.$$

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In Section 4, we need to use the *destabilization* process: if A is an elementary algebra (that is, A is isomorphic to the compact operators on a Hilbert space) and $\iota : A \to M(B)$ is a nondegenerate homomorphism, we define the *relative commutant* of A in B as

$$C(B, \iota) := \{c \in M(B) : c\iota(a) = \iota(a)c \in B \text{ for all } a \in A\}.$$

Then, B is a stabilization in a functorial way: there is a unique isomorphism

$$\theta : A \otimes C(B, \iota) \xrightarrow{\simeq} B$$

given on elementary tensors by $\theta(a \otimes c) = \iota(a)c$. We call this construction the destabilization process; it is probably folklore, and is described explicitly in [BKQT23, Proposition A.2], [Exe17, Lemma 27.2], [Fis04, Section 3], [HRr98, Theorem 2.1], and [KOQ16, Proposition 3.4], for example.

3. Fell bundles

We show that every coaction of a compact group on a finite-dimensional C^*-algebra A is associated with a Fell bundle. In fact, the argument works just as well for amenable groups, and we state Theorem 3.1 in this context, since we feel that it might be useful elsewhere.

We need to use the fact that every finite subset of G is a spectral set for $A(G)$; this is implied by [War76, Theorem 4], since the empty set is a Ditkin set (a fact that is explicitly mentioned in the paragraph preceding Warner's theorem 1).

THEOREM 3.1. *Let δ be a coaction of an amenable group G on a finite-dimensional C^*-algebra A. Then, there is a Fell bundle $\mathcal{A}$ over G such that (A, δ) is isomorphic to the dual coaction $(C^*(\mathcal{A}), \delta_{\mathcal{A}})$.*

PROOF. Since G is amenable, we have $C^*(G) = C^*_r(G)$, and it follows that the Fourier algebra $A(G)$ separates the points of $M(C^*(G))$. We use the associated duality of the locally convex spaces $M(C^*(G))$ and $A(G)$. Let $a \in A$ be nonzero and put

$$(a)^\circ = \{\phi \in A(G) : \phi \cdot a = 0\}.$$

Note that

$$\delta(a) = \sum_{i=1}^{p} e_i \otimes c_i,$$

where $\{e_1, \dots, e_p\}$ is a basis of A and $c_i \in M(C^*(G))$. To ease the notational burden a little, put $D = \{c_1, \dots, c_p\}$. Because the e_i are linearly independent, it follows by slicing that $\phi \in (a)^\circ$ if and only if ϕ is in the annihilator $D^\perp$ in $A(G)$.

Now, $(a)^\circ$ is an ideal of $A(G)$ of finite codimension, so is contained in only finitely many maximal ideals. By [Eym64, Théorèm 3.34], the maximal ideals of $A(G)$ are of the form $I_x = \{\phi \in A(G) : \phi(x) = 0\}$, so the zero set of $(a)^\circ$ can be identified with a

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finite subset $E \subseteq G$. As we mentioned before this proof, E is spectral because it is finite (see [War76, Theorem 4]); consequently, the ideal $(a)^\circ$ coincides with the intersection

$$I_E = \bigcap_{x\in E} I_x = \{\phi \in A(G) : \phi(x) = 0 \text{ for all } x \in E\}.$$

Using the duality of locally convex spaces, we can interpret the above as saying that the two subsets D, E of $M(C^*(G))$ have the same annihilator in $A(G)$, that is,

$$D^\perp = E^\perp.$$

It then follows by the bipolar theorem that D is contained in the double polar of E. Since E is finite, this double polar is just the linear span. Thus, we have

$$D \subseteq \operatorname{span} E.$$

Thus,

$$\delta(a) \in A \otimes \operatorname{span} E,$$

and so there are $b_x \in A$ such that

$$\delta(a) = \sum_{x\in E} b_x \otimes x. \tag{3-1}$$

We have

$$\begin{aligned}\sum_{x\in E} \delta(b_x) \otimes x &= (\delta \otimes \mathrm{id}) \circ \delta(a)\\ &= (\mathrm{id} \otimes \delta_G) \circ \delta(a)\\ &= \sum_{x\in E} b_x \otimes x \otimes x.\end{aligned}$$

Since E is linearly independent in $M(C^*(G))$, we conclude that

$$\delta(b_x) = b_x \otimes x \quad \text{for all } x \in E,$$

that is, b_x is in the spectral subspace A_x.

Slicing (3-1) by $1 \in A(G)$, we conclude that $a = \sum_{x\in E} b_x$. Letting a run through a basis of A, we conclude that A is the sum of the spectral subspaces $\{A_x\}_{x\in \operatorname{sp}\delta}$, where $\operatorname{sp}\delta$ is the spectrum of the coaction δ. Now, the family $\mathcal{A} := \{A_x\}_{x\in G}$ is a Fell bundle and since A is finite-dimensional, it must coincide with the full cross-sectional algebra $C^*(\mathcal{A})$. □

REMARK 3.2. The above argument would not work for arbitrary G, because for nonamenable G, we have $A(G) \subseteq B_r(G)$, the reduced Fourier algebra of G that kills the kernel of the regular representation of G.

REMARK 3.3. In the proof of Theorem 3.1, we only needed the spectral subspaces A_x for x in the finite subset $\operatorname{sp}\delta$ of G. This need not be a subgroup, but when it is, the rest of G is in some sense superfluous: δ restricts to a coaction ε of $\operatorname{sp}\delta$ and the

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given coaction of G coincides with the inflation from $\operatorname{sp}\delta$ to G, as indicated in the commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & M(A \otimes C^*(\operatorname{sp}\delta)) \\ & \searrow^{\delta} & \downarrow \\ & & M(A \otimes C^*(G)) \end{array}$$

4. Unitary implementation

We give an elementary proof of an implementation theorem (proved for arbitrary type I factors in [Was88, Theorem 3], using more sophisticated machinery).

THEOREM 4.1. *Every coaction of a compact group on M_n is unitarily implemented.*

PROOF. Let $\delta : A \to M(A \otimes C^*(G))$ be a coaction where $A = M_n$. Choose a decomposition of $C^*(G)$ into a c_0-direct sum of matrix algebras:

$$C^*(G) = \bigoplus_{i \in S} M_i.$$

For each i, we have a central projection $p_i \in C^*(G)$ such that $p_i C^*(G) \simeq M_i$, and we fix such an isomorphism and in fact identify $p_i C^*(G) = M_i$. Then, we get a homomorphism

$$\delta_i = (1 \otimes p_i)\delta : A \to A \otimes M_i.$$

Since A is elementary, we can apply the destabilization process: write C_i for the relative commutant of $\delta_i(A)$ in $M(A \otimes M_i)$, so that

$$A \otimes M_i = \operatorname{span}\{\delta(A)C_i\} \simeq A \otimes C_i.$$

Since $A = M_n$, we see that C_i must also be a matrix algebra and, by counting dimensions, we must have $C_i \simeq M_i$. Fixing such an isomorphism, by composition, we get an automorphism $\phi_i \in \operatorname{Aut}(A \otimes M_i)$ such that

$$\phi_i(A \otimes 1) = \delta_i(A) \quad \text{and} \quad \phi_i(1 \otimes M_i) = C_i.$$

Since $A \otimes M_i$ is elementary, we can choose a unitary U_i in $A \otimes M_i$ such that $\phi_i = \operatorname{Ad} U_i$. It follows that

$$\delta_i(a) = \operatorname{Ad} U_i(a \otimes 1) \quad \text{for all } a \in A.$$

Now, we put them together: define a unitary

$$U = \sum_i U_i \in M(C^*(G)).$$

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Then, we have

$$\delta(a) = \sum_i \delta_i(a) = \sum_i \operatorname{Ad} U_i(a \otimes 1_{M_i}) = \operatorname{Ad} U(a \otimes 1_{M(C^*(G))}),$$

so that U implements δ. □

LEMMA 4.2. *With the above notation, if U is a unitary implementing δ, then the set of all unitaries implementing δ is*

$$\{U(1 \otimes u) : u \in UM(C^*(G))\}.$$

PROOF. This follows quickly from finding a u_i in each quotient by $1 \otimes p_i$, then assembling them. □

5. Inner

In the preceding section, we saw that all coactions of G on M_n are implemented by unitaries $U \in M(M_n \otimes C^*(G))$. The inner coactions of G on M_n are precisely those for which we take $U = (\mu \otimes \mathrm{id})(w_G)$ for a unital homomorphism $\mu : C(G) \to M_n$.

We pause to see what μ must look like. By the spectral theorem, we choose orthogonal minimal projections $p_1, \ldots, p_n$ and $x_1, \ldots, x_n \in G$ such that

$$\mu(f) = \sum_i f(x_i) p_i.$$

Moreover, any such choice of p_i and x_i gives a homomorphism $\mu : C(G) \to M_n$.

For any coaction δ of G on M_n, observe that A^δ is a unital C^*-subalgebra of M_n; we get some information by looking at its maximal abelian C^*-subalgebras.

THEOREM 5.1. *Let δ be a coaction of G on $A = M_n$ with $n > 1$. Then, δ is inner if and only if its fixed-point algebra M_n^δ has an abelian C^*-subalgebra of dimension n.*

PROOF. First suppose that M_n^δ has an abelian C^*-subalgebra of dimension n. Then, we can find rank-one projections $p_1, \ldots p_n \in A^\delta$ such that $\sum_i p_i = 1$. Choose matrix units e_{ij} such that $e_{ii} = p_i$ for $i = 1, \ldots, n$. By Theorem 4.1, we can choose a unitary U implementing δ. We can write

$$U = \sum_{i,j} e_{ij} \otimes y_{ij} \quad \text{with } y_{ij} \in M(C^*(G)).$$

For each k,

$$\begin{aligned} p_k \otimes 1 &= \delta(p_k) \\ &= \sum_{i,r,j,s} e_{ir} p_k e_{sj} \otimes y_{ir} y_{js}^* \\ &= \sum_{i,j} e_{ij} \otimes y_{ik} y_{jk}^*. \end{aligned}$$

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Because the e_{ij} are linearly independent, all terms except when $i = j = k$ are zero, and $y_{kk}y_{kk}^* = 1$, so y_{kk} is a unitary. However, with $i \neq k$ and $j = k$,

$$0 = y_{ik}y_{kk}^*,$$

so $y_{ik} = 0$ since $y_{kk} \in UM(C^*(G))$. Thus,

$$U = \sum_i p_i \otimes y_i,$$

where we have written $y_i = y_{ii}$ to simplify the notation. For all i, j,

$$\delta(e_{ij}) = \sum_{r,s} p_r e_{ij} p_s \otimes y_r y_s^* = e_{ij} \otimes y_i y_j^*.$$

The coaction identity implies that $\delta_G(y_i y_j^*) = y_i y_j^* \otimes y_i y_j^*$, so $y_i y_j^* \in G$ and $e_{ij} \in A_{y_i y_j^*}$.

Now, we adjust U so that y_i itself is in G: recall that if U' is any unitary implementing δ, then $U' = U(1 \otimes u)$ for some $u \in UM(C^*(G))$ and, moreover, every choice of $u \in UM(C^*(G))$ gives a suitable U'. If we take $u = y_n^*$, then $x_i := y_i y_n^* \in G$ for $i = 1, \ldots, n$ (and $x_n = e$) and, therefore, we now can replace the original U by

$$U = \sum_i p_i \otimes x_i$$

with $x_i \in G$.

We define a homomorphism $\mu : C(G) \to M_n$ by

$$\mu(f) = \sum_i p_i f(x_i)$$

and then

$$U = (\mu \otimes \mathrm{id})(w_G),$$

so δ is inner.

Conversely, suppose that δ is inner, determined by a unital homomorphism $\mu : C(G) \to A$. Choose mutually orthogonal minimal projections $p_1, \ldots, p_n \in M_n$ and $x_1, \ldots, x_n \in G$ such that

$$\mu(f) = \sum_i f(x_i) p_i.$$

Thus, δ is implemented by

$$(\mu \otimes \mathrm{id})(w_G) = \sum_i p_i \otimes x_i,$$

and a quick computation shows that

$$\delta(a) = \sum_{i,j} p_i a p_j \otimes x_i x_j^{-1}.$$

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Choosing matrix units e_{ij} such that $e_{ii} = p_i$, we have $e_{ij} \in A_{x_i x_j^{-1}}$. In particular, $p_i = e_{ii} \in A_e = A^\delta$ for all i, so the fixed-point algebra A^δ has an abelian C^*-subalgebra of dimension n. □

Paraphrasing the above theorem and its proof, we see that an inner coaction δ on $A = M_n$ is implemented by

$$U = \sum_1^n p_i \otimes x_i,$$

where $p_i = e_{ii}$ and $x_i \in G$. For the matrix units,

$$e_{ij} \in A_{x_i x_j^{-1}}.$$

Moreover, the x_i terms are only unique up to multiplying (on the right) by any fixed $x \in G$. Here are some easy consequences.

PROPOSITION 5.2. *With the above notation:*

(1) $e_{ij} \in A^\delta$ *if and only if* $x_i = x_j$;
(2) *for any particular pair* i, j, *if* $e_{ij} \in A^\delta$, *then without loss of generality, we can modify the* x *terms so that* $x_i = x_j = e$, *and then for all* $k \neq i, j$,

$$e_{ik} \in A_{x_k^{-1}}$$
$$e_{kj} \in A_{x_k}.$$

EXAMPLE 5.3. For inner coactions δ of G on $A = M_2$, since $e_{22} \in A^\delta$, the only choice is the element $x \in G$ for which $e_{12} \in A_x$, and then the spectrum is $\operatorname{sp}\delta = \{e, x, x^{-1}\}$. If $x = x^{-1}$, then $\operatorname{sp}\delta = \mathbb{Z}_2$. In general, $\operatorname{sp}\delta$ may not be a subgroup of G, unless $x^2 = x^{-1}$, in which case, $\operatorname{sp}\delta = \mathbb{Z}_3$. Note that if $x^2 \neq x^{-1}$, we still 'regard' δ as coming from an effective coaction of $\mathbb{Z}_3$, but this is of questionable value since G might not have $\mathbb{Z}_3$ as a subgroup.

Of course, if $x = e$, then δ is trivial and $A^\delta = A$.

EXAMPLE 5.4. For inner coactions δ of G on $A = M_3$, if, for example,

$$A^\delta = \operatorname{span}\{e_{ij} : 1 \leq i, j \leq 2 \text{ or } i = j = 3\},$$

then δ is implemented by a corepresentation of the form

$$U = e_{11} \otimes 1 + e_{22} \otimes 1 + e_{33} \otimes x$$

for some $x \in G$. This is a routine application of Proposition 5.2 and the discussion immediately preceding it.

6. Ergodic

Focusing on coactions of our compact group G on M_n, there are some immediate, easy facts.

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LEMMA 6.1. *If G has an effective ergodic coaction on M_n with $n > 1$, then:*

(1) *G has order $|G| = n^2$;*
(2) *if G is nonabelian, then $n > 2$;*
(3) *G is noncyclic.*

PROOF. (1) If δ is an ergodic coaction of G on $A = M_n$, then (see [KS82]) every spectral subspace A_x is spanned by a unitary, and the subspaces $\{A_x : x \in G\}$ are linearly independent, so $|G| = n^2$.

(2) This follows immediately from statement (1).

(3) This follows quickly from the triviality of the 2-cohomology $H^2(G, \mathbb{T})$ when G is cyclic; see [FD88a, Proposition III.5.11] for example. However, here is a quick and dirty argument. Suppose G is cyclic with generator x, and pick a unitary $u \in A_x$. Then, $\{u^i : i = 1, \ldots, n\}$ is a choice of unitaries in the spectral subspaces, from which it follows that M_n is commutative, which is absurd. □

Lemma 6.1 is almost trivial. The (seemingly) slightly stronger Proposition 6.2 below is easy but relies on somewhat more sophisticated technology.

In our study of effective ergodic coactions on M_n, we need to make contact with finite groups of *central type* (we thank Erik Bédos for informing us of this terminology and the references [GS19, Sch16]), defined in the literature as groups having a simple twisted group algebra; the motivation for the terminology has to do with central group extensions. Groups of central type have been of increasing interest in the past few decades. The abelian groups of central type are quite well understood. The smallest nonabelian examples have order 16, where there are two nonisomorphic groups of central type. In Example 6.7, we discuss a couple of examples that we found in [Sch16].

We need a bit of the theory of 2-cocycles and we adopt the conventions of [Kle62]. Keep in mind that for any ergodic coaction δ of a finite group G on a unital C^*-algebra A, each spectral subspace A_x is spanned by a unitary, and any choice of a spanning set of unitaries $\{U_x\}_{x\in G}$, where we always choose $U_e = 1$, gives a projective representation of G in A that by definition has an associated 2-*cocycle* $\omega \in Z^2(G, \mathbb{T})$ determined by

$$U_x U_y = \omega(x, y)U_{xy} \quad \text{for } x, y \in G,$$

and that satisfies:

(1) $\omega(x, e) = \omega(e, x) = 1$ for all $x \in G$ and
(2) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ for all $x, y, z \in G$.

Here, U as above is called an *ω-representation*.

As explained in [FD88b, Propositions VIII.10.10 and VIII.16.2], ω-representations of G, representations of a cocycle bundle $\mathcal{A}$ with cocycle ω, and central extensions of $\mathbb{T}$ by G correspond up to isomorphism. Choosing the spanning unitaries U_x is almost the same as choosing the associated cocycle ω—the unitaries determine the

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cocycle uniquely, but a different choice of unitaries could lead to the same cocycle. Two cocycles ω and ω' are *cohomologous* if there is a map $m : G \to \mathbb{T}$ such that

$$\omega'(x, y) = m(x)m(y)\overline{m(xy)}\omega(x, y) \quad \text{for } x, y \in G,$$

and ω' is *trivial* if it is cohomologous to $\omega = 1$. Every cocycle is cohomologous to a *normalized* one, meaning that

$$\omega(x, x^{-1}) = 1 \quad \text{for all } x \in G,$$

and we always assume that our cocycle ω is normalized. Choosing ω to be normalized is the same as choosing the spanning unitaries so that

$$U_x^* = U_{x^{-1}} \quad \text{for all } x \in G.$$

A routine computation shows that for ω normalized,

$$\omega(x, y)^{-1} = \omega(y^{-1}, x^{-1}) \quad \text{for all } x, y \in G.$$

There, in general, are many ω-representations for any given cocycle ω. When G is finite, there are ω-representations U that are *universal* in the sense that for any ω-representation V, there is a unique homomorphism π_V making the diagram

$$\begin{array}{ccc} G & \xrightarrow{U} & C^*(U(G)) \\ & \searrow^{V} & \downarrow{\scriptstyle \pi_V} \\ & & C^*(V(G)) \end{array}$$

commute. As usual, the pair $(C^*(U(G)), U)$ is unique up to isomorphism and we write $C^*_\omega(G)$ for any such algebra $C^*(U(G))$, and call it the *twisted group C^*-algebra* for ω (although we do not, in general, give a special notation to the associated U). One choice of such a universal ω-representation is the *right regular ω-representation* [Kle62, page 556] R of G on $L^2(G)$, defined by

$$(R_x\xi)(y) = \omega(y, x)\xi(yx).$$

We have

$$C^*_\omega(G) = \operatorname{span} U(G)$$

for any universal U, so

$$\dim C^*_\omega(G) = |G|.$$

Also, any ω-representation V is universal if and only if the associated homomorphism $\pi_V : C^*_\omega(G) \to C^*(V(G))$ is faithful, if and only if the unitaries $\{V_x : x \in G\}$ are linearly independent.

Later, we are interested in the possibility that $C^*_\omega(G)$ is simple, that is, is isomorphic to M_n for some n. This obviously puts restrictions on G—for example, the order $|G|$ must be n^2. However, this alone is insufficient. It seems natural to say that G *has*

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a simple twisted C^*-algebra if there exists a cocycle ω for which $C^*_\omega(G)$ is simple; equivalently, there exists an irreducible projective representation U of G such that the unitaries $\{U_x\}_{x\in G}$ are linearly independent. In the literature, such a group is said to be of *central type*. There is a subtlety: the literature also contains numerous studies of *faithful* irreducible projective representations U, meaning that the composition of U with the quotient map $\mathrm{U}(n) \to \mathrm{PU}(n)$ is faithful; equivalently, the only $x \in G$ for which U_x is a scalar operator is $x = e$. This is weaker than the unitaries $\{U_x\}_{x\in G}$ being linearly independent. For example, it follows from [Ng76, Theorem 4.4] that there are groups of order 16 with faithful irreducible projective representations U on $\mathbb{C}^2$ and, trivially, $\{U_x\}_{x\in G}$ must be linearly dependent. Interestingly, these groups are nonabelian. This behavior cannot occur for abelian groups, because Frucht [Fru32] proved that a finite abelian group has a faithful irreducible projective representation if and only if it is isomorphic to $H \times H$ for some (abelian) group H, and by [Kle62, paragraph following Corollary 1], such a group does have a simple twisted C^*-algebra. The smallest nonabelian finite groups of central type have order 16.

PROPOSITION 6.2. *If G has an ergodic coaction on M_n with $n > 1$, then G has a finite abelian noncyclic subgroup.*

PROOF. Suppose that δ is an ergodic coaction on $A = M_n$. Then, the spectrum $\operatorname{sp}\delta$ is a subgroup of order n^2, and without loss of generality, we suppose that $G = \operatorname{sp}\delta$. Choose spanning unitaries U_x for the spectral subspaces A_x and let ω be the associated 2-cocycle. Then, U is an irreducible ω-representation of G. In [Kle62, page 559], Kleppner defines $x \in G$ to be *ω-regular* if

$$\omega(x, y) = \omega(y, x) \quad \text{whenever } xy = yx,$$

and it follows from [Kle62, Corollary 1] that $G \setminus \{e\}$ contains no ω-regular elements. Thus, picking any $x \neq e$, we find $y \neq e$ such that $xy = yx$ and $\omega(x, y) \neq \omega(y, x)$. Let $H = \langle x, y\rangle$ be the abelian subgroup generated by x and y, so ω restricted to $H \times H$ is a nontrivial cocycle. As we mentioned in the preceding proof, H must be noncyclic. □

REMARK 6.3. When G is abelian, Kleppner's characterization mentioned in the above proof can actually be reformulated as [OPT80, Theorem 5.9]. Even if G is not assumed to be abelian, it is possible to give a characterization generalizing that in [OPT80].

PROPOSITION 6.4. *If G has an irreducible projective representation U on $\mathbb{C}^p$ with p prime and $\{U_x : x \in G\}$ linearly independent, then $G = \mathbb{Z}_p \times \mathbb{Z}_p$. With $\gamma = \exp(2\pi i/p)$, then up to equivalence,*

$$U(1, 0) = \operatorname{diag}(\gamma^j : j = 1, \ldots, p) \quad \textit{and} \quad U(0, 1) = \{\delta(j, j + 1) : j \in \mathbb{Z}_p\}.$$

PROOF. From the assumptions, there are $x, y \in G$ with $yx = xy$ and

$$\omega(x, y) \neq \omega(y, x).$$

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Since $|G| = p^2$, the order of x and y must be 1, p, or p^2 (remember that we assume G is finite). One quickly rules out 1 and p^2. Then, one realizes that one can pick x, y such that

$$\omega(x,y)\overline{\omega(y,x)} = \exp(2\pi i/p) =: \gamma.$$

We must have $H \cap K = \{e\}$, so $|HK| = |H|\,|K| = p^2 = |G|$ and, hence, $G = \mathbb{Z}_p \times \mathbb{Z}_p$. We leave as an exercise to show that U as described is (up to equivalence) the only irreducible projective representation of $\mathbb{Z}_p \times \mathbb{Z}_p$ satisfying the theorem. □

EXAMPLE 6.5. It follows from Proposition 6.2 that SU(2) has no ergodic coactions on any M_n with $n > 1$, because every finite abelian subgroup is cyclic (folklore). Interestingly, the quotient group SO(3) has an ergodic coaction on M_2, because it contains the Klein-4 group $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a subgroup; for example, the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are suitable generators.

REMARK 6.6. Reference [Kle62, Discussion following Corollary 1] shows that if H is any abelian group of order n, then there is a 2-cocycle ω on $G = H \times \widehat{H}$ given by

$$\omega((x,\chi),(y,\sigma)) = \chi(y),$$

and no element of $G \setminus \{e\}$ is ω-regular, so G is of central type and hence has an effective ergodic coaction on M_n. Note that the set-up in Proposition 6.4 is a special case. In fact, Frucht proved in [Fru32] that a finite abelian group has a faithful irreducible projective representation if and only if it is isomorphic to the square of an abelian group, in which case, it is of central type, as we explained in the preliminaries. This was rediscovered somewhat later by Olesen *et al.* [OPT80, Theorem 5.9].

Ng [Ng76] generalized this to nonabelian groups; his setting was primarily metacyclic groups (extensions of cyclic by cyclic). Reference [Ng76, Proposition 4.1] shows that if G is such a group, with generators a and b, if there is a faithful irreducible projective representation, then by [Ng76, Theorem 5.2], G is the semidirect product of $\langle a\rangle$ and $\langle b\rangle$. Moreover, if $|G|$ is a power of an odd prime, then G has a faithful irreducible projective representation if and only if the subgroups $\langle a\rangle$ and $\langle b\rangle$ are isomorphic. So, for G to be nonabelian, the smallest such example would have order $81 = 3^4$.

EXAMPLE 6.7. By [Sch16, Theorems 3.8 and 3.9], there are two nonabelian groups of central type of order 16 (the smallest possible). They are as follows, with examples of irreducible projective representations of degree 4.

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- $G = \langle a, b, c \mid a^4 = b^2 = c^2 = e, ab = ba, bc = cb, cac^{-1} = a^{-1} \rangle$. For a suitable projective representation, take

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

  Then,

$$AB = -BA, \quad A^4 = B^2 = C^2 = I, \quad BC = CB, \quad CAC^{-1} = iA^{-1}.$$

- $G = \langle a, b, c \mid a^4 = b^2 = c^2 = e, ab = ba, bc = cb, cac^{-1} = ab \rangle$. Take the same A, B as above, but now take

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -i & 1 \\ 0 & 0 & 1 & -i \\ i & 1 & 0 & 0 \\ 1 & i & 0 & 0 \end{pmatrix}.$$

  The only new relation is $CAC^{-1} = AB$.

It seems there are rather few finite nonabelian groups of central type. For example, in [GS19, Theorem 3.21], it is shown that if G is nonabelian of central type and $|G| = p^2q^2$ with primes $p < q$, then the number of nonisomorphic possibilities is one if $pq \neq 6$ and two if $pq = 6$.

7. $G = S_3$

It is interesting to examine coactions δ of the smallest nonabelian group S_3 (the permutations of a set of three elements) on $A = M_n$. We fix a presentation

$$S_3 = \langle \alpha, \beta \mid \alpha^3 = \beta^2 = e, \alpha\beta = \beta\alpha^2 \rangle.$$

Since all proper subgroups of S_3 are cyclic, to avoid trivialities, we look for effective coactions. By Lemma 6.1, this requires $n \geq 3$, and we cannot expect δ to be ergodic.

Here is where we can make some progress: look for effective *inner* coactions. As we have observed, we need $n \geq 3$. However, even $n = 3$ will not work: in the notation of the proof of Theorem 5.1, we have $\operatorname{sp}\delta = \{x_i x_j^{-1}\}$ with $x_i \in S_3$, and we can assume $x_3 = 1$. So, we can choose x_1, x_2, and then we have $e_{12} \in A_{x_1 x_2^{-1}}$. Since S_3 has three self-inverse elements, trivial computations show that we can have at most three non-identity elements in $\operatorname{sp}\delta$, which is absurd. Therefore, we cannot have an effective inner coaction of S_3 on M_3.

Therefore, an effective inner coaction of S_3 on M_n requires $n \geq 4$. Then it is easy to generate examples: as we mentioned above, we might as well choose $x_4 = 1$. Then, for example, we can choose

$$x_1 = \alpha, \quad x_2 = \beta \quad \text{and} \quad x_3 = \alpha\beta.$$

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Then, $\{x_i x_j^{-1} : i,j = 1,\dots,3\}$ contains

$$x_1 x_2^{-1} = \alpha\beta \quad \text{and} \quad x_2 x_3^{-1} = \beta\alpha = \alpha^2\beta,$$

and we have accounted for all six elements of S_3. Trivially, we could have made other choices and we could do something similar for any $n \geq 5$ as well. We can conclude as follows.

PROPOSITION 7.1. *For any $n \geq 4$, the above procedure gives all effective inner coactions of S_3 on M_n, but there are none for $n < 4$.*

8. Conclusion

Clearly, we have only scratched the surface here; there are many aspects of compact coactions on M_n that need more investigation. For example, Section 5 leads to the following train of thought: the center of the fixed-point algebra A^δ is spanned by projections $\{p_i\}_{i\in S}$, and the coaction δ restricts to the corners $p_i A p_i$. However, a little more thought reveals that we have a 'coaction' on a Fell bundle $\{p_i A p_j\}_{i,j\in S}$ over the groupoid $S \times S$ with multiplication $(i,j)(j,k) = (i,k)$. It is tantalizing to think that this observation could yield significant further insight into the coaction δ.

Also, coactions of compact groups have been recently shown to be *Pedersen rigid* in the sense that the outer conjugacy class is completely determined by the isomorphism class of the dual action (see [KOQT23]). How is this related to our results in this paper?

A next step in the study of coactions of the compact group G is to 'ramp up' from M_n to $\mathcal{K}$ (the compact operators on a separable infinite-dimensional Hilbert space). Perhaps one should even investigate more general AF-algebras in this connection? In particular, how about coactions of G on UHF algebras, for example, $M_n \otimes M_n \otimes \cdots$, investigated by Masuda in [Mas08]?

If U is a unitary implementing the coaction δ as in Theorem 4.1, it is possible to show that there is a unique unitary $u \in M(C^*(G) \otimes C^*(G))$ such that

$$(\mathrm{id} \otimes \delta_G)(U) = U_{12} U_{13} (1 \otimes u).$$

This looks like some sort of '2-cocycle' for U, and perhaps it would be fruitful to regard U as a sort of 'twisted corepresentation'. This would perhaps lead to a cohomology theory of coactions. In fact, this is explored in [Lan92, Was88]. The ultimate goal here would be to classify coactions of a compact group G on matrix algebras M_n.

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