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# WHEN DO BIRKHOFF OBJECTS SEPARATE?

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*We dedicate this paper to the memory of Fatima Fenaroli Lawvere.*

**Abstract.** We show that, for any pre-cohesive presheaf topos, Birkhoff objects separate if, and only if, the *pseudo-constants* of every object in the site are jointly epic. In the examples where the sites are duals of categories of finitely generated algebras, the condition above holds by Birkhoff's theorem (as in the classifier of directly indecomposable complex algebras) or by restricting to algebras with 'suitable' subdirect representations (such as W-determined $C^\infty$-rings).

**§1. Introduction.** Lawvere's 2008 paper [9] outlines how the construction of 'gros' toposes in algebraic geometry may be generalized from rings to varieties of rigs explaining, for example, how the theory of (bounded) distributive lattices determines, just as the theory of rings does, a 'gros' topos such that the embedding of 'affine spaces' preserves, not only finite limits but also, finite coproducts. The key ingredient is *extensivity* of the relevant sites but, naturally, this is not the only property and the examples suggest the existence of further general concepts that deserve attention [10]:

> From Hilbert's theorem of zeroes, and from Noether's ideal theory, Birkhoff [2] derived certain algebraic concepts (as explained by Tholen [21]) that have a dual significance in general toposes, similar to their role in the original examples of algebraic geometry.

In order to make the significance explicit, Birkhoff objects (in toposes 'of spaces') are introduced in [10, Definition 4.1], and it is proved in [10, Proposition 5.1] that, in the pre-cohesive topos of reflexive graphs, an object is Birkhoff if, and only if, it is $\neg\neg$-separated. It follows from this result that Birkhoff objects form a separating subcategory, simply because representable reflexive graphs are $\neg\neg$-separated.

Our purpose is to further investigate Birkhoff objects in axiomatic cohesion [8], and their relation to Birkhoff's theorem [2] which implies that, in categories of algebras for a Lawvere theory, subdirectly irreducible (sdi) objects coseparate. More specifically, we propose to study, for a pre-cohesive geometric morphism $\mathcal{E} \to \mathcal{S}$, the condition that Birkhoff objects in $\mathcal{E}$ separate.

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The similar conditions

| sdi objects coseparate | Birkhoff objects separate |
|---|---|
| (in algebraic categories) | (in pre-cohesive toposes) |

are not dual to each other, but they are related, as illustrated by our results and examples.

Our main result characterizes when Birkhoff objects separate in pre-cohesive presheaf toposes. Through examples we illustrate how it relates to Birkhoff's theorem and similar phenomena, such as the consideration of algebras that arise as subobjects of products of a restricted class of algebras that need not be sdi.

In §2 we consider separating subcategories of presheaf toposes, and recall Birkhoff's theorem. §3 describes the basic ingredients of the examples that will be used to illustrate the main results. In §4 we introduce a generalization of Birkhoff objects that allow us to prove some simple but relevant closure conditions. In §5 we recall the canonical intensive quality of a hyperconnected and local geometric morphism. The two previous sections are combined in §6 to recall the definition of Birkhoff object and derive some of their basic properties. The main results are proved and illustrated in §7.

**§2. Separating subcategories and Birkhoff's theorem.** To avoid any confusion let us recall the well-known notion of separation.

DEFINITION 2.1. *Fix a full subcategory $\mathcal{G}$ of the category $\mathcal{C}$. A pair of maps $f, g : X \to Y$ in $\mathcal{C}$ is* equalized by $\mathcal{G}$ *if, for every $G$ in $\mathcal{G}$ and $x : G \to X$, $fx = gx : G \to Y$. The subcategory $\mathcal{G}$ is called* separating *if any two maps in $\mathcal{C}$ equalized by $\mathcal{G}$ are actually equal.* (*We may also say that the objects in $\mathcal{G}$* separate.)

For instance, the Yoneda functor, embedding a small category into the corresponding presheaf topos, is separating.

In what follows we consider a presheaf topos over a small category $\mathcal{D}$, and we distinguish here *classes* and *sets* in the usual way. It is a well-known fact that a family of morphisms $\{f_i : P_i \to P\}_{i\in I}$, $I$ a set, is jointly epic if for every $x \in PD$, $D \in \mathcal{D}$, there exists an $i \in I$ such that $x$ is in the image of $(\varphi_i)_D : P_iD \to PD$. This follows basically from the fact that the condition on the family is equivalent to the morphism $\sum_i P_i \to P$, induced by the $f_i$'s, being epi, and epis in a presheaf category are pointwise onto. The same result for classes is surely folklore, but since we were unable to locate an appropriate reference in the literature, we provide a proof.

LEMMA 2.2. *A family $\{f_i : P_i \to P\}_{i\in I}$ of morphisms in a presheaf topos indexed by a* class *$I$ is jointly epimorphic if and only if for every $x \in PD$, $D \in \mathcal{D}$, there exists an $i \in I$ such that $x$ is in the image of $(\varphi_i)_D : P_iD \to PD$.*

*Proof.* One direction is clear. For the other, assume that $F = \{f_i : P_i \to P\}_{i\in I}$ is jointly epic. Because we are in a presheaf topos there is, for each $i \in I$, a jointly epic set $E_i = \{g : R_g \to P_i\}$ of maps with representable domain. The class

$$H := \{f_i g : R_g \to P | i \in I, g \in E_i\}$$

consists of maps with domain in the Yoneda embedding $\mathcal{D} \to \widehat{\mathcal{D}}$, so the family $H$ is actually a set because $\mathcal{D}$ is small and $\widehat{\mathcal{D}}$ is locally small. Since jointly epic families compose, $H$ is set and it is jointly epic. The result then follows by the remarks previous to the statement of the lemma. □

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PROPOSITION 2.3. *A full subcategory of a presheaf topos is separating if, and only if, it contains all representables up to idempotent splitting.*

*Proof.* One direction follows from the well-known fact that representables separate. For the converse, let $\mathcal{D}$ be a small category and let $\mathcal{G} \to \widehat{\mathcal{D}}$ be a full and separating subcategory. Let $D$ be an object in $\mathcal{D}$. Since $\mathcal{G} \to \widehat{\mathcal{D}}$ is separating, the class

$$F = \{f : G_f \to \mathcal{D}(-, D) \mid G_f \in \mathcal{G}\}$$

of maps to $\mathcal{D}(-, D)$, with domain in the subcategory $\mathcal{G} \to \widehat{\mathcal{D}}$, is jointly epic. By Lemma 2.2, it follows that there exists an $f : G_f \to \mathcal{D}(-, D)$ in $F$ such that $\mathrm{id}_D$ is in the image of $f_D : G_f D \to \mathcal{D}(D, D)$. Thus $f$ is epi, and since $\mathcal{D}(-, D)$ is projective, it follows that $\mathcal{D}(-, D)$ is a splitting of an idempotent on $G_f$. □

Coseparating subcategories will also play a role in the paper.

Fix a category $\mathcal{A}$ and let B be a class of maps in $\mathcal{A}$.

DEFINITION 2.4. *A* B-representation *of an object $A$ in $\mathcal{A}$ is a jointly monic set of maps in* B *with domain $A$.*

To recall Birkhoff's theorem it is convenient to introduce some terminology.

DEFINITION 2.5 (Subdirect representations). *If we let* B *be the class of regular epimorphisms in $\mathcal{A}$, then* B*-representations will be called* subdirect representations.

DEFINITION 2.6. *An object $A$ in $\mathcal{A}$ is* sdi *if every subdirect representation of $A$ contains an isomorphism.*

If $\mathcal{A}$ has a terminal object then it is not sdi because the empty family of quotients is a subdirect representation of 1.

THEOREM 2.7 (Birkhoff [2]). *In any algebraic category, every object $A$ has a subdirect representation $F$ such that, for every $f : A \to A_f$ in $F$, $A_f$ is sdi.*

COROLLARY 2.8. *In any algebraic category, the full subcategory of sdi algebras is coseparating.*

We recall in passing that sdi algebras are directly indecomposable [3, Theorem 8.5].

**§3. Determination.** Birkhoff's theorem directs our attention to the class of regular epimorphisms with sdi codomain but, for several reasons, a different choice of maps could be interesting. Between the class of maps used in that result, and an arbitrary class of maps, there are classes that we could call 'principal'. Indeed, any full subcategory $\mathrm{P} \to \mathcal{A}$ determines the 'principal' class $\mathrm{B}_\mathrm{P}$ of maps in $\mathcal{A}$ with codomain in P.

DEFINITION 3.1. *An object of $\mathcal{A}$ is* P-determined *if it has a* $\mathrm{B}_\mathrm{P}$*-representation.*

EXAMPLE 3.2 (0-determined algebras). *Somewhat abusing the notation, let* 0 *be the full subcategory determined by the initial object. It follows from* [2, *lemma* 1] *that in the categories of Boolean algebras and of distributive lattices, all objects are* 0*-determined.*

In other words, in the categories mentioned above, the initial object is a coseparator.

Let **Ring** be the coextensive category of (commutative and unitary) rings.

EXAMPLE 3.3 (Regular rings). *Recall that a ring is* regular *if for any $a$ there is a $u$ such that $aua = a$. It follows from* [2, *Corollary* 4] *that regular rings are* P*-determined, where* $\mathrm{P} \to$ **Ring** *is the full subcategory of fields.*

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Birkhoff warns that the fact that sdi rings without nilpotents are fields [2, lemma 2], does not imply that a ring without nilpotents is a subdirect product of fields. "It is not necessarily true that a homomorphic image of rings without nilpotent elements is itself without nilpotent elements."

For the next example recall that the category of $C^\infty$-rings is the category of models of the Lawvere theory of finite powers of $\mathbb{R}$ and smooth maps between them. Since polynomials with coefficients in $\mathbb{R}$ determine smooth functions, there is an obvious forgetful functor from the category of $C^\infty$-rings to that of $\mathbb{R}$-algebras. See [6, 19] for detailed information.

EXAMPLE 3.4 (0-determined $C^\infty$-rings). *These are called* point determined *in* [6, *Definition III*.5.9] *and* [19, *Definition I*.4.1$(a)$]. *For instance, $C^\infty$-rings of the form $C^\infty(M)$ for a manifold $M$ are* 0*-determined by* [6, *corollary III*.5.10].

Relying on [19, Proposition I.3.27], we may say that a $C^\infty$-ring $A$ is *Weil* if, and only if, there is a unique map $A \to \mathbb{R}$ and $A$ is finite dimensional as a real vector space.[1]

EXAMPLE 3.5 (W-determined $C^\infty$-rings). *In the category of $C^\infty$-rings, let* P *be the full subcategory of Weil algebras. By definition, the* P*-determined $C^\infty$-rings are the W-determined ones as defined in* [6, *Definition III*.9.1] *and* [19, *Definition I*.4$(b)$], *although in the latter reference they are called* near-point determined.

W-determined $C^\infty$-rings are used in [6, III.9] to construct models of SDG embedding manifolds with boundary. See also the related [18], where variants of these models are constructed in order to exhibit a bidirectional ring of Euler reals.

Two other full subcategories P of $C^\infty$-rings, and their associated P-determined objects, are considered in [19, Definition I.4.1].

Let us consider now the category $\mathbb{C}/\mathbf{Ring}$ of $\mathbb{C}$-algebras. Relying on [13, lemma 6.2], we may say that a $\mathbb{C}$-algebra is *Weil* if it is finitely generated and local (i.e., it is the domain of a unique map with $\mathbb{C}$ as codomain). We then have the associated notion of *W-determined* $\mathbb{C}$-algebra.

LEMMA 3.6. *If a finitely generated $\mathbb{C}$-algebra is sdi as a ring then it is also Weil.*

*Proof.* Let $B$ be an algebra as in the statement. Since it is finitely generated, it is Noetherian (essentially by Hilbert's basis theorem). So $B$ is both Noetherian and sdi. Hence, it is a local Artinian ring (see [7, 12.3C] and [15]). By Hilbert's weak Nullstellensatz (say, in the form [1, 7.10]) the residue field is a finite algebraic extension of $\mathbb{C}$ and so, it is exactly $\mathbb{C}$. In other words, there exists a unique map from $B$ to $\mathbb{C}$. □

The next corollary of Birkhoff's theorem might be folk but we do not know a reference.

PROPOSITION 3.7. *Finitely generated $\mathbb{C}$-algebras are W-determined.*

*Proof.* Let $A$ be a ring. Birkhoff's theorem implies the existence of a subdirect representation $F = \{f : A \to A_f\}$ of $A$ such that $A_f$ is sdi for every $f \in F$. If $A$ is a $\mathbb{C}$-algebra then, simply by composition, $A_f$ is a $\mathbb{C}$-algebra and the projection $f : A \to A_f$ is a morphism of $\mathbb{C}$-algebras. If $A$ is moreover finitely generated then so is $A_f$. Hence, the result follows from Lemma 3.6. □

[1] Actually, the cited characterization says "at most one point" but Weil algebras are local by definition and finitely presentable by [19, Corollary I.3.18] and so, they have a unique point by [19, Corollary I.3.8].

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REMARK 3.8. *See* [12] *for examples of sdi rings with more than one maximal ideal.*

It should be remarked that, in contrast with the case of $k$-algebras for a field, a finitely generated $C^\infty$-ring need not be finitely presentable. Similarly, a finitely generated local $C^\infty$-ring need not be Weil, and *germ-determined* $C^\infty$-rings play an important role in the theory. There are indications, that similar distinctions seem to arise in the category of integral rigs [20].

Boolean algebras, distributive lattices, rings, $\mathbb{C}$-algebras, $C^\infty$-rings, and so on are all coextensive categories. So, as expected, some of the above concepts have a direct geometrical expression in extensive categories.

Let us consider an extensive category $\mathcal{E}$ with terminal object 1. As usual a *point* is a map with terminal domain. We may say that $1 \to X$ is a point *of* X.

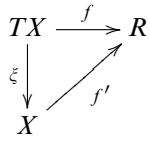
If we let $\mathcal{E}$ be the opposite of the category of Boolean algebras, or that of distributive lattices, then Example 3.2 implies that 1 is a separator in $\mathcal{E}$. Similarly, if $\mathcal{E}$ is the opposite of the category of $C^\infty$-rings, Example 3.4 shows that manifolds determine objects in $\mathcal{E}$ whose points are jointly epic. For future reference we emphasize the following.

REMARK 3.9 (Complex affine schemes with a unique point). *Proposition 3.7 implies that, in the extensive opposite of the category of finitely generated $\mathbb{C}$-algebras, objects with a unique point separate.*

**§4. Orthogonality against a natural monomorphism.** Here we introduce a slight generalization of Lawvere's notion of Birkhoff object, in order to prove two basic results about them that do not need the background on pre-cohesive toposes. Recall that a geometric morphism is *hyperconnected* if its inverse image functor is fully faithful, and its counit is monic [5, A4.6.6].

Let $t : \mathcal{E} \to \mathcal{F}$ be a hyperconnected geometric morphism between toposes and denote its (monic) counit by $\xi$. For brevity, let $T = t^* t_* : \mathcal{E} \to \mathcal{E}$.

DEFINITION 4.1. *An object $R$ in $\mathcal{E}$ is* orthogonal (to $\xi$) *if for every $X$ in $\mathcal{E}$ and every map $f : TX \to R$ there exists a unique $f' : X \to R$ such that the following diagram*




*commutes. Similarly, we say that $R$ is* weakly orthogonal (to $\xi$) *if for every $X$ and $f$ as above there is at most one $f'$ making the triangle commute.*

The definition is very similar to that of separated objects for a Lawvere–Tierney topology. The lemma below shows that weakly orthogonal objects share with separated objects many useful closure properties.

LEMMA 4.2. *The subcategory of weakly orthogonal objects is closed under products, subobjects, and it is an exponential ideal.*

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*Proof.* We prove only the exponential ideal property. Let $R$ be weakly orthogonal and let $Y$ be any object in $\mathcal{E}$. Let $g, h : X \to R^Y$ be such that

$$TX \xrightarrow{\xi} X \underset{h}{\overset{g}{\rightrightarrows}} R^Y$$

commutes. Then

$$T(X \times Y) \xrightarrow{\langle T\pi_0, T\pi_1\rangle} TX \times TY \xrightarrow{\xi_X \times \xi_Y} X \times Y \underset{h\times id}{\overset{g\times id}{\rightrightarrows}} R^Y \times Y \xrightarrow{ev} R$$

$$T(X \times Y) \xrightarrow{\xi_{X\times Y}} X \times Y$$

also commutes. Since $R$ is weakly orthogonal, $ev(g \times id) = ev(h \times id)$, so $g = h$. □

REMARK 4.3. *In particular, any limit of weakly orthogonal objects that exists in $\mathcal{E}$ is weakly orthogonal.*

But not everything can be weakly orthogonal. The following is an adaptation of the argument in the penultimate paragraph of [10].

PROPOSITION 4.4. *The subobject classifier of $\mathcal{E}$ is weakly orthogonal to $\xi$ if, and only if, $t : \mathcal{E} \to \mathcal{F}$ is an equivalence.*

*Proof.* If $t$ is an equivalence then $\xi$ is an isomorphism, so every object of $\mathcal{E}$ is, trivially, weakly orthogonal to $\xi$. In particular, the subobject classifier $\Omega$ is. Conversely, if $\Omega$ is weakly orthogonal, then for any $X$ in $\mathcal{E}$, $\Omega^X$ is weakly orthogonal by Lemma 4.2 so, as the singleton map $\{\ \}_X : X \to \Omega^X$ is monic, the same lemma implies that $X$ is weakly orthogonal. In other words, every object is weakly orthogonal. Hence, $\xi$ is epic as well as monic. So it is an iso and therefore the hyperconnected $t$ is an equivalence. □

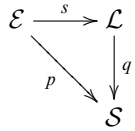
**§5. The Leibniz core.** Let $p : \mathcal{E} \to \mathcal{S}$ be a local and hyperconnected geometric morphism (recall that a geometric morphism is *local* if its direct image part has a fully faithful right adjoint [5, C3.6.1]). Denote the (monic) counit of $p^* \dashv p_*$ by $\beta$. As observed in [8], there is a canonical monic natural transformation $\phi : p^* \to p^!$ (see also [11, lemma 2.1]). We picture $\mathcal{E}$ as a topos 'of spaces', $p^* : \mathcal{S} \to \mathcal{E}$ as the full subcategory of discrete spaces, its right adjoint $p_*$ as sending a space to the corresponding discrete space of points, and the counit $\beta_X : p^*(p_*X) \to X$ as the discrete subspace of points of $X$. The (necessarily fully faithful) right adjoint $p^! : \mathcal{S} \to \mathcal{E}$ to $p_*$ is thought of as the subcategory of codiscrete spaces.

We say that $p$ is a *quality type* if the canonical $\phi : p^* \to p^!$ is an isomorphism. Also, an object $X$ in $\mathcal{E}$ will be called *Leibniz* if $\beta_X : p^*(p_*X) \to X$ has a retraction [10, Remark 4.2].

PROPOSITION 5.1. *The full subcategory $s^* : \mathcal{L} \to \mathcal{E}$ of Leibniz objects is the inverse image functor of a hyperconnected geometric morphism $s : \mathcal{E} \to \mathcal{L}$. Also, the composite $p_*s^* : \mathcal{L} \to \mathcal{S}$ is the direct image functor of a quality type $q : \mathcal{L} \to \mathcal{S}$ and the following*

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*diagram*




*commutes.*

*Proof.* This is essentially a strengthened version of [8, Theorem 2]. The details may be found in [14]. For the purposes of the present paper it is enough to borrow from there that the comonad $(L, \lambda)$ corresponding to $s^* \dashv s_*$ is defined by declaring that the square below

$$\begin{array}{ccc} LX & \longrightarrow & p^*(p_*X) \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\phi_{p_*X}} \\ X & \xrightarrow[\eta]{} & p^!(p_*X) \end{array}$$

is a pullback, where $\eta$ is the unit of $p_* \dashv p^!$. Since $\phi$ is monic, so is $\lambda$. □

We refer to $\lambda_X : LX \to X$ as the *Leibniz core* of $X$. The intuition is better described by a quotation from [10]: "The Leibniz Core of a space $X$ is the union $L(X)$ of all its generalized points; this is obtained as the right adjoint of the inclusion functor from the subcategory of those spaces that 'look like clouds of Leibnizian monads'. The more general figures that substantiate cohesion between points are omitted in the reduction from $X$ to $L(X)$, but each point may have self-cohesion (which is retained in $L(X)$)." This intuition also suggests that $X$ and $LX$ have the same points. More precisely we have the following lemma.

LEMMA 5.2. *For any $X$ in $\mathcal{E}$, $p_*\lambda : p_*(LX) \to p_*X$ is an isomorphism.*

*Proof.* The 'points' functor $p_* : \mathcal{E} \to \mathcal{S}$ preserves pullbacks and $p_*\phi$ is an isomorphism by [11, lemma 2.1]. So the result follows from the explicit description of $\lambda$ in Proposition 5.1. □

**§6. Birkhoff objects.** Let $p : \mathcal{E} \to \mathcal{S}$ be a local and hyperconnected geometric morphism and let $s : \mathcal{E} \to \mathcal{L}$ be the canonical hyperconnected geometric morphism built in Proposition 5.1. As in the proof of that result, denote $s^*s_*$ by $L$ and the (monic) counit of $s^* \dashv s_*$ by $\lambda$, the Leibniz core.

DEFINITION 6.1. *An object $R$ in $\mathcal{E}$ is called* Birkhoff *if it is weakly orthogonal to $\lambda$ (in the sense of Definition 4.1). More explicitly, $R$ is Birkhoff if, and only if, for every pair of maps $f, g : X \to R$, commutativity of the diagram*

$$LX \xrightarrow{\lambda} X \underset{g}{\overset{f}{\rightrightarrows}} R$$

*implies that $f = g$.*

Birkhoff objects were introduced in [10] as those $R$ such that "for any $X$, any 'infinitesimal' map $L(X) \to R$ can be integrated in at most one way to a global function

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$X \to R$." As an example, Proposition 5.1 loc. cit. considers the pre-cohesive topos $p : \widehat{\Delta_1} \to \mathbf{Set}$ of reflexive graphs (over **Set**), recalls that "The Leibniz graphs are those consisting entirely of loops, so that the inclusion $L(X) \to X$ omits those arrows from $X$ whose source and target are distinct," and states that Birkhoff objects coincide with the reflexive binary relations. (To see this notice that the Leibniz core of the reflexive graph $\to$ is the discrete graph with two nodes.) In other words, in the case of $p : \widehat{\Delta_1} \to \mathbf{Set}$, Birkhoff objects coincide with the separated objects for the subtopos $p_* \dashv p^!$. We will see that this is not true in general, but one implication lifts to a more general result.

COROLLARY 6.2. *If $X$ is separated with respect to the centre of $p$ (i.e., the subtopos $p_* \dashv p^!$) then $X$ is Birkhoff. In particular, discrete and codiscrete objects are Birkhoff.*

*Proof.* Since Birkhoff objects are closed under subobjects by Lemma 4.2, it is enough to show that codiscrete objects are Birkhoff. So let $g, h : X \to p^!S$ be such that

$$LX \xrightarrow{\lambda} X \underset{h}{\overset{g}{\rightrightarrows}} p^!S$$

commutes. Then

$$p_*(LX) \xrightarrow{p_*\lambda} p_*X \underset{p_*h}{\overset{p_*g}{\rightrightarrows}} p_*(p^!S) \xrightarrow{\varepsilon} S$$

commutes, where $\varepsilon$ is the counit $p_* \dashv p^!$. Since $p_*\lambda$ is an iso by Lemma 5.2, $\varepsilon(p_*g) = \varepsilon(p_*h)$ and so $g = h$, showing that $p^!S$ is Birkhoff.

Alternatively, using the explicit construction of $\lambda$ recalled in Proposition 5.1, it is easily shown that $\beta_X : p^*(p_*X) \to X$ factors through $\lambda_X$. Since $\beta_X$ is the least dense (w.r.t. the subtopos $p_* \dashv p^!$) subobject of $X$ by [17, lemma 3.2], the result follows. □

On the other hand, in a comment after [10, proposition 5.1], Lawvere states that the subobject classifier cannot have the Birkhoff property. Indeed, the next result implies that if $p$ is sufficiently cohesive then the subobject classifier is not Birkhoff.

COROLLARY 6.3. *For $p : \mathcal{E} \to \mathcal{S}$ as above, the subobject classifier of $\mathcal{E}$ is Birkhoff if, and only if, $p$ is a quality type.*

*Proof.* By Proposition 4.4, $\Omega$ is Birkhoff if and only if $s : \mathcal{E} \to \mathcal{L}$ is an equivalence, but this holds if and only if $p$ is a quality type. □

**§7. Birkhoff objects in pre-cohesive presheaf toposes.** In this section we consider a small category $\mathcal{C}$ with a terminal object and such that every object has a point, so that the canonical geometric morphism $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive, and we characterize the Birkhoff objects therein.

LEMMA 7.1. *An object $R$ in $\widehat{\mathcal{C}}$ is Birkhoff if, and only if, for every $C$ in $\mathcal{C}$, and every pair $s, t : \mathbf{y}C \to R$, commutativity of the following diagram*

$$L(\mathbf{y}C) \xrightarrow{\lambda} \mathbf{y}C \underset{t}{\overset{s}{\rightrightarrows}} R$$

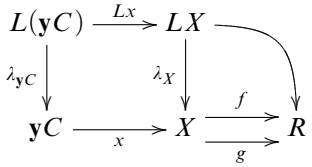
*implies that $s = t$.*

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*Proof.* One direction is immediate. For the converse assume that condition in the statement holds, and consider a pair of maps $f, g : X \to R$, such that the following diagram

$$LX \xrightarrow{\lambda} X \underset{g}{\overset{f}{\rightrightarrows}} R$$

commutes. We need to check that $f = g$. By Yoneda, it is enough to test that $fx = gx$ for every $x : \mathbf{y}C \to X$ and every object $C$ in $\mathcal{C}$. For any such $x$ we have that the following diagram commutes:




so, by hypothesis, $fx = gx$, as we needed to show. □

Roughly speaking, by the Yoneda lemma, to test that an object in $\widehat{\mathcal{C}}$ is Birkhoff it is enough to test against Leibniz cores of representables.

DEFINITION 7.2. *A morphism $f : D \to C$ in $\mathcal{C}$ is a* pseudo-constant *if, for any two points $a, b : 1 \to D$ in $\mathcal{C}$, $fa = fb : 1 \to C$.*

In other words, the pseudo-constants are those morphisms that are constant on points. We think of a pseudo-constant as a morphism that factors through an object that has exactly one point. Notice that if $D$ has exactly one point then every map $D \to C$ is a pseudo-constant.

LEMMA 7.3. *For any $C$ in $\mathcal{C}$, if we let $\mathbf{y}C = \mathcal{C}(-, C)$ and $\lambda_{\mathbf{y}C} : L(\mathbf{y}C) \to \mathbf{y}C$ be its Leibniz core, then for each $D$ in $\mathcal{C}$, the inclusion*

$$\lambda_{\mathbf{y}C,D} : (L(\mathbf{y}C))D \to (\mathbf{y}C)D = \mathcal{C}(D, C)$$

*may be identified with the subset of pseudo-constants from $D$ to $C$.*

*Proof.* As observed in [14], it follows from the explicit construction of the Leibniz core $\lambda : LX \to X$ that

$$(LX)D = \{x \in XD \mid \text{ for all } a, b : 1 \to D, x \cdot a = x \cdot b\} \to XD$$

for every $X$ in $\widehat{\mathcal{C}}$ and $D$ in $\mathcal{C}$. Taking $X = \mathbf{y}C = \mathcal{C}(-, C)$ completes the proof. □

In order to discuss a sufficient condition for Birkhoff objects to form a separating family we first need to characterize them more explicitly in the present context.

PROPOSITION 7.4. *An object $R$ in $\widehat{\mathcal{C}}$ is Birkhoff if, and only if, it satisfies the following condition: for every $C$ in $\mathcal{C}$ and $s, t \in RC$, if for every pseudo-constant $f : D \to C$ in $\mathcal{C}$, $s \cdot f = t \cdot f$, then $s = t$.*

*Proof.* By Lemma 7.1 we have that an object $R$ in $\widehat{\mathcal{C}}$ is Birkhoff if, and only if, for every $C$ in $\mathcal{C}$, and every $s, t : \mathcal{C}(-, C) \to R$, commutativity of the following diagram

$$L(\mathbf{y}C) \xrightarrow{\lambda} \mathbf{y}C \underset{t}{\overset{s}{\rightrightarrows}} R$$

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implies that $s = t$. Now the fork above commutes if, and only if, for every $D$ in $\mathcal{C}$, the following diagram

$$(L(\mathbf{y}C))D \xrightarrow{\lambda} \mathcal{C}(D, C) \underset{t}{\overset{s}{\rightrightarrows}} RD$$

commutes. By Lemma 7.3, $\lambda : (L(\mathbf{y}C))D \to \mathcal{C}(D, C)$ is the subset of pseudo-constants from $D$ to $C$, so the result follows. □

Compare Proposition 7.4 with the analogous characterization of $\neg\neg$-separated objects obtained by replacing 'pseudo-constant' by 'point'. (We will recall the details in the proof of Theorem 7.9). Sometimes, separated objects for the $\neg\neg$-topology are said to 'live on their points'. Similarly, we may say that Birkhoff objects are those that 'live on their Leibnizian monads'.

THEOREM 7.5. *Let $\mathcal{C}$ be a small category with terminal object and such that every object has a point so that $\widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive. Birkhoff objects separate in $\widehat{\mathcal{C}}$ if, and only if, for every $B$ in $\mathcal{C}$, the pseudo-constants of $B$ are jointly epic in $\mathcal{C}$.*

*Proof.* For $A$ in $\mathcal{C}$, Proposition 7.4 implies that: the representable $\mathcal{C}(-, A)$ is Birkhoff in $\widehat{\mathcal{C}}$ if, and only if, for every object $B$ in $\mathcal{C}$, and every pair $s, t \in \mathcal{C}(B, A)$ the following holds: if for every pseudo-constant $f : D \to B$, $sf = tf$, then $s = t$. (Roughly speaking, $A$ believes that pseudo-constants are jointly epic.) Hence, representables in $\widehat{\mathcal{C}}$ are Birkhoff if, and only if, for every $B$ in $\mathcal{C}$, the pseudo-constants of $B$ are jointly epic in $\mathcal{C}$. Equivalently, by Proposition 2.3, Birkhoff objects in $\widehat{\mathcal{C}}$ separate. □

The following corollary is more directly applicable to the examples at hand.

COROLLARY 7.6. *Let $\mathcal{C}$ be a small category with a terminal object and such that every object has a point. If objects with a unique point separate in $\mathcal{C}$ then Birkhoff objects separate in $\widehat{\mathcal{C}}$.*

*Proof.* If objects with a unique point separate then pseudo-constants are jointly epic. □

Let $\mathrm{Aff}_{\mathbb{C}}$ be the extensive category of affine schemes (over $\mathbb{C}$). That is, the opposite of the category of finitely presentable $\mathbb{C}$-algebras. (Objects in $\mathrm{Aff}_{\mathbb{C}}$ are sometimes called (complex-)affine schemes *of finite type*.) It is well-known that the topos of presheaves on $\mathrm{Aff}_{\mathbb{C}}$ classifies $\mathbb{C}$-algebras.

Let $\mathrm{cAff}_{\mathbb{C}} \to \mathrm{Aff}_{\mathbb{C}}$ be the full subcategory of connected affine schemes. It is also well-known that the topos $\mathcal{G}_{\mathbb{C}}$ of presheaves on $\mathrm{cAff}_{\mathbb{C}}$ is pre-cohesive (over **Set**), that it classifies $\mathbb{C}$-algebras without idempotents, and that the restricted Yoneda embedding $\mathrm{Aff}_{\mathbb{C}} \to \widehat{\mathrm{cAff}_{\mathbb{C}}} = \mathcal{G}_{\mathbb{C}}$ preserves, not only finite limits, but also finite coproducts.

COROLLARY 7.7. *Birkhoff objects separate in the pre-cohesive $\mathcal{G}_{\mathbb{C}}$.*

*Proof.* Objects with a unique point separate in $\mathrm{Aff}_{\mathbb{C}}$ by Remark 3.9 so, as objects with a unique point are connected in $\mathrm{Aff}_{\mathbb{C}}$, they also separate in $\mathrm{cAff}_{\mathbb{C}}$ and, therefore, Corollary 7.6 is applicable. □

REMARK 7.8. *It seems relevant to mention that the rings whose sdi factors are local have been characterized in* [4]. *They are sometimes called* Camillo rings. *With the construction of classifying toposes in mind, it is also natural to wonder about the finitely generated algebras whose sdi factors are local.*

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Notice that, for a $\mathcal{C}$ as in the statement of Theorem 7.5, we may extract the full subcategory $\mathcal{C}' \to \mathcal{C}$ consisting of the objects whose pseudo-constants are jointly epic. The category $\mathcal{C}'$ contains the terminal object, every object in $\mathcal{C}'$ has a point, and Theorem 7.5 implies that Birkhoff objects in the pre-cohesive $\widehat{\mathcal{C}'} \to \mathbf{Set}$ separate. Notice that this process is a dual analogue of extracting a class of B-determined algebras for some B. In fact, such a process is used in [18, sec. 5], in order to use the properties of W-determined $C^\infty$-rigs to show that the ring of Euler reals in the smaller topos has a 'good order'.

It is natural to wonder if there is an elementary version of the process sketched above. Given a pre-cohesive geometric morphism $\mathcal{E} \to \mathcal{S}$, can we extract a subtopos $\mathcal{E}^\sharp \to \mathcal{E}$ such that the composite $\mathcal{E}^\sharp \to \mathcal{S}$ is pre-cohesive and $\mathcal{E}^\sharp$ has 'enough' Birkhoff objects?

THEOREM 7.9. *If $\mathcal{C}$ is a small category with a terminal object and such that every object has a point then the following are equivalent*:

1. *The terminal object is a separator in $\mathcal{C}$.*
2. *Representables are $\neg\neg$-separated in $\widehat{\mathcal{C}}$.*
3. *Birkhoff objects in $\widehat{\mathcal{C}}$ separate and they are $\neg\neg$-separated.*

*In this case, Birkhoff objects coincide with $\neg\neg$-separated objects.*

*Proof.* The subtopos $p_* \dashv p^! : \mathbf{Set} \to \widehat{\mathcal{C}}$ coincides with the subtopos of $\neg\neg$-sheaves by [11, Corollary 4.5]. So $X$ in $\widehat{\mathcal{C}}$ is $\neg\neg$-separated if, and only if, the unit $\eta_X : X \to p^!(p_* X)$ is monic [5, A4.3.6]. This happens when for every $C \in \mathcal{C}$, the function $XC \to (X1)^{\mathcal{C}(1,C)}$ that sends $x \in XC$ to $(x \cdot a \mid a : 1 \to C) \in (X1)^{\mathcal{C}(1,C)}$ is injective. That is, any two elements of $XC$ that come out equal when acted on by any point are equal.

The above characterization implies that the first two items in the statement are equivalent. The third item implies the second by Proposition 2.3. So, to complete the proof of the equivalence between the three items, it is enough to show that the first two items imply the third. Certainly, the first item, together with Corollary 7.6, implies that Birkhoff objects separate. Hence, it remains to prove that they are $\neg\neg$-separated. So assume that $R$ is Birkhoff. That is, it is 'weakly orthogonal' to Leibniz cores $L(\mathbf{y}C) \to \mathbf{y}C$ of representable objects. As $\mathbf{y}C$ is $\neg\neg$-separated, [14, corollary 5.1] implies that its Leibniz core coincides with the discrete core $p^*(p_*(\mathbf{y}C)) \to \mathbf{y}C$. So $R$ is weakly orthogonal to discrete cores of representables, which means that $R$ is $\neg\neg$-separated.

Finally, we already know that $\neg\neg$-separated objects are Birkhoff by Corollary 6.2. So an object is Birkhoff if, and only if, it is $\neg\neg$-separated. □

Theorem 7.9 provides a general explanation of why $\neg\neg$-separated and Birkhoff objects coincide in certain pre-cohesive toposes such as that of reflexive graphs considered in [10], and many other pre-cohesive toposes of combinatorial spaces: the classifier of distributive lattices without complemented elements, its subtopos of simplicial sets, the classifier of non-trivial Boolean algebras, the classifier of strictly bipointed objects and the topos of ball complexes. See [16, Examples 4.2 and 4.3] and references therein. (Notice that in order to prove that the initial object coseparates in the category of finitely presentable distributive lattices we are tacitly invoking Birkhoff.)

It seems worth mentioning that some of the examples, such as reflexive graphs, or simplicial sets, are *weakly generated* [8] by Birkhoff objects in the sense that the smallest subtopos containing the Birkhoff objects is the whole topos.

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Finally, many of the examples satisfy that Birkhoff objects coincide with the separated objects for the 'infinitesimal dimension' level $\varepsilon$ [13]; but we will prove this elsewhere.

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