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Spaces of maximal points

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This paper shows that it is precisely the complete metrizable separable metric spaces that can be realized as the set of maximal points of an ω-continuous dcpo, where the set of maximal points is topologized with the relative Scott topology.

1. Introduction

In an early paper D. Scott (Scott 1970) suggested that an appropriate directed complete partially ordered set (dcpo) of intervals should be useful for the study of continuous and computable functions on real intervals. Related ideas of computability on metric spaces had surfaced much earlier, for example, in Lacombe (1959) (see also Martin-Löf (1970)). Scott's suggestion was developed along various lines in Weihrauch and Schreiber (1981) and Kamimura and Tang (1984). Two recent developments have given new impetus to such considerations. One of these is the introduction and ripening of the theory of the probabilistic power domain of a continuous poset. If a continuous poset can be appropriately associated with a topological space, one has at hand, via the probabilistic power domain, what can be a very effective and powerful tool for the study of a variety of problems associated with integration, measure and probability. In addition, A. Edalat has effectively used such structures in a wide variety of applications involving measures, dynamical systems, and iterated function systems and fractals (Edalat 1995; Edalat 1996). The central purpose of this paper is to consider in some detail how spaces arise as the set of maximal points of a continuous poset with a countable base, and to characterize such spaces as the class of Polish spaces, where the maximal points are endowed with the relative Scott topology. We introduce the idea of requiring that the set of maximal points should inherit the same relative topology from the Lawson topology. This turns out to be a very fruitful notion that permits great generality, but at the same time permits the derivation of many important structure results.

After becoming aware of the results of this paper, A. Edalat and R. Heckmann (Edalat and Heckmann 1996) were able to give a much more explicit and natural construction for realizing a Polish space as the set of maximal points of an ω-continuous dcpo. The dcpo that they associate to a complete metric space is the space of formal closed balls.

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2. Maximal-point spaces and hulls

We quickly recall some basic notions concerning continuous posets (see, for example, Gierz *et al.* (1980), Lawson (1979), Lawson (1988), Davey and Priestley (1990) and Abramsky and Jung (1995)). A subset D of a partially ordered set X is *directed* if given $x, y \in D$, there exists $z \in D$ such that $x, y \leqslant z$. A *directed complete partially ordered set* or dcpo is a partially ordered set $(X, \leqslant)$ such that every directed subset of X has a least upper bound in X. Frequently, one postulates a least element $\perp$ also.

For $x, y \in X$, a dcpo, we write $x \ll y$ if for every directed set D with $y \leqslant \sup D$, we have $x \leqslant d$ for some $d \in D$. We set

$$\Downarrow y := \{x \in X : x \ll y\}.$$

A dcpo X is called a *continuous poset* if

(i) $y = \sup \Downarrow y$ for all $y \in X$ and
(ii) each $\Downarrow y$ is a directed set.

One can show directly that if $\Downarrow y$ contains a directed set with supremum y, then both (i) and (ii) hold.

A *base* for a continuous poset P is a set $B \subseteq P$ such that for all $x \in P$,

$$x = \sup\{y \in P : y \ll x,\ y \in B\} = \sup(\Downarrow x \cap B),$$

and the supremum is taken over a directed set. A continuous poset P is *ω-continuous* if it has a countable base B. This turns out to be equivalent to the existence of a countable base for the Scott topology (see Gierz *et al.* (1980, Chapter III.4) or Abramsky and Jung (1995)).

Example 2.1. (1) Let X be a locally compact space with a countable base $\mathscr{B}$ for the topology that is closed under finite unions. Then $\mathscr{B}$ is a base for the continuous lattice of open sets $\Omega(X)$.
(2) Let X be a locally compact sober space with a countable base $\mathscr{B}$ for the topology. The dcpo $\kappa(X)$ of non-empty compact saturated (*i.e.*, an intersection of open sets) subsets of X ordered by reverse inclusion admits a countable base. Indeed one may consider all pairs $(U, V) \in \mathscr{B} \times \mathscr{B}$ such that $U \ll V$, and pick $K_{(U,V)} \in \kappa(X)$ such that $U \subseteq K_{(U,V)} \subseteq V$. (The existence of such a compact set is equivalent to $U \ll V$ – see, for example, Hofmann and Lawson (1978)). One shows directly that the $K_{(U,V)}$ form a countable base for $\kappa(X)$.

A subset A of a dcpo X is *Scott-closed* if

(i) $A = \downarrow A$, and
(ii) if $D \subseteq A$ is a directed set and $d = \sup D$, then $d \in A$.

A function between two dcpo's X and Y is *Scott continuous* if it preserves suprema of directed sets, or, equivalently, if it is continuous with respect to the Scott topologies on X and Y.

The *weak* topology on a poset P is the weakest topology such that the *specialization order*

$$x \leqslant y :\Leftrightarrow x \in \overline{\{y\}}$$

agrees with the original order. The weak^d is the order-dual version of this, and has $\uparrow x$,

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$x \in P$ as a subbase for the closed sets. In the case of continuous posets, the weak^d topology agrees with the *cocompact* topology, which is generated by complements of saturated compact sets.

The *Lawson* or *λ-topology* is the join of the Scott and weak^d topologies (the symmetrization topology of the bitopological space). In the case of continuous posets, it arises, alternatively, as the *patch* topology of the Scott topology, which is defined as the join of the cocompact and Scott topologies. For continuous posets, the λ- topology is always Haudorff and completely regular, and for ω-continous posets it is separable metrizable.

We consider ω-continuous posets P satisfying

$$(\dagger) \qquad p \in P \Rightarrow \exists A \text{ Scott closed in } P,\ \uparrow p \cap \mathrm{Max}(P) = A \cap \mathrm{Max}(P),$$

where $\mathrm{Max}(P)$ is the set of maximal points in P. Alternatively

$$(\ddagger) \qquad \text{Scott topology}|\mathrm{Max}(P) = \lambda - \text{topology}|\mathrm{Max}(P),$$

that is, the two topologies restricted to the set of maximal elements agree. Indeed the subbasic closed sets in the λ-topology on P are either Scott closed or of the form $\uparrow p$ for some $p \in P$, and from this it follows easily that (†) and (‡) are equivalent. In this case, $\mathrm{Max}(P)$ is a separable metric space in either relative topology, since the λ-topology is separable metric for ω-continuous posets.

Definition 2.1. A separable metric space X is called an *MP-space* ('maximal point space') if there exists an ω-continuous poset P satisfying condition (†) (equivalently, (‡)) such that X is homeomorphic to $\mathrm{Max}(P)$. In this case $X \leftrightarrow \mathrm{Max}(P) \hookrightarrow P$ is called an *MP-hull* for X.

Example 2.2. (*The Upper Space*) For a locally compact separable metric space X, there is a standard MP- hull for X called the *upper space* UX, which consists of the set of all non-empty compact subsets of X ordered by reverse inclusion. As in Example 2.1, UX is an ω- continuous poset, and the homeomorphic injection $x \mapsto \{x\}: X \to UX$ is an MP-hull for X.

MP-spaces were studied by Kamimura and Tang (Kamimura and Tang 1984) for the case when the hulls were retracts of Scott domains, that is, ω-continuous posets in which any finite set that is bounded above has a least upper bound. They called such spaces 'total spaces'.

The next proposition and its corollary are the main results of the paper. The proposition gives a general construction of an MP-hull for a large class of spaces – this is a modification of the upper space construction.

Proposition 2.1. Let X be a dense G_δ-subset of a compact metric space Y, and let UY be the upper space of non-empty compact subsets of Y. Then there exists an MP-hull $x \mapsto \{x\}: X \to P \subseteq UY$ for X satisfying

(i) P is a subset of UY and is closed under directed sups in the inherited order;
(ii) P is a continuous poset, and the relation $\ll_P$ of P is the restriction of $\ll_{UY}$ to P;
(iii) for any countable base $\mathscr{B}$ of UY, the members of $\mathscr{B}$ belonging to P form a countable base for P.

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Proof. Since Y is compact, the metric on Y is bounded. Let $\{U_n\}_{n\in\mathbf{N}}$ be a descending collection of dense open subsets of Y such that $X = \bigcap_n U_n$. We define collections $\mathscr{K}$ and $\{\mathscr{K}_n : n \in \mathbf{N}\}$ of compact subsets of Y. We set $\mathscr{K}$ equal to the set of all compact subsets of X and

$$\mathscr{K}_n := \left\{ K \text{ compact} \subseteq U_n : \exists x \in U_n \text{ such that } \overline{B}\left(x, \frac{1}{n} d(x, Y \setminus U_n)\right) \subseteq K \right\},$$

where $d(x, Y \setminus U_n)$ is the distance from x to $Y \setminus U_n$ and $\overline{B}(x, \epsilon)$ is the closed ϵ-ball consisting of all points of distance less than or equal to ϵ from x. We define the hull P by

$$P := \mathscr{K} \cup \bigcup_{n\in\mathbf{N}} \mathscr{K}_n$$

ordered by reverse inclusion. The associated embedding is given by $x \mapsto \{x\} : X \to P$. Note that P is a subset of the upper space UY and that its order is that inherited from the upper space. We henceforth identify X with the corresponding singleton subsets, which are members of P. The proof now proceeds by establishing in a sequence of steps that this is indeed an MP-hull for X.

Step 1: *P is a sub-dcpo of the upper space UY, that is, it is closed under directed intersections.*

Let $\{K_j : j \in \mathbf{N}\}$ be a descending family of members of P. If any K_j is a compact subset of X, then their intersection will be also, and hence the intersection is again in P. Set $J := \{n : \exists\ j \text{ such that } K_j \in \mathscr{K}_n\}$. If J is unbounded in $\mathbf{N}$, one has directly from the descending property that $\bigcap K_j \subseteq \bigcap U_n = X$, and thus the intersection is a compact subset of X. If J is bounded, there is a subsequence, again denoted K_j, and $n \in \mathbf{N}$ such that $K_j \in \mathscr{K}_n$ for all j. Note that the subsequence has the same intersection, so without loss of generality we restrict our attention to it. Then, for each j, there exists $x_j \in U_n$ such that K_j contains $\overline{B}\big(x_j, (1/n)d(x_j, Y \setminus U_n)\big)$. For some subsequence, $x_j \to x \in K_1 \subseteq U_n$. It follows directly from the continuity of the distance function and the fact that the $\{K_j\}$ are descending that $\overline{B}\big(x, (1/n)d(x, Y \setminus U_n)\big)$ is contained in each K_j, and hence in their intersection. Thus the intersection $\bigcap K_j$ is also in $\mathscr{K}_n \subseteq P$.

Step 2: *P is a continuous poset, and $\ll_P$ is the restriction of $\ll_{UY}$.*

Let $K \in \mathscr{K}_n$. Then there exists a compact set K' that contains K in its interior and is contained in U_n. It follows that $K' \in \mathscr{K}_n$. Also $K' \ll_P K$, since this is true in the upper space UY, and P is a sub-dcpo. Since K is the directed intersection of all such K', we have established continuity of P at K. Suppose now that K is a compact subset of X. For any $n \geqslant 2$, we define K_n by

$$K_n := \left\{ x \in U_n : \exists y \in K \text{ such that } d(x, y) \leqslant \frac{1}{n} d(y, Y \setminus U_n) \right\}.$$

Then K_n is a compact neighbourhood of K, so $K_n \ll K$, and the $\{K_n\}$ form a decreasing sequence with intersection K. Thus P is again continuous at K, since to establish continuity of a dcpo, it suffices to find a directed set of elements way below each element with supremum that element. Finally, if $K, K' \in P$ with $K' \ll K$, then, since we have just seen that there exists a descending sequence $\{K_n\}$ in P with intersection K, each containing K

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in its interior, we conclude $K \subseteq K_n \subseteq K'$ for some n. Thus K is contained in the interior of K' in Y, and $K' \ll_{UY} K$.

Step 3: *The members of any countable base of UY that belong to P form a base for P, and hence P is an ω-continuous poset.*

Let $\mathscr{B}$ be a countable base for UY. Let K be a compact subset of X and let $L_1, \ldots, L_j \ll K$ in P. Pick a descending sequence of compact neighbourhoods $\{K_n\}$ of K that are in P as in the preceding step. Then there exists m such that $L_i \ll K_m$ for each i, from the definition of $\ll$ and the interpolation property. Then $K_m \in \mathscr{K}_m$ is contained in the interior of L_i for each i. Pick a compact neighbourhood K' of K_m such that $K' \subseteq U_m \cap \bigcap_i \mathrm{int}(L_i)$. From the definition of a basis, there exists $C \in \mathscr{B}$ such that $K_m \subseteq C \subseteq K'$. Then $C \in \mathscr{K}_m \subseteq P$, since $K_m \subseteq C \subseteq U_m$, and $L_i \ll C \ll K$. For $K \in \mathscr{K}_n$, it is straightforward to argue that the members of $\mathscr{B}$ that are contained in U_n and each containing K in its interior are again in $\mathscr{K}_n$ and form a directed set with supremum K. This establishes that the members of $\mathscr{B}$ in P form a base for P.

Step 4: *The set X forms the set of maximal points of P.*

Let $K \in P$. If K is a compact subset of X, then clearly K must be a singleton in order to be maximal. If $K \in \mathscr{K}_n$, then from the definition of $\mathscr{K}_n$ and the density of X, it follows that K contains a member of X, and hence is not maximal in P.

Step 5: *The continuous poset P satisfies* (†).

Let $K \in P$. We show for each $x \in X \setminus K = \mathrm{Max}(P) \setminus \uparrow K$, that there exists a Scott open set in P whose intersection with X misses the set K. Now by Step 2, $\{x\}$ is the descending intersection of a sequence of compact neighbourhoods $\{K_n\}$, where each $K_n \in P$. Since $x \in Y \setminus K$, there exists K_n such that $K_n \cap K = \varnothing$. Then $\Uparrow K_n \cap \mathrm{Max}(P) = \mathrm{int} K_n \cap X$, and thus the claim is established, since $\Uparrow K_n$ is Scott open.

Step 6: *The relative Scott topology on X agrees with the metric topology.* Let $x \in U$, an open set in the given metric topology on X. Then there exists an open set W in Y such that $W \cap X = U$. Again, pick the descending sequence, as in Step 2, of compact neighbourhoods $\{K_n\}$ with intersection $\{x\}$. Then some K_n is a subset of W, and hence $\Uparrow K_n \cap X = \mathrm{int}(K_n) \cap X$ is a relative Scott open set contained in U. This establishes continuity of the identity mapping in one direction. Conversely, pick a Scott open set V in P containing $x \in X \subseteq P$. Then some $K_n \in V$, since K_n is a directed sequence converging to x. Thus $\mathrm{int}(K_n) \cap X \subseteq V \cap X$ and the set $\mathrm{int}(K_n) \cap X$ is metric open in X. Thus continuity in the other direction holds. □

Recall that a Polish space is a separable topological space for which the topology is given by a complete metric.

Corollary 2.2. Let X be a Polish space. Then X admits an MP-hull, and hence is an MP-space.

Proof. It is a standard topological result that a Polish space is homeomorphic to a G_δ subset of I^ω, a countable product of intervals $I = [0, 1]$ (see, for example, Bourbaki (1989)). Thus we assume $X \subseteq I^\omega$, and $\{U_n\}_{n \in \mathbf{N}}$ is a descending collection of open sets in I^ω such that $X = \bigcap_n U_n$. We then set $Y := \overline{X}$ in I^ω, and immediately see that X is a dense G_δ in the compact metric space Y. We now apply Proposition 2.1. □

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3. Order compactifications

In this section we continue our consideration of ω-continuous posets P that satisfy

$$(\dagger) \qquad p \in P \Rightarrow \exists A \text{ Scott closed in } P,\ \uparrow p \cap \mathrm{Max}(P) = A \cap \mathrm{Max}(P).$$

We first recall some additional background material (see, for example, Gierz *et. al.* (1980, Chapter VI)). A *partially ordered topological space* (or *pospace*) is a topological space X equipped with a partial order that is closed considered as a subset of $X \times X$ (equipped with the product topology). In this case the topology is automatically Hausdorff. For a continuous poset P, we let $(P, \lambda(P))$ denote the ordered topological space P equipped with the Lawson topology; P is then a partially ordered topological space.

Let X be a partially ordered topological space. An *order compactification* for X is a continuous order-preserving mapping $j: X \to Y$, where Y is a compact partially ordered space such that $j(X)$ is dense in Y. For a continuous poset there is a canonical order compactification. This embedding has been shown to be equivalent to what has been called the Fell compactification of the space P endowed with the Scott topology (see Hoffmann (1985) and Lawson (1991)). We shall refer to it as the *Fell order compactification.*

Lemma 3.1. Let L be the lattice of Scott closed sets of a continuous poset P. Then L is a completely distributive (hence continuous) lattice, and hence a compact ordered space with respect to the Lawson topology. The poset P embeds as the set of coprime elements via $p \mapsto \downarrow p$, and this injection is an order compactification into the closure $F(P)$ of the image of P. Furthermore, the embedding is a homeomorphism and order isomorphism onto its image in $F(P)$.

Proof. See Lawson (1979), Gierz *et al.* (1980, Proposition V.5.12), Hoffmann (1985) and Lawson (1991). □

Lemma 3.2. Let P be an ω-continuous poset. For the Fell order compactification $P \hookrightarrow F(P)$, the compact pospace $F(P)$ is metrizable and the maximal elements of P embed as maximal elements of $F(P)$ if P satisfies (†).

Proof. It is standard that if P is ω-continuous, the lattice of Scott closed sets L is also dually ω-continuous, hence second countable in the Lawson topology, hence metrizable (see, for example, Gierz *et al.* (1980, Chapter III.4)). We identify P with the coprimes of the lattice L of closed subsets (see Lemma 3.1). Let $x \in \mathrm{Max}(P)$and suppose that there exists $y \in \overline{P} = F(P)$ with $x < y$. Since L is completely distributive, it is continuous, and hence there exists $z \in L$ such that $z \ll y$ and $z \not\leqslant x$. Since L is completely distributive, every element is a supremum of coprimes (see Lawson (1979), for example), and thus there exists $q \in P$ such that $q \leqslant z$, but $q \not\leqslant x$. By (†) (applied to q), there exists a Scott open set U in P such that $x \in U$ and $U \cap \mathrm{Max}(P)$ misses $\uparrow q \cap \mathrm{Max}(P)$. There exists $w \in U$ such that $w \ll x$ in P; but then $w \ll x$ in L (see Lawson (1979)). Thus $\Uparrow w$ is Scott open in L. Since $\Uparrow w \cap \Uparrow q$ is a λ-neighbourhood of $y \in \overline{P}$, we conclude there exists an $s \in P$ that is in this neighbourhood. Since P is a dcpo, there exists $m \in P$ that is maximal in P such that $s \leqslant m$. But then $s \in (U \cap \mathrm{Max}(P)) \cap (\uparrow q \cap \mathrm{Max}(P))$, which is a contradiction. □

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Theorem 3.3. Let P be an ω-continuous poset. Then the following conditions are equivalent:

(1) P satisfies (†).
(2) There exists a compact metrizable pospace Y and an injective order compactification $j: P \to Y$ such that j is an order isomorphism and homeomorphism from $(P, \lambda(P))$ onto $j(P)$, and j carries maximal elements of P to maximal elements in Y.
(3) There exists a topological and order embedding of $(P, \lambda(P))$ into I^ω equipped with the product topology and coordinatewise order such that the image of each maximal element remains maximal in the closure of the image of P.
(4) There is a continuous order-preserving injection i from $(P, \lambda(P))$ into a compact pospace Q such that the maximal elements of P are mapped into the maximal elements of Q.

Proof. (1) $\Rightarrow$ (2): This follows immediately from Lemma 3.2 with $Y = F(P)$.

(2) $\Rightarrow$ (3): Urysohn-type arguments go over to the ordered setting to obtain the order-embedding of Y into I^ω, a countable product of intervals (see Nachbin (1965) and Gierz *et al.* (1980, Exercise VI.1.17)). The composition from P into Y with Y into I^ω gives the desired embedding.

(3) $\Rightarrow$ (4): Take the corestriction of the embedding in (3) to the closure of the image in I^ω.

(4) $\Rightarrow$ (1): Let $i: P \to Q$ be the given injection. Let $x \in P$ and let $y \in \mathrm{Max}(P)$ such that $x \not\leqslant y$. Then $\uparrow i(x)$ is a compact subset of Q missing $i(y)$. By properties of compact partially ordered spaces, $\downarrow(\uparrow i(x))$ is a compact lower set, and it misses $i(y)$, since $i(y)$ is maximal. Thus the complement of $\downarrow(\uparrow i(x))$ in Q is an open upper set containing $i(y)$. Its inverse image in P must be an open upper set in P containing y, and hence is a Scott open set (see Gierz *et al.* (1980, Proposition III.1.6); the proof there is for lattices but carries over). Thus the complement of $\uparrow x$ in $\mathrm{Max}(P)$ is open in the relative Scott topology, and hence condition (†) is satisfied. □

Corollary 3.4. If an ω-continuous poset has compact Lawson topology, it satisfies (†). In particular, this applies to ω-continuous Scott domains, bifinite domains and their retracts, and the FS-domains.

Proof. One can apply the identity map for the order compactification in Condition (4) of Theorem 3.3. The various special classes of domains mentioned are known to be compact in the Lawson topology (see Abramsky and Jung (1995)). □

Corollary 3.5. Let P be an ω-continuous poset satisfying (†). Then there exists a descending sequence $\{U_n\}$ of Scott open sets such that $\mathrm{Max}(P) = \bigcap_n U_n$.

Proof. Let $j: P \to Y$ be the injective order compactification given in Condition (2) of Theorem 3.3. There exists a metric d on Y compatible with the topology such that if $x \leqslant y \leqslant z$, then $d(x, y) + d(y, z) = d(x, z)$ (see Gierz *et al.* (1980, Exercise VI.1.17)). For each n, set

$$K_n := \left\{ x \in Y : \exists y \in Y \text{ such that } x < y \text{ and } d(x, y) \geqslant \frac{1}{n} \right\}.$$

It follows from the special order property of the metric d that K_n is a lower set, and

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from the compactness of Y and the continuity of d that it is closed. Furthermore, one sees directly that $\mathrm{Max}(Y) = Y \setminus \bigcup_n K_n$. We set $U_n = P \setminus j^{-1}(K_n)$. Then U_n is open and increasing in P, hence Scott open in X, as we saw in the proof of Theorem 3.3. Each U_n contains $\mathrm{Max}(P)$ since $j(\mathrm{Max}\, X) \subseteq \mathrm{Max}\, Y$. Finally, since j is injective, non-maximal elements map to non-maximal elements, and thus $\mathrm{Max}(P) = \bigcap_n U_n$. □

Corollary 3.6. For an ω-continuous poset P satisfying (†), the space $\mathrm{Max}(P)$ is a Polish space.

Proof. The poset P equipped with the Lawson topology, the patch of the Scott topology, embeds as the spectrum of its lattice Ω of Scott open sets equipped with the Lawson topology (this is dual to the embedding in the closed sets we have been considering). Since P is ω- continuous, the lattice Ω has a countable base, and hence is compact metric in its Lawson topology. The set P, the spectrum, is a countable intersection of open sets (see Gierz *et al.* (1980, Exercise III.4.21)), and is thus a Polish space in the relative topology, which is its Lawson topology. By Corollary 3.5, $\mathrm{Max}(P)$ is a also a G_δ-set in P, and is thus also a Polish space in the relative Lawson topology, which is the relative Scott topology by (‡). □

Theorem 3.7. A space is an MP-space if and only if it is a Polish space.

Proof. The theorem is a direct corollary of Corollaries 2.2 and 3.6. □

4. Extending continuous functions

The results and methods of this section were suggested by those of Weihrauch and Schreiber in Weihrauch and Schreiber (1981).

Throughout this section we work in the following setting: P denotes an ω-continuous poset that satisfies (†). By Theorem 3.3 there exists a compact metrizable ordered space Y and an injective order compactification $j: P \to Y$ such that j is an order isomorphism and a homeomorphism from $(P, \lambda(P))$ onto $j(P)$, equipped with the relative order and topology from Y, and such that j carries maximal elements of P into maximal elements of Y. We henceforth identify P with its image under j in Y. By Gierz *et al.* (1980, Exercise VI.1.17), there exists a metric d on Y compatible with the topology that is *order convex*, that is, if $x \leqslant y \leqslant z$ in Y, then $d(x,z) = d(x,y) + d(y,z)$, and its restriction to P is an order convex metric on P compatible with the Lawson topology.

We define a *weight function* $|\cdot| : Y \to [0,\infty)$ by

$$|y| := \sup\{d(y,z) : z \in Y,\ y \leqslant z\}.$$

We list some elementary properties of the weight function.

Lemma 4.1. The weight function $|\cdot|$ satisfies

(i) $\forall y \in Y$ (respectively, $\forall y \in P$), $|y| \geqslant 0$ and $|y| = 0 \Leftrightarrow y$ is a maximal element.

(ii) The inverse image under $|\cdot|$ of $[a,\infty)$ is a closed lower set. Hence the restriction $|\cdot| : P \to [0,\infty)$ is upper semicontinuous with respect to the Scott topology on P.

(iii) If $x < y$, then $|y| + d(x,y) \leqslant |x|$.

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Proof.

(i) This follows directly from the definition of $|\cdot|$.

(iii) Let $x < y$ and let $y < w$. Then from order convexity, $d(y,w) = d(x,w) - d(x,y) \leqslant |x| - d(x,y)$. Taking the supremum over all $w > y$, we obtain $|y| \leqslant |x| - d(x,y)$, and thus assertion (iii) follows.

(ii) Let $|y| \geqslant a$ for $0 \leqslant a$. By (iii), if $x < y$, then $|x| > |y|$, so the set $A := \{u \in P : a \leqslant |u|\}$ is a lower set. Suppose that $\{x_n\}$ is a sequence converging to x and suppose that $|x_n| \geqslant a$ for all n. For each x_n, pick $y_n \geqslant x_n$ such that $d(x_n, y_n) > |x_n| - (1/n) \geqslant a - (1/n)$. By compactness of Y, we may pass to a convergent subsequence of the $\{y_n\}$, again labelled y_n, such that $y_n \to y$. By continuity of d, $d(x,y) \geqslant a$ and by closedness of $\leqslant$, $x \leqslant y$. Thus $|x| \geqslant a$, and $x \in A$. Thus A is a closed lower set. Its intersection with P is a closed lower set in P, and since directed sets converge in the Lawson topology to their suprema (as one easily sees using subbasic open sets), it follows that $A \cap P$ is closed under directed sups, and hence Scott closed. □

Definition 4.1. Let P be an ω-continuous poset satisfying (†) and equipped with an order convex metric d compatible with the Lawson topology and with a weight function $|\cdot| : P \to [0,\infty)$ satisfying

(i) $\forall y \in P$, $|y| \geqslant 0$ and $|y| = 0 \Leftrightarrow y$ is a maximal element;

(ii) the mapping $|\cdot| : P \to [0,\infty)$ is upper semicontinuous with respect to the Scott topology on P;

(iii) if $x < y$ in P, then $|y| + d(x,y) \leqslant |x|$.

Then $(P, d, |\cdot|)$ is called a *stratified* ω-continuous poset.

Corollary 4.2. Every ω-continuous poset satisfying (†) admits a metric d and a weight function $|\cdot|$ such that $(P, d, |\cdot|)$ is a stratified ω-continuous poset.

Proof. The corollary follows directly from the considerations preceding Definition 4.1. □

Theorem 4.3. Let P_1 and P_2 be ω-continuous posets, where P_1 satisfies (†) and P_2 is a complete semilattice (*i.e.*, every non-empty subset has an infimum). Let X be a non-empty subset of $\mathrm{Max}(P_1)$, and let $f : X \to P_2$ be a function such that $f(X) \subseteq \mathrm{Max}(P_2)$. Then there exists a Scott continuous function $g : P_1 \to P_2$ such that $g \leqslant f$ on X and $g(x) = f(x)$ at each point of continuity of f.

Proof. We may assume without loss of generality that P_1 has a least element $\perp$, since by adjoining one we preserve property (†).

By Corollary 4.2, we may pick a metric d on P_1 and $|\cdot| : P_1 \to [0,\infty)$ such that a stratification results. Let $B(x,\epsilon)$ denote the open ball of radius ϵ around x with respect to the metric d.

We consider first the case that there exists a base B for P_1 such that $|b| > 0$ for all $b \in B$. For $c \in B$, we set

$$q(c) := \inf\{\epsilon > 0 : \exists b \in B(c,\epsilon) \cap B \cap {\downarrow} c \text{ such that } {\uparrow} b \cap X \neq \varnothing\}.$$

Note that the set on the right is non-empty for $\epsilon > d(c,\perp)$. For $c \in B$, pick an arbitrary

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but fixed member $r(c)$ of P_1 such that

$$r(c) \in \{b \in B : b \in \downarrow c \cap N(c, q(c) + |c|),\ \uparrow b \cap X \neq \varnothing\}.$$

Note that the right-hand set is non-empty by definition of q and the fact that $|c| > 0$ in the case we are considering. Define $g : P_1 \to P_2$ by

$$g(x) := \bigvee_{b \in B \cap \Downarrow x} \inf\{f(y) : y \in X \cap \uparrow r(a) \text{ for some } a \in B,\ b \ll a\}.$$

Note that since B is a base, the supremum on the right is a directed supremum, and hence exists. We verify in steps various properties of g.

Step 1: *g is monotone.*

This is straightforward since for larger y we take supremum over a larger set.

Step 2: *g is Scott continuous.*

Let $\{x_n\}$ be a totally ordered sequence with supremum x. Pick $w \ll g(x)$. Then there exist $b \in B$, $b \ll x$ such that

$$w \leqslant \inf\{f(y) : y \in X \cap \uparrow r(a) \text{ for some } a \in B,\ b \ll a\}.$$

There exists some x_m such that $b \ll x_m$ (using the definition of $\ll$ and the interpolation property). Then

$$w \leqslant \inf\{f(y) : y \in X \cap \uparrow r(a) \text{ for some } a \in b,\ b \ll a\} \leqslant g(x_m).$$

Since $w \ll g(x)$ was arbitrary, we conclude that $\sup_n g(x_n) \geqslant g(x)$, and this coupled with the monotonicity of g implies g is Scott continuous.

Step 3: *$g(x) \leqslant f(x)$ for all $x \in X$.*

Let $b \in B$ with $b \ll x$. It follows from the interpolation property and the definition of a base that there exists $a \in B$ such that $b \ll a \ll x$. Then $r(a) \leqslant a \leqslant x$ by definition of $r(a)$; thus $x \in \uparrow r(a) \cap X$. It follows that

$$\inf\{f(w) : w \in X \cap \uparrow r(a) \text{ for some } a \in B, b \ll a\} \leqslant f(x),$$

and hence $g(x) \leqslant f(x)$.

Step 4: *$g(x) = f(x)$ at any point x where f is continuous.*

Let f be continuous at x and let $\{b_n\}$ be a totally ordered sequence in B with supremum x and $b_n \ll x$ for each n. Let $w \ll f(x)$. By continuity of f at x, there exists a Scott open set V containing x such that $f(V \cap X) \subseteq \Uparrow w$. Choose b_n such that $b_n \in V$, and choose $\epsilon > 0$ such that $B(x, \epsilon) \subseteq \Uparrow b_n$ (the latter is possible, since $\Uparrow b_n$ is Scott, hence metric, open). Note that $f(\Uparrow b_n \cap X) \subseteq \Uparrow w$.

Choose a Scott open set W containing x such that $w \in W$ implies $|w| < \epsilon/3$ (this is possible, since $|\cdot|$ is upper semicontinuous and $|x| = 0$). There exists $m \geqslant n$ such that $b_m \in W$. Then

$$\inf\{f(y) : y \in X \cap \uparrow r(a) \text{ for some } a \in B,\ b_m \ll a\} \leqslant g(x).$$

Suppose that $r(a) \leqslant y \in X$ and $b_m \ll a \in B$. Then from the choice of $r(a)$,

$$\begin{aligned} d(r(a), x) &\leqslant d(r(a), a) + d(a, b_m) + d(b_m, x) && (1)\\ &\leqslant q(a) + |a| + (|b_m| - |a|) + (|b_m| - |x|) && (2)\\ &\leqslant q(a) + 2|b_m| < q(a) + \tfrac{2}{3}\epsilon. && (3) \end{aligned}$$

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Now $d(b_m, a) \leqslant |b_m| - |a| \leqslant |b_m| < \epsilon/3$. Since $b_m \leqslant a$, it follows from the definition that $q(a) \leqslant \epsilon/3$. Thus $q(a) + (2/3)\epsilon \leqslant \epsilon$. We conclude that $d(r(a), x) < \epsilon$. From this it follows that $b_n \ll r(a)$, and hence $b_n \ll y$ if $r(a) \leqslant y$. We conclude

$$b_n \ll y \qquad \text{if} \qquad y \in X \cap \uparrow r(a) \text{ for some } a \in B,\ b_m \ll a,$$

and hence $f(y) \in f(\Uparrow b_n \cap X) \subseteq \Uparrow w$ for such y. It follows that

$$w \leqslant \inf\{f(y) : y \in X \cap \uparrow r(a) \text{ for some } a \in B,\ b_m \ll a\} \leqslant g(x).$$

Since $w \ll f(x)$ was arbitrary, it follows that $f(x) \leqslant g(x)$, and hence $f(x) = g(x)$ from the previous step. This concludes the special case when $|b| > 0$ for each $b \in B$.

We turn now to the general case. Let $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$ denote the ω-continuous poset ordered with its usual order with ∞ the largest element. Define $|n| = 1/n$, $|\infty| = 0$, $d(m, n) = d(n, m) = (1/m) - (1/n)$ for $m \leqslant n$ and $d(n, \infty) = 1/n$. We take the basis $\mathbf{N}$ for $\mathbf{N}^*$. Let P_1, P_2, X, and f be as in the hypothesis of the theorem. Then $P_1 \times \mathbf{N}^*$ with the coordinatewise order is an ω-continuous poset with set of maximal elements $\mathrm{Max}(P_1) \times \{\infty\}$, which we may identify with $\mathrm{Max}(P_1)$. The fact that the product of the Scott topologies is the Scott topology of the products (Gierz *et al.* 1980, Lemma III.1.3) ensures that with respect to the relative Scott topology, $\mathrm{Max}(P_1)$ and $\mathrm{Max}(P_1 \times \mathbf{N}^*)$ are homeomorphic with respect to this identification. We may pick a metric d on P_1 and $|\cdot| : P_1 \to [0, \infty)$ such that a stratification of P_1 results. Then the sum distance $d\big((x, m), (y, n)\big) := d(x, y) + d(m, n)$ and the sum $|(x, n)| := |x| + |n|$ yield a stratification of $P_1 \times \mathbf{N}^*$, as can be easily verified (the upper semicontinuity of $|\cdot|$ depends on continuity of each summand and the fact that the Scott topology of the product is the product of the Scott topologies, and similarly for the compatibility of the metric with the Lawson topology; see Gierz *et al.* (1980, Exercise II.4.23) and the modification of Gierz *et al.* (1980, Lemma III.1.3) to posets with $\perp$). If B is a basis for P_1, then $B \times \mathbf{N}$ is a basis for $P_1 \times \mathbf{N}^*$. If P_1 embeds in a compact pospace Y, then $P_1 \times \mathbf{N}^*$ embeds in $Y \times \mathbf{N}^*$, and thus by Theorem 2.3, $P_1 \times \mathbf{N}^*$ has property (†). If $X \subseteq \mathrm{Max}(P_1)$ and $f : X \to P_2$ satisfies the hypothesis of this theorem, then, identifying X with $X \times \{\infty\}$, we obtain a function from a subset of the maximal elements of $P_1 \times \mathbf{N}^*$ to P_2. The poset $P_1 \times \mathbf{N}^*$ has $|(b, n)| > 0$ for each basis element $(b, n) \in B \times \mathbf{N}$, since $|n| = 1/n$. Thus we may apply the previous special case to the poset $P_1 \times \mathbf{N}^*$ and $f : X \times \{\infty\}$, and obtain a Scott continuous function $g : P_1 \times \mathbf{N}^* \to P_2$ such that $g(x) \leqslant f(x)$ at all $x \in X$, and the two agree at points of continuity. When we restrict g to $P_1 = P_1 \times \{\infty\}$, we obtain the desired function g with domain P_1. □

The next examples give some standard cases in which MP-hulls that are complete semilattices arise, and thus where Theorem 3.4 applies when these MP-hulls are the codomains.

Example 4.1. (*Locally compact spaces*) If Y is a separable metric locally compact space, one can consider the complete semilattice P of all non-empty closed subsets ordered by reverse inclusion as an MP-hull for Y, where Y is embedded as the singleton sets. Alternatively, one may choose P to be all compact subsets of Y ordered by reverse inclusion, with an element $\perp$ adjoined when Y is not compact ($\perp$ is needed to have a complete semilattice).

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Example 4.2. (*Convex structures*) Consider a compact convex subset X of a locally convex space. Let P be the set of all non-empty compact convex subsets of X ordered by reverse inclusion, and embed X in P as the singletons. In particular, for a real interval (or the extended reals) one obtains for P the set of closed subintervals.

Example 4.3. (*Spaces of infinite trees*) For each n, let X_n be a countable discrete space, and let $f_n: X_{n+1} \to X_n$ be a surjective mapping. Then one can form the inverse limit X_∞, and an MP-hull for it by $P := X_\infty \cup \bigcup_n X_n$ with $x \leqslant y$ for $x \in X_n$ and $y \in X_{n+1}$ if and only if $f_n(y) = x$. The members of X_∞ are larger than the elements they project on. This ordering generates a partial order for an ω-continuous poset satisfying (†), indeed an infinite tree with X_∞ the endpoints 'at infinity', which is an MP-hull for X_∞, and it is a complete semilattice since it is a tree. By choosing $X_n = \mathbf{N}^n$ and $f_n: \mathbf{N}^{n+1} \to \mathbf{N}^n$ to be projection onto the first n-coordinates, we obtain an MP-hull for $\mathbf{N}^{\mathbf{N}}$ ('Baire space'), which is homeomorphic to the irrationals. A similar construction with a fixed finite set replacing $\mathbf{N}$ yields the Cantor set as the set of points at infinity.

We close by giving an alternate derivation in our setting of a result that may be found in Weihrauch and Schreiber (1981).

Proposition 4.4. Let X, Y be Polish spaces and let $f: A \to Y$ be a continuous function from a subset A of X into Y. Then there exists a G_δ-set D of X containing A such that f extends continuously to a function from D to Y.

Proof. As in the proof of Corollary 2.2, we may find a compact metric space Z that contains Y as a G_δ-set. Let P be an MP-hull for X and let UZ denote the upper space hull for Z (Example 2.2). By Theorem 4.3, the mapping f extends to a Scott continuous mapping $g: P \to UZ$. Now Z is a G_δ in UZ (either directly or by Corollary 3.5), and Y is a G_δ-subset of Z, so Y is a G_δ-subset of UZ. Hence its inverse image under g is a G_δ subset of P. By Corollary 3.5, X is a G_δ-subset of P, and the intersection of the inverse image set with X is a G_δ-subset of X for which the restriction of g is a continuous extension of f. □

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