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UNIVERSAL GENERALISED QUADRATIC FORMS REDUCE TO THEIR QUADRATIC SUBFORMS

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Abstract

Chwiedziuk *et al.* ['No proper generalized quadratic forms are universal over quadratic fields', *Ramanujan J.* **69** (2026), Article no. 92] recently proved that over a real quadratic field, a totally positive definite universal generalised quadratic form must contain a universal quadratic subform and asked whether this fails in higher degree. We prove it never does. We establish a sharper statement, valid over every totally real field: if such a form represents a value using a nonzero proper variable, that value is bounded below in at least two Archimedean embeddings. Universality follows at once, since a target can be made small in $d-1$ of the d embeddings simultaneously, the unit rank of a totally real field of degree d being $d-1$. The bound two is sharp. We analyse examples to explain the mechanisms governing the result in degrees 2, 3 and 4.

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1. Introduction and main results

A quadratic form over the ring of integers O_K of a totally real number field K is *universal* if it represents every totally positive integer. Universal quadratic forms have a long history over $\mathbb{Z}$, beginning with Lagrange's theorem that $x_1^2+x_2^2+x_3^2+x_4^2$ represents every positive integer and culminating in the 290-theorem of Bhargava and Hanke [1], by which a positive definite integral quadratic form is universal as soon as it represents an explicit finite list of integers. Over a totally real field K, one replaces positivity by total positivity and asks for forms representing every totally positive element of O_K. Already, sums of squares behave subtly in this setting: Siegel [10] proved that $x_1^2+\cdots+x_n^2$ is universal over O_K only for $K=\mathbb{Q}$ and $K=\mathbb{Q}(\sqrt{5})$.

Over real quadratic fields $\mathbb{Q}(\sqrt{D})$, the minimal number of variables in a universal form and related invariants are controlled by the continued fraction expansion of $\sqrt{D}$ and by the totally positive *indecomposable* integers, those totally positive elements of O_K that cannot be written as a sum of two totally positive integers (see Blomer

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and Kala [2] and the survey [7]). Generalised quadratic forms, introduced over totally real fields by Browning *et al.* [3], enlarge the class of quadratic forms by allowing the Galois conjugates of a variable to occur alongside the variable and encode the arithmetic of systems of quadrics. Their universality was studied by the Prague school [4, 5].

In [4], Chwiedziuk *et al.* proved the following important dichotomy over real quadratic fields. Call a generalised form *proper* if it has a variable z with both z and its conjugate present. Then, under a positive-definiteness hypothesis, a universal generalised form must contain a universal ordinary quadratic subform; the definiteness hypothesis is necessary, with a semidefinite counterexample over $\mathbb{Q}(\sqrt{2})$ [4, Theorem 7, Example 9]. That proof is special to degree two. As noted in [4, Remark 8], its engine is a lower bound on the symmetric expression $z^2 + \tau(z)^2 = \mathrm{Tr}(z^2)$, available because a degree two field has exactly two conjugates, so a proper variable carries both. In degree $d \geqslant 3$, a proper variable may carry only two of d conjugates, with no symmetric expression to bound from below, and on this basis, the authors conjecture that a counterexample should exist over a cyclic cubic field.

We settle this. The phenomenon they isolate is real; a partial proper variable can be made small in one embedding, but it yields no counterexample, because it cannot be made small in two embeddings at once. This is the content of our main result, proved in Section 3 as Theorem 3.1 and sharpening [4, Lemma 6]: if a totally positive definite generalised form represents a value through a nonzero proper variable, then that value is bounded below in at least two of the d Archimedean embeddings. A proper variable can therefore contribute to representing a totally positive integer only when that integer is large in at least two embeddings, and from this, the universality theorem is immediate, in full generality.

THEOREM 1.1 (Universality). *Let K be any totally real number field of degree $d \geqslant 2$ and let G be a totally positive definite generalised quadratic form over O_K built from $H = \mathrm{Aut}(K/\mathbb{Q})$. If G is universal, then G contains a universal quadratic subform. Consequently, G is universal if and only if its quadratic subform is so.*

In particular, the higher-degree counterexample conjectured in [4, Remark 8] does not exist. Theorem 1.1 removes both the degree restriction and any Galois assumption; for K Galois of degree d, one has $|H| = d$, recovering the setting of Remark 8.

Theorem 1.1 follows from the estimate of Section 3 by one observation: given $\alpha \in O_K^+$, a unit ε can be chosen so that $\varepsilon^2\alpha$ is small in any prescribed $d - 1$ embeddings simultaneously, possible because the unit group of a totally real field of degree d has rank $d - 1$. Thus, the totally real hypothesis serves twice, supplying real embeddings for definiteness and the unit rank $d - 1$ for the shrinking. Beyond Dirichlet's unit theorem, the argument only requires that a nonzero algebraic integer cannot have all Archimedean absolute values below 1.

This bound is sharp. Over the smallest cyclic cubic field $\mathbb{Q}(\zeta_7)^+$, we exhibit in Section 5 a definite proper form $B(z) = z^2 + \sigma(z)^2$ whose value is large in two embeddings and tends to 0 in the third, so the count two cannot be improved to three for

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$d = 3$ and the single-embedding argument of [4] fails verbatim for $d \geqslant 3$. A companion witness for a partial proper variable over the non-Galois quartic field $\mathbb{Q}(\sqrt{3+\sqrt{2}})$ is given in Example 4.4.

Finally, total positive-definiteness cannot be relaxed in degree two: [4, Example 9] constructs a totally positive semidefinite universal generalised form over $\mathbb{Q}(\sqrt{2})$ with no universal quadratic subform. Whether the same failure occurs in degree $d \geqslant 3$ is the natural remaining question and we record it as an open problem in Section 6.

2. The setup

Let K be a totally real number field of degree d, with ring of integers $\mathcal{O}_K$, distinct real embeddings $\sigma_1, \dots, \sigma_d \colon K \hookrightarrow \mathbb{R}$ and automorphism group $H = \mathrm{Aut}(K/\mathbb{Q})$. For $\alpha \in K$, write $\mathrm{N}(\alpha) = \prod_k \sigma_k(\alpha)$ and $\mathrm{Tr}(\alpha) = \sum_k \sigma_k(\alpha)$; call α totally positive, $\alpha \gg 0$, if $\sigma_k(\alpha) > 0$ for all k, and let $\mathcal{O}_K^+$ be the totally positive integers and $\mathcal{O}_K^\times$ the units in $\mathcal{O}_K$.

2.1. Generalised quadratic forms. We follow [4, Definitions 1–3] in the degree d formulation of [4, Remark 8]. To each variable z_i $(1 \leqslant i \leqslant r)$, attach a nonempty support $P_i \subseteq H$ with a formal variable $w_{i,\rho}$ for each $\rho \in P_i$. A *generalised quadratic form* is

$$G(z_1, \dots, z_r) = Q((w_{i,\rho})_{i,\,\rho \in P_i})\big|_{w_{i,\rho} = \rho(z_i)},$$

where Q, the *associated form*, is an ordinary quadratic form with coefficients in $\mathcal{O}_K$ (that is, G is *integral*). A variable z_i is *cosmetic* if $|P_i| = 1$ and *proper* if $|P_i| \geqslant 2$; a proper variable is *full* if $P_i = H$. The *quadratic subform* is $\widetilde{Q} := G|_{z_i = 0 \text{ for proper } i}$, an ordinary quadratic form in the cosmetic variables. As in [4, Definition 2], G is *totally positive definite* if $\sigma_k(Q)$ and the real form, obtained by applying σ_k to the coefficients, is positive definite for every k; equivalently, $Q \gg 0$ on $K^n \setminus \{0\}$. It is *universal* if it represents every $\alpha \in \mathcal{O}_K^+$. Since each $\rho \in H$ is a bijection of $\mathcal{O}_K$, the cosmetic conjugations do not affect which integers are represented, so $\widetilde{Q}$ is universal as a generalised form if and only if the underlying ordinary form is so [4, Section 2].

2.2. Two lemmas. The first is [4, Lemma 5] and is the only use of definiteness.

LEMMA 2.1. *If G is totally positive definite with associated form Q in n variables, there is a rational $\delta = \delta(K, G) > 0$ with $\sigma_k(Q)(w_1, \dots, w_n) \geqslant \delta(w_1^2 + \cdots + w_n^2)$ for every embedding σ_k and all real w.*

PROOF. For each k, take $\delta_k > 0$ to be the least eigenvalue of the positive-definite form $\sigma_k(Q)$, so $\sigma_k(Q)(w) \geqslant \delta_k \|w\|^2$, and set $\delta = \min_k \delta_k$. □

The second is where the degree d enters; in degree two, it is [4, Lemma 4], shrinking the single nondistinguished embedding. We deduce it from the statement of Dirichlet's unit theorem.

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LEMMA 2.2. *Let K be totally real of degree d and fix any $d-1$ of its embeddings. For every $C>0$, there is a unit $\varepsilon \in O_K^\times$ with $|\sigma_k(\varepsilon)| < C$ for each of the chosen $d-1$ embeddings.*

PROOF. By Dirichlet's unit theorem [9, Ch. I, Section 7], [8, Ch. 5], the logarithmic embedding $\mathrm{Log}\colon O_K^\times \to \mathbb{R}^d$, $\eta \mapsto (\log|\sigma_1\eta|, \dots, \log|\sigma_d\eta|)$, maps $O_K^\times$ onto a full lattice in the trace-zero hyperplane $H_0 = \{x \in \mathbb{R}^d : \sum_k x_k = 0\}$. Projection of H_0 onto any $d-1$ of the coordinates is a linear isomorphism onto $\mathbb{R}^{d-1}$, so this lattice contains a point all of whose chosen $d-1$ coordinates lie below $\log C$; any unit mapping to such a point satisfies the conclusion. □

We use two further facts. If $z \in O_K$ has $|\sigma_k(z)| < 1$ for all k, then $|\mathrm{N}(z)| = \prod_k |\sigma_k(z)| < 1$ and as $\mathrm{N}(z) \in \mathbb{Z}$, this forces $z = 0$. Further, H acts freely on the embeddings by $\sigma \mapsto \sigma \circ \rho$: if $\sigma \circ \rho = \sigma$, then injectivity of σ gives $\rho = \mathrm{id}$, so for distinct $\rho, \rho' \in H$, the embeddings $\sigma \circ \rho, \sigma \circ \rho'$ are distinct.

3. The two-embedding bound

THEOREM 3.1. *Let K be a totally real number field of degree d, let G be a totally positive definite generalised quadratic form over O_K and let $\delta > 0$ be a definiteness constant for its associated form (Lemma 2.1). If $z_1, \dots, z_r \in O_K$ are such that some proper variable z_i takes a nonzero value, then*

$$\#\{\, k \in \{1, \dots, d\} : \sigma_k(G(z_1, \dots, z_r)) \geqslant \delta\} \geqslant 2.$$

If z_i is full, meaning all d conjugates of z_i occur, then $\sigma_k(G(\mathbf{z})) \geqslant \delta \operatorname{Tr}(z_i^2) \geqslant \delta d$ for every k.

PROOF. Let δ be as in Lemma 2.1 and let z_i be a proper variable with $z_i \neq 0$. For each embedding σ_k, writing the associated variables as $w_{i',\rho'} = \rho'(z_{i'})$ and discarding all nonnegative terms but those attached to z_i,

$$\sigma_k(G(\mathbf{z})) = \sigma_k(Q)((\sigma_k(\rho'(z_{i'})))) \geqslant \delta \sum_{\rho \in P_i} \sigma_k(\rho(z_i))^2. \tag{3.1}$$

Suppose, for contradiction, that $\sigma_k(G(\mathbf{z})) \geqslant \delta$ for at most one index k; then, at least $d-1$ embeddings have $\sigma_k(G(\mathbf{z})) < \delta$. Choose $d-1$ of them and relabel as $\sigma_1, \dots, \sigma_{d-1}$, leaving σ_d as the only possibly excluded embedding. By (3.1), for $k \leqslant d-1$,

$$\delta \sum_{\rho \in P_i} \sigma_k(\rho(z_i))^2 \leqslant \sigma_k(G(\mathbf{z})) < \delta \quad \Longrightarrow \quad |\sigma_k(\rho(z_i))| < 1 \quad \text{for all } \rho \in P_i.$$

Fix $m \in \{1, \dots, d\}$ and pick distinct $\rho, \rho' \in P_i$, possible since $|P_i| \geqslant 2$. By freeness, $\sigma_m \circ \rho^{-1}$ and $\sigma_m \circ \rho'^{-1}$ are distinct, so not both equal σ_d; choose $\rho'' \in \{\rho, \rho'\}$ with $\sigma_m \circ \rho''^{-1} = \sigma_k$, $k \leqslant d-1$. Then, $\sigma_m = \sigma_k \circ \rho''$ gives $|\sigma_m(z_i)| = |\sigma_k(\rho''(z_i))| < 1$. As m was arbitrary, all d absolute values of z_i lie below 1, forcing $z_i = 0$, which is a contradiction. Hence, at least two embeddings satisfy $\sigma_k(G(\mathbf{z})) \geqslant \delta$.

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If the variable z_i is full, then for each k, the multiset $\{\sigma_k(\rho(z_i)) : \rho \in H\}$ is equal to $\{\sigma_m(z_i) : 1 \leqslant m \leqslant d\}$ by simple transitivity, so $\sum_{\rho\in H} \sigma_k(\rho(z_i))^2 = \mathrm{Tr}(z_i^2)$, and (3.1) gives $\sigma_k(G(\mathbf{z})) \geqslant \delta \,\mathrm{Tr}(z_i^2)$. Finally, $z_i^2 \in O_K^+ \setminus \{0\}$ has $\mathrm{N}(z_i^2) \geqslant 1$, so by the arithmetic-geometric mean inequality, $\mathrm{Tr}(z_i^2) \geqslant d\,\mathrm{N}(z_i^2)^{1/d} \geqslant d$. □

4. Universality and an equivalence

PROOF OF THEOREM 1.1. Let δ be the constant of Lemma 2.1 and $\alpha \in O_K^+$ arbitrary. Put $M = 1 + \max_k \sigma_k(\alpha)$, choose $C > 0$ with $C^2M < \delta$, and by Lemma 2.2, take a unit ε with $|\sigma_k(\varepsilon)| < C$ for $k \leqslant d-1$. Set $\beta := \varepsilon^2\alpha \in O_K$. Then, $\beta \gg 0$ and $\sigma_k(\beta) = \sigma_k(\varepsilon)^2\sigma_k(\alpha) < C^2M < \delta$ for $k \leqslant d-1$, so at most one embedding has $\sigma_k(\beta) \geqslant \delta$. As G is universal and $\beta \gg 0$, write $\beta = G(z_1, \ldots, z_r)$. By Theorem 3.1, no proper variable can be nonzero, since that would force two embeddings of β to be $\geqslant \delta$. Hence, $\beta = \widetilde{Q}(\mathbf{y})$ for cosmetic values $\mathbf{y} \in O_K^s$, and as $\widetilde{Q}$ is an ordinary quadratic form, $\widetilde{Q}(\varepsilon^{-1}\mathbf{y}) = \varepsilon^{-2}\beta = \alpha$ with $\varepsilon^{-1}\mathbf{y} \in O_K^s$; the cosmetic conjugations being bijections of O_K, this is a representation of α by $\widetilde{Q}$. So, $\widetilde{Q}$ is universal. The converse is trivial, since $\widetilde{Q}$ arises from G by setting variables to zero, so every value of $\widetilde{Q}$ is a value of G. □

COROLLARY 4.1. *Let G be totally positive definite over a totally real field K, with constant δ. If $\beta \in O_K^+$ is represented by G and $\sigma_k(\beta) < \delta$ for at least $d-1$ embeddings, then β is represented by the quadratic subform $\widetilde{Q}$. Equivalently, proper variables can enlarge the represented set only by integers that are $\geqslant \delta$ in at least two embeddings.*

PROOF. Write $\beta = G(\mathbf{z})$. By Theorem 3.1, no proper variable is nonzero, so $\beta = \widetilde{Q}(\mathbf{y})$. □

REMARK 4.2. In degree two, 'at least two embeddings' means both, so Theorem 3.1 recovers and slightly strengthens [4, Lemma 6], which records only the distinguished embedding. The strengthening is invisible there, as in degree two, every proper variable is full and full variables are controlled in all embeddings. More generally, the full case of Theorem 3.1 gives $\sigma_k(G(\mathbf{z})) \geqslant \delta\,\mathrm{Tr}(z_i^2)$ in every embedding, the exact analogue of the symmetric bound of the quadratic case. For a partial proper variable, the trace stays large, but can concentrate in $d-1$ embeddings, and Theorem 3.1 shows it can never concentrate to leave fewer than two.

REMARK 4.3 (Non-Galois fields). The proof of Theorem 3.1 uses only the facts that $H = \mathrm{Aut}(K/\mathbb{Q})$ acts freely on the embeddings and that proper means $|P_i| \geqslant 2$; transitivity is never used. Hence, Theorems 3.1 and 1.1 hold over any totally real field with the generalised form notion built from $\mathrm{Aut}(K/\mathbb{Q})$. When $\mathrm{Aut}(K/\mathbb{Q})$ is trivial, as for a non-Galois cubic, there are no proper variables and the statements are vacuous. The first nontrivial non-Galois case is a quartic with $\mathrm{Aut}(K/\mathbb{Q}) \cong C_2$, treated in the following example.

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EXAMPLE 4.4 (A non-Galois quartic). Let $K = \mathbb{Q}(t)$ with $t = \sqrt{3+\sqrt{2}}$, the field of $x^4 - 6x^2 + 7$. Its four roots $\pm\sqrt{3 \pm \sqrt{2}}$ are real, so K is totally real and, by the standard criterion for biquadratic quartics, with $p = -6$, $q = 7$, neither $q = 7$ nor $q(p^2 - 4q) = 56$ being a square, the Galois closure is D_4. Thus, K is non-Galois with $\mathrm{Aut}(K/\mathbb{Q}) = \{1, \tau\} \cong C_2$, where $\tau(t) = -t$, and the unique quadratic subfield is $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(t^2 - 3)$. The involution τ acts on the four embeddings $\sigma_k : t \mapsto \pm\sqrt{3 \pm \sqrt{2}}$ without fixed points, pairing $\{\sigma_1, \sigma_2\}$ and $\{\sigma_3, \sigma_4\}$.

The proper form $B(z) = z^2 + \tau(z)^2$ has associated form $u^2 + v^2$, positive definite, so B is totally positive definite with $\delta = 1$ and uses two of the four conjugates. Theorem 3.1 predicts that at least two of the four embeddings of $B(z)$ are $\geqslant 1$ for every nonzero z and the covering step reads concretely here: among the two distinct embeddings $\sigma_m, \sigma_m \circ \tau$, at least one survives the exclusion of any single embedding, so controlling three embeddings forces $|\sigma_m(z)| < 1$ for all m; hence, $z = 0$. The bound two is attained: for $z = 2 - t^2$, the four values of $\sigma_k(B(z))$ are $11.66\ldots$, $11.66\ldots$, $0.343\ldots$, $0.343\ldots$, large in exactly two embeddings. A search over $z = a + bt + ct^2 + dt^3$ with $|a|, |b|, |c|, |d| \leqslant 3$ confirms that the number of embeddings with $\sigma_k(B(z)) \geqslant 1$ is never below two.

5. Sharpness over $\mathbb{Q}(\zeta_7)^+$

Let $K = \mathbb{Q}(\zeta_7)^+ = \mathbb{Q}(\theta)$ with $\theta = 2\cos(2\pi/7)$ the smallest cyclic cubic field. Here, θ has minimal polynomial $x^3 + x^2 - 2x - 1$, the ring of integers is $\mathbb{Z}[\theta]$ and the generator σ of $\mathrm{Gal}(K/\mathbb{Q}) \cong C_3$ acts by $\sigma(\theta) = \theta^2 - 2$. Consider the proper form

$$B(z) = z^2 + \sigma(z)^2,$$

which uses the two conjugates $z, \sigma(z)$ and not $\sigma^2(z)$, with associated form $u^2 + v^2$ positive definite. Thus, B is totally positive definite, with $\delta = 1$ admissible, and $B(z) \gg 0$ for $z \neq 0$.

PROPOSITION 5.1. *Theorem 3.1 is sharp: the count two cannot be replaced by three for $d = 3$. With $z = 227 + 328\,\theta - 409\,\theta^2$, one has $B(z) = 2417977 - 1247022\,\theta - 554977\,\theta^2 \in O_K^+$ and*

$$\sigma_1(B(z)) = 1.3416\ldots \times 10^{-3} < 1 = \delta, \quad \sigma_2(B(z)),\ \sigma_3(B(z)) = 2.8630\ldots \times 10^{6} \geqslant \delta,$$

so exactly two embeddings are $\geqslant \delta$. Moreover,

$$\inf\{\sigma_1(B(z)) : 0 \neq z \in O_K, B(z) \gg 0\} = 0.$$

PROOF. The displayed values are computed in $\mathbb{Z}[\theta]$ and verified to 40 digits. For the infimum, $\sigma_1(B(z)) = \sigma_1(z)^2 + \sigma_2(z)^2$; with $z = u^n$ for the unit $u = -2 - 2\theta + 3\theta^2$, whose embeddings are $(0.1709\ldots, -0.5157\ldots, 11.34\ldots)$, both $|\sigma_1(u)|, |\sigma_2(u)| < 1$, so $\sigma_1(B(u^n)) = \sigma_1(u)^{2n} + \sigma_2(u)^{2n} \to 0$ while $B(u^n) \gg 0$ throughout. □

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This is the configuration anticipated in [4, Remark 8]: a definite proper variable carrying two of three conjugates makes $\sigma_1(B(z))$ arbitrarily small, so the single-embedding argument of [4, Lemma 6, Theorem 7] fails verbatim for $d \geqslant 3$. What survives is that $B(z)$ cannot be small in two embeddings at once, since $\sigma_1(B(z)), \sigma_2(B(z)) < 1$ would force $\sigma_1(z)^2, \sigma_2(z)^2, \sigma_3(z)^2 < 1$; hence, $|\mathrm{N}(z)| < 1$ and $z = 0$. Shrinking the target in $d - 1 = 2$ embeddings, as in Theorem 1.1, exploits exactly this and is available because the unit rank of a cubic field is 2.

6. An open problem

Total positive-definiteness cannot be relaxed in Theorem 1.1, at least in degree two. Over $\mathbb{Q}(\sqrt{2})$, [4, Example 9] builds a semidefinite universal generalised form $G = S + H$ with no universal quadratic subform. By a theorem of Cohn [6], the sum of four squares S represents all totally positive elements of the index-two sublattice $\mathbb{Z}[2\sqrt{2}]$ of $\mathbb{Z}[\sqrt{2}]$, while H, a sum of suitably weighted squares $\ell(z)^2$ of the proper linear form $\ell(z) = 2z - \tau(z)$, supplies the missing coset. The squares $\ell(z)^2$ are rank one, so G is only semidefinite and every quadratic subform of G is a subform of S; hence, not universal. In our argument, definiteness is used exactly once, in Lemma 2.1; for a merely semidefinite form, the constant δ is 0 and Theorem 3.1 collapses, in any degree. This leaves the following question.

PROBLEM 6.1. Let K be a totally real number field of degree $d \geqslant 3$ with $\mathrm{Aut}(K/\mathbb{Q}) \neq \{1\}$ so that proper variables exist over K. Does there exist a totally positive semidefinite generalised quadratic form over O_K that is universal but has no universal quadratic subform? More generally, determine the totally real fields over which such a form exists.

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