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# BILATERAL RELEVANT LOGIC

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**§1. Introduction.** The purpose of this paper is to introduce and study a Bilateral proof-theory of (one variant of) *Relevant Logic* (RL), based on a *bilateral* natural-deduction ($ND$) proof-system. There is a fairly wide agreement about the *grounds for asserting* a relevant implication $\varphi \rightarrow_{\mathbf{R}} \psi$, embodied in its introduction rule ($I$-rule) ($\rightarrow_{\mathbf{R}} I$), namely, the *use* of the assumption $\varphi$ in the derivation of $\psi$. Similarly, there is a general agreement about the presence of two conjunctions in Relevant Logic, due to the fact that it is a substructural logic, lacking the weakening structural rule ($W$). Finally, there is an agreement that (full) Relevant Logic (i.e., containing also negation) is *paraconsistent*, in that it rejects *explosion* (namely, $\varphi \wedge \neg\varphi \vdash \psi$), the latter exhibiting irrelevance of consequence to assumptions. In a bilateral proof-system for a positive (negation-free) logic, *incoherence* (see below) is the counterpart of inconsistency.

The driving questions of this paper are the following.

$Q_1$**:** What are the *grounds for denial* of the relevant implication?
$Q_2$**:** How do the grounds for denials of the two conjunctions differ, and why?
$Q_3$**:** What can be asserted/denied based on an incoherent context?

Recall that *Bilateralism* Rumfitt (2000) is a proof-system which takes both *assertion* and *denial* on a par, as primitive attitudes. While typically bilateralism is introduced in order to "salvage" Classical Logic (CL) from attacks based on the lack of harmony (explained below) of its traditional $ND$-rules, in Francez (2013a) I studied bilateralism in a more general setting, in a negation-free object language, investigating its effect on *Proof-Theoretic Semantics* (PTS), in particular on a central concept of PTS, *harmony*, viewed as a condition on a natural-deduction ($ND$) system to qualify as meaning-conferring. Here, I want to draw on this effect, expanding the argument to apply to a positive fragment of Relevant Logic, and propose answers to $Q_1$, $Q_2$, and $Q_3$.

My interest is purely proof-theoretical, in accordance to PTS, and no model-theory is discussed.

**§2. Preliminaries.** In this section, I briefly review the basic notions to be used in the paper, starting with the reification of the proof-theoretically determined meaning of a logical constant in general. Then, I briefly review the $ND$-presentation of the (implicational fragment of) intuitionistic propositional logic **I**. Then, the unilateral $ND$-presentation of the (implicational fragment of the) relevant logic **R** is reviewed. Finally, bilateralism is briefly reviewed, together with the notion of harmony associated with it.

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***2.1. Proof-theoretic reified sentential meanings.*** Under the (mainstream) proof-theoretic semantics program, the meaning of a logical constant is determined by the $I$-rules of a meaning-conferring $ND$-system. In section 2.7., a condition the satisfaction of which qualifies an $ND$-system, say $N$, as meaning-conferring is discussed. In Francez and Ben-Avi (2011) we propose a definition of a reification of the meaning determined by the meaning-conferring system, further elaborated upon in Francez (2014). I briefly delineate this proposal, needed in the sequel. Suppose $N$ is such a meaning-conferring system. I assume the usual tree-like $N$-derivation. A *context* $\Gamma$ is a structure (depending on the underlying logic) of object-Language formulas. I use $\Gamma : \varphi$ for a (single-conclusion) sequent. The usual abbreviations $\Gamma, \varphi$ and $\Gamma, \Gamma'$ are used for extending contexts.

DEFINITION 2.1 (canonical derivation). ***I*-canonicity:** *An $N$-derivation $\mathcal{D}$ of $\varphi$ from $\Gamma$ is* $I$-canonical *iff the last $N$-rule applied in $\mathcal{D}$ is an $I$-rule (of the main operator of $\varphi$). The collection of all $I$-canonical derivations of $\varphi$ from $\Gamma$ is denoted by $[\![\varphi]\!]^{Ic}_{\Gamma}$. I use $\vdash^{I_c}_{N}$ for $I$-canonical derivability.*

***E*-canonicity:** *An $N$-derivation $\mathcal{D}$ of $\psi$ from $\varphi, \Gamma$ is $E$-*canonical *iff the first $N$-rule applied to $\varphi$ (as a major premise) in $\mathcal{D}$ is an $E$-rule (of the main operator of $\varphi$). The collection of all $E$-canonical derivations from $\varphi$ and $\Gamma$ is denoted by $[\![\varphi]\!]^{Ec}_{\Gamma}$. I use $\vdash^{E_c}_{N}$ for $E$-canonical derivability.*

*When $N$ is clear from context or is immaterial it is omitted.*

The reified meaning of a compound sentences $\varphi$, based on $I$-canonical derivations, is defined below.

DEFINITION 2.2 (reified meaning).

$$[\![\varphi]\!] = \lambda\Gamma.[\![\varphi]\!]^{Ic}_{\Gamma}$$

In other words, the meaning of $\varphi$ is taken to be a mapping from contexts $\Gamma$ to all $I$-canonical derivations (if any) of $\varphi$ from $\Gamma$. This is an *explicit* definition, contrasting the more traditional view in PTS that the rules *implicitly* define the meaning of the logical constants. While $[\![\varphi]\!]$ can be viewed as denotational, the denotation is purely proof-theoretical, not involving any model-theoretic appeal to models with entities, reference, and truth. This definition is discussed in more detail in Francez (2014).

The definition of reified meaning gives rise to a definition of grounds for assertion and outcomes of assertion of a compound sentence $\varphi$.

DEFINITION 2.3 (grounds and outcomes).
***grounds for assertion:*** *A context $\Gamma$ is a* ground for assertion *of $\varphi$ iff $\vdash^{Ic}\Gamma : \varphi$. Let*

$$GA[\![\varphi]\!] = \{\Gamma \mid \vdash^{Ic}\Gamma : \varphi\} \tag{1}$$

***outcomes from assertion:*** *An ordered pair $\langle\Gamma, \psi\rangle$ is an* outcome of assertion *of $\varphi$ iff $\vdash^{Ec}\Gamma, \varphi : \psi$. Let*

$$OA[\![\varphi]\!] = \{\langle\Gamma, \psi\rangle \mid \vdash^{Ec}\Gamma, \varphi : \psi\} \tag{2}$$

The adaptation of those definitions to bilateralism is presented below.

***2.2. Intuitionistic Logic: the implicational fragment.*** The object language consists of a countable set $\mathcal{P} = \{p_i \mid i \geq 0\}$ of *propositional variables* (or *atomic propositions*), and the binary connective '$\rightarrow$' indicating implication. Formulas are defined recursively, as usual, and are ranged over by $\varphi, \psi$ (possibly indexed). A context $\Gamma$ here is a (finite) set of

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$$\frac{}{\varphi \ : \varphi}\ (Ax)$$

$$\frac{\Gamma, [\varphi]_i \ : \psi}{\Gamma : (\varphi \rightarrow \psi)}\ (\rightarrow I^i) \qquad \frac{\Gamma : \varphi \quad \Gamma : (\varphi \rightarrow \psi)}{\Gamma : \psi}\ (\rightarrow E)$$

Fig. 1. The implicational fragment of **I**.

$$\frac{\Gamma : \varphi}{\Gamma, \Gamma' : \varphi}\ (W) \quad \frac{\Gamma, \varphi, \varphi : \psi}{\Gamma, \varphi : \psi}\ (C) \quad \frac{\Gamma_1, \varphi_2, \varphi_1, \Gamma_2 : \psi}{\Gamma_1, \varphi_1, \varphi_2, \Gamma_2 : \psi}\ (E)$$

Fig. 2. The structural rules of **I**.

formulas. The operational rules for $\mathbf{I}_{\rightarrow}$ are presented in Figure 1 (in Gentzen's "logistic" style, with explicit contexts). To avoid notational clutter in presenting examples, a simple, nonlogistic presentation, suppressing contexts, is used. Square brackets[1] indicate *discharge* of an assumption, and the index of the assumption is used to indicate the instance of the rule discharging it. In $\mathbf{I}_{\rightarrow}$ both *vacuous* discharge—discharging an assumption not used in the subderivation—and *multiple discharge*—discharging more than one instance of the assumption, are permitted. A *general elimination (GE)* rule for intuitionistic implication is given by

$$\frac{\Gamma : (\varphi \rightarrow \psi) \quad \Gamma : \varphi \quad \Gamma, [\psi]_i : \chi}{\Gamma : \chi}\ (\rightarrow_{\mathbf{I}} GE^i) \tag{3}$$

The structural rules of $\mathbf{I}_{\rightarrow}$ (applicable to the whole **I**) are presented in Figure 2. As usual, in view of those rules, a context $\Gamma$ for **I** can be viewed as a set instead of a sequence, and the axiom can be taken as $\frac{}{\Gamma, \varphi \ : \varphi}\ (Ax)$. I assume the usual notions of *premise, major/minor premise*, and *conclusion* of a rule (either operational or structural). A *derivation* (in **I**), ranged over by $\mathcal{D}$, is a tree of sequents, where leaves are instances of $(Ax)$, and every intermediate node is the conclusion of a rule, having above it the premise(s) of the rule. The notation $\begin{matrix}\Gamma \\ \mathcal{D} \\ \varphi\end{matrix}$ indicates a derivation of $\varphi$ from $\Gamma$. The root is the conclusion of the whole derivation. By $\vdash_{\mathbf{I}} \Gamma \ : \psi$ is indicated the derivability of $\Gamma \ : \varphi$ in **I**. The following two derivations in $\mathbf{I}_{\rightarrow}$, of $\vdash_{\mathbf{I}} \varphi \rightarrow (\psi \rightarrow \varphi)$ (the positive "paradox of material implication"), using vacuous discharge (of $[\psi]_2$), and weakening (with $[\psi]_2$), respectively, are the main $\mathbf{I}_{\rightarrow}$-derivations rejected by relevant logics. Their blocking is discussed in Section 2.3.

$$\dfrac{\dfrac{\dfrac{}{[\varphi]_1 : \varphi}\ (Ax)}{\varphi : (\psi \rightarrow \varphi)}\ (\rightarrow I^2)}{: (\varphi \rightarrow (\psi \rightarrow \varphi))}\ (\rightarrow I^1) \tag{4}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{[\varphi]_1 : \varphi}\ (Ax)}{\varphi, [\psi]_2 : \varphi}\ (W)}{\varphi : (\psi \rightarrow \varphi)}\ (\rightarrow I^2)}{: (\varphi \rightarrow (\psi \rightarrow \varphi))}\ (\rightarrow I^1) \tag{5}$$

A central notion of what follows is *derivation composition*, defined below.

[1] Strictly speaking, the bracketing of discharged assumptions, used in traditional nonlogistic presentation of $ND$-rules, is superfluous in logistic presentations. Still, I find it as enhancing readability.

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DEFINITION 2.4 (**derivation composition**). *Let* $\overset{\Gamma,\varphi}{\underset{\psi}{\mathcal{D}}}$ *be a derivation of* $\psi$ *from* $\Gamma, \varphi$ *(in* **I***), having* $\varphi$ *as one of its leaves, the rest of the leaves constituting* $\Gamma$. *Let* $\overset{\Gamma'}{\underset{\varphi}{\mathcal{D}'}}$ *be a derivation (in* **I***) of* $\varphi$ *from* $\Gamma'$. *Then* $\mathcal{D}[\varphi := \overset{\mathcal{D}'}{\varphi}]$ $\psi$ *is the tree resulting by replacing the leaf* $\varphi$ *of* $\mathcal{D}$ *with the subtree* $\mathcal{D}'$ *(identifying the root of the latter with the leaf of the former).*

The following is well-known.

PROPOSITION 2.5 (**closure of I under derivation composition**). *For* $\mathcal{D}$ *and* $\mathcal{D}'$ *as in Definition 2.4,* $\mathcal{D}[\varphi := \overset{\mathcal{D}'}{\varphi}]$ $\psi$ *is a derivation (in* **I** *of* $\psi$ *from* $\Gamma, \Gamma'$*).*

Note that when composing the "body" of $\mathcal{D}$ remains unaltered—the only changes take place at the leaves. This proposition reflects the transitivity of '$\vdash_{\mathbf{I}}$'.

EXAMPLE 2.6. *Consider the following two one-step derivations.*

$$\mathcal{D}: \dfrac{\varphi \quad \varphi\to\psi}{\psi}\,(\to E) \qquad \mathcal{D}': \dfrac{\chi \quad \chi\to\varphi}{\varphi}\,(\to E) \tag{6}$$

*Then,*

$$\mathcal{D}[\varphi := \overset{\mathcal{D}'}{\varphi}]\ \psi = \dfrac{\dfrac{\chi \quad \chi\to\varphi}{\varphi}\,(\to E) \quad \varphi\to\psi}{\psi}\,(\to E) \tag{7}$$

***2.3. The Relevant Logic* R*: the implicational fragment.*** The "relevant logician" insists that in a derivable relevant implication $\varphi\to_{\mathbf{R}}\psi$, $\varphi$ must be "relevant" in some sense to $\psi$ in order to imply it. One such sense of relevance, typical to $ND$ presentations of Relevant Logic, is that $\varphi$ must be *used* in the derivation of $\psi$ to allow for $(\to I)$ to apply (and discharge $\varphi$). As a result, both vacuous discharge and the weakening structural rule $(W)$ are rejected.

I shall not concern myself in this paper with negation in relevant logic and its relatedness to *paraconsistency*. Rather, I concentrate on the (positive) implicational-conjunctive fragment of the relevant logic **R** introduced in Anderson and Belnap (1975) (see also Mares (2004)). In spite of being a positive fragment, paraconsistency issues emerge here too. As argued by Dutilh-Novaes (2008), explosion need not be associated with negation; rather, it is associated with contradiction. As already mentioned, in a bilateral presentation of a positive fragment of a logic, *incoherence* is the counterpart of contradiction. So, I extend Dutilh-Novaes' argument further, relating paraconsistency to incoherence (see Section 3.3.).

The main two features of **R** are excluding the structural rule of *weakening* (known also as *Dilution*), and restricting the introduction rule of the conditional in its natural deduction presentation. As shown in Francez (2013c), this restriction gives rise to a restriction on composing derivations. Such derivation compositions lie at the heart of *derivation reduction* (Prawitz (1965)), a central tool in establishing intrinsic harmony (see subsection 2.4.1.). The central topic of Francez (2013c) is overcoming the restriction on composability of derivations, showing that harmony and stability in $\mathbf{R}_{\to}$ obtain in spite of an apparent lack thereof.

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$$\frac{}{\varphi : \varphi}\ (Ax)$$

$$\frac{\Gamma, [\varphi]_i : \psi_\alpha\ (i{\in}\alpha)}{\Gamma : (\varphi{\rightarrow_{\mathbf{R}}}\psi)_{\alpha-\{i\}}}\ (\rightarrow_{\mathbf{R}} I^i) \qquad \frac{\Gamma : \varphi_\alpha \quad \Gamma : (\varphi{\rightarrow_{\mathbf{R}}}\psi)_\beta}{\Gamma : \psi_{\alpha\cup\beta}}\ (\rightarrow_{\mathbf{R}} E)$$

Fig. 3. The implicational fragment of **R**.

$$\frac{\Gamma, \varphi, \varphi : \psi}{\Gamma, \varphi : \psi}\ (C) \quad \frac{\Gamma_1, \varphi_2, \varphi_1, \Gamma_2 : \psi}{\Gamma_1, \varphi_1, \varphi_2, \Gamma_2 : \psi}\ (E)$$

Fig. 4. The structural rules of **R**.

The operational rules for the implication fragment $\mathbf{R}_{\rightarrow}$ of **R** (in context-sharing logistic tree-style) are presented in Figure 3, and its structural rules in Figure 4. I refer to this system as the *unilateral* proof system for $\mathbf{R}_{\rightarrow}$. Lower case Greek letter range over sets of indices. To avoid notational clutter, in examples the rules will be used in shorthand notation, without contexts.

$$\frac{\begin{array}{c}[\varphi]_i \\ \vdots \\ \psi_\alpha\end{array}}{(\varphi{\rightarrow_{\mathbf{R}}}\psi)_{\alpha-\{i\}}}\ (\rightarrow_{\mathbf{R}} I^i)\ \ (i{\in}\alpha) \qquad \frac{\varphi_\alpha \quad (\varphi{\rightarrow_{\mathbf{R}}}\psi)_\beta}{\psi_{\alpha\cup\beta}}\ (\rightarrow_{\mathbf{R}} E)$$

Derivations (in tree-form) are defined recursively as usual, where application of $(\rightarrow I)$ always adheres to the side-condition assuring use. Consequently, relevant derivations are restricted so as to satisfy the conditions on indices. I skip the tedious details of an explicit definition. To formulate the definition of *relevant derivability* $\vdash_{\mathbf{R}_{\rightarrow}}$, I use the following convention, following Mares (2012): a context is always indexed with consecutive indices, starting from 1; namely $\Gamma = \varphi_1, \cdots, \varphi_n$. Let $i(\Gamma) = \{1, \cdots, n\}$. However, if some formula occurs twice (or more) in $\Gamma$, all occurrences carry the same index, the index of the leftmost occurrence by convention.

DEFINITION 2.7 (**relevant derivability**). $\vdash_{\mathbf{R}_{\rightarrow}} \Gamma : \varphi$ *iff there exists a derivation in* $\mathbf{R}_{\rightarrow}$ *of* $\Gamma : \varphi_\alpha$ *where* $\alpha = i(\Gamma)$.

**Remarks:**

1. Assumptions are always introduced with a *fresh* index (not used elsewhere). When $C$ (contraction) is applied, the two copies of $\varphi$ carry the same index, the index of $\varphi$ in the conclusion. Actually, there is no need to assume $(C)$ as primitive, since it is admissible due to the possibility of multiple discharge. Note that $(W)$ (weakening) is *not* present, rendering $\mathbf{R}_{\rightarrow}$ substructural.
2. Relevant implication can be introduced only if its discharged assumption has been used in the subderivation in the premise (indicated by the side condition $i{\in}\alpha$). The discharge is indicated by the removal of the index from the conclusion's dependence set. The application of $(\rightarrow_{\mathbf{R}} E)$ constitutes such a use.
3. Note that the derivation of the first paradox of material implication above, presented in (4), is blocked, as no vacuous discharge is possible. The side condition on the unused assumption $[\psi]_2$ is violated, as $2 \not\in \{1\}$.

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$$\dfrac{\dfrac{\dfrac{}{[\varphi]_1 : \varphi_{\{1\}}}\,(Ax)}{\varphi : (\psi \rightarrow_{\mathbf{R}} \varphi)_{\{?\}}}\,(\rightarrow_{\mathbf{R}} I^2)}{: (\varphi \rightarrow_{\mathbf{R}} (\psi \rightarrow_{\mathbf{R}} \varphi))_{\{??\}}}\,(\rightarrow_{\mathbf{R}} I^1) \tag{8}$$

Similarly, the second derivation in (5) is blocked by the unavailability of the weakening rule $(W)$.

4. Below is the $GE$-rule for $\rightarrow_{\mathbf{R}}$.

$$\dfrac{\begin{array}{c}[\psi]_i \\ \vdots \\ (\varphi \rightarrow_{\mathbf{R}} \psi)_\alpha \quad \varphi_\beta \quad \chi_\gamma \quad (i \in \gamma)\end{array}}{\chi_{\alpha \cup \beta \cup \gamma - \{i\}}}\,(\rightarrow_{\mathbf{R}} E^i) \tag{9}$$

The standard $(\rightarrow E)$ (in Figure 3) is obtained from $(\rightarrow GE)$ as a special case by choosing the arbitrary conclusion $\chi$ as $\psi$ itself.

Note that the restriction on the $(\rightarrow_{\mathbf{R}} I)$ rule, namely the use of $\varphi$ in the derivation of $\psi$, propagates into the definition of $I$-canonicity of a derivation, and thereby into the definition of the grounds for assertion $GA[\![\varphi]\!]$. Now $\Gamma \in GA[\![\varphi]\!]$ iff $\Gamma$ *relevantly* derives $\varphi$. For example, in $\mathbf{I}_{\rightarrow}$ it holds that $\psi \in GA[\![\varphi \rightarrow \psi]\!]$, which does not hold in $\mathbf{R}_{\rightarrow}$.

***2.4. Harmony and Stability.*** In this section, I consider the criteria to be imposed on an $ND$-system to qualify as meaning-conferring, and principles guaranteeing those criteria are met. In Prior (1960), Prior's famously attacked the PTS program by exhibiting a connective *tonk* with an $I$-rule of '$\vee$' (disjunction) and an $E$-rule of '$\wedge$' (conjunction), that trivialized the derivability relation '$\vdash$' by rendering $\vdash \varphi : \psi$ for every $\varphi$ and $\psi$. Ever since that attack, it became clear that not *every* collection of $I/E$-rules qualifies as meaning-conferring. The main two criteria that were proposed for the $I/E$-rules for qualifying as meaning-conferring are the following.

**Conservative extension:** According to Belnap (1962), the rules should conservatively extend the underlying derivability relation, as determined by the structural rules assumed. I will not further pursue this criterion here.

**Harmony and stability:** According to Dummett (e.g., in Dummett (1991)), Prawitz (e.g., Prawitz (1971, 1978)), Tennant (e.g., Tennant (1997)), and many others, there should be a *balance* between the $I$-rules and $E$-rules of each connective: the $E$-rules should no be neither too strong nor too weak with respect to (w.r.t.) the $I$-rules.

*2.4.1. Harmony.* Traditionally (in a unilateral logic), harmony is associated with a principle, referred to as the *inversion principle* Prawitz (1965). Basically, it states that every conclusion derived from (a compound) $\varphi$ is also a conclusion derivable from any ground for asserting $\varphi$. To distinguish this principle from a second one, not present in the unilateral case, I will refer to it as the *vertical inversion* principle.

This principle can be "implemented" in two ways, leading to two ways of capturing the informal notion of harmony.

***harmony in form:*** According to this view, the $E$-rules have a *specific form*, known as *general elimination* rules (von Plato (2001), Schroeder-Heister (1984), among others), allowing the derivation of an *arbitrary conclusion* from the major premise of the $E$-rule. In Francez and Dyckhoff (2012), a general construction is provided for the $E$-rules in this form, given the $I$-rules. It is proved there that the rules generated by this construction satisfy also *intrinsic harmony* (see next item).

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***intrinsic harmony:*** The inversion principle gives rise to proof-reductions, eliminating "detours" of introducing a formula and immediately eliminating it (a maximal formula). Whenever such reductions exist, the $E$-rules (for the main operator of $\varphi$) are established as not being too strong w.r.t. the corresponding $I$-rules.

Following Pfenning and Davies (2001); Davies and Pfenning (2001), intrinsic harmony is formalized by *local-soundness*, existence of reductions removing maximal formulas.

DEFINITION 2.8 (**maximal formula**). *A maximal formula in an $ND$-derivation $\mathcal{D}$ is a node in $\mathcal{D}$ that is the conclusion of an application of an $I$-rule, and, simultaneously, a major premise of an application of an $E$-rule.*

DEFINITION 2.9. *An $ND$-system $N$ is* locally-sound *iff every derivation $\mathcal{D}$ having an occurrence of a maximal formula can be transformed into an equivalent derivation $\mathcal{D}'$ (with the same conclusion and open assumptions, but without that occurrence of the maximal formula). Such a transformation is called a* (proof) reduction.

If $N$ is locally-sound, it means that the $E$-rules in $N$ are *not too strong* w.r.t. the $I$-rules, as nothing can be "gained" by first introducing and then immediately eliminating. Clearly, the rules for *tonk fail* local-soundness, as the maximal formula $\varphi\ tonk\ \psi$ in (10) cannot be removed.

$$\dfrac{\dfrac{\varphi}{\varphi\ tonk\ \psi}\ (tonkI)}{\psi}\ (tonkE) \tag{10}$$

As is well-known, $\mathbf{I}_{\rightarrow}$ (actually, the whole of $\mathbf{I}$) is locally-sound (Prawitz (1965)). The removal of an implicative maximal formula is by means of the reduction in (11) (with all lateral assumptions omitted).

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]_i\\ \mathcal{D}\\ \psi\end{array}}{\varphi\rightarrow\psi}\ (\rightarrow I^i) \quad \begin{array}{c}\mathcal{D}'\\ \varphi\end{array}}{\psi}\ (\rightarrow E) \qquad \rightsquigarrow_r \qquad \begin{array}{c}\mathcal{D}[\varphi_i := \begin{array}{c}\mathcal{D}'\\ \varphi\end{array}]\\ \psi\end{array} \tag{11}$$

The reduction is well-defined due to the closure of $\mathbf{I}_{\rightarrow}$ under derivation composition (Proposition 2.5).

The harmony of $\mathbf{R}_{\rightarrow}$ is shown in Section 2.7.4.

*2.4.2. Stability.* The notion of stability did not receive a precise definition in Dummett (1991), but the intention is clear: the $E$-rules should not be *too weak* w.r.t. the $I$-rules. Again, following Pfenning and Davies (2001); Davies and Pfenning (2001), *an approximation* to this property can be formalized by means of the existence of *expansions*.

DEFINITION 2.10. *An $ND$-system $N$ is* locally-complete *iff for every derivation $\mathcal{D}$ in $N$ for $\varphi : \varphi$ (for a compound $\varphi$) there exists a derivation $\mathcal{D}'$ for $\varphi : \varphi$ that contains an application of an $E$-rule to $\varphi$ as a major premise, as well as application of all the $I$-rules (for the operator dominating $\varphi$). $\mathcal{D}'$ is called* an expansion *of $\mathcal{D}$.*

The local-completeness implies that the $I$-rules are strong enough (w.r.t. the $E$-rules) to allow a "reconstruction" of $\varphi$ after it was "taken apart". Local-completeness is only an approximation to stability. For example (from Dummett (1991)), *quantum logic disjunction* the same $I$-rules as the intuitionistic disjunction but has an $E$-rule more restricted than the intuitionistic $(\vee E)$-rule in allowing no lateral assumptions in the minor premises, but still having an expansion. Thus, the quantum disjunction does not allow exploiting all

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the consequences of its ground for introduction. For example, the law of *distributivity* (of conjunction over disjunction) can be proved in the intuitionistic logic, but not in the quantum logic.

As is also well-known, $\mathbf{I}_{\rightarrow}$ (and, actually, full $\mathbf{I}$) is locally-complete. The expansion for '$\rightarrow$ is shown below.

$$\frac{\mathcal{D}}{(\varphi\rightarrow\psi)} \quad \rightsquigarrow_e \quad \frac{\dfrac{\mathcal{D}}{(\varphi\rightarrow\psi)} \quad [\varphi]_i}{\dfrac{\psi}{(\varphi\rightarrow\psi)}\ (\rightarrow I^i)}\ (\rightarrow E) \tag{12}$$

The stability of $\mathbf{R}_{\rightarrow}$ is shown in Section 2.7.4.

***2.5. The effect of substructurality.*** One of the greatest achievements of proof-theory is the discovery of substructural logics Schroeder-Heister and Došen (1993), Paoli (2002), and Restall (2000). This discovery leads to a refinement of the meanings of traditional connectives (see Paoli (2007) for a view of this refinement as a disambiguation of the operators of Classical Logic). The distinction goes under several names, e.g., additive/multiplicative, shared/independent context, lattice/group, and extensional/intensional.

As a typical example, consider the case of conjunction, considered below for the special case of Relevant Logic. I'll use additive/multiplicative as names here, and denote the two conjunctions by '$\wedge$' and '$\circ$', respectively.

$$\frac{\Gamma : \varphi \quad \Gamma : \psi}{\Gamma : \varphi\wedge\psi}\ (\wedge I) \qquad \frac{\Gamma_1 : \varphi \quad \Gamma_2 : \psi}{\Gamma_1\Gamma_2 : \varphi \circ \psi}\ (\circ I) \tag{13}$$

These are also the respective '$(\wedge R)$' and '$(\circ R)$' rules in the Sequent Calculus presentation. This distinction effects the form of the harmoniously induced $GE$-rules embodying the harmony in form. Their logistic presentation is the following.

$$\frac{\Gamma : \varphi\wedge\psi \quad \Gamma, \varphi : \chi}{\Gamma : \chi}\ (\wedge_1 GE) \quad \frac{\Gamma : \varphi\wedge\psi \quad \Gamma, \psi : \chi}{\Gamma : \chi}\ (\wedge_2 GE) \quad \frac{\Gamma_1 : \varphi \circ \psi \quad \Gamma_2, \varphi, \psi : \chi}{\Gamma_1\Gamma_2 : \chi}\ (\circ GE) \tag{14}$$

The attribution of these rules[2] to the additive/multiplicative distinction was recently observed in Read (2010, 2014) (in connection with the construction of harmoniously induced $GE$-rules in Francez and Dyckhoff (2012)).

In order to explain this impact of substructurality, we need to have a closer look at the definition of local-soundness. In the absence of Weakening, two derivations of some conclusion are equivalent iff they depend on *exactly* the same open assumptions. Thus, when implementing the inversion principle[3] via a reduction, one cannot use *any* grounds; rather, *exhaustive* grounds are needed, depending on exactly the same open assumptions as the conclusion. This amounts to generating harmoniously induced $GE$-rules that have

[2] With the clear connection to the sequent calculus '$(\wedge R)$' and '$(\circ R)$ rules

$$\frac{\varphi, \Gamma : \chi}{\varphi\wedge\psi, \Gamma : \chi}\ (\wedge_1 L) \quad \frac{\psi, \Gamma : \chi}{\varphi\wedge\psi, \Gamma : \chi}\ (\wedge_2 L) \quad \frac{\varphi, \psi, \Gamma : \chi}{\varphi \circ \psi, \Gamma : \chi}\ (\circ L)$$

[3] This observation was already formulated in Francez (2013b) in a multiconclusion $ND$ setting, formulated in terms of the right context $\Delta$ (in a generic sequent $\Gamma : \Delta$).

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exhaustive grounds as dischargeable assumption(s) for drawing the arbitrary conclusion. In additive rules, all premises depend on the *same* context $\Gamma$, so each premise counts as exhaustive. However, in a multiplicative rule, only the collection of *all* premises is exhaustive. This difference underlies the difference in the $GE$-rules induced by '$(\wedge I)$' and '$(\circ I)$'.

First, observe that the $GE$-rules in (14) do support reductions establishing local-soundness.

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi & \psi\end{array}}{\varphi\wedge\psi}(\wedge I) \quad \begin{array}{c}[\varphi]_i\\ \mathcal{D}_3\\ \chi\end{array}}{\chi}(\wedge_1 GE^i) \quad \rightsquigarrow_r \quad \begin{array}{c}\mathcal{D}_1\\ \mathcal{D}_3[\varphi := \varphi]\\ \chi\end{array} \tag{15}$$

(where the reduction using '$(\wedge_2 GE)$' is similar).

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi & \psi\end{array}}{\varphi\circ\psi}(\circ I) \quad \begin{array}{c}[\varphi]_i, [\psi]_j\\ \mathcal{D}_3\\ \chi\end{array}}{\chi}(\circ GE^{i,j}) \quad \rightsquigarrow_r \quad \begin{array}{c}\mathcal{D}_1 \quad \mathcal{D}_2\\ \mathcal{D}_3[\varphi := \varphi, \psi := \psi]\\ \chi\end{array} \tag{16}$$

Attempting to (erroneously) reduce using '$(\circ I)$' and '$(\wedge GE)$' would result in a derivation in which the arbitrary conclusion $\chi$ incorrectly does not depend on $\Gamma_2$, hence not equivalent to the original derivation.

The simple, nonlogistic, presentation of the $I$-rules for the two conjunctions, adapted below to Relevant Logic, is the following.

$$\dfrac{\begin{array}{c}[\varphi]_i\\ \vdots\\ \varphi\wedge\psi \quad \chi\end{array}}{\chi}(\wedge_1 GE^i) \qquad \dfrac{\begin{array}{c}[\psi]_i\\ \vdots\\ \varphi\wedge\psi \quad \chi\end{array}}{\chi}(\wedge_2 GE^i) \qquad \dfrac{\begin{array}{c}[\varphi]_i, [\psi]_j\\ \vdots\\ \varphi\circ\psi \quad \chi\end{array}}{\chi}(\circ GE^{i,j}) \tag{17}$$

***2.6. Bilateralism.*** I adopt the notation of Rumfitt (2000) and add a *force marker*, '+' or '−', in front of each sentence of the object language. Those markers do not belong to the object language in which the asserted/denied sentences are expressed. Rather, they belong to the structural language in which $I/E$-rules are formulated—similarly to the comma separating assumptions in a context. I retain meta-variables $\varphi$, $\psi$ (possibly subscripted) as ranging over object language sentences, and let $\sigma$ (possibly subscripted) to range over force-marked sentences. Also, $\overline{\sigma}$ indicates the *reversal* of the force marker of $\sigma$, where it is understood that $\overline{\overline{\sigma}} = \sigma$. A positively marked sentence of the form '$+\varphi$' is to be understood as indicating assertion of $\varphi$, while a negatively marked sentence '$-\varphi$' is to be understood as indicating denial of $\varphi$. In a bilateral $ND$-system, an operator, say '$*$', has two kinds of $I$-rules, marked $(*^+I)$ and $(*^-I)$, and two kinds of $E$-rules, marked $(*^+E)$ and $(*^-E)$. The positive $I$-rules introduce $+(\varphi * \psi)$, while the negative $I$-rules introduce $-(\varphi * \psi)$. Similarly for elimination. Note that this kind of bilateral $ND$-system is different from $ND$-systems for *refutation* (see, for example, Tranchini (2010)), where the rules serve to draw a contradiction from premises.

When constructing a bilateral extension of a given unilateral $ND$-system, the positive rules for any introduced expression are what is called in Humberstone (2000) "the positive sign analogs" of the unilateral rules for that expression, that is, the given $I$-rules, but with a '+' preceding each sentence in the premises and in the conclusion. An immediate conclusion of this convention is that positive conclusions cannot have negative premises, while negative conclusions may have positive as well as negative premises. However, when the bilateral system is *not* an extension of a unilateral system, the above conclusion need

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not hold: a positive conclusion *may* have a negative premise, as shown by the following example.

EXAMPLE 2.11. *Consider the familiar (binary) connective xor, having the following positive I-rules.*

$$\frac{\Gamma : +\varphi \quad \Gamma : -\psi}{\Gamma : +(\varphi \; xor \; \psi)} \; (xor_1^+ I) \qquad \frac{\Gamma : -\varphi \quad \Gamma : +\psi}{\Gamma : +(\varphi \; xor \; \psi)} \; (xor_2^+ I) \tag{18}$$

The notions of a derivation and composition of derivations are naturally adapted to bilateral rules.

*2.6.1. Coherence (of contexts).* The objects over which a bilateral $ND$-rules are defined are sequents of one of the forms $\Gamma : +\varphi$ and $\Gamma : -\varphi$. A context $\Gamma$ is here taken to contain both positively and negatively marked sentences. An important property of contexts is that of *coherence*, formally defined below. It is the counterpart of *consistency* in a unilateral logic with explicit sentential negation. It relies on the two attitudes of assertion and denial being mutually exclusive.

DEFINITION 2.12 (Coherent context). *A context* $\Gamma$ *is* coherent*, denoted by* $\mathcal{C}(\Gamma)$*, iff for every sentence* $\varphi$ *(in the object language), at most one of* $\Gamma : +\varphi$ *and* $\Gamma : -\varphi$ *is derivable. Otherwise,* $\Gamma$ *is* incoherent.

In particular, in the presence of the usual axiom $\Gamma, \sigma : \sigma$ (for both positively and negatively signed $\sigma$), any context $\Gamma, +\varphi, -\varphi$, containing an assertion and a denial of the same sentence, is incoherent. The incoherence of such contexts reflects the role of the signs as indicating forces that cannot be held together as forces of the same content, an important attribute of bilateralism.

However, the distinction between coherent and incoherent context is not sufficient for endowing an $ND$-system with the expression of bilaterality, based on the two forces of assertion and denial forming a contrariety. The latter effect is established by means of appropriate *structural rules*.

$$(a)\;\; \Gamma, +\varphi, -\varphi : \sigma \;\; (INC) \qquad (b) \;\; \frac{\Gamma, [+\varphi]_i : +\psi \quad \Gamma, [+\varphi]_j : -\psi}{\Gamma : -\varphi} \; (CO^{i,j}) \tag{19}$$

The $(INC)$ rule establishes the incoherence of an incoherent context, that cannot be coherently used in deductions, by letting *every* sentence be derivable from it. The second rule, $(CO)$, a *coordination* rule in the terminology of Rumfitt (2000), allows denial of an assumption the assertion of which leads to derivability of the same content but with reversed forces. To obtain classical strength, the rule $(CO)$ needs to be strengthened, allowing also the reversal of force from assuming a denied sentence $-\varphi$ leading to incoherence.

In Rumfitt (2000), dealing explicitly with classical logic, two structural rules are added (under the name of *coordination rules*). However, those structural rules involve the *falsum* constant '$\bot$', halfway to sentential negation.

$$\frac{\Gamma, [\sigma]_i : \bot}{\Gamma : \overline{\sigma}} \; (RED^i) \qquad \frac{\Gamma : \sigma \quad \Gamma : \overline{\sigma}}{\Gamma : \bot} \; (LNC)$$

There are two issues here.

1. What justifies the inclusion of those coordination rules?
2. How is the required effect to be achieved in a "positive language", having neither '$\bot$' nor '$\neg$'?

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$$\frac{}{+\varphi : +\varphi}\ (Ax^{+})$$

$$\frac{\Gamma, [+\varphi]_i : +\psi}{\Gamma : +(\varphi{\rightarrow}\psi)}\ (\rightarrow^{+}I^{i}) \qquad \frac{\Gamma : +(\varphi{\rightarrow}\psi) \quad \Gamma{\rightarrow} + \varphi \quad \Gamma, [+\psi]_i : +\xi}{\Gamma : +\xi}\ (\rightarrow^{+}E^{i})$$

Fig. 5. The $\mathbf{BI}_{\rightarrow}$ rules for assertion.

$$\frac{}{-\varphi : -\varphi}\ (Ax^{-})$$

$$\frac{\Gamma : +\varphi \quad \Gamma : -\psi}{\Gamma : -(\varphi{\rightarrow}\psi)}\ (\rightarrow^{-}I) \qquad \frac{\Gamma\vdash - (\varphi{\rightarrow}\psi) \quad \Gamma, [+\varphi]_i, [-\psi]_j : \sigma}{\Gamma : \sigma}\ (\rightarrow^{-}E^{i,j})$$

Fig. 6. The $\mathbf{BI}_{\rightarrow}$ rules for denial.

As admitted in Rumfitt (2008), ($RED^{i}$) does not originate from bilateralism per se; rather, its role is to enable a bilateral formalization of classical logic, a formalization enjoying some favorable proof-theoretic properties. Since my interest here is not in the rivalry between classical and intuitionistic logics (and several other nonclassical ones), I will not attend to this coordination rule. As for "positive languages", the required effect of bilateralism is obtained via the structural rules in (19) and the vertical and horizontal balance as specified in Section 2.7.

The above discussion of structural bilateral rules fits Classical/Intuitionistic Logic. The bilateral structural rules fitting bilateral Relevant Logic, and particularly the role of ($INC$), are discussed in Section 3.3.

*2.6.2. A Bilateral ND-system for* **I**. To illustrate bilateral $ND$-systems and for comparison with the intended bilateral $ND$-system fot $\mathbf{R}_{\rightarrow}$, Figures 5 and 6 present the bilateral $ND$-system for $\mathbf{I}_{\rightarrow}$. The logic can be viewed as an extension of the (unilateral) positive implicational fragment of **I**, but with $E$-rules in $GE$ form, referred to as $\mathbf{BI}_{\rightarrow}$.

Note that discharged assumptions are either positive or negative. Note again that all the $E$-rules (both positive and negative) are in the form known as *general elimination* rules, allowing to draw *arbitrary conclusions* from the major premise, whether positive or negative. This rule-format exhibits vertical harmony in form (see below) for both the positive and the negative rules.

PROPOSITION 2.13. $\mathbf{BI}_{\rightarrow}$ *is closed under derivation composition.*

***2.7. Bilateral harmony and stability.***

*2.7.1. Bilateral harmony.* In the unilateral setting, intrinsic harmony is 'vertical', in relating $E$-rules to $I$-rules. In a bilateral setting, the vertical criteria split into two groups, for the respective positive and negative groups of rules. However, as claimed in Francez (2013a), there is in the bilateral case a need to relate positive $I$-rules to negative $I$-rules (of the same operator), to guarantee those rules justify being rules for assertion and denial; clearly, a certain dependency among those two groups is required, so that $+\varphi$ and $-\varphi$ turn to be a *contrariety*.

*2.7.2. Vertical balance.* In a bilateral logic, if unilateral rules are equated with the corresponding positive rules, the above principle guarantees that the *positive* $E$-rules are not too strong w.r.t. the *positive* $I$-rules. Clearly (see Francez (2013a)), the negative $E$-rules also need to be shown not to be too strong w.r.t. the *negative* $I$-rules. To see this need, consider the following example.

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EXAMPLE 2.14. *Suppose the following two negative rules constitute the denial rules for 'tonk'.*

$$\frac{\Gamma\vdash -\varphi}{\Gamma\vdash -(\varphi\ tonk\ \psi)}\ (tonk^{-}I) \qquad \frac{\Gamma\vdash -(\varphi\ tonk\ \psi)}{\Gamma\vdash -\psi}\ (tonk^{-}E) \tag{20}$$

Clearly, similarly to the unilateral/positive case, those rules trivialize the derivability relation in letting $\vdash -\varphi : -\psi$ for any $\varphi,\ \psi$. This observation leads to an enhancement of the vertical inversion principle to bilateralism, introduced in Francez (2013a), encompassing negative rules too.

**The bilateral vertical inversion principle:**

1. Any conclusion of an assertion of (a compound) $\varphi$ is also a conclusion of any $\Gamma\in GA[\![\varphi]\!]$.
2. Any conclusion of a denial of (a compound) $\varphi$ is also a conclusion of any $\Gamma\in GD[\![\varphi]\!]$.

An example of a reduction w.r.t. the negative rules in $\mathbf{I}_{\rightarrow}$ is presented below.

EXAMPLE 2.15.

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +\varphi & -\psi\end{array}}{-(\varphi\rightarrow\psi)}\ (\rightarrow^{-}I) \qquad \begin{array}{c}[+\varphi]_i,[-\psi]_j\\ \mathcal{D}_3\\ \sigma\end{array}}{\sigma}\ (\rightarrow^{-}E^{i,j}) \quad \rightsquigarrow_r \quad \begin{array}{c}\mathcal{D}_3[+\varphi :=\overset{\mathcal{D}_1}{+\varphi}, -\psi :=\overset{\mathcal{D}_2}{-\psi}]\\ \sigma\end{array} \tag{21}$$

The reduction is well-defined by the closure of $\mathbf{BI}_{\rightarrow}$ under derivation composition (Proposition 2.13).

For harmony in form, the bilateral vertical inversion principle requires that negative $E$-rules are also of the $GE$-form. A simple argument can establish that the construction in Francez and Dyckhoff (2012) can be applied in the negative case too.

For intrinsic harmony, the bilateral vertical inversion principle requires the existence of both positive and negative detour reductions.

In the other direction, for local-completeness in the bilateral setting, we need to require the existence of both positive and negative expansion, balancing also the negative $E$-rules w.r.t. the negative $I$-rules. Below is the negative expansion in $\mathbf{BI}_{\rightarrow}$.

EXAMPLE 2.16.

$$\begin{array}{c}\mathcal{D}\\ -(\varphi\rightarrow\psi)\end{array} \quad \rightsquigarrow_e \quad \dfrac{\begin{array}{c}\mathcal{D}\\ -(\varphi\rightarrow\psi)\end{array} \qquad \dfrac{[+\varphi]_i \quad [-\psi]_j}{-(\varphi\rightarrow\psi)}\ (\rightarrow^{-}I)}{-(\varphi\rightarrow\psi)}\ (\rightarrow^{-}E^{i,j}) \tag{22}$$

*2.7.3. Horizontal balance.* Horizontal balance should relate positive and negative rules in such a way that assertion and denial are rendered contrary forces; that is, assertion and denial of the same contents leads to a contrariety[4]. It also accounts for a certain symmetry found amongst those rules. In continuation to Gentzen's remark to the effect that $I$-rules are self-justifying, and $E$-rules are justified by the $I$-rules by the vertical inversion principle, one can see the positive $I$-rules as a definition that should justify also the negative $I$-rules. Also, in the same way that the role of the $I/E$-rules as to which is self-justifying and which group justifies the other group, in the bilateral case one can view the *negative*

[4] For the classical case, contradictory forces are needed.

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$I$-rules as being self-justifying, and the positive $I$-rules being justified by the negative ones.

The overall relationship induced by bilateral harmony on the rules of a bilateral $ND$-system can be summarized by the following *commuting diagram* for a generic operator '$*$'. I use $hh$ and $vh$ to abbreviate 'horizontal harmony' and 'vertical harmony', respectively. The arrows marked '$i$' indicate 'inducing', thus the relationship holds by construction. The relationships shown by the other arrows need to be verified to establish the commuting of the diagram.

$$\begin{array}{ccc} (*^{+}I) & \xrightarrow[i]{hh} & (*^{-}I) \\ {\scriptstyle vh}\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle vh} \\ (*^{+}E) & \xrightarrow{hh} & (*^{-}E) \end{array} \tag{23}$$

This diagram reflects the view that the positive $I$-rules are those determining meaning (according to the PTS inferentialism approach). Similar diagrams, differing in the identity and direction of the '$i$'-marked arrows, can summarize the pragmatism approach of PTS, where the $E$-rules are given as meaning determining. In addition, similar diagrams fit the less common view, the point of departure of which are the negative rules. I'll not pursue further those extra possibilities here.

I now show how to bring about the relationship expressed by the above commuting diagram. Clearly, grounds for denial should be related to denial of grounds for assertion. However, a finer distinction is needed to get this relationship right. The approach proposed is more similar to harmony in form than to intrinsic harmony.
Consider first a simple example.

EXAMPLE 2.17. *Consider the '$\wedge$'-rules of* **I***, to constitute the positive rules in a bilateral $ND$-system for* **I***:*

$$\frac{\Gamma : +\varphi \quad \Gamma : +\psi}{\Gamma : +(\varphi \wedge \psi)}\ (\wedge^{+}I) \qquad \frac{\Gamma : +(\varphi \wedge \psi) \quad \Gamma, [+\varphi]_i, [+\psi]_j : \xi}{\Gamma : \xi}\ (\wedge^{+}E^{i,j}) \tag{24}$$

*Now suppose, that the following single negative $I$-rule is added, supposedly capturing (on its own) the introduction of the denial of conjunction.*

$$\frac{\Gamma : -\varphi}{\Gamma : -(\varphi \wedge \psi)}\ (\wedge^{-}I) \tag{25}$$

*Pretheoretically, this seems wrong! This single negative $I$-rule is* too weak *w.r.t. the positive $I$-rules of conjunction. A second rule is needed, namely*

$$\frac{\Gamma : -\psi}{\Gamma : -(\varphi \wedge \psi)}\ (\wedge^{-}I) \tag{26}$$

*Similarly,*

$$\frac{\Gamma : -\varphi \quad \Gamma : -\psi}{\Gamma : -(\varphi \wedge \psi)}\ (\wedge^{-}I) \tag{27}$$

*seems, pretheoretically, too strong w.r.t. the positive $I$-rules of conjunction.*

A conjunction should be deniable upon the denial of each conjunct (separately). But which principle would bring this about? For the formulation of such a principle, I adopt (part of) the rule classification in Francez and Dyckhoff (2012).

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DEFINITION 2.18 (**rule classification**).

1. *An ND-rule is* hypothetical *if it allows for at least one premise with* assumptions discharge*; otherwise, it is* categorical. *The discharged assumptions are referred to as the* supports *of the rule.*
2. *An ND-rule is* combining *if it has more than one premise; otherwise it is noncombining (referred to also as splitting).*

The definitions below are taken from Francez (2013a), where it was implicitly assumed that the structural rules '$(W)$' and '$(C)$' are present. Below, the definition is adapted to the substructural setting (here the absence of '$(W)$').

DEFINITION 2.19 (**The horizontal inversion principle—constants with a categorical $I$-rule**:). *Let $\varphi$ have a constants with a categorical $I$-rule as its main operator.*

1. *Any denial of a ground of assertion of a categorical combining (compound) $\varphi$ is a ground for denial of $\varphi$.*
2. *The collection of denials of a ground of assertion of a categorical splitting (compound) $\varphi$ is a ground for denial of $\varphi$.*
3. *Any conclusion of a denial of a categorical combining (compound) $\varphi$ is also a conclusion of each of the grounds for assertion of $\varphi$.*
4. *Any conclusion of a denial of a categorical splitting (compound) $\varphi$ is also a joint conclusion of the grounds for assertion of $\varphi$.*

The first clause indicates that the negative $I$-rules (of the main operator of a categorical combining $\varphi$) are neither too weak nor too strong relative to the positive $I$-rules (of the same operator). The second clause assures the same for a categorical splitting logical constant. This balance can be seen as a generalization of (classical) De-Morgan's laws, specifically relating negation and the dual pair of conjunction/disjunction (see below). The other two clauses relate the positive and negative $E$-rules to each other.

As for constants with a hypothetical $I$-rule, the positive $I$-rule of such a constant discharges assumptions, its support. The support is "temporarily" added to form parts of the grounds for asserting the conclusions of the premises of the rule. To deny the conclusion of the rule (with the constant as its main operator), one has, therefore, to accept the support while denying the conclusions of the premises.

DEFINITION 2.20 (**The horizontal inversion principle—constants with a hypothetical $I$-rule**:). *Let $\varphi$ have a constant with a hypothetical $I$-rule as its main operator.*

1. *Any assertion of the support together with the denial of the conclusion of the premise is a ground for denial of $\varphi$.*
2. *Any conclusion of the denial of $\varphi$ is also a joint conclusion of the assertion of the support and denial of the conclusion of the premise.*

To accommodate the absence of '$(W)$', according to the explanation in Section 2.5., any appeal to 'grounds' in the above definition has to be replaced by an appeal to 'exhaustive grounds'.

Returning to Example 2.17, since the grounds for assertion of $+(\varphi \wedge \psi)$ are $\Gamma = +\varphi, +\psi$ (a categorical combining constant), having only the negative $I$-rule in (25) is too weak, as both the denials of the separate conjuncts have to be, by the first horizontal inversion principle, grounds for denial of the conjunction. What is needed is the second negative $I$-rule for conjunction (26).

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Similarly, the rule (27) is seen to be too strong.

EXAMPLE 2.21. *For an example of denial $I/E$-rules for an operator with an hypothetical rule, consider the intuitionistic implication. Here the (single) support is $\varphi$, and the conclusion of the (single) premise is $\psi$. By the horizontal inversion principle, the following negative $I/E$-rules for implication are justified.*

$$\frac{\Gamma : +\varphi \quad \Gamma : -\psi}{\Gamma : -(\varphi \to \psi)}\ (\to^{-}I) \qquad \frac{\Gamma : -(\varphi \to \psi) \quad \Gamma, [+\varphi]_i, [-\psi]_j : \xi}{\Gamma : \xi}\ (\to^{-}E^{i,j}) \tag{28}$$

DEFINITION 2.22 (**bilateral harmony**). *A bilateral ND-system is* bilaterally harmonious *iff it satisfies both the vertical and the horizontal inversion principles.*

The following proposition (Francez (2013a)), following directly from the horizontal inversion principle, relates denial and duality.

PROPOSITION 2.23. *The premises of a negative $I$-rule for a connective, say '$*$', are the premises of the positive $I$-rule of its dual connective $*^{\circ}$, but with a reversed force marker.*

For conjunction and disjunction, this amounts to the usual (classical) De-Morgan laws. For implication, the premises of the $I$-rule of its dual are $\Gamma : -\varphi$ and $\Gamma : +\psi$. Hence, its negative $I$-rule is

$$\frac{\Gamma : +\varphi \quad \Gamma : -\psi}{\Gamma : -(\varphi \to \psi)}\ (\to^{-}I) \tag{29}$$

The main theorem characterizing a bilaterally harmonious $ND$-system was shown in Francez (2013a).

THEOREM 2.24 (**coordination admissibility**). *The structural rule $CO$ (see (19 (b))) is* admissible *in any bilaterally harmonious $ND$-system.*

*2.7.4. The unilateral harmony and stability of $\mathbf{R}_{\to}$.* Next, I answer, briefly reviewing Francez (2013c), the question: is $\mathbf{R}_{\to}$ (unilaterally) locally-sound and -complete?

On first sight, the answer to the local-soundness question is negative, because, for derivation composition as defined (a "blind" replacement), $\mathbf{R}_{\to}$ is *not* closed under composition, hence disallows implication reduction. To see this, consider the $\mathbf{R}_{\to}$ analog of example (2.6). This will look like the following:

$$\mathcal{D}: \frac{\varphi_1 \quad (\varphi \to_{\mathbf{R}} \psi)_2}{\psi_{1,2}}\ (\to_{\mathbf{R}}E) \qquad \mathcal{D}': \frac{\chi_1 \quad (\chi \to_{\mathbf{R}} \varphi)_2}{\varphi_{1,2}}\ (\to_{\mathbf{R}}E) \tag{30}$$

and

$$\mathcal{D}[\varphi_1 := \overset{\mathcal{D}'}{\varphi_{1,2}}]_{\psi_{1,2}} = \frac{\dfrac{\chi_1 \quad (\chi \to_{\mathbf{R}} \varphi)_2}{\varphi_{1,2}}\ (\to_{\mathbf{R}}E) \quad (\varphi \to_{\mathbf{R}} \psi)_2}{\psi_{1,2}}\ (\to_{\mathbf{R}}E) \tag{31}$$

Clearly, (31) is *not* a legal $\mathbf{R}_{\to}$ derivation. There are two violations.

1. The first, more trivial, violation is, that in $\Gamma, \Gamma'$ there are two assumptions indexed '2', contrary the definition. This can be amended easily, by modifying the definition of derivation composition so that the indices in $\Gamma$ and in $\Gamma'$ are renamed before

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the transformation as follows: change in $\Gamma$ $(\varphi \to_{\mathbf{R}} \psi)_2$ to $(\varphi \to_{\mathbf{R}} \psi)_3$ (and keep $\Gamma'$ unchanged in this special case).

$$\mathcal{D}[\varphi_1 := \overset{\mathcal{D}'}{\varphi_{1,2}}]_{\psi_{1,2}} = \dfrac{\dfrac{\chi_1 \quad (\chi \to_{\mathbf{R}} \varphi)_2}{\varphi_{1,2}}\,(\to E) \quad (\varphi \to_{\mathbf{R}} \psi)_3}{\psi_{1,2}}\,(\to E) \tag{32}$$

The resulting $\Gamma$, $\Gamma'$ is now properly indexed.
This is still incorrect, as the dependency of the final conclusion is wrong!

2. The more essential change is in the "body" of $\mathcal{D}$, where all the dependencies have to be modified, to be consistent with the renaming above. What we want for the example is

$$\mathcal{D}[\varphi_1 := \overset{\mathcal{D}'}{\varphi_{1,2}}]_{\psi_{1,2,3}} = \dfrac{\dfrac{\chi_1 \quad (\chi \to_{\mathbf{R}} \varphi)_2}{\varphi_{1,2}}\,(\to_{\mathbf{R}} E) \quad (\varphi \to_{\mathbf{R}} \psi)_3}{\psi_{1,2,3}}\,(\to E) \tag{33}$$

which a legal $\mathbf{R}_{\to}$ derivation.

A generalization of this special case leads to the following definition of *relevant derivation composition*.

DEFINITION 2.25 (**relevant derivation composition**). *Let* $\overset{\mathcal{D}}{\psi_\gamma}{}^{\Gamma_1, \varphi_i, \Gamma_2}$ *be a derivation of* $\psi_\gamma$ *from* $\Gamma_1, \varphi_i, \Gamma_2$ *(in* $\mathbf{R}_{\to}$*), having* $\varphi_i$ *as one of its leaves, the rest of the leaves constituting* $\Gamma_1, \Gamma_2$. *Let* $\overset{\mathcal{D}'}{\varphi_\alpha}{}^{\Gamma'}$ *be a derivation (in* $\mathbf{R}_{\to}$*) of* $\varphi_\alpha$ *from* $\Gamma'$. *Then* $\mathcal{D}[\varphi_i := \overset{\mathcal{D}'}{\varphi_\alpha}]^{\mathbf{R}}_{\psi_\gamma}$ *is the tree resulting by replacing the leaf* $\varphi_i$ *of* $\mathcal{D}$ *with the subtree* $\mathcal{D}'$ *(identifying the root of the latter with the leaf of the former), subject to the following modifications.*

1. *Let*

$$\hat{j} = \begin{cases} j & \chi_j \in \Gamma_1 \\ j + |\Gamma_1| & \chi_j \in \Gamma' \\ j + |\Gamma'| & \chi_j \in \Gamma_2 \end{cases} \tag{34}$$

*Now, replace every* $\chi_j \in \Gamma_1, \Gamma', \Gamma_2$ *by* $\chi_{\hat{j}}$.

2. *For a collection of indices* $\gamma$, *let* $\hat{\gamma} = \{\hat{j} \mid j \in \gamma\}$. *Replace, in the resulting tree, every occurrence* $\chi_\gamma$ *by*

$$\begin{cases} \chi_{\hat{\gamma}} & i \notin \gamma \\ \chi_{\hat{\gamma} \cup \alpha - \{i\}} & i \in \gamma \end{cases}$$

What is the significance of the proposed definition regarding the understanding of relevance in this logic? The resulting system can be viewed as assigning (the restricted) derivation composition the role of "use propagation":

- A use of an assumption $\varphi$ is converted by the composition to the use of the assumptions the derived $\varphi$ depends on.
- A justified discharge of $\varphi$, having been used in a subderivation, is converted by the composition to a justified discharge of the assumptions the derived $\varphi$ depends on.

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Another question raised by the definition of relevant composition is whether it still can be considered as expressing transitivity of relevant deducibility. The answer depends on whether the two derivations, after reindexing, remain *the same* derivations. I tend to a positive answer. The situation is very similar to the identification of $\alpha$ variants, renaming bound variables, on the term level, usually regarding as coding the same derivation on the logical side of the Curry–Howard correspondence.

The following proposition follows directly from the definition of relevant derivation composition.

PROPOSITION 2.26 (**closure of $\mathbf{R}_{\to}$ under relevant derivation composition**). *For $\mathcal{D}$ and $\mathcal{D}'$ as in Definition 2.25,* $\begin{matrix}\mathcal{D}[\varphi_i :=\overset{\mathcal{D}'}{\varphi_\alpha}]^{\mathbf{R}} \\ \psi_\gamma\end{matrix}$ *is a derivation (in $\mathbf{R}_{\to}$).*

A consequence of Proposition (2.26) is the following theorem, ensuring the harmony of $\mathbf{R}_{\to}$.

THEOREM 2.27 (**local-soundness of $\mathbf{R}_{\to}$**). $\mathbf{R}_{\to}$ *is locally-sound.*

*Proof.* The required reduction is

$$\dfrac{\dfrac{\begin{matrix}[\varphi]_i \\ \mathcal{D} \\ \psi_\alpha \quad (i\in\alpha)\end{matrix}}{(\varphi\to_{\mathbf{R}}\psi)_{\alpha-\{i\}}}\,(\to I^i_{\mathbf{R}}) \quad \begin{matrix}\mathcal{D}' \\ \varphi_\beta\end{matrix}}{\psi_{\alpha\cup\beta-\{i\}}}\,(\to E_{\mathbf{R}}) \quad \rightsquigarrow_r \quad \begin{matrix}\mathcal{D}[\varphi_i :=\overset{\mathcal{D}'}{\varphi_\beta}]^{\mathbf{R}} \\ \psi_{\alpha\cup\beta-\{i\}}\end{matrix} \tag{35}$$

which is well-defined by Proposition (2.26). □

Let me consider further the interpretation of the reduction in (35). One might wonder how does the $(\to_R E)$-rule (with its arbitrary $\alpha$ and $\beta$ index sets) record the fact that, when applying the $(\to_R I)$-rule, the assumption $\varphi$ was really used in the subderivation of $\psi$? Such a recording seems to be needed if local-soundness here "implements" the inversion principle. To realize that the rule *does* record the use, consider the following argument. By having the major premise $\varphi\to_{\mathbf{R}}\psi$ (of the application of $(\to_{\mathbf{R}}E)$) indexed $\alpha-\{i\}$, we may infer the existence of a subderivation $\mathcal{D}$ of $\psi_\alpha$ (for some fresh $i$, $i\in\alpha$) from a used discharged assumption $\varphi_i$. When $\mathcal{D}$ is relevantly composed with the derivation $\mathcal{D}'$ of the minor premise $\varphi_\beta$ (of that application of $(\to_{\mathbf{R}}E)$), the result is a derivation of $\psi_{\alpha\cup\,\beta-\{i\}}$, which shows the effect of propagation of use discussed above.

With regards the question of the local-completeness of $\mathbf{R}_{\to}$, the positive answer here is much easier, as no derivation compositions are involved. The expansion for '$\to_{\mathbf{R}}$' is a slight variation of that for the intuitionistic '$\to$' in (12).

$$\begin{matrix}\mathcal{D} \\ (\varphi\to_{\mathbf{R}}\psi)_\alpha\end{matrix} \quad \rightsquigarrow_e \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ (\varphi\to_{\mathbf{R}}\psi)_\alpha\end{matrix} \quad [\varphi]_i}{\psi_{\alpha\cup\{i\}}}\,(\to_{\mathbf{R}}E)}{(\varphi\to_{\mathbf{R}}\psi)_\alpha}\,(\to_{\mathbf{R}}I^i) \tag{36}$$

Note that the expansion restores the conclusion with the *same* set of dependency indices.

**§3. A bilateral proof-system for $\mathbf{R}_{\to}$.** I now come to the main part of the paper, devising and studying a bilateral $ND$-system $\mathbf{BR}_{\to}$ for $\mathbf{R}_{\to}$.

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$$\frac{}{+\varphi : +\varphi}\ (Ax^{+})$$

$$\frac{\Gamma, [+\varphi]_i : +\psi_\alpha \quad (i \in \alpha)}{\Gamma : +(\varphi \rightarrow_{\mathbf{R}} \psi)_{\alpha - \{i\}}}\ (\rightarrow^{+}_{\mathbf{R}} I^i) \qquad \frac{\Gamma : +\varphi_\alpha \quad \Gamma : +(\varphi \rightarrow_{\mathbf{R}} \psi)_\beta}{\Gamma : +\psi_{\alpha \cup \beta}}\ (\rightarrow^{+}_{\mathbf{R}} E)$$

Fig. 7. $\mathbf{BR}_{\rightarrow}$: bilateral rules for assertion.

***3.1. The positive rules.*** Since $\mathbf{BR}_{\rightarrow}$ is devised as an extension of the unilateral system $\mathbf{R}_{\rightarrow}$ in Section 2.3., its rules of assertion, presented in Figure 7, are the positive sign analogs of the rules in Figure 3. Clearly, the rule $(\rightarrow^{+}_{\mathbf{R}} I)$ preserve the relevant logic principle that the rule for assertion of an implication depends on the use of the assumption to derive the the consequent when deriving the implication.

***3.2. The negative rules.*** First, note that both positive and negative formulas can serve as assumptions, in particular as dischargeable assumptions.

**The negative $I$-rule:** From inspecting the rule (28) for denying an intuitionistic implication (in Figure 6), the denial of a relevant implication, namely an $(\rightarrow^{-}_{\mathbf{R}} I)$-rule for $-(\varphi \rightarrow_{\mathbf{R}} \psi)$, certainly has to involve asserting $\varphi$ and denying $\psi$.
The question is, *what is the role of the use of $+\varphi$ in $(\rightarrow^{+}_{\mathbf{R}} I)$ in an $(\rightarrow^{-}_{\mathbf{R}} I)$-rule of denying a relevant implication?*

By the classification of rules above, '$\rightarrow_{\mathbf{R}}$' is a constant with a hypothetical $I$-rule. So by the appropriate clause of Definition 2.20, a ground for denial of $\varphi \rightarrow_{\mathbf{R}} \psi$ is any assertion of the support of $(\rightarrow^{+}_{\mathbf{R}})$, namely $+\varphi$, together with the denial of the conclusion of the premise, namely $-\psi$. Recall, however, that the grounds do not mention explicitly any indices, the latter "hidden" within a derivation. Therefore, in the required $(\rightarrow^{-}_{\mathbf{R}} I)$, the schematic indices that need to be assumed are such that the use of $+\varphi$ is reflected. Those considerations lead to the following negative $I$-rule for the relevant implication.

$$\frac{+\varphi_\alpha \quad -\psi_\beta}{-(\varphi \rightarrow_{\mathbf{R}} \psi)_{\alpha \cup \beta}}\ (\rightarrow^{-}_{\mathbf{R}} I) \tag{37}$$

The use of $+\varphi$ by the positive $I$-rule is reflected in that the negative $I$-rule has the index set of the conclusion including $\alpha$.

**The negative $E$-rule:** Again, by the appropriate clause of Definition 2.20, a consequence of denial of $\varphi \rightarrow_{\mathbf{R}} \psi$ is any conclusion of both asserting the support of $(\rightarrow^{+}_{\mathbf{R}} I)$, namely $+\varphi$, and the denial of the conclusion of the premise, namely $-\psi$. Cast as a $GE$-rule for the sake of harmony in form, we get the following $E$-rule.

$$\frac{\begin{array}{ccc} & [+\varphi]_i, [-\psi]_j & \\ & \vdots & \\ -(\varphi \rightarrow_{\mathbf{R}} \psi)_\alpha & \xi_\beta & (i, j \in \beta) \end{array}}{\xi_{\alpha \cup \beta - \{i,j\}}}\ (\rightarrow^{-}_{\mathbf{R}} E^i) \tag{38}$$

The regular negative $E$-rules are derivable from the negative $GE$-rule.

$$\frac{-(\varphi \rightarrow_{\mathbf{R}} \psi)_\alpha}{+\varphi_\alpha}\ (\rightarrow^{-} E_1) \qquad \frac{-(\varphi \rightarrow_{\mathbf{R}} \psi)_\alpha}{-\psi_\alpha}\ (\rightarrow^{-}_{\mathbf{R}} E_1) \tag{39}$$

The negative rules of $\mathbf{BR}_{\rightarrow}$ are summarized in Figure 8.

***3.3. The structural rules.*** Recall that a major concern of Relevant Logic is to preserve the relevance of assumptions to conclusion. Thus, the structural rule $(INC)$, directly

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$$\frac{+\varphi_\alpha \quad -\psi_\beta}{-(\varphi{\rightarrow_{\mathbf{R}}}\psi)_{\alpha\cup\beta}}\ (\rightarrow_{\mathbf{R}}^{-}I) \qquad \frac{-(\varphi{\rightarrow_{\mathbf{R}}}\psi)_\alpha \quad \begin{array}{c}[+\varphi]_i,[-\psi]_j\\ \vdots\\ \xi_\beta\end{array} \quad (i,j\in\beta)}{\xi_{\alpha\cup\beta-\{i,j\}}}\ (\rightarrow_{\mathbf{R}}^{-}E^{i,j})$$

Fig. 8. $\mathbf{BR}_{\rightarrow}$: bilateral rules for denial.

leading to explosion, must be abandoned. This raises the question, what *is* inferable from an incoherent context. An observation in Priest (1999) can be adapted to the current setting. Priest asks the question, what is the contents of a contradiction $\varphi\wedge\neg\varphi$, in the context of model-theoretic consequence. He identifies three possible answers, two of which are relevant here.

**The *total* account:** According to this account, a contradiction entails *everything*.
**The *partial* account:** According to this account, a contradiction entails *something* (but not everything).
**The *null* account:** According to this account, a contradiction entails *nothing*.

If incoherence is viewed as the bilateral counterpart of contradiction, then, moving to proof-theoretic consequence as conveying contents, those three accounts can be adapted to the bilateral setting to determine the contents of an incoherent context.

- The total account is embodied in the adoption of the $(INC)$ structural rule, leading to explosion. This account is suitable for Classical/Intuitionistic bilateral logic, as seen above.
- For Relevant bilateral logic, the null account seems the most suitable: nothing should be derivable from an incoherent context. This can be viewed as "relevant coherence". This naturally leads to the rejection of $(INC)$ as a structural rule in **BR** (and fragments thereof). I adopt this approach here.

***3.4. Relevant derivation and derivability.*** The rules of $\mathbf{BR}_{\rightarrow}$ give rise to the natural definition of a $\mathbf{BR}_{\rightarrow}$-derivation.

PROPOSITION 3.1 (closure of $\mathbf{BR}_{\rightarrow}$ under derivation composition). $\mathbf{BR}_{\rightarrow}$ *is closed under relevant derivation composition.*

This is the analog of Proposition 2.26, resting on the same argument involving index renaming.

Now, relevant bilateral derivability can be defined, enforcing the null account.

DEFINITION 3.2. $\vdash_{\mathbf{BR}_{\rightarrow}}\Gamma:\varphi$ *iff* $\mathcal{C}(\Gamma)$ *and there is a* $\mathbf{BR}_{\rightarrow}$ *derivation of* $\varphi$ *from* $\Gamma$.

A remark about bilateral coherence is in order here. In Definition 2.12, both $+\varphi$ and $-\varphi$ are taken to carry the same index, say $\alpha$, where $\alpha = i(\Gamma)$ by the definition of derivability.

EXAMPLE 3.3. *A derivation establishing*

$$\vdash_{\mathbf{R}\rightarrow} +(\varphi\rightarrow\psi), -(\varphi\rightarrow\chi) : -(\psi\rightarrow\chi) \tag{40}$$

$$\frac{\dfrac{-(\varphi\rightarrow\chi)_{\{2\}}}{+\varphi_{\{2\}}}\ (\rightarrow^{-}E) \quad +(\varphi\rightarrow\psi)_{\{1\}}}{+\psi_{\{1,2\}}}\ (\rightarrow^{+}E) \quad \frac{-(\varphi\rightarrow\chi)_{\{2\}}}{-\chi_{\{2\}}}\ (\rightarrow^{-}E)}{-(\psi\rightarrow\chi)_{\{1,2\}}}\ (\rightarrow^{-}I) \tag{41}$$

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*Note the implicit use of (C) (contraction) by using the open assumption* $-(\varphi\rightarrow\chi)_{\{2\}}$ *twice.*

The extension of the definitions of grounds for assertion and outcomes of assertion to relevant bilateral reification of meaning is presented below.

***grounds for assertion:***

$$GA[\![\varphi]\!] = \{\Gamma \mid \mathcal{C}(\Gamma) \text{ and } \vdash^{Ic} \Gamma : +\varphi\} \tag{42}$$

***grounds for denial:***

$$GD[\![\varphi]\!] = \{\Gamma \mid \mathcal{C}(\Gamma) \text{ and } \vdash^{Ic} \Gamma : -\varphi\} \tag{43}$$

***outcomes from assertion:***

$$OA[\![\varphi]\!] = \{\langle \Gamma, \sigma\rangle \mid \mathcal{C}(\Gamma, +\varphi) \text{ and } \vdash^{Ec} \Gamma, +\varphi : \sigma\} \tag{44}$$

***outcomes from denial:***

$$OD[\![\varphi]\!] = \{\langle \Gamma, \sigma\rangle \mid \mathcal{C}(\Gamma, -\varphi) \text{ and } \vdash^{Ec} \Gamma, -\varphi : \sigma\} \tag{45}$$

***3.5. Vertical harmony of*** $\mathbf{BR}_{\rightarrow}$**.** The vertical harmony and stability of the positive rules, establishing local-soundness and local-completeness, hold, as was shown for the rules of $\mathbf{R}_{\rightarrow}$, of which they are the positive sign analogs. On the other hand, the negative rules where shown to emerge of considerations imposed by the horizontal harmony, so their vertical harmony has to be verified. The reduction establishing local-soundness of the negative rules is presented below.

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ +\varphi_\alpha & -\psi_\alpha\end{array}}{-(\varphi\rightarrow_{\mathbf{R}}\psi)_\alpha}\,(\rightarrow^{-}_{\mathbf{R}}I) \qquad \begin{array}{c}[+\varphi]_i, [-\psi]_j\\ \mathcal{D}'\\ \xi_\beta\end{array}}{\xi_{\alpha\cup\beta-\{i,j\}}}\,(i,j\in\beta)(\rightarrow_{\mathbf{R}}E^{i,j}) \quad \rightsquigarrow_r \quad \mathcal{D}'[+\varphi_i :=\underset{\xi_\alpha}{+\varphi_\alpha}, -\psi_j :=\underset{\xi_\beta}{-\psi_\alpha}]^{\mathbf{R}}_{\xi_{\alpha\cup\beta-\{i,j\}}} \tag{46}$$

The reduction is well-defined due to the closure of $\mathbf{BR}_{\rightarrow}$ under relevant derivation composition.

The expansion establishing the local-completeness of the negative rules is presented below.

$$\underset{-(\varphi\rightarrow_{\mathbf{R}}\psi)_\alpha}{\mathcal{D}} \rightsquigarrow_e \dfrac{\dfrac{\underset{-(\varphi\rightarrow_{\mathbf{R}}\psi)_\alpha}{\mathcal{D}} \quad \dfrac{[+\varphi]_i \quad [-\psi]_j}{-(\varphi\rightarrow_{\mathbf{R}}\psi)_{\{i,j\}}}\,(\rightarrow^{-}_{\mathbf{R}}I)}{-(\varphi\rightarrow_{\mathbf{R}}\psi)_\alpha}\,(\rightarrow^{-}_{\mathbf{R}}E^{i,j})} \tag{47}$$

**§4. Adding Conjunction(s).** One of the highlights of Relevant Logic, as a special case of substructural logics, is the distinction between two conjunctions; for recent discussions with emphasis on Relevant Logic, see Paoli (2007) and Mares (2012). In the context of **R**, those conjunction are denoted by '$\wedge$' (preserving the name 'conjunction'), and 'fusion', denoted by '$\circ$'. The unilateral logic obtained by adding them is $\mathbf{R}_{\{\rightarrow,\wedge,\circ\}}$.

The unilateral $I$-rules for the two conjunctions are the following.

$$\dfrac{\varphi_\alpha \quad \psi_\alpha}{(\varphi\wedge\psi)_\alpha}\,(\wedge I) \qquad \dfrac{\varphi_\alpha \quad \psi_\beta}{(\varphi\circ\psi)_{\alpha\cup\beta}}\,(\circ I) \tag{48}$$

The notable difference between the rules is indexing on the premises and the conclusion. This indexing reflects the following general substructural logistic presentation of the rules in (13).

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By (the unilateral) vertical harmony, the (unilateral) harmoniously induced $GE$-rules are the following.

$$\frac{\begin{array}{c}[\varphi]_i\\ \vdots\\ (\varphi\wedge\psi)_\alpha \quad \chi_\beta\ (i\in\beta)\end{array}}{\chi_{\alpha\cup\beta-\{i\}}}(\wedge_1 GE^i) \qquad \frac{\begin{array}{c}[\psi]_i\\ \vdots\\ (\varphi\wedge\psi)_\alpha \quad \chi_\beta\ (i\in\beta)\end{array}}{\chi_{\alpha\cup\beta-\{i\}}}(\wedge_2 GE^i) \qquad \frac{\begin{array}{c}[\varphi]_i,[\psi]_j\\ \vdots\\ (\varphi\circ\psi)_\alpha \quad \chi_\beta \quad (i,j\in\beta)\end{array}}{\chi_{\alpha\cup\beta-\{i,j\}}}(\circ GE^{i,j}) \tag{49}$$

again in analogy to the general substructural rules in (17).

PROPOSITION 4.1 (closure of $\mathbf{R}_{\{\rightarrow,\wedge,\circ\}}$ under relevant derivation composition). *$\mathbf{R}_{\{\rightarrow,\wedge,\circ\}}$ is closed under relevant derivation composition.*

To see the local-soundness, here are the respective reductions.

$$\frac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi_\alpha & \psi_\alpha\end{array}}{(\varphi\wedge\psi)_\alpha}(\wedge I) \qquad \begin{array}{c}[\varphi]_i\\ \mathcal{D}\\ \chi_\beta\ (i\in\beta)\end{array}}{\chi_{\alpha\cup\beta-\{i\}}}(\wedge_1 GE^i) \rightsquigarrow_r \begin{array}{c}\mathcal{D}_1\\ \mathcal{D}[\varphi_i:=\varphi_\alpha]^R\\ \chi_{\alpha\cup\beta-\{i\}}\end{array} \tag{50}$$

The second reduction with '$(\wedge_2 GE)$' is similar.

$$\frac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi_\alpha & \psi_\beta\end{array}}{(\varphi\circ\psi)_{\alpha\cup\beta}}(\circ I) \qquad \begin{array}{c}[\varphi]_i,[\psi]_j\\ \mathcal{D}\\ \chi_\gamma \quad (i,j\in\beta)\end{array}}{\chi_{\alpha\cup\beta\cup\gamma-\{i,j\}}}(\circ GE^{i,j}) \rightsquigarrow_r \begin{array}{c}\mathcal{D}_1 \quad \mathcal{D}_2\\ \mathcal{D}[\varphi_i:=\varphi_\alpha,\ \psi_j:=\psi_\beta]^R\\ \chi_{\alpha\cup\beta\cup\gamma-\{i,j\}}\end{array} \tag{51}$$

The reductions are well-defined by Proposition 4.1.

For the bilateral $ND$-system $\mathbf{BR}_{\{\rightarrow,\wedge,\circ\}}$, the positive rules are the positive sign analogs of the unilateral rules in $\mathbf{R}_{\{\rightarrow,\wedge,\circ\}}$. I now turn to the determination of the negative $I$-rules from the positive ones by the bilateral horizontal harmony.

$$\frac{-\varphi_\alpha}{-(\varphi\wedge\psi)_\alpha}(\wedge_1^- I) \qquad \frac{-\psi_\alpha}{-(\varphi\wedge\psi)_\alpha}(\wedge_2^- I) \qquad \frac{-\varphi_\alpha \quad -\psi_\beta}{-(\varphi\circ\psi)_{\alpha\cup\beta}}(\circ^- I) \tag{52}$$

And by the bilateral vertical harmony, the negative $GE$-rules are the following.

$$\frac{\begin{array}{ccc}[-\varphi]_i & & [-\psi]_i\\ \vdots & & \vdots\\ -(\varphi\wedge\psi)_\alpha \quad \sigma_\beta\ (i\in\beta) & & \sigma_\beta\ (i\in\beta)\end{array}}{\sigma_{\alpha\cup\beta-\{i,j\}}}(\wedge^- GE^i) \qquad \frac{\begin{array}{c}[-\varphi]_i,[-\psi]_j\\ \vdots\\ -(\varphi\circ\psi)_\alpha \quad \sigma_\beta \quad (i,j\in\beta)\end{array}}{\sigma_{\alpha\cup\beta-\{i,j\}}}(\circ GE^{i,j}) \tag{53}$$

PROPOSITION 4.2 (closure of $\mathbf{R}_{\{\rightarrow,\wedge,\circ\}}$ under relevant derivation composition). *$\mathbf{BR}_{\{\rightarrow,\wedge,\circ\}}$ is closed under relevant derivation composition.*

Below are the negative reductions, required to establish the bilateral vertical harmony.

$$\frac{\begin{array}{ccc}\mathcal{D}_1 & [-\varphi]_i & [-\psi]_i\\ -\varphi_\alpha & \mathcal{D}_2 & \mathcal{D}_3\\ -(\varphi\wedge\psi)_\alpha & \sigma_\beta\ (i\in\beta) & \sigma_\beta\ (i\in\beta)\end{array}}{\sigma_{\alpha\cup\beta-\{i\}}}(\wedge_1^- GE^i) \rightsquigarrow_r \begin{array}{c}\mathcal{D}_1\\ \mathcal{D}_2[-\varphi_i:=-\varphi_\alpha]^R\\ \sigma_{\alpha\cup\beta-\{i\}}\end{array} \tag{54}$$

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(and similarly for '$(\wedge_2^- I)$').

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ -\varphi_\alpha & -\psi_\beta\end{array}}{-(\varphi\circ\psi)_{\alpha\cup\beta}}\,(\circ^- I) \qquad \begin{array}{c}[-\varphi]_i, [-\psi]_j\\ \mathcal{D}_3\\ \sigma_\gamma\end{array} \quad (i,j\in\beta)}{\sigma_{\alpha\cup\beta\cup\gamma-\{i,j\}}}\,(\circ^- GE^{i,j}) \quad \rightsquigarrow_r \quad \begin{array}{c}\mathcal{D}_1 \qquad\qquad \mathcal{D}_2\\ \mathcal{D}_3[-\varphi_i :=-\varphi_\alpha, -\psi_j :=-\psi_\beta]^R\\ \sigma_{\alpha\cup\beta\cup\gamma-\{i,j\}}\end{array} \tag{55}$$

The reductions are well-defined by Proposition 4.2.

I leave the reader to verify that negative expansions exist, so local-completeness holds for $\mathbf{R}_{\{\rightarrow,\wedge,\circ\}}$.

**§5. Conclusions.** In this paper, I studied some aspects of applying bilateralism to (a positive fragment of) the relevant logic **R**. The main issues considered, as reflected in the questions $Q_1$, $Q_2$, and $Q_3$ in the Introduction section, are the impact on denial of relevance (mandatory use of assumptions in derivations), substructurality (the absence of Weakening) and paraconsistency (the absence of Explosion). The framework used in this study is that of proof-theoretic semantics, with its emphasis on harmony and stability. The emerging notion of "relevant denial" is substructural and paraconsistent, as one would expect when dealing with Relevant Logic.

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