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# A perturbation method for the Vlasov–Poisson system

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A perturbation method for the Vlasov–Poisson system is presented. It is self-consistent and entirely based on Lie transformations, which are considered as active transformations, generating the dynamics of the particle distribution function in the space of distribution functions. The main result is a set of three equations that forms a good starting point for a wide variety of problems concerning nonlinear wave propagation. Besides being efficient, the new perturbation method is systematic and therefore also suited for the use of computer algebra.

## 1. Introduction

A major advance in theoretical plasma physics at the beginning of the 1980s was the formulation of nondissipative plasma equations as continuous Hamiltonian systems that, when viewed in terms of Eulerian variables, have noncanonical form. Examples include ideal magnetohydrodynamics (Morrison and Greene 1980), the Vlasov equation (Morrison 1980; Marsden and Weinstein 1982) and the two-fluid equations (Spencer and Kaufman 1982). This was further developed by Larsson (1992, 1993, 1996*a–c*) and Flå (1994), who found Hamiltonian action principles for the above-mentioned systems.

There are fundamental as well as practical reasons for adopting Hamiltonian field-theoretical treatments in plasma physics. A deeper understanding of plasma processes can be obtained from formulations that exhibit the Hamiltonian structure underlying those processes. An underlying Hamiltonian structure also automatically gives the dynamical equations a certain general mathematical structure and provides self-consistent dynamical equations with conservation laws.

However, one important area of these field-theoretical treatments of plasma physics is still poorly developed: there are no systematic, structure-preserving perturbation methods that deal efficiently with various approximations of interest (i.e. eikonal and long-wavelength approximations).

Since linearization and various nonlinear approximations are cornerstones in almost every field of theoretical plasma physics, progress in this area would be of great importance – practically as well as theoretically. The perturbation methods used today are very general and do not take advantage of any underlying structure of the equations to which they are applied. As a result, they are inefficient and cumbersome to work with. The fact that underlying Hamiltonian structures have

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now been obtained for the above-mentioned systems changes the situation dramatically. It should now be possible to develop a systematic and efficient perturbation method that preserves and utilizes these newly found Hamiltonian structures. In the same way as Hamiltonian perturbation methods in classical mechanics (e.g. the Poincare–Von Zeipel method (Goldstein 1980) or the Lie transformation method (Cary 1981; Lichtenberg and Lieberman 1992)) are superior to non-Hamiltonian methods, we expect such a method to be superior to the general field-theoretical methods used today.

In this preliminary work we introduce a new perturbation method that in a systematic way derives a set of equations describing the slow amplitude modulations of electrostatic waves in a Vlasov–Poisson plasma. We consider a nonrelativistic electron plasma (including ions is straightforward), neglect wave–particle interactions and limit ourself to fields of the form

$$\phi(\mathbf{r},t) = \varepsilon\left[\phi_1(\varepsilon\mathbf{r},\varepsilon t)e^{i\theta} + \text{c.c.}\right] + \varepsilon^2\left[\phi_0(\varepsilon\mathbf{r},\varepsilon t) + \left(\phi_2(\varepsilon\mathbf{r},\varepsilon t)e^{2i\theta} + \text{c.c.}\right)\right] + \sum_{n=3}^{N}\varepsilon^n\left[\phi_n(\varepsilon\mathbf{r},\varepsilon t)e^{ni\theta} + \text{c.c.}\right], \tag{1}$$

where $\varepsilon$ is a small parameter and $\theta = \mathbf{k}\cdot\mathbf{r} - \omega t$. That is, we assume the existence of two well-separated time and length scales. Given this ansatz, we proceed in a systematic way by introducing a transformation that relates the particle distribution function $f$ to another function $F$, the oscillating-centre distribution. The result is a set of three equations ((9)–(11) below) that forms a good starting point for theoretical considerations of a large variety of problems concerning nonlinear wave propagation. The perturbation method is general in the sense that it formally works in the same way whether or not we have an external magnetic field. (External magnetic fields will give rise to an additional time scale due to gyromotion, which is not taken care of in our 'two-generator' approach here. It is straightforward to include, though.) Furthermore, the noncanonical Poisson bracket formulation makes it invariant under changes of coordinates. In this paper we have sought simplicity rather then mathematical sophistication, since we think that a 'good' perturbation method, besides being efficient and systematic, should also be easy to use and understand. We have therefore tried to emphasize the method rather than the theory behind it. However, in order to understand the derivation of the equations (9)–(11), some basic knowledge of Lie transformations and symplectic structures is necessary.

The fact that the perturbation method is systematic makes it well suited for the use of computational symbolic algebra programs like Maple and Mathematica. This feature makes this new perturbation method very attractive, since the calculations required for higher orders are rather cumbersome even in simple cases like the one we shall present in Sec. 4. In more complex cases – for example taking into account an external magnetic field or a nonhomogeneous plasma – the work required for higher-order calculations without using symbolic algebra programs would be tremendous.

From a theoretical viewpoint, this paper also has some novel features. In previous papers concerning Hamiltonian perturbation theory in connection with the Vlasov equation, Lie transformations have been used to make a change from particle coordinates to oscillating-centre coordinates in order to obtain a kinetic description of oscillating centres. In the present paper we do not interpret the Lie transforma-

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tions as a change of coordinates, but rather as active transformations, generating the dynamics of the particle distribution function $f$ in the space of distribution functions $\mathscr{F}$. This point of view is discussed further below. Another novel feature is the decomposition of the dynamics of $f$ in $\mathscr{F}$ into two parts by making use of a composition of two generators, each consisting of a large but finite number of near identity Lie transformations: one responsible for the slow time-averaged dynamics and one responsible for the rapid oscillations. This approach has the advantage that the oscillating-centre distribution function $F$ can be introduced in a natural way without any ad hoc assumptions as in Dewar's action principle based on a low-action principle in material particle coordinates (Dewar 1972).

The outline of this paper is as follows. In Sec. 2 we introduce the notation and basic equations to be used. In Sec. 3 we derive the set of perturbation equations, which is the main result of the paper. We also discuss its relation to previous perturbation methods. In Sec. 4 we illustrate the new perturbation method by deriving a set of equations describing the slow amplitude modulations of a Langmuir wave. Finally, conclusions are drawn in Sec. 5.

## 2. Notation and basic equations

The Vlasov–Poisson equations

$$\frac{\partial f}{\partial t} + \{f, H\} = 0, \tag{2}$$

$$\nabla^2 \phi = 4\pi e \left( \int_{-\infty}^{\infty} f \, d\mathbf{v} - N_0 \right) \tag{3}$$

describe the dynamics of a plasma with mobile electrons moving under the influence of an electrostatic potential $\phi(\mathbf{r}, t)$ with a fixed background of positive charge of density $N_0$. Here $f(\mathbf{z}, t)$ is the electron distribution function, and $(\mathbf{z}, t) = (\mathbf{r}, \mathbf{v}, t)$ are local coordinates on the seven-dimensional extended phase space $\mathscr{M}$ upon which the distribution function $f(\mathbf{z}, t)$ is defined. $H(\mathbf{z}, t) = \frac{1}{2} m |\mathbf{v}|^2 - e\phi(\mathbf{r}, t)$ is the Hamiltonian for a single electron in an electrostatic field, and $\{\cdot\,, \cdot\}$ is the Poisson bracket, defined as

$$\{F, G\} = \frac{1}{m} [\boldsymbol{\nabla}_{\mathbf{r}} F \cdot \boldsymbol{\nabla}_{\mathbf{v}} G - \boldsymbol{\nabla}_{\mathbf{v}} F \cdot \boldsymbol{\nabla}_{\mathbf{r}} G] - \frac{e}{m^2 c} \mathbf{B}(\mathbf{r}) \cdot (\boldsymbol{\nabla}_{\mathbf{v}} F \times \boldsymbol{\nabla}_{\mathbf{v}} G), \tag{4}$$

where $\mathbf{B}(\mathbf{r})$ is an external magnetic field. In the absence of an external magnetic field the Poisson bracket (4) reduces to the usual canonical Poisson bracket. Note that we have assumed the magnetic field to be time-independent. Although it is possible to include time-dependent magnetic fields and nonelectrostatic fields, this is beyond the scope of this paper.

The fact that in a collisionless plasma we have conservation of the phase-space density $f(\mathbf{z}, t)\, d\mathbf{z}$ restricts the dynamics of $f$ in the space of distribution functions $\mathscr{F}$ to certain so-called symplectic leaves $\mathscr{O}_{f^0}$. Here $\mathscr{O}_{f^0}$ denotes an infinite-dimensional submanifold of $\mathscr{F}$ that contains a given reference distribution function $f^0$ and is constructed by composing $f^0$ with all phase space-conserving Hamiltonian transformations. For our purposes (perturbation theory) it is beneficial to think of $f^0$ as the unperturbed particle distribution function satisfying

$$\frac{\partial f^0}{\partial t} + \{f^0, H_0\} = 0,$$

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where $H_0 = \frac{1}{2}m|\mathbf{v}|^2 - e\phi_0$ and $\mathbf{E} = -\boldsymbol{\nabla}\phi_0$ represents an unperturbed electric field.

The transformations used throughout this paper are Lie transformations generated by Hamiltonian vector fields. Not only have they proven expedient to work with, but they also seem to be the natural choice when dealing with Hamiltonian systems since they preserve the phase-space density. Since we are dealing with perturbations that are supposed to be small in some sense, we represent the canonical transformation as a near-identity Lie transformation: $\exp(L_W)f^0$, where $W: \mathscr{M} \mapsto \mathbb{R}$ is the so-called generating function of the transformation. Here $\exp(L_W)f^0$ represents the Lie series:

$$\exp(L_W)f^0 = f^0 + \{f^0, W\} + \frac{1}{2!}\{\{f^0, W\}, W, \} + \frac{1}{3!}\{\{\{f^0, W\}, W\}, W\} + \dots \quad .$$

The totality of all generating functions traces out $\mathscr{O}_{f^0}$ in $\mathscr{F}$ labelled by the given reference distribution $f^0$. This leads us to the following definition of the symplectic leaves $\mathscr{O}_{f^0}$:

$$\mathscr{O}_{f^0} = \{f \in \mathscr{F} \mid f = \exp(L_W)f^0 \quad \forall W: \mathscr{M} \mapsto \mathbb{R}\}.$$

Another way to look at $W$ is as a parameter that through the relation

$$f = \exp(L_W)f^0 \tag{5}$$

generates the dynamics of $f$ upon $\mathscr{O}_{f^0}$ (hence the name 'generating function'). Using this vocabulary, we can say that the symplectic leaves are parametrized by the generating functions $W$. The Vlasov equation (2) puts additional constraints on $W$. In fact, replacing $f$ in the Vlasov equation with $\exp(L_W)f^0$ leads to an equation (the symplectic-leaf Vlasov equation) that determines $W$. This approach has been taken by Ye *et al.* (1991).

## 3. The perturbation equations

So far we have not used the a priori assumption that we only consider perturbations in $\phi$ of a certain kind, namely (1). The fact that we have limited ourselves to a certain type of field variations will enable us to simplify the problem. We proceed as follows. Owing to the ansatz (1), we expect the dynamics of $f$ upon the symplectic leaf $\mathscr{O}_{f^0}$ to consist of two parts: a rapid oscillating part (of the form $e^{in\theta}$) superimposed upon a slow part with a characteristic time scale $\varepsilon t$. We also expect the generating function $W$ to consist of a rapid and a slow part. Knowing this, we replace $\exp(L_W)$ (the generator of $f$) with a composition of two generators $T_R^+$ and $T_S^+$, each consisting of a large but finite number of Lie transformations. Here $T_S^+$ ($S$ for 'slow') is responsible for the slow time-averaged dynamics of $f$, and $T_R^+$ ($R$ for 'rapid') takes care of the rapid oscillations. Instead of (5), we then have

$$f(\mathbf{z}, t) = T_R^+ T_S^+ f^0(\mathbf{z}, t),$$

where

$$T_R^+ = \exp(\varepsilon L_{R_1}) \exp(\varepsilon^2 L_{R_2}) \cdots \exp(\varepsilon^n L_{R_n}),$$
$$T_R^- = \exp(-\varepsilon^n L_{R_n}) \cdots \exp(-\varepsilon^2 L_{R_2}) \exp(-\varepsilon L_{R_1}),$$

and similarly for $S$. This approach has the benefit that we are now able to control the transformation order by order. We also introduce the notation $F = T_S^+ f^0$, which

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legitimately can be referred to as the oscillating-centre distribution function since it has no rapid oscillations and therefore is the time-averaged part of $f$. Given that $F = T_S^+ f^0$ and that $f = T_R^+ T_S^+ f^0$, we find that $f$ and $F$ must be related as

$$f = T_R^+ F.$$

In order to incorporate $F$ into the Vlasov–Poisson equations, we replace $f$ in (2) with $T_R^+ F$ to obtain

$$\frac{\partial}{\partial t}(T_R^+ F) + \{T_R^+ F, H\} = 0. \tag{6}$$

Using the identities (A4)–(A6) in Appendix A, it is straightforward to rewrite (6) as

$$T_R^+ \left[ \frac{\partial F}{\partial t} + \left\{ F, \boldsymbol{\alpha}_R^- \cdot \frac{\partial \mathbf{R}}{\partial t} + T_R^- H \right\} \right] = 0, \tag{7}$$

which clearly implies that the expression inside the square brackets equals zero. (Here we have introduced the vector $\mathbf{R} = (R_1, R_2, \ldots, R_n)$ and the operator $\boldsymbol{\alpha}_R^-$. The definition of $\boldsymbol{\alpha}_R^-$ is given in Appendix A.) The fact that we have chosen to replace the single generator function $\exp(L_W)$ with a composition of two generators $T_R^+$ and $T_S^+$ have given us a certain degree of freedom. We can choose one of the generators $\mathbf{R}$ and $\mathbf{S}$ at will. However, the requirement that $F$ should have no rapid oscillations implies (from (7)) that $\mathbf{R}$ should be chosen in such a way that

$$\boldsymbol{\alpha}_R^- \cdot \frac{\partial \mathbf{R}}{\partial t} + T_R^- H \tag{8}$$

should have no rapid oscillations. That is, we chose $\mathbf{R}$ to eliminate all the rapidly varying terms of (8). What is left over (terms with no rapid oscillations) is our oscillating-centre Hamiltonian $K$. This is a straightforward procedure that can be performed order by order. This procedure is exactly the same as if we make a Lie transformation from particle coordinates to oscillating-centre coordinates in order to obtain a kinetic description of oscillating centres (see e.g. Ekiel-Jezewska *et al.* 1995). The interpretation is, however, quite different. In our case we do *not* interpret $\mathbf{R}$ as a generator of a change of coordinates but instead as the generator of the dynamics of $f$ upon $\mathcal{O}_{f^0}$ that is responsible for the rapid variations. This point of view has the advantage that we avoid many of the difficulties connected with oscillating-centre coordinates – for example, whether the change of coordinates should be considered as a passive or active transformation and the difficulties involved in transforming to an accelerated reference system. Another advantage with our approach is that no statistical ansatz is needed to obtain a dynamical equation for the oscillating-centre distribution function as in Ekiel-Jezewska *et al.* (1995). Also, no additional ansatz about $F$ is needed as in Dewar's ad hoc action principle based on a low-action principle in material particle coordinates (Dewar 1972). The advantage of a lab frame as in our case is simply that no additional ansatz is needed except the a priori information introduced about the prescribed composite transformation (i.e. that $f$ can be written as $T_R^+ T_S^+ f^0$).

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This leads us to the following set of equations which is the main result of this paper:

$$K = \left\langle \boldsymbol{\alpha}_R^- \cdot \frac{\partial \mathbf{R}}{\partial t} + T_R^- H \right\rangle_0, \tag{9}$$

$$\frac{\partial F}{\partial t} + \{F, K\} = 0, \tag{10}$$

$$\nabla^2 \phi = 4\pi e \left( \int T_R^+ F \, d\mathbf{v} - N_0 \right), \tag{11}$$

where $\langle . \rangle_0$ means all nonoscillating terms in the expansion. Here (11) is found by replacing $f$ with $T_R^+ F$ in Poisson's equation. This set of equations forms an efficient algorithm for deriving equations describing slow variations of the oscillating-centre distribution function $F$ and the potentials $\phi_i$, $i = 1, 2, \ldots, n$. The first equation together with the requirement that $K$ should have no rapid oscillations determine both $K$ and $\mathbf{R}$ to every order. The advantage of (9) compared with the use of oscillating-centre coordinates is that here (9) is a set of equations for the rapidly oscillating generators $\mathbf{R}$. These can be solved by any method we choose, and will in principle include linear Landau damping effects – although these are neglected in Sec. 4, where an adiabatic approximation is used. Having thereby obtained $K$ and $\mathbf{R}$, we proceed to (10) and (11). Equation (10) determines the slow dynamics of the oscillating-centre distribution function $F$. Each order in the expansion parameter $\varepsilon$ results in a separate equation. In (11) we go the other way around. We do not separate it order by order, but instead let terms having the same harmonic dependences $\mathrm{e}^{in\theta}$, $n = 0, 1, 2, \ldots$, form separate equations. The factors $\mathrm{e}^{in\theta}$ merely works as a 'placeholders' which can later be removed. This results in a set of equations describing the slow amplitude modulations of the potentials $\phi_i$.

One of the main features of this perturbation method is that it is very systematic. In finding $\mathbf{R}$, we solve the same kind of equation again and again. And once $\mathbf{R}$ (and thereby $K$) is found, we immediately obtain the dynamical equations for $F$ and $\phi$ from (10) and (11). The fact that the perturbation method is systematic makes it well suited for the use of computational symbolic algebra programs like Maple and Mathematica. This feature makes this new perturbation method very attractive, since the calculations required for higher orders are rather cumbersome even in simple cases like the one we shall present in the next section. In fact, most of the calculations in the next section have been done on a computer (Maple worksheets containing the calculations in Sec. 4 can be obtained by e-mail on request from Jonas.Lundberg@mt.luth.se).

In order to illustrate the perturbation method described above, we proceed with an example. We consider the simplest possible case: the slow amplitude modulations of Langmuir waves in a plasma with no external magnetic field.

## 4. Slow amplitude modulations of Langmuir waves

In this section we consider the slow amplitude modulations of a Langmuir wave, taking into account interactions with higher harmonics. This is probably the simplest possible nontrivial case of slow amplitude modulations in a Vlasov–Poisson plasma. This case has been studied previously by Dewar (1972), and we refer to that paper for a physical interpretation of the results. Our goal here is merely to

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illustrate the usefulness of the new perturbation method. We end up with the same result as in Dewar (1972). However, the amount of work needed to achieve the result is significantly reduced.

Besides neglecting wave–particle interactions, we also assume the plasma to be homogenous and time-independent, and with no external magnetic field. We can therefore (without any further loss of generality) assume that the carrier wave with wavenumber $k$ and frequency $\omega$ propagates along the $x$-axis. Given the ansatz (1), we find that the single-particle Hamiltonian now takes the form

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \varepsilon^3 H_3,$$

where

$$\begin{aligned} H_0 &= \tfrac{1}{2}mv^2 \\ H_1 &= -e[\phi_1(\varepsilon x, \varepsilon t)e^{i\theta} + \text{c.c.}], \\ H_2 &= -e[\phi_2(\varepsilon x, \varepsilon t)e^{2i\theta} + \text{c.c.}], \\ H_3 &= -e[\phi_3(\varepsilon x, \varepsilon t)e^{3i\theta} + \text{c.c.}] \end{aligned}$$

and $\theta = kx - \omega t$. Since we have no external magnetic field, the noncanonical Poisson bracket (4) reduces to the usual canonical Poisson bracket

$$\{F, G\} = \frac{1}{m}(F_{,x}G_{,v} - G_{,x}F_{,v}).$$

In the above expression and in what follows we use the index notation '$,q$' to denote partial derivatives with respect to the variable $q$: for example, $\phi_{1,xx} = \partial^2\phi_1/\partial x^2$.

We now proceed to (9). Using the expansions in Appendix B, we find that

$$K = K_0 + \varepsilon K_1 + \varepsilon^2 K_2 + \varepsilon^3 K_3,$$

where

$$K_0 = H_0, \tag{12a}$$

$$K_1 = DR_1 + H_1, \tag{12b}$$

$$K_2 = DR_2 + H_2 - \tfrac{1}{2}\{H_1, R_1\} - \tfrac{1}{2}\{K_1, R_1\}, \tag{12c}$$

$$\begin{aligned} K_3 = {} & DR_3 + H_3 - \{H_2, R_1\} + \tfrac{1}{3}\{\{H_1, R_1\}, R_1\} \\ & -\{K_1, R_2\} + \tfrac{1}{6}\{\{K_1, R_1\}, R_1\}, \end{aligned} \tag{12d}$$

where we have used (12$b$) in the right-hand side of (12$c$) and (12$d$), and $D$ denotes the differential operator

$$D = \frac{\partial}{\partial t} + \{\cdot, H_0\}.$$

The requirement that $K$ should have no rapid oscillations determines both $K$ and $\mathbf{R}$. We find that

$$K_0 = \tfrac{1}{2}mv^2, \qquad K_1 = 0, \qquad K_2 = K_{20} + \varepsilon K_{21}, \qquad K_3 = 0, \tag{13}$$

where

$$\begin{aligned} K_{20} &= \frac{e^2k^2|\phi_1|^2}{m(\omega - kv)^2}, \\ K_{21} &= i\frac{e^2k^2(\phi_1\phi_{1,t}^* - \phi_1^*\phi_{1,t})}{m(\omega - kv)^3} + i\frac{e^2k\omega(\phi_1\phi_{1,x}^* - \phi_1^*\phi_{1,x})}{m(\omega - kv)^3}, \end{aligned}$$

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and that

$$R_1 = i\frac{e\phi_1 e^{i\theta}}{\omega - kv} + \text{c.c} + \varepsilon\left[\left(\frac{eD\phi_1}{(\omega-kv)^2} - \frac{e\phi_{1,x}}{k(\omega-kv)}\right)e^{i\theta} + \text{c.c.}\right]$$
$$+\varepsilon^3\left[\left(2i\frac{eD\phi_{1,x}}{k(\omega-kv)^2} - i\frac{eD^2\phi_1}{(\omega-kv)^3} - i\frac{e\phi_{1,xx}}{k^2(\omega-kv)}\right)e^{i\theta} + \text{c.c.}\right],$$
$$R_2 = \left(i\frac{e^2k^2\phi_1^2}{4m(\omega-kv)^3} + i\frac{e\phi_2}{2(\omega-kv)}\right)e^{2i\theta} + \text{c.c.}$$
$$+\varepsilon\left[\left(\frac{eD\phi_2}{4(\omega-kv)^2} - \frac{e\phi_{2,x}}{4k(\omega-kv)} - \frac{e^2k\phi_1\phi_{1,x}}{4m(\omega-kv)^3} + \frac{3e^2k^2\phi_1 D\phi_1}{4m(\omega-kv)^4}\right)e^{2i\theta} + \text{c.c.}\right],$$
$$R_3 = -i\frac{2e^2k^2\phi_1^*\phi_2 e^{i\theta}}{m(\omega-kv)^3} + i\frac{2e^2k^2\phi_1\phi_2 e^{3i\theta}}{3m(\omega-kv)^3} + i\frac{e\phi_3 e^{3i\theta}}{3(\omega-kv)} + \text{c.c.}$$

The fact that both $K_1$ and $K_3$ in (13) are zero is generic to the ansatz (1). A closer look at (12$b$, $c$) shows that $R_1$ and $R_2$ always have the harmonic dependences $\mathrm{e}^{\pm i\theta}$ and $\mathrm{e}^{\pm 2i\theta}$ respectively given the harmonic dependences of $H_i$, $i = 1, 2, 3$. We then find that all terms on the right-hand sides of (12$b$, $d$) will have harmonic dependences, and hence $K_1$ and $K_3$ are always zero to all orders.

Note also that our assumption that we can neglect effects due to resonant particles is essential. The systematic process described above will not work for particle velocities close to the phase velocity of the wave. The ordering of terms will break down and singularities will appear. In order to include resonant particles, we must give them a separate treatment where the ordering of terms will be quite different. Progress in this area of research (nonlinear Landau damping) has recently been reported by Brodin (1997), who divided the velocity space into a resonant part $\{v: \omega/k - \delta v < v < \omega/k + \delta v$, where $\gamma_l/k \ll \delta v \ll kv_{\text{th}}^2/\omega\}$ and a nonresonant part $||v - \omega/k|| > \delta v$ in order to give a separate treatment of the resonant particles. Here $\gamma_l$ is the linear damping rate and $v_{\text{th}}$ is the thermal speed. However, such an approach (although possible) is beyond the scope of this paper.

Having thereby obtained $K$ and $\mathbf{R}$ we proceed to (10). Assuming $F$ to have the form

$$F = F_0(v) + \varepsilon^2 F_2(v) + \varepsilon^3 F_3(v),$$

we find from (10) the following equations:

$$O(1): \qquad F_{0,t} + \{F_0, K_0\} = 0 \tag{14a}$$
$$O(\varepsilon^2): \qquad F_{2,t} + \{F_2, K_0\} + \{F_0, K_{20}\} = 0, \tag{14b}$$
$$O(\varepsilon^3): \qquad F_{3,t} + \{F_3, K_0\} + \{F_0, K_{21}\} = 0, \tag{14c}$$

and from (11) we obtain

$$e^0: \qquad N_0 - \int F_0(v)\,dv - \int F_2(v)\,dv - \int F_3(v)\,dv = 0, \tag{15a}$$
$$e^{i\theta}: \qquad \varepsilon\omega(1-\Pi_0)\phi_1 + \varepsilon^2\left[2i\Pi_1\phi_{1,t} - 2i\frac{\omega}{k}(1-\Pi_1)\phi_{1,x}\right]$$
$$-\varepsilon^3\left[\frac{\omega}{k^2}(1 + 2\Pi_1 - 3\Pi_2)\phi_{1,xx} + \frac{2}{k}(\Pi_1 - 3\Pi_2)\phi_{1,xt} - \frac{3}{\omega}\Pi_2\phi_{1,tt}\right.$$
$$\left. + \frac{5e^2k^4}{2m^2\omega^3}\Pi_4|\phi_1|^2\phi_1 - \frac{3ek^2}{m\omega}\pi_2\phi_2\phi_1^* + \frac{4\pi e^2\omega\phi_1}{m}\int\frac{F_2(v)}{(\omega-kv)^2}\,dv\right] = 0, \tag{15b}$$

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$$e^{2i\theta}: \qquad \varepsilon^2 k\left[(4-\Pi_0)\phi_2+\frac{3ek^2}{2m\omega^2}\Pi_2\phi_1^2\right] \tag{15c}$$

$$-\varepsilon^3\left[i(4-\Pi_1)\phi_{2,x}-i\frac{k\Pi_1}{\omega}\phi_{2,t}+i\frac{6ek^2\Pi_3}{m\omega^2}\phi_1\phi_{1,x}+i\frac{6ek^3\Pi_3}{m\omega^3}\phi_1\phi_{1,t}\right]=0,$$

$$e^{3i\theta}: \qquad \varepsilon^3\left[(9-\Pi_0)\phi_3+\frac{3ek^2\Pi_2}{m\omega^2}\phi_2\phi_1-\frac{5e^2k^4\Pi_4}{2m^2\omega^4}\phi_1^3\right]=0, \tag{15d}$$

where we have introduced the notation

$$\Pi_n=\frac{4\pi e^2\omega^n}{m}\int\frac{F_0(v)}{(\omega-kv)^{2+n}}\,dv.$$

Equations (14*a*) and (15*a*) reflect the facts that the oscillating-centre distribution function to the lowest order equals the unperturbed particle distribution function $f^0$ and that the number of oscillating centres equals the number of electrons. Equations (14*b*, *c*) and (15*b*–*d*) form a closed system of equations that determines the slow amplitude modulations of a Langmuir wave up to $O(\varepsilon^3)$. This set of equations can be written in a more attractive form using lower-order approximations in the higher-order terms. From (15*b*, *c*) we obtain

$$\phi_{1,t}=\frac{\omega}{k}\frac{1-\Pi_1}{\Pi_1}\phi_{1,x}+O(\varepsilon),$$

$$\phi_2=-\frac{ek^2\Pi_2}{2m\omega^2}\phi^2+O(\varepsilon),$$

given that $\omega$ is defined by the dispersion relation $1-\Pi_0=0$ for a given wavenumber $k$. Using these approximations in the terms $O(\varepsilon^3)$ in (15*b*–*d*), we find the following system of equations:

$$F_{2,t}+\{F_2,K_0\}+\{F_0,K_{20}\}=0, \tag{16}$$

$$i(\phi_{1,t}+v_g\phi_{1,x})+\frac{v_g'}{2}\phi_{1,xx}-\frac{e^2k^4}{4m^2\omega^3}\frac{5\Pi_4+3\Pi_2^2}{\Pi_1}|\phi_1|^2\phi_1 - \frac{2\pi e^2\omega\phi_1}{m\Pi_1}\int\frac{F_2\,dp}{(\omega-kv)^2}=0, \tag{17}$$

$$\phi_2=-\frac{ek^2\Pi_2}{2m\omega^2}\phi_1^2-\varepsilon\frac{iek^3}{3m\omega^3}\left[(\Pi_1\Pi_2+6\Pi_3)\,\phi_1\phi_{1,t}+\frac{\omega}{k}(\Pi_1\Pi_2+6\Pi_3-4\Pi_2)\,\phi_1\phi_{1,x}\right], \tag{18}$$

$$\phi_3=\frac{e^2k^4}{m^2\omega^4}\left(\frac{3\Pi_2^2}{8}+\frac{5\Pi_4}{16}\right)\phi_1^3. \tag{19}$$

Here $v_g$ ($=\partial\omega/\partial k$) is the group velocity and $v_g'$ ($=\partial^2\omega/\partial k^2$) is the group dispersion of the carrier wave, both of which can be obtained from the dispersion relation $1-\Pi_0=0$. Equations (16) and (17) now form a closed Zakharov-like system of equations that determines the slow amplitude modulations of a Langmuir wave. Once $\phi_1$ and $F_2$ is obtained, (18) and (19) yield explicit expressions for the higher harmonics. The two equations (16) and (17) are equivalent to equations (29) and (50) in Dewar (1972), and we refer to that paper for a physical analysis.

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## 5. Conclusions

In conclusion, then, we have presented a new perturbation method for the Vlasov–Poisson system. It is self-consistent and entirely based on Lie transformations, which in this paper are considered as active transformations, generating the dynamics of the particle distribution function $f$ in the space of distribution functions $\mathscr{F}$. The dynamics of $f$ in $\mathscr{F}$ is decomposed into two parts by making use of a composition of two Lie transformations: one responsible for the slow time-averaged dynamics and one responsible for the rapid oscillations. This approach has the benefit that the oscillating-centre distribution function $F$ can be introduced in a natural way without any ad hoc assumptions.

Besides being efficient, the new perturbation method is systematic and therefore suited for the use of computer algebra. Furthermore, the perturbation method is general in the sense that it works formally the same way whether or not we have an external (time independent) magnetic field, and the noncanonical Poisson bracket formulation makes it invariant under changes of coordinates.

The main result is a set of three equations (9)–(11) that forms a good starting point for theoretical considerations of a wide variety of problems concerning nonlinear wave propagation.

We have deliberately avoided a rigorous mathematical treatment, and instead have tried to emphasize the perturbation *method* rather then the theory behind it. In a forthcoming paper a more thorough investigation of dynamics in terms of active Lie transformations will be presented. We fully believe that this approach offers a better physical insight than the repeated changes-of-coordinates approach frequently used in previous works dealing with Hamiltonian perturbation theory.

## Appendix A. Definitions and identities

In this appendix we define the two operators $T_W^{\pm}$ and $\boldsymbol{\alpha}_{\mathbf{W}}^{\pm}$ and present/derive some identities involving these two operators that are used in the derivation of (9)–(11).

The noncanonical Poisson bracket associated with the dynamics of a single electron with charge $-e$ in a electrostatic field is

$$\{F, G\} = \frac{1}{m}\left(\boldsymbol{\nabla}_{\mathbf{r}} F \cdot \boldsymbol{\nabla}_{\mathbf{v}} G - \boldsymbol{\nabla}_{\mathbf{v}} F \cdot \boldsymbol{\nabla}_{\mathbf{r}} G\right) - \frac{e}{m^2 c}\mathbf{B}(\mathbf{r}) \cdot (\boldsymbol{\nabla}_{\mathbf{v}} F \times \boldsymbol{\nabla}_{\mathbf{v}} G),$$

where $\mathbf{B}(\mathbf{r})$ is an external magnetic field. It is straightforward to show that this Poisson bracket satisfies the relation

$$\int A\{B, C\}\, d\mathbf{z}\, dt = \int B\{C, A\}\, d\mathbf{z}\, dt \qquad \forall A, B, C : \mathscr{M} \mapsto \mathbb{R}. \tag{A 1}$$

Given any function $W \in C^\infty(\mathscr{M})$, we define a near-identity Lie transformation $\exp(\pm L_W) : C^\infty(\mathscr{M}) \mapsto C^\infty(\mathscr{M})$ as

$$\exp(\pm L_W) G = G \pm \{G, W\} + \frac{1}{2!}\{\{G, W\}, W, \} \pm \frac{1}{3!}\{\{\{G, W\}, W\}, W\} + \dots\ .$$

Lie transformations generated by Hamiltonian vector fields commute with the Poisson bracket (Abraham and Marsden 1980); that is,

$$\exp(\pm L_W)\{F, G\} = \{\exp(\pm L_W) F, \exp(\pm L_W) G\}. \tag{A 2}$$

In perturbation theory it is desirable to control the transformation order by order

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in a small parameter $\varepsilon$. We therefore introduce the operators $T_W^{\pm}$ consisting of a large but finite number of Lie transformations

$$T_W^{+} \equiv \exp(\varepsilon L_{W_1}) \exp(\varepsilon^2 L_{W_2}) \cdots \exp(\varepsilon^n L_{W_n}),$$
$$T_W^{-} \equiv \exp(-\varepsilon^n L_{W_n}) \cdots \exp(-\varepsilon^2 L_{W_2}) \exp(-\varepsilon L_{W_1}),$$

having the property that

$$T_W^{+} T_W^{-} G = T_W^{-} T_W^{+} G = G \qquad \forall G: \mathscr{M} \mapsto \mathbb{R}.$$

By repeated use of the identity (A 1), we find that

$$\int\int F T_W^{+} G \, d\mathbf{z} \, dt = \int\int G T_W^{-} F \, d\mathbf{z} \, dt \qquad \forall F, G: \mathscr{M} \mapsto \mathbb{R}. \tag{A 3}$$

Another property of the operator $T_W^{\pm}$ is

$$T_W^{\pm}\{F, G\} = \{T_W^{\pm} F, T_W^{\pm} G\} \qquad \forall F, G: \mathscr{M} \mapsto \mathbb{R}, \tag{A 4}$$

which follows from repeated use of (A 2).

Let $\mathbf{W}$ be the vector $\mathbf{W} = (W_1, W_2, \ldots, W_n)$. We then define the operator $\boldsymbol{\alpha}_{\mathbf{W}}^{\pm}$ by the relation

$$\frac{\partial}{\partial t}(T_{\mathbf{W}}^{\pm} G) = \pm \left\{ T_W^{\pm} G, \boldsymbol{\alpha}_w^{\pm} \cdot \frac{\partial \mathbf{W}}{\partial t} \right\} + T_W^{\pm} \frac{\partial G}{\partial t}, \qquad \forall G: \mathscr{M} \mapsto \mathbb{R}. \tag{A 5}$$

The above definition of $\boldsymbol{\alpha}_{\mathbf{W}}^{\pm}$ is implicit, and we prefer it this way. We could of course define it by the explicit expression (B 2) derived in Appendix B and then derive (A 5) as a resulting identity. However, to start from the infinte series (B 2) and then derive (A 5) is much more difficult then the other way around.

By using

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial t}(T_W^{+} T_W^{-} G) = \frac{\partial}{\partial t}(T_W^{-} T_W^{+} G) \qquad \forall G: \mathscr{M} \mapsto \mathbb{R}$$

and (A 5), we can deduce the identities

$$\boldsymbol{\alpha}_{\mathbf{W}}^{\pm} \cdot \frac{\partial \mathbf{W}}{\partial t} = T_W^{\pm} \left( \boldsymbol{\alpha}_{\mathbf{W}}^{\mp} \cdot \frac{\partial \mathbf{W}}{\partial t} \right). \tag{A 6}$$

## Appendix B. Expansions

In this appendix we present the expansions of $T_{\mathbf{W}}^{-} A$ and $\boldsymbol{\alpha}_W^{-} \cdot \partial \mathbf{W}/\partial t$ defined in Appendix A and used in Sec. 4.

From the definition of $T_W^{\pm}$ in Appendix A and given that $A: \mathscr{M} \mapsto \mathbb{R}$ can be ordered as

$$A = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \varepsilon^4 A_4,$$

we find the expansion of $T_W^{-} A$ up to $O(\varepsilon^3)$:

$$\begin{aligned} T_W^{-} A = {} & A_0 + \varepsilon(A_1 - \{A_0, W_1\}) \\ & + \varepsilon^2 (A_2 - \{A_1, W_1\} + \tfrac{1}{2}\{\{A_0, W_1\}, W_1\} - \{A_0, W_2\}) \\ & + \varepsilon^3 (A_3 - \{A_2, W_1\} + \tfrac{1}{2}\{\{A_1, W_1\}, W_1\} - \{A_1, W_2\} \\ & - \tfrac{1}{6}\{\{\{A_0, W_1\}, W_1\}, W_1\} + \{\{A_0, W_1\}, W_2\} - \{A_0, W_3\}) \end{aligned} \tag{B 1}$$

From the definition of $\boldsymbol{\alpha}_{\mathbf{W}}^{\pm}$ in Appendix A, we find that

$$\boldsymbol{\alpha}_{\mathbf{W}}^{-} = (\tilde{\alpha}_1^{-}, \tilde{\alpha}_2^{-}, \ldots, \tilde{\alpha}_n^{-}),$$

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where

$$\tilde{\alpha}_j^- = \varepsilon^j \exp(-\varepsilon^n L_{W_n}) \cdots \exp(-\varepsilon^{j+1} L_{W_{j+1}})\, \alpha(-\varepsilon^j L_{W_j})$$

and $\alpha$ is an operator defined as

$$\alpha(x) = \frac{\exp(x) - 1}{x}.$$

Given that we obtain the expansion of $\boldsymbol{\alpha}_{\mathbf{W}}^- \cdot \partial \mathbf{W}/\partial t$ up to $O(\varepsilon^3)$,

$$\begin{aligned} \boldsymbol{\alpha}_{\mathbf{W}}^- \cdot \frac{\partial \mathbf{W}}{\partial t} = {} & \varepsilon \frac{\partial W_1}{\partial t} + \varepsilon^2 \left( \frac{\partial W_2}{\partial t} - \frac{1}{2} \left\{ \frac{\partial W_1}{\partial t}, W_1 \right\} \right) \\ & + \varepsilon^3 \left( \frac{\partial W_3}{\partial t} + \frac{1}{6} \left\{ \left\{ \frac{\partial W_1}{\partial t}, W_1 \right\}, W_1 \right\} - \left\{ \frac{\partial W_1}{\partial t}, W_2 \right\} \right). \end{aligned} \tag{B 2}$$

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