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Integrals with a large parameter. Several nearly coincident saddle-points

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Abstract. In the contour integral

$$\int g(z, \alpha_1, \alpha_2, \ldots, \alpha_l) \exp\{Nf(z, \alpha_1, \alpha_2, \ldots, \alpha_l)\} dz$$

the functions $g(\ \ )$ and $f(\ \ )$ are analytic functions of their arguments, and N is a large positive parameter. When N tends to ∞, asymptotic expansions can usually be found by the method of steepest descents, which shows that the principal contributions arise from the saddle-points, i.e. the values of z at which $\partial f/\partial z = 0$. The position of the saddle-points varies with the l parameters, denoted by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, and it may happen that two or more saddle-points coincide at a certain value of $\boldsymbol{\alpha}$. The asymptotic expansions given by the ordinary method of steepest descents are then non-uniform. The case of a single parameter and two nearly coincident saddle-points was studied earlier and leads to Airy functions. Here we are concerned with the case of l parameters and of m saddle-points which are nearly coincident when $\boldsymbol{\alpha}$ is near $\mathbf{0}$. It is then known that $f(z, \boldsymbol{\alpha})$ can be transformed locally into a polynomial of degree $m+1$, and accordingly we here restrict $f(z, \boldsymbol{\alpha})$ to be such a polynomial. The form of the resulting asymptotic expansion had already been conjectured (correctly as we now see), and involves certain special functions of $m-1$ variables. To establish the validity of the expansion we must show that the remainder after any number of terms is of smaller magnitude than the last term retained in the expansion. In other words, we must establish an inequality between integrals which is to hold for sufficiently small $\boldsymbol{\alpha}$ and sufficiently large N, and which involves l parameters and m nearly coincident saddle-points. This inequality is given in equation (1·3) below.

In the present paper a proof is given of this inequality, along the following lines. The inequality is proved if a certain set of m stronger inequalities can be proved. These latter are conditions of boundedness: $|P_j(N; \boldsymbol{\alpha})| \leqslant A$, for $j = 1, 2, \ldots, m$. They are not difficult to prove in regions of $\boldsymbol{\alpha}$-space where the saddle-points are well separated, but difficult to prove elsewhere. Let an additional complex variable τ be introduced. This splits the saddle-points when τ is bounded away from zero, and for such τ we can therefore prove the corresponding inequalities $|P_j^*(N; \boldsymbol{\alpha}, \tau)| \leqslant A$ where the P_j^* are regular functions of τ. It now follows from the maximum-modulus theorem for the variable τ, that $|P_j(N; \boldsymbol{\alpha})| = |P_j^*(N; \boldsymbol{\alpha}, 0)| \leqslant A$, which is the desired result. The region of validity in $\boldsymbol{\alpha}$-space contains an open domain independent of N. (It is probable that this property can be used to extend the validity to a still larger region.)

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The calculation is given in detail for two parameters and three saddle-points. (A fuller outline of the mathematical argument is given at the beginning of section 5.) The generalization to l parameters and m saddle-points is immediate.

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1. *Introduction.* The present paper is concerned with the asymptotic expansion of contour integrals of the form

$$I(N;\alpha_1,\alpha_2,\ldots,\alpha_l)=\int g(z,\alpha_1,\ldots,\alpha_l)\exp\{Nf(z,\alpha_1,\ldots,\alpha_l)\}\,dz, \tag{1·1}$$

where N is a large real positive parameter, and $g(z,\alpha_1,\ldots,\alpha_l)$ and $f(z,\alpha_1,\ldots,\alpha_l)$ are analytic functions of z and of the parameters $\alpha_1,\ldots,\alpha_l$. The asymptotic expansion can often be found by the method of steepest descents. The contour of integration is deformed to pass through saddle-points of $f(z)$, which are the zeros of the derivative $f'(z)$, and the asymptotic expansion can then be found from the local expansions of $g(z)$ and $f(z)$ near the saddle-points. The saddle-points in the integral (1·1) depend on $\alpha_1,\ldots,\alpha_l$, and vary when these parameters vary. (For brevity we shall write

$$\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_l),\quad |\boldsymbol{\alpha}|=|\alpha_1|+\ldots+|\alpha_l|.)$$

It may happen that at a certain value of $\boldsymbol{\alpha}$ two or more saddle-points coincide; the steepest-descents expansion then becomes non-uniform. In an earlier paper (2) the simplest case of a single parameter and of two nearly coincident saddle-points was treated. In that work a new local variable of integration $v(z,\alpha)$ was introduced by means of the relation

$$f(z,\alpha)=\tfrac{1}{3}v^3-\zeta(\alpha)v+B(\alpha); \tag{1·2}$$

the resulting asymptotic series involves Airy functions of the variable $N^{\frac{2}{3}}\zeta(\alpha)$, and it was found that successive terms decrease rapidly, like powers of N^{-1}. The region of validity in the α-plane was found to be independent of N, and it was shown later (5) that by means of this property the asymptotic series could be continued into a larger region in which (1·2) need no longer be applicable. These results were much stronger than earlier ones (for references see (5), section 1), and it was natural to inquire whether analogous results can be obtained for three or more nearly coincident saddle-points. For three saddle-points (for m saddle-points) the right-hand side of the relation (1·2) must be replaced by a quartic polynomial (a polynomial of degree $m+1$), and it was shown by Levinson (3) that locally regular transformations with this property do indeed exist. The form of the resulting asymptotic expansions was first studied by Bleistein (1). To prove the validity of the expansion an inequality was needed for the remainder after n terms, such as the following.

Suppose that $G(u,\boldsymbol{\alpha})$ is regular in the disc $|u|\leqslant r$ when $\boldsymbol{\alpha}$ is near $\mathbf{0}$. Suppose also that

$$p_{m+1}(u,\boldsymbol{\alpha})=\frac{u^{m+1}}{m+1}+\sum_{k=1}^{m-1}\xi_k(\boldsymbol{\alpha})\,u^{m-k}$$

is a polynomial of degree $m+1$, and that $\xi_k(\mathbf{0})=0$ for all k. (The term in u^m is absent.)

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Then there is a positive constant A (depending on G) and a positive constant N_0, such that, for all $N > N_0$ and all $\boldsymbol{\alpha}$ in a certain neighbourhood of $\boldsymbol{\alpha} = \mathbf{0}$, we have

$$\left|\int G(u,\boldsymbol{\alpha})\exp\{Np_{m+1}(u,\boldsymbol{\alpha})\}\,du\right| \leqslant A\sum_{k=0}^{m-1}\left|\int u^k\exp\{Np_{m+1}(u,\boldsymbol{\alpha})\}\,du\right|, \tag{1·3}$$

where the integration is taken along an arbitrary path (independent of N) which lies in the disc $|u| \leqslant r$ and is such that the contributions from the ends of the range are negligible.

Bleistein proposed an inductive method of proof in which the results for

$$m = 1, 2, \ldots, m'$$

were to be used to prove the result for $m = m' + 1$. Difficulties were found to arise near those points in $\boldsymbol{\alpha}$-space which give rise to two or more coincident saddle-points. These difficulties become more severe as the number of dimensions increases, and were not fully resolved by Bleistein. Furthermore, Bleistein's method of proof gives a region of validity in $\boldsymbol{\alpha}$-space which contracts to a thin region as $N \to \infty$, although the actual region of validity was expected to contain at least an open domain independent of N. There is moreover a gap in Bleistein's argument as he himself points out. (For a discussion see section 7 below.)

In the present paper a proof is given of the validity of the asymptotic expansion in an open domain in $\boldsymbol{\alpha}$-space (i.e. of the inequality (1·3)) by a method which is applicable to any number of parameters and any number of nearly coincident saddle-points. This domain is independent of N. The idea of the proof is simple. A certain bound in $\boldsymbol{\alpha}$-space is to be obtained, and it is observed that this can be done in those regions where the saddle-points are well separated. (We are, however, concerned with the case of nearly coincident saddle-points.) Let an additional complex variable τ be introduced. When $|\tau|$ is bounded away from 0 and $|\boldsymbol{\alpha}|$ is small enough the saddle-points are well separated and a bound is obtained; the maximum-modulus principle is then used in the τ-plane to obtain the same bound for $\tau = 0$ when the saddle-points are nearly coincident. For reasons of clarity the proof of the theorem is given for the special case of three nearly coincident saddle-points, and of two independent parameters which will henceforth be denoted by α and β. It is not difficult to see, however, that all the arguments are applicable to any number of parameters and any number of nearly coincident saddle-points. These arguments are not related to the arguments given in (1) or in (2).

2. *Transformation to standard form.* We consider the integral

$$I(N;\alpha,\beta) = \int g(z,\alpha,\beta)\exp\{Nf(z,\alpha,\beta)\}\,dz, \tag{2·1}$$

where it is supposed that $g(z,\alpha,\beta)$ and $f(z,\alpha,\beta)$ are regular functions for all z in a region $|z| \leqslant \rho_1$, and for all (α,β) in a region $D_0(\alpha,\beta)$. (We introduce the convention that all the regions $D_k(\alpha,\beta)$ defined in the present paper are open domains in (α,β)-space and contain the point $(0,0)$.) It follows that $f(z,\alpha,\beta)$ has a convergent expansion

$$f(z,\alpha,\beta) = \sum_{k=0}^{\infty} a_k(\alpha,\beta)\,z^k, \tag{2·2}$$

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where the coefficients $a_k(\alpha, \beta)$ are regular functions of α and β. We are concerned with the case of three nearly coincident saddle-points

$$a_1(0,0) = a_2(0,0) = a_3(0,0) = 0; a_4(0,0) \neq 0.$$

When (α, β) varies near $(0,0)$ there are always three saddle-points near $z = 0$. We are not here concerned with the contributions from distant saddle-points, and we may therefore confine our attention to the part of the path lying within a fixed disc, $|z| \leqslant \rho_2$, say, where ρ_2 is the radius defined in the statement of Theorem 1 below. It is now possible to reduce $I(N; \alpha, \beta)$ to a standard form by means of

THEOREM 1. *In the expansion* (2·2) *let it be supposed that*

$$a_1(0,0) = a_2(0,0) = a_3(0,0) = 0; a_4(0,0) \neq 0.$$

Then there exists a transformation $u = u(z, \alpha, \beta)$ *(and an inverse transformation*

$$z = z(u, \alpha, \beta))$$

uniformly regular and (1, 1) *for all* z *in a domain containing a disc* $|z| \leqslant \rho_1$ *and for all* (α, β) *in a domain* $D_1(\alpha, \beta)$, *and such that*

$$f(z, \alpha, \beta) = A(\alpha, \beta) - \eta(\alpha, \beta)\, u - \xi(\alpha, \beta)\, u^2 + \tfrac{1}{4}u^4, \tag{2·3}$$

an exact quartic polynomial. The functions $A(\alpha, \beta)$, $\eta(\alpha, \beta)$ *and* $\xi(\alpha, \beta)$ *are regular in* $D_1(\alpha, \beta)$; *also* $\xi(0,0) = \eta(0,0) = 0$. *The image domain in the* u*-plane contains a disc* $|u| \leqslant r$ for *all* $(\alpha, \beta) \in D_1(\alpha, \beta)$, *and the inverse image of* $|u| \leqslant r$ *contains a disc* $|z| \leqslant \rho_2$.

Proof. This follows at once from Levinson's Theorem ((3), Theorem 1·2) after a trivial change of origin in the u-plane (to remove the cubic term) and a trivial change of scale (to normalize the quartic term). This concludes the proof of Theorem 1.

Let the transformation $z = z(u, \alpha, \beta)$ be applied in the disc $|z| \leqslant \rho_2$. On writing

$$g(z, \alpha, \beta)\, \frac{dz}{du} = G(u, \alpha, \beta)$$

we obtain, with an exponentially negligible error, the equation

$$\exp(-NA(\alpha, \beta))\, I(N; \alpha, \beta) = \int G(u, \alpha, \beta) \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\}\, du, \tag{2·4}$$

where the (finite) path of integration in the u-plane is the image of the (finite) path in the z-plane, and may usually (with an exponentially negligible error) be deformed into a fixed path in the u-plane when $|\alpha| + |\beta|$ is small enough;

$$(\alpha, \beta) \in D_2(\alpha, \beta) \subseteq D_1(\alpha, \beta),$$

say. Equation (2·4) gives the standard form of the integral $I(N; \alpha, \beta)$ defined by (2·1).

3. *The form of the asymptotic expansion.* The problem has now been reduced to the asymptotic expansion of

$$\int G(u, \alpha, \beta) \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\}\, du, \tag{3·1}$$

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where the integration may be restricted to a fixed path lying in a disc $|u| \leqslant r$, with an error that is exponentially negligible provided that (α, β) is restricted to a domain $D_2(\alpha, \beta)$. Let us now, following Bleistein ((1), p. 540), introduce the function sequences

$$G_k(u, \alpha, \beta), \quad H_k(u, \alpha, \beta), \quad P_k(\alpha, \beta), \quad Q_k(\alpha, \beta), \quad R_k(\alpha, \beta).$$

These functions are regular for all u in the disc $|u| \leqslant r$ and all (α, β) in $D_2(\alpha, \beta)$, and are defined by the following equations.

$$G(u, \alpha, \beta) \equiv G_0(u, \alpha, \beta) = P_0 + Q_0 u + R_0 u^2 + (u^3 - 2\xi u - \eta) H_0(u, \alpha, \beta), \quad (3{\cdot}2)$$

$$-(\partial/\partial u) H_0(u, \alpha, \beta) \equiv G_1(u, \alpha, \beta) = P_1 + Q_1 u + R_1 u^2 + (u^3 - 2\xi u - \eta) H_1(u, \alpha, \beta), \quad (3{\cdot}3)$$

$$-(\partial/\partial u) H_k(u, \alpha, \beta) \equiv G_{k+1}(u, \alpha, \beta)$$
$$= P_{k+1} + Q_{k+1} u + R_{k+1} u^2 + (u^3 - 2\xi u - \eta) H_{k+1}(u, \alpha, \beta). \quad (3{\cdot}4)$$

Explicit expressions for these functions in the form of contour integrals are given by Bleistein. By repeated integration by parts it is seen that

$$\int G_0(u, \alpha, \beta) \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\} du, \quad (3{\cdot}5)$$

$$-\left(P_0 + \frac{P_1}{N} + \ldots + \frac{P_n}{N^n}\right) \int \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\} du, \quad (3{\cdot}6)$$

$$-\left(Q_0 + \frac{Q_1}{N} + \ldots + \frac{Q_n}{N^n}\right) \int u \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\} du, \quad (3{\cdot}7)$$

$$-\left(R_0 + \frac{R_1}{N} + \ldots + \frac{R_n}{N^n}\right) \int u^2 \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\} du, \quad (3{\cdot}8)$$

$$= \frac{1}{N^n} \int (u^3 - 2\xi u - \eta) H_n(u, \alpha, \beta) \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\} du$$
$$= \frac{1}{N^{n+1}} \int G_{n+1}(u, \alpha, \beta) \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\} du, \quad (3{\cdot}9)$$

where integrated terms from the ends of the range of integration have been formally omitted. The integral (3·9) is seen to be of the same form as the original integral (3·5). (Finite limits of integration will be discussed in section 6 of the present paper.) The foregoing calculation suggests the following theorem, cf. ((1), p. 540).

THEOREM 2. *There is a positive number N_0 and a domain $D_3(\alpha, \beta)$, such that, for all real $N > N_0$ and for all $(\alpha, \beta) \in D_3(\alpha, \beta)$ we have*

$$\int G(u, \alpha, \beta) \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\} du$$
$$= \frac{1}{N^{\frac{1}{4}}} \mathfrak{A}_0(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta) \sum_{k=0}^{n} \frac{P_k(\alpha, \beta)}{N^k} - \frac{1}{N^{\frac{1}{2}}} \mathfrak{A}_1(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta) \sum_{k=0}^{n} \frac{Q_k(\alpha, \beta)}{N^k}$$
$$+ \frac{1}{N^{\frac{3}{4}}} \mathfrak{A}_2(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta) \sum_{k=0}^{n} \frac{R_k(\alpha, \beta)}{N^k} + \mathfrak{R}_{n+1}(N; \alpha, \beta) \quad (3{\cdot}10)$$

where by definition

$$\mathfrak{A}_m(X, Y) = \int (-U)^m \exp(\tfrac{1}{4}U^4 - XU^2 - YU) dU, \quad (m = 0, 1, 2), \quad (3{\cdot}11)$$

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where $\xi(\alpha,\beta)$ and $\eta(\alpha,\beta)$ are defined by (2·3), *where P_k, Q_k and R_k are defined by* (3·4), *where the limits of integration in* (3·11) *are at ∞ in the appropriate valleys (the paths being extended from the finite limits to ∞ along paths of descent), where $D_3(\alpha,\beta)$ is defined in Lemma 3 below, and where*

$$|\mathfrak{R}_{n+1}(N;\alpha,\beta)| \leqslant \frac{M_{n+1}}{N^{n+1}}\left\{\frac{1}{N^{\frac{1}{4}}}|\mathfrak{A}_0(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta)| + \frac{1}{N^{\frac{1}{2}}}|\mathfrak{A}_1(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta)| + \frac{1}{N^{\frac{3}{4}}}|\mathfrak{A}_2(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta)|\right\} \quad (3\cdot12)$$

for some constant M_{n+1}. (We may, e.g. take $M_{n+1} = A\max|G_{n+1}(u,\alpha,\beta)|$, taken over the disc $|u| \leqslant r$ and $D_3(\alpha,\beta)$.)

The proof of this theorem will be given in sections 5 and 6, where it will be seen to follow at once from a much stronger result, stated here as

THEOREM 3. *With the same assumptions and the same notation as in Theorem 2 we have*

$$\int G(u,\alpha,\beta)\exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\}\,du = \frac{1}{N^{\frac{1}{4}}}\mathfrak{A}_0(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta)\,P(N;\alpha,\beta) - \frac{1}{N^{\frac{1}{2}}}\mathfrak{A}_1(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta)\,Q(N;\alpha,\beta) + \frac{1}{N^{\frac{3}{4}}}\mathfrak{A}_2(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta)\,R(N;\alpha,\beta), \quad (3\cdot13)$$

where P, Q, R, the functions defined in section 4 below, satisfy the inequalities

$$\left|P(N;\alpha,\beta) - \sum_{k=0}^{n}\frac{P_k(\alpha,\beta)}{N^k}\right| \leqslant \frac{M_{n+1}}{N^{n+1}},$$
$$\left|Q(N;\alpha,\beta) - \sum_{k=0}^{n}\frac{Q_k(\alpha,\beta)}{N^k}\right| \leqslant \frac{M_{n+1}}{N^{n+1}}, \quad (3\cdot14)$$
$$\left|R(N;\alpha,\beta) - \sum_{k=0}^{n}\frac{R_k(\alpha,\beta)}{N^k}\right| \leqslant \frac{M_{n+1}}{N^{n+1}}.$$

The proof of Theorem 3 is given for infinite limits of integration in section 5 below; and for finite limits in section 6 below.

4. *Integrals with infinite limits. Definitions.* It is convenient to begin with the case of infinite limits which avoids certain complications of detail. The case of finite limits will be studied in section 6. It will therefore be assumed that $G_0(u,\alpha,\beta), \ldots, G_{n+1}(u,\alpha,\beta)$ are regular for all finite values of u, and suitably bounded at ∞ so that all the integrals of Theorem 2 exist. It will be assumed that the path of integration goes from one valley at ∞ to another valley at ∞. (The valleys are the directions

$$u = \infty\exp(\tfrac{1}{4}\pi i), \quad \infty\exp(\tfrac{3}{4}\pi i), \quad \infty\exp(\tfrac{5}{4}\pi i), \quad \infty\exp(\tfrac{7}{4}\pi i) \quad \text{at} \quad \infty.)$$

Let paths $C(1)$, $C(2)$, $C(3)$ between three arbitrary distinct pairs of valleys be chosen in the u-plane, and let us write

$$f(1,N;\alpha,\beta) = \int_{C(1)} G(u,\alpha,\beta)\exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\}\,du, \quad (4\cdot1)$$

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$$N^{-\frac{1}{4}(m+1)}\,\mathfrak{A}_m(1, N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta) = \int_{C(1)} (-u)^m \exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - \eta u)\}\,du, \quad (m = 0, 1, 2), \tag{4·2}$$

for integrals along $C(1)$, with similar expressions for integrals along $C(2)$ and $C(3)$. The following definitions are suggested by the form of Theorem 2, when it is noted that the theorem is to hold for any path from ∞ to ∞ in the u-plane. Actually, if the functions $P(N; \alpha, \beta)$, $Q(N; \alpha, \beta)$, $R(N; \alpha, \beta)$ satisfy the equations

$$f(j) = \frac{1}{N^{\frac{1}{4}}}\mathfrak{A}_0(j)\,P - \frac{1}{N^{\frac{1}{2}}}\mathfrak{A}_1(j)\,Q + \frac{1}{N^{\frac{3}{4}}}\mathfrak{A}_2(j)\,R, \quad (j = 1, 2, 3) \tag{4·3}$$

for the three paths $C(1)$, $C(2)$, $C(3)$, then by linear combination they satisfy the corresponding equation for any path from ∞ to ∞. We therefore introduce the

Definition. $P(N; \alpha, \beta)$, $Q(N; \alpha, \beta)$, $R(N; \alpha, \beta)$ are the solutions of (4·3), given by the expressions

$$P = N^{\frac{1}{4}}\Delta_0/\Delta, \quad Q = -N^{\frac{1}{2}}\Delta_1/\Delta, \quad R = N^{\frac{3}{4}}\Delta_2/\Delta,$$

$$\text{where } \Delta_0(N; \alpha, \beta) = \begin{vmatrix} f(1) & \mathfrak{A}_1(1) & \mathfrak{A}_2(1) \\ f(2) & \mathfrak{A}_1(2) & \mathfrak{A}_2(2) \\ f(3) & \mathfrak{A}_1(3) & \mathfrak{A}_2(3) \end{vmatrix}, \tag{4·4}$$

$$= [f(1), \quad \mathfrak{A}_1(2), \quad \mathfrak{A}_2(3)] \tag{4·5}$$

(in an abbreviated notation identifying determinants by their diagonal elements), and where similarly

$$\Delta_1(N; \alpha, \beta) = [f(1), \quad \mathfrak{A}_2(2), \quad \mathfrak{A}_0(3)], \tag{4·6}$$

$$\Delta_2(N; \alpha, \beta) = [f(1), \quad \mathfrak{A}_0(2), \quad \mathfrak{A}_1(3)], \tag{4·7}$$

and

$$\Delta(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta) = [\mathfrak{A}_0(1), \quad \mathfrak{A}_1(2), \quad \mathfrak{A}_2(3)]. \tag{4·8}$$

It will be seen later (see Lemma 6) that Δ can be found explicitly,

$$\Delta(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}\eta) = \Delta(0, 0)\exp(-2N\xi^2).$$

For brevity, let us write

$$F(u, \alpha, \beta) = \tfrac{1}{4}u^4 - \xi(\alpha, \beta)\,u^2 - \eta(\alpha, \beta)\,u. \tag{4·9}$$

Let τ be an additional complex variable. We define a modified function

$$\begin{aligned} F^*(u, \alpha, \beta, \tau) &= F(u, \alpha, \beta) - \tau u \\ &= \tfrac{1}{4}u^4 - \xi u^2 - (\eta + \tau)\,u, \end{aligned} \tag{4·10}$$

the asterisk indicating dependence on τ, and similarly we define determinants

$$\Delta_0^* = \Delta_0^*(N; \alpha, \beta, \tau), \quad \Delta_1^* \quad \text{and} \quad \Delta_2^*$$

as in (4·4), (4·6) and (4·7), except that the modified function $F^*(u, \alpha, \beta, \tau)$ takes the place of $F(u, \alpha, \beta)$ in the integrand. Thus

$$\Delta_k(N; \alpha, \beta) = \Delta_k^*(N; \alpha, \beta, 0), \quad (k = 0, 1, 2), \tag{4·11}$$

Similarly we define $f^*(1, N; \alpha, \beta, \tau)$, $\mathfrak{A}_m^*(1, N; \alpha, \beta, \tau)$. We shall often write these and other functions in an abbreviated form, such as $f^*(1, \tau)$, $\mathfrak{A}_m^*(1, \tau)$. Since the limits of

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integration are in fixed valleys at ∞ it follows that $f^*(1,\tau)$ and $\mathfrak{U}_m^*(1,\tau)$ are regular functions of τ for all τ.

5. *Integrals with infinite limits. Proofs.* The crucial intermediate result at which we are aiming is

THEOREM 4. *There is a constant A, such that, for all real $N > N_0$ and for all*

$$(\alpha,\beta)\in D_3(\alpha,\beta)$$

we have
$$|P(N;\alpha,\beta)| \leqslant A, \quad |Q(N;\alpha,\beta)| \leqslant A, \quad |R(N;\alpha,\beta)| \leqslant A. \tag{5·1}$$

Evidently this is a weak form of Theorem 3. To establish Theorem 4, a number of lemmas will be proved. Lemma 1 shows that the variable τ causes the saddle-points to separate when τ is suitably bounded away from 0, and α and β are sufficiently small. Lemma 2 is concerned with determinants $\Delta_0^*, \Delta_1^*, \Delta_2^*$ with well-separated saddle-points. It is shown that such determinants can be rearranged so that each row involves only a single saddle-point. In Lemma 3 bounds are obtained for Δ_0^*, Δ_1^* and Δ_2^*; these bounds are independent of τ as is shown in Lemmas 4 and 5. We now use the observation that Δ_0^*, Δ_1^* and Δ_2^* are regular functions of the complex variable τ. In Lemma 7 the maximum-modulus theorem is applied to these determinants, and it is concluded that the original determinants Δ_0, Δ_1 and Δ_2 (obtained by setting $\tau = 0$ in Δ_0^*, Δ_1^* and Δ_2^*) have the bounds found in Lemma 3. It now follows that P, Q and R are bounded, as is asserted by Theorem 4.

Theorem 3 can then be deduced by applying Theorem 4 to the function

$$G_{n+1}(u,\alpha,\beta)$$

defined in equation (3·4), and Theorem 2 is an immediate consequence of Theorem 3. These results will now be given in detail.

LEMMA 1. *Let r be the radius defined in Theorem 1, and let r_1 by an arbitrary fixed radius less than r. Then there is an $\epsilon > 0$, and a domain $D_3(\alpha,\beta) \subseteq D_1(\alpha,\beta)$, such that the three zeros $u_{\mathrm{I}}^*(\tau)$, $u_{\mathrm{II}}^*(\tau)$, $u_{\mathrm{III}}^*(\tau)$ of $u^3 - 2\xi u - \eta - \tau$ lie inside $|u| = \frac{1}{2}(r_1+r)$ and are separated by at least $\frac{1}{10}r_1$, whenever τ lies in the annulus $r_1^3(1-\epsilon) \leqslant |\tau| \leqslant r_1^3(1+\epsilon)$, and (α,β) lies in $D_3(\alpha,\beta)$.*

Proof of Lemma 1. Suppose first that $(\alpha,\beta) = (0,0)$, and that $\epsilon = 0$; then the saddle-points u_{I}^*, u_{II}^*, u_{III}^* are the zeros of $u^3 - \tau$ which are separated by $3^{\frac{1}{2}}r_1$ and lie on

$$|u| = r_1 < r.$$

It is now obvious from continuity that the saddle-points remain near $|u| = r_1$ (e.g. inside $|u| = \frac{1}{2}(r_1+r)$) and that they remain separated (e.g. by at least $\frac{1}{10}r_1$), provided only that ϵ and $|\alpha| + |\beta|$ are chosen small enough. This concludes the proof of Lemma 1. Saddle-points satisfying the conditions of Lemma 1 will be said to be *well separated.*

Suppose, then, that the saddle-points $u_{\mathrm{I}}^*(\tau)$, $u_{\mathrm{II}}^*(\tau)$, $u_{\mathrm{III}}^*(\tau)$ are well separated. For a general value of τ in the annulus the two paths of steepest descent from any saddle-point do not pass through a second saddle-point, and we can then define the path $C^*(\mathrm{I},\tau)$ composed of the two paths of steepest descent from $u_{\mathrm{I}}^*(\tau)$ to ∞. Similarly we

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can define $C^*(\mathrm{II}, \tau)$ and $C^*(\mathrm{III}, \tau)$. When τ varies in the annulus the saddle-points and the paths of integration also vary. We denote by $f^*(\mathrm{I}, \tau)$ the integral along $C^*(\mathrm{I}, \tau)$; there is an ambiguity of sign in this definition, let the sign be fixed arbitrarily at a general value $\tau = \tau_0$, and elsewhere by continuity. Similarly, we define $f^*(\mathrm{II}, \tau)$, $f^*(\mathrm{III}, \tau)$ and $\mathfrak{A}_m^*(\mathrm{I}, \tau)$, $\mathfrak{A}_m^*(\mathrm{II}, \tau)$, $\mathfrak{A}_m^*(\mathrm{III}, \tau)$. Corresponding to the definition of Δ_0^* (see equation (5·3) below) we now define

$$\delta_0^*(N; \alpha, \beta, \tau) = [f^*(\mathrm{I}), \mathfrak{A}_1^*(\mathrm{II}), \mathfrak{A}_2^*(\mathrm{III})], \tag{5·2}$$

and similarly the determinants δ_1^*, δ_2^* and δ^*. Note that these are defined only when the saddle-points are well separated.

Now suppose that τ varies in the annulus. Then it is well known that the elements of δ_0^* change discontinuously across certain curves in the τ-plane (the *Stokes lines*) because the (infinite) limits of integration of e.g. $f^*(\mathrm{I}, \tau)$ change discontinuously (see (5), Table 1). When τ lies on a Stokes line then at least one of the paths $C^*(\mathrm{I}, \tau)$, $C^*(\mathrm{II}, \tau)$, $C^*(\mathrm{III}, \tau)$ passes through more than one saddle-point. In other words, the integral $f^*(\mathrm{I}, \tau)$ represents different functions of τ in different parts of the τ-plane. It is not difficult to see (cf. (5), section 4) that the asymptotic expansions (as $N \to \infty$) of these different functions are analytic continuations of each other across a Stokes line.

Consider now the path $C^*(\mathrm{I}, \tau_0)$ for an arbitrarily chosen general value τ_0 lying on $|\tau| = r_1^3$. This path joins two distinct valleys at ∞. We denote by $C(1)$ any path joining these same two valleys; this path is to be kept fixed when τ is varied. Similarly, we define $C(2)$ and $C(3)$. We define $f^*(1, \tau)$ as the integral along $C(1)$ (with similar definitions for $\mathfrak{A}_m^*$ and for other paths) and can then, as in (4·4) above, define

$$\Delta_0^*(N; \alpha, \beta, \tau) = [f^*(1), \quad \mathfrak{A}_1^*(2), \quad \mathfrak{A}_2^*(3)] \tag{5·3}$$

where the integrands depend on τ but the limits of integration of each element are independent of τ. Evidently Δ_0^* is a regular function of τ, since each element has this property. Evidently also $\Delta_0^* = \delta_0^*$ when $\tau = \tau_0$, by our construction. Similar equations hold for Δ_1^*, Δ_2^* and Δ^*. We shall now prove

LEMMA 2. *Suppose that τ varies in the annulus $r_1^3(1-\epsilon) \leqslant |\tau| \leqslant r_1^3(1+\epsilon)$. Then, in this annulus, we have*

$$\Delta_0^*(N; \alpha, \beta, \tau) = \delta_0^*(N; \alpha, \beta, \tau), \tag{5·4}$$

with a similar result for Δ_1^*, Δ_2^* and Δ^*.

Proof of Lemma 2. We shall prove the result for Δ^*; the proof for the other determinants is similar since all the integrals have the same Stokes lines. Evidently the result holds when both Δ^* and δ^* are regular near $\tau = \tau_0$, and it is therefore sufficient to show that δ^* remains a regular function of τ when τ crosses a Stokes line. On either side of a Stokes line the path $C^*(\mathrm{I}, \tau)$ (say) is well defined, and so are its two limits as τ tends to a point on the Stokes line from one side or the other. Let the two limit paths be denoted by $C^*(\mathrm{I}, +)$ and $C^*(\mathrm{I}, -)$ respectively. $C^*(\mathrm{I}, +)$ can differ from $C^*(\mathrm{I}, -)$ only when the path of steepest descent from u_{I}^* passes through a *lower* saddle-point. Let us (in the present argument only) renumber the saddle-points so that u_{I}^* is the highest, u_{III}^* the lowest saddle-point. Then we have

$$\mathfrak{A}_m^*(\mathrm{III}, +) = \mathfrak{A}_m^*(\mathrm{III}, -), \tag{5·5}$$

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$$\mathfrak{A}_m^*(\mathrm{II}, +) = \mathfrak{A}_m^*(\mathrm{II}, -) + \mathfrak{r}(\mathrm{II}, \mathrm{III})\, \mathfrak{A}_m^*(\mathrm{III}, -), \tag{5·6}$$

$$\mathfrak{A}_m^*(\mathrm{I}, +) = \mathfrak{A}_m^*(\mathrm{I}, -) + \mathfrak{r}(\mathrm{I}, \mathrm{II})\, \mathfrak{A}_m^*(\mathrm{II}, -) + \mathfrak{r}(\mathrm{I}, \mathrm{III})\, \mathfrak{A}_m^*(\mathrm{III}, -) \tag{5·7}$$

where the coefficients $\mathfrak{r}$ can take only the values -1, 0 or $+1$. (This last property is not needed.) Similar equations hold for f^*. Thus

$$\begin{vmatrix} 1 & \mathfrak{r}(\mathrm{I}, \mathrm{II}) & \mathfrak{r}(\mathrm{I}, \mathrm{III}) \\ 0 & 1 & \mathfrak{r}(\mathrm{II}, \mathrm{III}) \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \mathfrak{A}_0^*(\mathrm{I}, -) & \mathfrak{A}_1^*(\mathrm{I}, -) & \mathfrak{A}_2^*(\mathrm{I}, -) \\ \mathfrak{A}_0^*(\mathrm{II}, -) & \mathfrak{A}_1^*(\mathrm{II}, -) & \mathfrak{A}_2^*(\mathrm{II}, -) \\ \mathfrak{A}_0^*(\mathrm{III}, -) & \mathfrak{A}_1^*(\mathrm{III}, -) & \mathfrak{A}_2^*(\mathrm{III}, -) \end{vmatrix}$$

$$= \begin{vmatrix} \mathfrak{A}_0^*(\mathrm{I}, +) & \mathfrak{A}_1^*(\mathrm{I}, +) & \mathfrak{A}_2^*(\mathrm{I}, +) \\ \mathfrak{A}_0^*(\mathrm{II}, +) & \mathfrak{A}_1^*(\mathrm{II}, +) & \mathfrak{A}_2^*(\mathrm{II}, +) \\ \mathfrak{A}_0^*(\mathrm{III}, +) & \mathfrak{A}_1^*(\mathrm{III}, +) & \mathfrak{A}_2^*(\mathrm{III}, +) \end{vmatrix}$$

and, since the first determinant has the value 1, it follows that

$$\delta^*(-) = \delta^*(+). \tag{5·8}$$

This argument can be applied to a continuous range of values of τ on the Stokes line. The two analytic functions $\delta^*(-)$ and $\delta^*(+)$ thus take the same values on the Stokes line, and are therefore analytic continuations of each other ((4), section 4·51). It follows that δ^* is regular in the annulus. This concludes the proof of Lemma 2 for Δ^*. The same argument is applicable to Δ_0^*, Δ_1^* and Δ_2^*.

We next obtain bounds for δ_0^*, δ_1^* and δ_2^*. We write

$$F_{\mathrm{I}}^*(\tau) = F^*(u_{\mathrm{I}}^*(\tau)), \quad S_1^*(\tau) = F_{\mathrm{I}}^*(\tau) + F_{\mathrm{II}}^*(\tau) + F_{\mathrm{III}}^*(\tau),$$

where all these functions depend also on α and β. Then we have

LEMMA 3. *There is a constant A, and a positive number N_0, such that, for all $N > N_0$, the inequalities*

$$|\delta_0^*(N; \alpha, \beta, \tau) \exp\{-NS_1^*(\alpha, \beta, \tau)\}| . N^{\frac{1}{2}} \leqslant A, \tag{5·9}$$

$$|\delta_1^*(N; \alpha, \beta, \tau) \exp\{-NS_1^*(\alpha, \beta, \tau)\}| . N^{\frac{1}{2}} \leqslant A, \tag{5·10}$$

$$|\delta_2^*(N; \alpha, \beta, \tau) \exp\{-NS_1^*(\alpha, \beta, \tau)\}| . N^{\frac{3}{2}} \leqslant A \tag{5·11}$$

are satisfied for all τ on the circle $|\tau| = r_1^3$ and for all $(\alpha, \beta) \in D_3(\alpha, \beta)$.

Proof of Lemma 3. Consider, for example, the determinant $\delta_0^*(N; \tau)$. The saddle-points are well separated, and each element of $\delta_0^*(N; \tau)$ involves only a single saddle-point. The ordinary method of steepest descents is therefore applicable in its simplest form. (This was the motive for introducing δ_0^*.) We write

$$f^*(\mathrm{I}, \tau) = \frac{\exp\{NF_{\mathrm{I}}^*(\tau)\}}{N^{\frac{1}{2}}} f^{**}(\mathrm{I}, \tau), \tag{5·12}$$

$$\mathfrak{A}_m^*(\mathrm{I}, \tau) = \frac{\exp\{NF_{\mathrm{I}}^*(\tau)\}}{N^{\frac{1}{2}}} \mathfrak{A}_m^{**}(\mathrm{I}, \tau), \tag{5·13}$$

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and similar expressions for the other two saddle-points. The factors are chosen so as to give rise to the steepest-descents expansions

$$f^{**}(\mathrm{I},\tau) \sim \sum_{k=0}^{\infty} \frac{b_k(\mathrm{I},\tau)}{N^k}$$

$$\mathfrak{A}_m^{**}(\mathrm{I},\tau) \sim \sum_{k=0}^{\infty} \frac{\mathfrak{A}_{mk}(\mathrm{I},\tau)}{N^k}$$

in inverse integral powers of N, actually with coefficients analytic across Stokes lines as we have seen. On substituting (5·12) and (5·13) it is seen that

$$\delta^*(N;\tau)\exp\{-N(F_{\mathrm{I}}^*+F_{\mathrm{II}}^*+F_{\mathrm{III}}^*)\}.N^{\frac{1}{4}} = [f^{**}(\mathrm{I}),\quad \mathfrak{A}_1^{**}(\mathrm{II}),\quad \mathfrak{A}_2^{**}(\mathrm{III})], \tag{5·14}$$

where each element of (5·14) has an asymptotic expansion in inverse powers of N. It follows that the determinant (5·14) also has an expansion of this form. In particular it is bounded when N is large enough, $N > N_0$, say. This concludes the proof of Lemma 2.

We note incidentally that, for $N > N_0$, we have

$$|f^{**}(\mathrm{I},\tau)| < \text{const.} \max_{\substack{|u|\leqslant r \\ (\alpha,\beta)\in D_3}} |G(u,\alpha,\beta)|. \tag{5·15}$$

This follows from the usual steepest-descents argument when it is observed that the saddle-points lie inside $|u| = \frac{1}{2}(r_1+r) < r$, i.e. at a finite distance from $|u| = r$.

LEMMA 4. *$S_1^*(\alpha,\beta,\tau) = F_{\mathrm{I}}^*(\tau)+F_{\mathrm{II}}^*(\tau)+F_{\mathrm{III}}^*(\tau)$ is independent of τ. (Actually $S_1^*(\alpha,\beta,\tau) = -2\xi^2$.)*

Proof. (The following method is applicable to polynomials of any degree.)

On evaluating the residues at the poles we have (omitting for convenience the asterisks)

$$S_1^*(\tau) = \frac{1}{2\pi i}\int \frac{F''(u)}{F'(u)} F(u)\,du, \tag{5·16}$$

where

$$\begin{aligned} F(u) &= \tfrac{1}{4}u^4 - \xi u^2 - (\eta+\tau)u, \\ F'(u) &= u^3 - 2\xi u - \eta - \tau, \\ F''(u) &= 3u^2 - 2\xi, \end{aligned} \tag{5·17}$$

and where the contour of integration encloses all the zeros of $F'(u)$. On subtracting a regular integrand which makes a vanishing contribution we obtain

$$S_1^*(\tau) = \frac{1}{2\pi i}\int \frac{F''(u)}{F'(u)}\,(F(u)-uF'(u))\,du. \tag{5·18}$$

Thus

$$S_1^*(\tau) = -\frac{1}{2\pi i}\int \frac{(1+a_2u^{-2})(1+b_2u^{-2})}{1+c_2u^{-2}-(\eta+\tau)u^{-3}}\cdot\tfrac{9}{4}u^3\,du \tag{5·19}$$

where a_2, b_2 and c_2 are independent of τ. Thus

$$\frac{\partial}{\partial\tau}S_1^*(\tau) = -\frac{1}{2\pi i}\int \frac{(1+a_2u^{-2})(1+b_2u^{-2})}{(1+c_2u^{-2}-(\eta+\tau)u^{-3})^2}\cdot\tfrac{9}{4}\,du, = 0$$

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since the expansion of the integrand contains no term in u^{-1}. Thus $S_1^*(\tau) = S_1^*(0)$. This concludes the proof of Lemma 4.

The result $S_1^*(0) = -2\xi^2$ can be obtained by evaluating the coefficient of u^{-1} in (5·19).

LEMMA 5. $\Delta^*(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}(\eta+\tau))$ *is independent of* τ.

Proof. This is the usual proof for a Wronskian determinant. By column-by-column differentiation we have

$$N^{-\frac{3}{4}}\frac{\partial}{\partial\tau}\Delta^* = N^{-\frac{3}{4}}\frac{\partial}{\partial\tau}[\mathfrak{A}_0^*(1), \mathfrak{A}_1^*(2), \mathfrak{A}_2^*(3)]$$

$$= -[\mathfrak{A}_1^*, \mathfrak{A}_1^*, \mathfrak{A}_2^*] - [\mathfrak{A}_0^*, \mathfrak{A}_2^*, \mathfrak{A}_2^*] \tag{5·20}$$

$$-[\mathfrak{A}_0^*, \mathfrak{A}_1^*, \mathfrak{A}_3^*]. \tag{5·21}$$

The terms in (5·20) vanish because each term has two columns identical; the term (5·21) vanishes because

$$\mathfrak{A}_3^* - 2\xi\mathfrak{A}_1^* + (\eta+\tau)\,\mathfrak{A}_0^*$$

$$= -\int (u^3 - 2\xi u - (\eta+\tau))\exp\{N(\tfrac{1}{4}u^4 - \xi u^2 - (\eta+\tau)u)\}\,du = 0,$$

whence $$[\mathfrak{A}_0^*, \mathfrak{A}_1^*, \mathfrak{A}_3^*] = 2\xi[\mathfrak{A}_0^*, \mathfrak{A}_1^*, \mathfrak{A}_1^*] - (\eta+\tau)[\mathfrak{A}_0^*, \mathfrak{A}_1^*, \mathfrak{A}_0^*] = 0,$$

because again two columns are identical. Thus $(\partial/\partial\tau)\Delta^* = 0$; this concludes the proof of Lemma 5.

LEMMA 6. $$\Delta(N^{\frac{1}{2}}\xi, N^{\frac{3}{4}}(\eta+\tau)) = \exp\{NS_1(\alpha,\beta)\}\Delta(0,0). \tag{5·22}$$

Proof. (The following method is applicable to polynomials of any degree.)

It is convenient to introduce the parameters $X = N^{\frac{1}{2}}\xi$, $Y = N^{\frac{3}{4}}(\eta+\tau)$, and the polynomial

$$F^\dagger(U, X, Y) = \tfrac{1}{4}U^4 - XU^2 - YU.$$

As in (3·11) we write

$$\mathfrak{A}_m(X, Y) = \int (-U)^m \exp(\tfrac{1}{4}U^4 - XU^2 - YU)\,dU.$$

The zeros of $U^3 - 2XU - Y$ are denoted by $U_{\mathrm{I}}(X, Y)$, $U_{\mathrm{II}}(X, Y)$, $U_{\mathrm{III}}(X, Y)$, and we write

$$S_1^\dagger(X, Y) = F^\dagger(U_{\mathrm{I}}) + F^\dagger(U_{\mathrm{II}}) + F^\dagger(U_{\mathrm{III}}).$$

It can be shown, as in Lemma 4, that

$$S_1^\dagger(X, Y) = S_1^\dagger(X, 0); \tag{5·23}$$

as in Lemma 2, that $$\delta(X, Y) = \Delta(X, Y) \tag{5·24}$$

where $$\delta(X, Y) = [\mathfrak{A}_0(\mathrm{I}), \mathfrak{A}_1(\mathrm{II}), \mathfrak{A}_2(\mathrm{III})];$$

and as in Lemma 5 that $$\Delta(X, Y) = \Delta(X, 0). \tag{5·25}$$

Let X be kept fixed. As $Y \to \infty$, each element of $\delta(X, Y)$ can be evaluated asymptotically by the ordinary method of steepest descents. It is sufficient to take Y real and positive.

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Consider, e.g. $\mathfrak{A}_0(I, X, Y)$. Put $U = Y^{\frac{1}{3}}v$, then

$$\mathfrak{A}_0(I, X, Y) = Y^{\frac{1}{3}}\int \exp\{Y^{\frac{4}{3}}(\tfrac{1}{4}v^4 - (X/Y^{\frac{2}{3}})\,v^2 - v)\}dv$$
$$\sim \exp\{F^\dagger(U_{\mathrm{I}}(X, Y))\}\,a_0^\dagger(I, X/Y^{\frac{2}{3}})\,Y^{-\frac{1}{3}}$$

where it is not difficult to see that

$$a_0^\dagger\left(I, \frac{X}{Y^{\frac{2}{3}}}\right) \to a_0^\dagger(I, 0) \neq 0 \quad \text{when} \quad \frac{X}{Y^{\frac{2}{3}}} \to 0.$$

Similarly $$\mathfrak{A}_0(I, 0, Y) \sim \exp\{F^\dagger(U_{\mathrm{I}}(0, Y))\}\,a_0^\dagger(I, 0)\,Y^{-\frac{1}{3}}.$$

On comparing these expressions it is seen that

$$\delta(X, Y)\exp(-S^\dagger(X, Y)) \sim \delta(0, Y)\exp(-S^\dagger(0, Y))$$

as $Y \to +\infty$. From (5·23), (5·24) and (5·25) it is seen that each factor is independent of Y. It follows that

$$\Delta(X, Y)\exp(-S^\dagger(X, Y)) = \Delta(0, Y)\exp(-S^\dagger(0, Y))$$

and since each factor is now defined for all Y we can put $Y = 0$. Also $S^\dagger(0, 0) = 0$. It follows that

$$\Delta(X, Y) = \exp(S^\dagger(X, Y))\,\Delta(0, 0).$$

This is equivalent to (5·22), and concludes the proof of Lemma 6. It follows from Lemma 2 and 3 that, e.g.

$$|\Delta_0^*(N; \alpha, \beta, \tau)|\,.\,N^{\frac{1}{4}} \leqslant A\,|\exp\{NS_1(\alpha, \beta)\}| \tag{5·26}$$

when $|\tau| = r_1^3$. We can now obtain bounds for $\Delta_0(N; \alpha, \beta) = \Delta_0^*(N; \alpha, \beta, 0)$, and similarly for Δ_1 and Δ_2.

Lemma 7.
$$|\Delta_0(N; \alpha, \beta)|\,.\,N^{\frac{1}{4}} \leqslant A\,|\exp\{NS_1(\alpha, \beta)\}|, \tag{5·27}$$
$$|\Delta_1(N; \alpha, \beta)|\,.\,N^{\frac{1}{2}} \leqslant A\,|\exp\{NS_1(\alpha, \beta)\}|, \tag{5·28}$$
$$|\Delta_2(N; \alpha, \beta)|\,.\,N^{\frac{3}{4}} \leqslant A\,|\exp\{NS_1(\alpha, \beta)\}|. \tag{5·29}$$

Proof of Lemma 7. The determinants $\Delta_0^*(\tau)$, $\Delta_1^*(\tau)$, $\Delta_2^*(\tau)$ are regular functions of τ since all their elements have this property. (Note once again that $f^*(1, \tau)$ is regular for all finite τ whereas $f^*(\mathrm{I}, \tau)$ is discontinuous across Stokes lines.) Thus, by the maximum-modulus principle ((4), chapter 5),

$$|\Delta_0(N; \alpha, \beta)| = |\Delta_0^*(N; \alpha, \beta, 0)|$$
$$\leqslant \max_{|\tau| = r_1^3} |\Delta_0^*(N; \alpha, \beta, \tau)|, \tag{5·30}$$

and so (5·27) follows from (5·26). The inequalities (5·28) and (5·29) are proved in the same way. This concludes the proof of Lemma 7 for integrals with infinite limits.

We are now in a position to proceed to the

Proof of Theorem 4. To show that, e.g. $P(N; \alpha, \beta)$ is bounded, combine equation (5·27) with the result of Lemma 6. Similarly it can be shown that $Q(N; \alpha, \beta)$ and

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$R(N;\alpha,\beta)$ are bounded. This concludes the proof of Theorem 4 for integrals with infinite limits.

Proof of Theorem 3.

To prove that
$$\left|P(N;\alpha,\beta)-\sum_{k=0}^{n}\frac{P_k(\alpha,\beta)}{N^k}\right|<\frac{M_{n+1}}{N^{n+1}}, \tag{5·31}$$

consider $\Delta_0(N;\alpha,\beta)$. Let the expressions (3·5) to (3·9) be substituted for the elements of the first column of Δ_0. We obtain

$$\Delta_0(N;\alpha,\beta)=\sum_{k=0}^{n}\frac{P_k(\alpha,\beta)}{N^k}[\mathfrak{A}_0(1),\,\mathfrak{A}_1(2),\,\mathfrak{A}_2(3)].N^{-\frac{1}{4}} \tag{5·32}$$

$$+\sum_{k=0}^{n}\frac{Q_k(\alpha,\beta)}{N^k}[\mathfrak{A}_1(1),\,\mathfrak{A}_1(2),\,\mathfrak{A}_2(3)].N^{-\frac{1}{2}} \tag{5·33}$$

$$+\sum_{k=0}^{n}\frac{R_k(\alpha,\beta)}{N^k}[\mathfrak{A}_2(1),\,\mathfrak{A}_1(2),\,\mathfrak{A}_2(3)].N^{-\frac{3}{4}} \tag{5·34}$$

$$+\frac{1}{N^{n+1}}\Delta_0(N;n+1;\alpha,\beta), \tag{5·35}$$

where $\Delta_0(N;n+1;\alpha,\beta)$ is defined like $\Delta_0(N;\alpha,\beta)$, except that G_{n+1} takes the place of G_0. We now observe that the determinants in (5·33) and (5·34) vanish since they have two identical columns. On dividing by $\Delta(N^{\frac{1}{4}}\xi, N^{\frac{1}{4}}\eta)$ we obtain

$$P(N;\alpha,\beta)-\sum_{k=0}^{n}\frac{P_k(\alpha,\beta)}{N^k}=\frac{1}{N^{n+1}}P(N;n+1;\alpha,\beta), \tag{5·36}$$

where $P(N;n+1;\alpha,\beta)=N^{\frac{1}{4}}\Delta_0(N;n+1;\alpha,\beta)/\Delta$ is bounded, by Theorem 4. And equation (5·15) shows that we may take $M_{n+1}=\text{const.}\max|G_{n+1}|$. Similar arguments are applicable to $Q(N;\alpha,\beta)$ and $R(N;\alpha,\beta)$. This concludes the proof of Theorem 3 for integrals with infinite limits.

Proof of Theorem 2.

Since from equation (4·3), we have

$$f(1,N;\alpha,\beta)=N^{-\frac{1}{4}}\mathfrak{A}_0(1)\,P(N;\alpha,\beta)-N^{-\frac{1}{2}}\mathfrak{A}_1(1)\,Q(N;\alpha,\beta)+N^{-\frac{3}{4}}\mathfrak{A}_2(1)\,R(N;\alpha,\beta),$$

and since, from equation (5·1), we have $|P|\leqslant A$, $|Q|\leqslant A$, $|R|\leqslant A$, it follows that

$$|f(1)|\leqslant A\{N^{-\frac{1}{4}}|\mathfrak{A}_0(1)|+N^{-\frac{1}{2}}|\mathfrak{A}_1(1)|+N^{-\frac{3}{4}}|\mathfrak{A}_2(1)|\}.$$

Equation (3·12) is obtained in the same way, after replacing G_0 by G_{n+1}. This concludes the proof of Theorem 2 for integrals with infinite limits.

6. *Integrals with finite limits.* It has hitherto been supposed that $G(u,\alpha,\beta)$ is defined in the whole of the u-plane, and that all the limits of integration are infinite. In general this will not be the case. Suppose now that $G(u,\alpha,\beta)$ is defined only when $|u|\leqslant r$. We shall see that our definitions and arguments can be modified to deal with this case.

The path of integration of f is now restricted to the circle $|u|\leqslant r$. We consider a

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specific case which is however typical. Let us take four fixed points (the *terminal points*) on the circle $|u| = r$, one in each valley, e.g. the points

$$u = r\exp(\tfrac{1}{4}\pi i), \quad r\exp(\tfrac{3}{4}\pi i), \quad r\exp(\tfrac{5}{4}\pi i), \quad r\exp(\tfrac{7}{4}\pi i).$$

The path $C(1)$ defined in section 4 joins the points at ∞ in two of the valleys, let the path $c(1')$ be defined as a path joining the terminal points in the same two valleys; let $c(2')$, $c(3')$ be defined similarly. Let $f(1', N; \alpha, \beta)$ be defined as the integral (4·1) along the finite path $c(1')$, let $f(2'), f(3')$ be defined similarly. Let $\Delta_0, \Delta_1, \Delta_2$ be redefined, with $f(1'), f(2'), f(3')$ replacing $f(1), f(2), f(3)$ in the first column. (The elements $\mathfrak{A}_m$ remain unchanged). Similarly let $\Delta_0^*, \Delta_1^*, \Delta_2^*$ be redefined. The identities (4·3) then remain valid, with $f(1'), f(2'), f(3')$ replacing $f(1), f(2), f(3)$ on the left-hand side. Lemma 1 is unaltered. In Lemma 2 the integrals $\mathfrak{A}_m^*$ continue to be taken along the paths $C^*(\mathrm{I}, \tau)$, $C^*(\mathrm{II}, \tau)$, $C^*(\mathrm{III}, \tau)$, but the integrals f^* are now taken along modified paths $c^*(\mathrm{I}, \tau)$, $c^*(\mathrm{II}, \tau)$, $c^*(\mathrm{III}, \tau)$ between the corresponding terminal points. The paths c^* evidently cannot be paths of steepest descent along their whole length, since the terminal points do not in general lie on the paths of steepest descent from the saddle-points. We must therefore re-examine the role of the paths of steepest descent in the proof of Lemma 2.

It is well known that the so-called method of steepest descents does not in fact require paths to be paths of *steepest* descent. It is sufficient if they are paths of descent near the dominant relevant saddle-point, if they lie well below the saddle-point elsewhere (where they may even be paths of ascent locally), and if they satisfy convergence conditions when they extend to ∞. Paths of steepest descent have, however, the merit of being defined uniquely (except on Stokes lines). If in the proof of Lemma 2 paths of steepest descent are replaced by more general paths of descent, it is seen that the change-over from (e.g.) $C^*(\mathrm{I}, +)$ to $C^*(\mathrm{I}, -)$ in equations (5·5) to (5·7) need not take place precisely at a Stokes line. There is usually a whole region, near the Stokes line in the τ-plane, where both $C^*(\mathrm{I}, +)$ and $C^*(\mathrm{I}, -)$ are paths of descent (though not of steepest descent), and where either $C^*(\mathrm{I}, +)$ or $C^*(\mathrm{I}, -)$ is admissible. Our argument for infinite paths can be modified in this way, and remains valid for finite paths provided only that, in determinants involving f^*, the change-over from $C^*(\mathrm{I}, +)$ to $C^*(\mathrm{I}, -)$ *can be made at the same value of* τ as the change-over from $c^*(\mathrm{I}, +)$ to $c^*(\mathrm{I}, -)$. It is possible to choose the paths c^* so that they have this property, for instance, the path $c^*(\mathrm{I}, \tau)$ may be chosen to coincide with $C^*(\mathrm{I}, \tau)$ inside the circle $|u| = r$, and then to continue along the circumference of $|u| = r$ to the nearest terminal point; it is not difficult to show that this has the required property provided that first r_1 and then $D_3(\alpha, \beta)$ are chosen sufficiently small. (With this choice the change-over can take place at the Stokes line.)

Lemmas 3–7 and Theorem 4 are unchanged, but the proof of Theorem 3 now involves a finite range of integration in equations (3·5) to (3·9), and also involves integrated terms from the ends of the range of integration. The differences due to such terms are evidently exponentially small (of order e^{-Nd}, say, when $(\alpha, \beta) = (0, 0)$, and therefore by continuity of order $e^{-\frac{1}{2}Nd}$ when $D_3(\alpha, \beta)$ is made small enough). The contribution to P is therefore also exponentially small, as is shown by the following crude argument.

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The contribution to Δ_0 is at most of order $\exp(-\frac{1}{2}Nd).\exp(2NM(\alpha,\beta))$ where $M(\alpha,\beta)$ is the height of the highest saddle-point. (The second exponential factor comes from $\mathfrak{A}_1$ and $\mathfrak{A}_2$.) The contribution to P is therefore at most of order

$$\exp[-\tfrac{1}{2}Nd+2NM(\alpha,\beta)-NS_1(\alpha,\beta)];$$

this is $\exp(-\frac{1}{2}Nd)$ when $(\alpha,\beta)=(0,0)$, and therefore remains exponentially small (by continuity) when $|\alpha|+|\beta|$ is small enough. The proof of Theorem 2 is unchanged. This concludes our discussion of finite paths of integration.

7. *Discussion.* We have now studied in detail the case of three saddle-points and two parameters but it will be clear that all our arguments remain applicable to the general case of arbitrarily many saddle-points and arbitrarily many parameters. The necessary modifications are obvious and will not be discussed here.

No effort has been made to determine the largest possible domain of validity in $\boldsymbol{\alpha}$-space. The domain of validity obtained above is independent of N. In the case of two nearly coincident saddle-points this property was used (5) to extend the validity by another method, which may be applicable to the general case. A preliminary study suggests that it is Theorem 3 rather than Theorem 2 that will then be needed. This problem is not considered further here.

Bleistein's earlier treatment (1) of the present problem proceeds differently. We consider again the case of three saddle points (u_1, u_2, u_3, say) and two parameters (α,β). Bleistein treats u_1, u_2, u_3 as the parameters in Theorem 1 (but since they are evidently not independent it is not clear that Theorem 1 is applicable). The saddle-points are treated as varying in (u_1, u_2, u_3) space. It is also supposed in Bleistein's argument that one of the parameters (u_1, say) is always less than $\epsilon(N)=O(N^{-\frac{1}{4}})$. This restricts the region of validity in (α,β) space, for, since $u_1^3-2\xi u_1-\eta=0$, we must have

$$|\eta(\alpha,\beta)| \leqslant 2|\xi(\alpha,\beta)u_1|+|u_1|^3 \to 0 \quad \text{as} \quad N\to\infty.$$

Thus the region of validity shrinks to the two-dimensional surface $\eta(\alpha,\beta)=0$ as $N\to\infty$, and this is not likely to be sufficient for analytic continuation; see (5). Bleistein actually considers the case of an arbitrary number of nearly coincident saddle-points. The reduction to polynomial form used by Bleistein ((1), eqn. 4·31) rests on an unproved postulate. It is supposed that in a certain sub-region containing $q<m$ of the m nearly coincident saddle points the function $f(z,\boldsymbol{\alpha})$ can be reduced to a polynomial of degree $q+1$. This postulate is not a consequence of Levinson's theorem. The latter asserts the weaker result that the function $f(z,\boldsymbol{\alpha})$ can be reduced to a polynomial of degree $m+1$ in a region containing all the m nearly coincident saddle points.

Our asymptotic expansion (which is related to Bleistein's) involves the functions $\xi(\alpha,\beta)$, $\eta(\alpha,\beta)$ and $A(\alpha,\beta)$ which appear in the transformation (2·3). Levinson's existence proof (3) constructs these in principle, in the form of convergent power series. Thus when

$$f(z,\alpha,\beta)=a_0(\alpha,\beta)-\beta z-\alpha z^2+\tfrac{1}{4}z^4+\sum_{k=5}^{\infty}a_k(\alpha,\beta)z^k$$

it can be shown that

$$\xi=\alpha-a_5(0,0)+O(|\alpha|^2+|\beta|^2); \quad \eta=\beta+O(|\alpha|^2+|\beta|^2).$$

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The method is not very practical when more than a few terms are required. The functions ξ, η and A can also be defined without explicit reference to the transformation (2·3), but this will not be pursued further in the present paper.

In the introduction the functions $g(\ )$ and $f(\ )$ of equation (1·1) were assumed to be *analytic* functions of the parameters; analyticity is actually used only in Levinson's proof of Theorem 1. In the remaining theorems continuity is sufficient, as can easily be seen.

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