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Vertices of Mather's beta function, II

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Abstract. If the β-function of a time-periodic Lagrangian on a manifold M has a vertex at a k-irrational homology class h, then $2k < \dim M$, so if $\dim M = 2$, then h is rational.

1. *Introduction*

This paper addresses the problem of the differentiability of Mather's β-function for time-periodic Lagrangian systems. The setting is the dynamics of time-periodic Lagrangian systems as introduced by Mather in [**Mr91**]. In the following, M is a closed, connected manifold. A Tonelli Lagrangian on M is a C^2 function on $TM \times \mathbb{T}$, where $\mathbb{T}$ is the unit circle, satisfying the following conditions:

(1) for every $(x, t) \in M \times \mathbb{T}$, the function $v \mapsto L(x, v, t)$ is superlinear;
(2) for every $(x, v, t) \in TM \times \mathbb{T}$, the bilinear form $\partial^2 L(x, v, t)/\partial v^2$ is positive definite;
(3) the local flow Φ_t defined on $TM \times \mathbb{T}$ by the Euler–Lagrange equation for extremals of the action of curves is complete.

A good example to keep in mind is the sum of a Riemann metric, viewed as a quadratic function on TM, and a time-periodic potential (a function on $M \times \mathbb{T}$). See [**F, Be08**] for further background and references.

Define $\mathcal{M}_{\text{inv}}$ to be the set of Φ_t-invariant, compactly supported, Borel probability measures on $TM \times \mathbb{T}$. Mather showed that the function (called the action of the Lagrangian on measures)

$$\begin{aligned} \mathcal{M}_{\text{inv}} &\longrightarrow \mathbb{R} \\ \mu &\longmapsto \int_{TM\times\mathbb{T}} L \, d\mu \end{aligned}$$

is well defined and has a minimum. A measure achieving this minimum is called L-minimizing.

When $M = \mathbb{T}$, by Mather's graph theorem [**Mr91**] an invariant measure has a rotation number just like an invariant measure of a circle homeomorphism. For other manifolds Mather proposed in [**Mr91**] the following generalization. First he observed that if ω is a closed 1-form on M and $\mu \in \mathcal{M}_{\text{inv}}$, then the integral $\int_{TM\times\mathbb{T}} \omega \, d\mu$ is well defined, and

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only depends on the cohomology class of ω. By duality this defines a homology class $[\mu]$. This $[\mu]$ is the unique $h \in H_1(M, \mathbb{R})$ such that

$$\langle h, [\omega] \rangle = \int_{TM \times \mathbb{T}} \omega \, d\mu$$

for any closed 1-form ω on M. As Mather showed in [**Mr91**], for any $h \in H_1(M, \mathbb{R})$, the set

$$\mathcal{M}_{h,\mathrm{inv}} := \{\mu \in \mathcal{M}_{\mathrm{inv}} \mid [\mu] = h\}$$

is not empty. Again the action of the Lagrangian on this smaller set of measures has a minimum, which is a function of h, called the β-function of the system:

$$\begin{aligned} \beta\colon\ H_1(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\ h &\longmapsto \min\left\{ \int_{TM \times \mathbb{T}} L \, d\mu \;\middle|\; \mu \in \mathcal{M}_{h,\mathrm{inv}} \right\}. \end{aligned}$$

A measure μ such that $\int L \, d\mu = \beta([\mu])$ is called (L, h)-minimizing.

There is a dual construction to that of β: if ω is a closed 1-form on M, then $L - \omega$ is a Lagrangian to which Mather's theory applies, and furthermore $L - \omega$ has the same Euler–Lagrange flow as L. The minimum over $\mathcal{M}_{\mathrm{inv}}$ of $\int (L - \omega) \, d\mu$ is actually a function of the cohomology class of ω, the opposite of which is called the α-function of the system:

$$\begin{aligned} \alpha\colon\ H^1(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\ c &\longmapsto -\min\left\{ \int_{TM \times \mathbb{T}} (L - \omega) \, d\mu \;\middle|\; \mu \in \mathcal{M}_{\mathrm{inv}}, [\omega] = c \right\}. \end{aligned}$$

An $(L - \omega)$-minimizing measure is also called (L, ω)-minimizing or (L, c)-minimizing if c is the cohomology of ω.

Mather proved that α and β are convex, superlinear and Fenchel dual of one another, that is,

$$\begin{aligned} \text{for all } h \in H_1(M, \mathbb{R}), \quad \beta(h) &= \sup_{c \in H^1(M,\mathbb{R})} (\langle c, h \rangle - \alpha(c)), \\ \text{for all } c \in H^1(M, \mathbb{R}), \quad \alpha(c) &= \sup_{h \in H_1(M,\mathbb{R})} (\langle c, h \rangle - \beta(h)). \end{aligned}$$

In particular, $\min \alpha = -\beta(0)$. The main geometric features of a convex function are its smoothness and strict convexity, or lack thereof. In the present setting they turn out to have interesting dynamical meanings as well. The prototype of all theorems in this subject is as follows.

THEOREM 1. [**Mr90**] *If $M = \mathbb{T}$, then β is differentiable at every irrational homology class. It is differentiable at a rational homology class if and only if periodic orbits in this class fill up $\mathbb{T}$.*

Since $H_1(\mathbb{T}, \mathbb{R}) = \mathbb{R}$ the word rational is self-explanatory. For other manifolds we need a bit of terminology. The torsion-free part of $H_1(M, \mathbb{Z})$ embeds as a lattice Γ in $H_1(M, \mathbb{R})$. A class $h \in H_1(M, \mathbb{R})$ is called integer if it lies in Γ, and rational if $nh \in \Gamma$ for some non-zero integer n. A subspace of $H_1(M, \mathbb{R})$ is called integer if it is generated by integer classes.

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A convex function has a tangent cone at every point. We say it has a vertex at x if its tangent cone at x contains no straight line. An enticing question, suggested by Mather's theorem, is whether or not vertices of β only occur at rational homology classes. As yet the best known result is as follows.

THEOREM 2. [**O05**] *If the β-function of a Lagrangian L has a vertex at a homology class h, then either h is rational or the support of every (L, h)-minimizing measure has Hausdorff dimension at least three.*

Note that this contains Theorem 1 since by Mather's graph theorem [**Mr91**] the support of a minimizing measure may be viewed as a subset of $M \times \mathbb{T}$; and if $M = \mathbb{T}$, then $M \times \mathbb{T}$ cannot contain a subset of dimension three.

We need to give a quantitative meaning to the irrationality of a homology class. The quotient $H_1(M, \mathbb{R})/\Gamma$ is a torus $\mathbb{T}^b$, where b is the first Betti number of M. For h in $H_1(M, \mathbb{R})$, the image of $\mathbb{Z}h$ in $\mathbb{T}^b$ is a subgroup of $\mathbb{T}^b$, hence its closure $\mathcal{T}(h)$ is a finite union of tori of equal dimension. This dimension is called the irrationality $I_{\mathbb{Z}}(h)$ of h. It is zero if h is rational. We say a class h is completely irrational if its irrationality is maximal, i.e. is equal to b. In the same way, if v is a vector of $\mathbb{R}^n$, we call the irrationality of v the dimension of the closure of the image of $\mathbb{Z}v$ in $\mathbb{R}^n/\mathbb{Z}^n$. Note that the irrationality of h is equal to that of nh for $n \in \mathbb{Z}$, $n \neq 0$ since the quotient of $\mathcal{T}(h)$ by $\mathcal{T}(nh)$ is a group of cardinality n.

Our main contribution is the following.

THEOREM 3. *If the β-function of a Lagrangian L on a manifold M has a vertex at a homology class h and p is the irrationality of h, then $2p < \dim M$. If, in particular,* $\dim M = 2$*, then h is rational.*

We say that β is differentiable in k directions at h if the tangent cone to β at h contains a linear space of dimension k. This thus leads us to ask whether β is always differentiable in k directions at a k-irrational homology class. Mather conjectures that it is true for C^∞ Lagrangians. The conjecture is verified for all C^2 Lagrangians in the case where $M = \mathbb{T}$ by Theorem 1.

For autonomous Lagrangians the relevant notion of irrationality is slightly different. Let $I_{\mathbb{R}}(h)$ be the dimension of the closure of the image in $\mathbb{T}^b$ of $\mathbb{R}h$ instead of $\mathbb{Z}h$. For further information regarding the relationship between the two notions of irrationality, see Appendix A.1. Note that the function $I_{\mathbb{R}}(h)$ is zero-homogeneous, that is, $I_{\mathbb{R}}(th) = I_{\mathbb{R}}(h)$ for all $h \in H_1(M, \mathbb{R})$ and $t \neq 0$.

In this context the problem is to show that β is always differentiable in $I_{\mathbb{R}}(h)$ directions at any homology class h. This cannot be true in full generality by [**BIK97**], but it is true for Finsler metrics on a compact, orientable surface, see [**Ba94**] for $M = \mathbb{T}^2$ and [**Mt03**, Corollary 3] for other genera. In [**AB06**] this result is generalized to stable norms in codimension one.

Note that for autonomous Lagrangians, by [**C95**], β is always differentiable in the radial direction, that is, for a given $h \in H_1(M, \mathbb{R}) \setminus \{0\}$, the function

$$\begin{aligned}]0, +\infty[&\longrightarrow]0, +\infty[\\ t &\mapsto \beta(th) \end{aligned}$$

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is C^1. So the β-function of an autonomous Lagrangian cannot have vertices except at the zero class. The closest thing to a vertex is a tangent cone which contains no plane, that is, β is differentiable in no direction other than radial. So far the only positive result for manifolds of dimension greater than two is as follows.

THEOREM 4. [**BIK97**] *The β-function of a Finsler metric on a manifold M is always differentiable in at least one non-radial direction at a completely irrational homology class.*

Note that this theorem contains the result of [**Ba94**] because on a two-torus for any non-zero homology class h we have either $I_{\mathbb{R}}(h) = 1$, in which case β is differentiable in at least one direction, namely the radial, or $I(h)_{\mathbb{R}} = 2$, that is, h is completely irrational.

We call an energy level E supercritical if $E > \min \alpha = -\beta(0)$. We extend Theorem 4 as follows.

THEOREM 5. *If the β function of an autonomous Lagrangian on a manifold M is not differentiable in any direction other than radial at h, and the energy level that contains the supports of the (L, h)-minimizing measures is supercritical, then $2I_{\mathbb{R}}(h) - 1 < \dim M$. In particular, if the dimension of M is three, then $I_{\mathbb{R}}(h) \leq 1$.*

Note that this theorem contains Theorem 4 because if the Lagrangian is a Finsler metric the critical energy level is the zero level, and it contains only fixed points, whose homology class is zero.

The paper is organized as follows. In §2.3 we introduce the main tool of this paper, which is the fibration of M over the circle defined by an integer 1-form. In §3 we prove Theorem 3 . In §4 we give another application of our main tool to the differentiability problem. In §5 we prove Theorem 5.

2. *Preliminaries*

In this section we briefly recall a few definitions, referring the reader to [**F**] (and [**Be08**] for the time-periodic case) for more information. Let L be a Tonelli Lagrangian on a closed manifold M.

2.1. *The minimal action functional.*

Since the α-function of L is convex, at every point its graph has a supporting hyperplane. We call a face of α the projection to $H^1(M, \mathbb{R})$ of the intersection of the graph of α with one of its supporting hyperplane. By Fenchel (or 'convex') duality it is equivalent to study the differentiability of β or to study the faces of α. If c is a cohomology class, we call F_c the largest face of α containing c in its relative interior. See [**Mt97**] for more information about faces. We denote by V_c the underlying vector space of the affine space that F_c generates in $H^1(M, \mathbb{R})$. We call $\tilde{V}_c$ the vector subspace of $H^1(M, \mathbb{R}) \times \mathbb{R}$ generated by pairs $(c' - c, \alpha(c) - \alpha(c'))$ where $c' \in F_c$. Beware of the sign of $\alpha(c) - \alpha(c')$. Note that $\tilde{V}_c$ is a linear graph above V_c. We have

$$\tilde{V}_c = \{\lambda(c' - c, \alpha(c) - \alpha(c')) \mid \lambda \in \mathbb{R},\ c' \in F_c\}$$

since c lies in the interior of F_c, F_c is convex, and α is affine on F_c.

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Replacing, if necessary, L by $L - \omega$ where $[\omega] = c$, we only need to consider the case when $c = 0$. Likewise, replacing L with $L - \alpha(0)$ we assume throughout the paper that $\alpha(0) = 0$.

2.2. *Weak KAM preliminaries.* Define, for all $n \in \mathbb{N}$,

$$\begin{aligned} h_n\colon\ (M \times \mathbb{T}) \times (M \times \mathbb{T}) &\longrightarrow \mathbb{R} \\ ((x, t), (y, s)) &\longmapsto \min \int_t^{s+n} L(\gamma, \dot\gamma, t)\, dt \end{aligned}$$

where the minimum is taken over all absolutely continuous curves $\gamma\colon\ [t, s + n] \longrightarrow M$ such that $\gamma(t) = x$ and $\gamma(s + n) = y$. Note that this is a slight abuse of notation, since we denote by the same t an element of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or the corresponding point in $[0, 1[$. The Peierls barrier is then defined as

$$\begin{aligned} h\colon\ (M \times \mathbb{T}) \times (M \times \mathbb{T}) &\longrightarrow \mathbb{R} \\ ((x, t), (y, s)) &\longmapsto \liminf_{n\to\infty} h_n((x, t), (y, s)). \end{aligned}$$

The Aubry set is

$$\mathcal{A}_0 := \{(x, t) \in M \times \mathbb{T} \mid h((x, t), (x, t)) = 0\}.$$

2.3. *Main tool.* Let c be an element of F_0, the largest face of α containing zero in its interior. Let ω be a smooth 1-form on M such that $[\omega] = c$. To emphasize the dependence on the Lagrangian we write $h(L)$ or $h(L - \omega)$ the corresponding Peierls barriers. For brevity we use the notation

$$\begin{aligned} u_1(x, t) &:= h(L - \omega)((x_0, 0), (x, t)), \\ u_0(x, t) &:= h(L)((x_0, 0), (x, t)). \end{aligned}$$

Call $p\colon\ \overline{M} \longrightarrow M$ the Abelian cover of M, that is, the covering whose group is Γ. Fix once and for all an origin $(x_0, 0) \in p^{-1}(\mathcal{A}_0 \cap \{t = 0\})$. Consider the map

$$\begin{aligned} \phi_\omega\colon\ \overline{M} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, t) &\longmapsto \int_{(x_0,0)}^{(x,t)} [\omega - \alpha(c)\, dt] - u_0 \circ p(x, t) + u_1 \circ p(x, t) \end{aligned}$$

where the integral is taken along any path in $\overline{M} \times \mathbb{R}$ joining $(x_0, 0)$ to (x, t).

The key argument of this paper is as follows.

PROPOSITION 6. *The map ϕ_ω satisfies a Hölder condition of order two along $p^{-1}(\mathcal{A}_0)$.*

Proof. Since $c \in F_0$, we have $\mathcal{A}_0(L - \omega) = \mathcal{A}_0(L)$ by Lemma A.4. In addition, the maps

$$\begin{aligned} M \times \mathbb{T} &\longrightarrow \mathbb{R} \\ (x, t) &\longmapsto h(L - \omega)((x_0, 0), (x, t)) \\ (x, t) &\longmapsto h(L)((x_0, 0), (x, t)) \end{aligned}$$

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are weak KAM solutions, so at every point (x, t) of $\mathcal{A}_0$ there exists a $v \in T_x M$ such that

$$\frac{\partial}{\partial x} h(L-\omega)((x_0, 0), (x, t)) = \frac{\partial L}{\partial v}(x, v, t) - \omega_x,$$
$$\frac{\partial}{\partial x} h(L)((x_0, 0), (x, t)) = \frac{\partial L}{\partial v}(x, v, t).$$

Furthermore (see [**Mt07**, 4.3])

$$\frac{\partial}{\partial t} h(L)((x_0, 0), (x, t)) - \frac{\partial}{\partial t} h(L-\omega)((x_0, 0), (x, t)) = -\alpha(c).$$

Let (U, f) be a coordinate chart for $M \times \mathbb{T}$. We have the following result (see [**F**, 4.5.5] for the autonomous case).

LEMMA 7. *There exists K such that for any weak KAM solution u for L, for any X in U such that $f(X) \in \mathcal{A}_0$, for any Y in U, we have*

$$|u(f(Y)) - u(f(X)) - d_X u \circ f(Y - X)| \leq K \|Y - X\|^2.$$

Proof. Take X, Y in U such that $f(X) \in \mathcal{A}_0$. Take a weak KAM solution u. If u is a backward weak KAM solution, we rename it u_-, and let u_+ be its conjugate, so u_+ is a forward weak KAM solution for L. If u is a forward weak KAM solution, we rename it u_+, and let u_- be its conjugate, so u_- is a backward weak KAM solution for L. By [**Be08**] there exists a real number K such that all backward (respectively forward) weak KAM solutions are K-semi-concave (respectively semi-convex). Since $f(X) \in \mathcal{A}_0$, u_- and u_+ are differentiable, and have the same derivative, at $f(X)$. So by semi-concavity (respectively semi-convexity)

$$u_-(f(Y)) - u_-(f(X)) \leq d_X u_- \circ f(Y - X) + K\|Y - X\|^2,$$
$$u_+(f(Y)) - u_+(f(X)) \geq d_X u_+ \circ f(Y - X) + K\|Y - X\|^2.$$

On the other hand, with u_- and u_+ being weak KAM conjugate, $u_-(f(X)) = u_+(f(X))$ because $f(X) \in \mathcal{A}_0$, and $u_-(f(Y)) \geq u_+(f((Y))$ so

$$u_-(f(Y)) - u_-(f(X)) \geq u_+(f(Y)) - u_+(f(X))$$

which combines with the former two inequalities to prove the lemma. □

Now let us finish the proof of Proposition 6. By our previous lemma there exist K, K' such that, for any X, Y with $f(X)$, $f(Y) \in \mathcal{A}_0$, we have

$$|u_1(f(Y)) - u_1(f(X)) - d_X u_1 \circ f(Y - X)| \leq K\|Y - X\|^2,$$
$$|u_0(f(Y)) - u_0(f(X)) - d_X u_0 \circ f(Y - X)| \leq K'\|Y - X\|^2,$$

where, by what we have obtained above,

$$d_{f(X)} u_1 = \left(\frac{\partial L}{\partial v}(X, v, t)\right) dx - \omega_{f(X)} + \left(\frac{\partial}{\partial t} h(L)((x_0, 0), (x, t)) + \alpha(c)\right) dt,$$
$$d_{f(X)} u_0 = \left(\frac{\partial L}{\partial v}(X, v, t)\right) dx + \left(\frac{\partial}{\partial t} h(L)((x_0, 0), (x, t))\right) dt,$$

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thus by subtraction

$$|u_1(f(Y)) - u_0(f(Y)) - u_1(f(X)) + u_0(f(X)) + (\omega_{f(X)} - \alpha(c)\,dt)\cdot d_X f(Y-X)| \\ \leq (K+K')\|Y-X\|^2.$$

Let us take lifts of $f(X)$, $f(Y)$ to $\overline{M}\times\mathbb{R}$, and denote them by $f(X)$, $f(Y)$ to avoid heavy notation. Since ω is smooth, by Taylor's formula we have for some K'':

$$\left|\int_{f(X)}^{f(Y)} [\omega_{f(X)} - \alpha(c)\,dt]\cdot d_X f(Y-X)\right| \leq K''\|Y-X\|^2,$$

thus

$$|\phi_\omega(f(Y)) - \phi_\omega(f(X))| \leq (K+K'+K'')\|Y-X\|^2$$

which proves the proposition. □

We obtain lower bounds on the Hausdorff dimension of the Aubry set by combining Proposition 6 with [**FFR**, Lemma A.1], which is a variation of a lemma of Ferry [**F75**].

LEMMA 8. (Ferry) *Let Φ be a map from a subset E of $\mathbb{R}^n$ to a metric space (X,d). Suppose that there exist $p>1$ and C such that*

$$\text{for all } x,\, y\in E, \quad d(\Phi(x),\Phi(y)) \leq C\|x-y\|^p.$$

Then the n/p-dimensional Hausdorff measure of $(\Phi(E),d)$ is zero.

3. *Proof of Theorem 3*

LEMMA 9. *Let L be a Tonelli Lagrangian on a closed manifold M. Assume that the vector space V_0 contains an integer subspace V'. Take cohomology classes $c_1,\dots,c_k$ in the face F_0 and $\omega_1,\dots,\omega_k$ smooth closed 1-forms on M such that:*

- *$[\omega_i]=c_i$, $i=1,\dots,k$;*
- *for all $i=1,\dots,k$, there exists $n_i\in\mathbb{Z}$, $n_i c_i \in H^1(M,\mathbb{Z})$;*
- *the classes c_i, $i=1,\dots,k$ form a basis of V' as a vector space.*

Then if the vector $(\alpha(c_i))_{i=1,\dots,k}$ is p-irrational, we have $\dim M > 2p$.

Proof. Since $n_i c_i$ is integer the restriction to $\overline{M}\times\{0\}$ of the map ϕ_{ω_i} defined in §2.3 is invariant under Γ, modulo $n_i^{-1}\mathbb{Z}$. We denote by Φ_i the quotient map, so we have

$$\begin{aligned}\Phi_i:\ M &\longrightarrow \mathbb{R}\Big/\frac{1}{n_i}\mathbb{Z}\\ x &\longmapsto \int_{x_0}^{x}\omega_i - u_0(x) + u_{1,i}(x).\end{aligned}$$

where the integral is taken along any path in M joining x_0 to x, and

$$u_{1,i}(x) := h(L-\omega_i)((x_0,0),(x,0)),$$
$$u_0(x) := h(L)((x_0,0),(x,0)).$$

Now define

$$\begin{aligned}\Phi:\ M &\longrightarrow \mathbb{T}^k = \bigoplus_{i=1}^{k}\mathbb{R}\Big/\frac{1}{n_i}\mathbb{Z}\\ x &\longmapsto (\Phi_i)_{i=1,\dots,k}.\end{aligned}$$

By Proposition 6, Φ satisfies a Hölder condition of order two along $\mathcal{A}_0\cap\{t=0\}$.

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Take $\gamma : \mathbb{R} \longrightarrow M$ an extremal such that $(\gamma(t), \dot{\gamma}(t), t)$ is contained in $\mathcal{A}_0$ for all t. We have, for all n in $\mathbb{Z}$,

$$\int_{t=0}^{n} (L - \omega_i + \alpha(c_i))(\gamma(t), \dot{\gamma}(t), t)\, dt = u_{1,i}(\gamma(n)) - u_{1,i}(\gamma(0)),$$
$$\int_{t=0}^{n} (L)(\gamma(t), \dot{\gamma}(t), t)\, dt = u_0(\gamma(n)) - u_0(\gamma(0)).$$

Taking the difference of the last two equations we obtain

$$\Phi(\gamma(n)) - \Phi(\gamma(0)) = n\left(\alpha(c_i) \bmod \frac{1}{n_i}\mathbb{Z}\right)_{i=1,\dots,k}.$$

Now if the vector $(\alpha(c_i))_{i=1,\dots,k}$ is p-irrational the set

$$\left\{ n\left(\alpha(c_i) \bmod \frac{1}{n_i}\mathbb{Z}\right)_{i=1,\dots,k} \;\middle|\; n \in \mathbb{Z} \right\}$$

is dense in a subtorus of dimension p. In particular, the p-dimensional Hausdorff measure of $\Phi(\mathcal{A}_0 \cap \{t = 0\})$ is positive. Since Φ is 2-Hölder, by Ferry's Lemma 8 we have $\dim M > 2p$. □

An obvious consequence of Lemma 9 is the following.

COROLLARY 10. *Let L be a Tonelli Lagrangian on a closed manifold of dimension two. If V_0 is integer, then so is $\tilde{V}_0$.*

3.1. *Vertices.* Suppose that β has a vertex at some homology class h. Then there exists a cohomology class c such that c is a subderivative to β at h and $V_c = H^1(M, \mathbb{R})$. In particular, V_c is an integer subspace so this situation is contained in the case we have just considered. As already explained, by subtracting a closed 1-form from L we can assume $c = 0$ and $\alpha(c) = 0$. Then we have $V_0 = H^1(M, \mathbb{R})$ so we may take n_i, c_i, $i = 1, \dots, b$ as in Lemma 9, with the additional requirement that $n_i c_i$, $i = 1, \dots, b$ is a basis of the $\mathbb{Z}$-module $H^1(M, \mathbb{Z})$. Then its dual basis is an integer basis of $H_1(M, \mathbb{R})$. The coordinates of h in the dual basis of $n_i c_i$, $i = 1, \dots, b$ are $n_i \langle c_i, h\rangle$, $i = 1, \dots, b$ so the irrationality of h is just that of the vector $\langle c_i, h\rangle$, $i = 1, \dots, b$. Pick an h-minimizing measure μ.

We have

$$\langle c_i, h\rangle = \alpha(c_i) + \beta(h),$$
$$0 = \langle 0, h\rangle = \alpha(0) + \beta(h) = \beta(h),$$

so $\langle c_i, h\rangle - \alpha(c_i) = 0$ for all $i = 1, \dots, b$. Thus, the irrationality p of h is exactly that of $(\alpha(c_i))_{i=1,\dots,k}$, which is $< \dim M/2$ by Lemma 9. This finishes the proof of Theorem 3.

4. *Integer points in $\tilde{V}_0$*

DEFINITION 11. Let $\tilde{E}_0$ be the set of $(c, \tau) \in H^1(M \times \mathbb{T}, \mathbb{R}) = H^1(M, \mathbb{R}) \times H^1(\mathbb{T}, \mathbb{R})$ such that there exists a smooth closed 1-form ω on $M \times \mathbb{T}$ with $[\omega] = (c, \tau)$ and $\operatorname{supp}(\omega) \cap \mathcal{A}_0 = \emptyset$. Let E_0 be the canonical projection of $\tilde{E}_0$ to $H^1(M, \mathbb{R})$.

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The relevance of this definition to the problem of the differentiability of Mather's β-function is given by the following theorem proved in [**Mt07**] (see [**Mt03**, Theorem 1], for the autonomous case).

THEOREM 12. *The following inclusion holds:*

$$E_0 \subset V_0.$$

In this section we look for sufficient conditions for the converse inclusion to be true. Our main result in this direction is Proposition 14.

4.1. Assume that $\tilde{V}_0$ contains an integer point, that is, a non-zero element of $H^1(M, \mathbb{Z}) \times \mathbb{Z}$.

Therefore, there exists a non-zero c in the interior of F_0 and a non-zero real number λ such that $\lambda(c, -\alpha(c)) \in H^1(M, \mathbb{Z}) \times \mathbb{Z}$. Let ω be a smooth 1-form representing c. The map ϕ_ω from $\overline{M} \times \mathbb{R}$ to $\mathbb{R}$ defined in §2.3 quotients as a map Φ_ω from $M \times \mathbb{T}$ to $\mathbb{R}/\lambda^{-1}\mathbb{Z}$.

Consider the equivalence relation on $\mathcal{A}_0$ defined by $(x, t) \approx (y, s)$ if and only if $h((x, t), (y, s)) + h((y, s), (x, t)) = 0$. The classes are called static classes. The quotient $\mathcal{A}_0/\approx$ is a metric space with distance

$$d(\overline{(x, t)}, \overline{(y, s)}) = h((x, t), (y, s)) + h((y, s), (x, t)),$$

where $\overline{(x, t)}$ is the equivalence class of (x, t). It is called a quotient Aubry set and denoted A_0 after [**Mr02**].

PROPOSITION 13. *The map Φ_ω from $M \times \mathbb{T}$ to $\mathbb{R}/\lambda^{-1}\mathbb{Z}$ is constant on the static classes of L.*

Proof. Take (x_1, t_1), (x_2, t_2) in the same static class. By Proposition A.4 we need not specify whether we mean static classes of L or $L - \omega$. So we may take curves $\gamma_n : [t_1, t_2 + k_n] \longrightarrow M$, where k_n is an integer sequence that tends to infinity, $\gamma_n(t_1) = x_1$, $\gamma_n(t_2 + k_n) = x_2$, and

$$h(L)((x_1, t_1), (x_2, t_2)) = \lim_{n\to\infty} \int_{t_1}^{t_2+k_n} L(\gamma_n, \dot{\gamma}_n, t)\, dt,$$

$$h(L - \omega)((x_1, t_1), (x_2, t_2)) = \lim_{n\to\infty} \int_{t_1}^{t_2+k_n} (L - \omega + \alpha(c))(\gamma_n, \dot{\gamma}_n, t)\, dt,$$

thus by subtraction

$$\lim_{n\to\infty} \int_{t_1}^{t_2+k_n} [\omega(\gamma_n, \dot{\gamma}_n, t) - \alpha(c)]\, dt = h(L)((x_1, t_1), (x_2, t_2)) - h(L - \omega)((x_1, t_1), (x_2, t_2)).$$

On the other hand, since (x_1, t_1) and (x_2, t_2) are in the same static class, we have (cf. [**Be08**, Lemma 4.1])

$$h(L)((x_0, 0), (x_2, t_2)) = h(L)((x_0, 0), (x_1, t_1)) + h(L)((x_1, t_1), (x_2, t_2)),$$

$$h(L - \omega)((x_0, 0), (x_2, t_2)) = h(L - \omega)((x_0, 0), (x_1, t_1)) + h(L - \omega)((x_1, t_1), (x_2, t_2)),$$

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that is,

$$h(L)((x_1, t_1), (x_2, t_2)) = u_0(x_2, t_2) - u_0(x_1, t_1),$$
$$h(L - \omega)((x_1, t_1), (x_2, t_2)) = u_1(x_2, t_2) - u_1(x_1, t_1),$$

and, finally,

$$\int_{(x_0,0)}^{(x_2,t_2)} [\omega - \alpha(c)\, dt] - u_0(x_2, t_2) + u_1(x_2, t_2)$$
$$= \int_{(x_0,0)}^{(x_1,t_1)} [\omega - \alpha(c)\, dt] - u_0(x_1, t_1) + u_1(x_1, t_1) \bmod \frac{1}{\lambda}\mathbb{Z}. \qquad \square$$

4.2. In this section we prove a partial converse to Theorem 12.

PROPOSITION 14. *If the quotient Aubry set A_0 is totally disconnected, and $\tilde{V}_0$ contains an integer point (c, τ), then*

$$(c, \tau) \in \tilde{E}_0.$$

In particular, if A_0 is totally disconnected, and $\tilde{V}_0$ is integer, then

$$E_0 = V_0.$$

REMARK 15. By [**Mn96**] for a generic Lagrangian there exists a unique minimizing measure, and in particular a unique static class, so A_0 is a point. Also, by [**Mr02**], if $\dim M = 2$, A_0 is always totally disconnected.

Proof. Take c in F_0, such that some multiple of $(c, -\alpha(c))$ is integer. Take a 1-form ω such that $[\omega] = c$ and define Φ_ω as above.

First assume that

$$\Phi_\omega(\mathcal{A}_0) \neq \mathbb{T}.$$

Since $\mathcal{A}_0$ is compact, so is $\Phi_\omega(\mathcal{A}_0)$. Therefore, $\Phi_\omega(\mathcal{A}_0)$ misses an open set V_1. Then we can find a map Φ which is C^1, homotopic to Φ_ω and misses an open set $V \subset V_1$. Take a closed 1-form η generating the integral cohomology of the circle $\mathbb{R}/\lambda^{-1}\mathbb{Z}$, supported in V. The pull-back $\Phi^*(\eta)$ of this form by Φ defines a closed 1-form in $M \times \mathbb{T}$, supported away from $\mathcal{A}_0$. Since η is cohomologous to the constant 1-form dx on the circle, we have

$$[\Phi^*(\eta)] = [\Phi^*(dx)]$$

but $\Phi^*(dx)$ is cohomologous to $\Phi_\omega^*(dx)$ since Φ is homotopic to Φ_ω. By the definition of Φ_ω, $\Phi_\omega^*(dx)$ is cohomologous to $(\omega, -\alpha(c)\, dt)$, which implies $c \in E_0$.

So what remains is to assume that Φ_ω restricted to $\mathcal{A}_0$ is surjective and use the hypothesis of total disconnectedness to modify Φ_ω. Assume, up to a translation, that $\Phi_\omega(x_0, 0) = 0$. We are looking for a map Φ which has the same effect as Φ_ω on cohomology classes and misses a neighborhood of the point $(2\lambda)^{-1}$ in the circle.

By [**HY**, Theorem 2.15], in a totally disconnected metric space, every point has a basis of open-closed neighborhoods. Thus, since the canonical projection from $\mathcal{A}_0$ to $\mathcal{A}_0/\approx$ is continuous, for every static class S contained in $\Phi_\omega^{-1}(1/2\lambda) \cap \mathcal{A}_0$ there exists an open

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(in $\mathcal{A}_0$) and closed neighborhood V_S of S such that $\Phi_\omega(V_S) \subset [1/4\lambda, 3/4\lambda]$. Cover the compact set $\Phi_\omega^{-1}(1/2\lambda) \cap \mathcal{A}_0$ with the V_S and take a finite subcover $V_1, \ldots, V_n$. Then the reunion $V_1 \cup \cdots \cup V_n$ is an open (in $\mathcal{A}_0$) and closed neighborhood of $\Phi_\omega^{-1}(1/2\lambda) \cap \mathcal{A}_0$. So its complement in $\mathcal{A}_0$ is closed. This shows that $\mathcal{A}_0$ splits as the disjoint union of two compact subsets K_1 and K_2 such that:

- $K_1 \supset \Phi_\omega^{-1}(1/2\lambda) \cap \mathcal{A}_0$;
- $\Phi_\omega(K_1) \subset [1/4\lambda, 3/4\lambda]$; and
- $1/2\lambda \notin \Phi_\omega(K_2)$.

Thus, there exists a C^1 function f on $M \times \mathbb{T}$ such that f is zero on K_2 and one on K_1.

Replace ω with $\omega - (2\lambda)^{-1}\, df$ in the definition of Φ_ω (without changing u_1), and call Φ the resulting map. Then, if $x \in K_2$, $\Phi(x) = \Phi_\omega(x) \neq 1/2\lambda$, and if $x \in K_1$,

$$\Phi(x) = \Phi_\omega(x) - \frac{1}{2\lambda} f(x) = \Phi_\omega(x) - \frac{1}{2\lambda} \neq \frac{1}{2\lambda}$$

which shows that Φ restricted to K is not surjective and brings us back to the first case. This proves Proposition 14. □

Now, putting Lemma 9, Remark 15 and Proposition 14 together, we obtain the following result.

COROLLARY 16. *If the dimension of M is two and V_0 contains an integer point c, then $c \in E_0$. In particular, if the dimension of M is two and V_0 is integer, then $E_0 = V_0$.*

5. *Proof of Theorem 5*

Let $h_0 \in H_1(M, \mathbb{R})$ be such that the tangent cone to β at h_0 contains no plane. Let $c \in H^1(M, \mathbb{R})$ be such that $\langle c, h_0 \rangle = \alpha(c) + \beta(h_0)$, that is, c defines a supporting hyperplane to β at h_0. Modifying L by a closed 1-form if $c \neq 0$, we may assume that $c = 0$. Then V_0 has codimension one or zero. We now show that the second case only happens when $h_0 = 0$. For convenience we also assume that $\alpha(0) = 0$.

LEMMA 17. *We have $\langle V_0, h_0 \rangle = 0$. Therefore, if $h_0 \neq 0$, then V_0 has codimension one and V_0 is the orthogonal of h_0.*

Proof. Since $0 \in H^1(M, \mathbb{R})$ is a subderivative to β at h_0, and L is autonomous, by [**C95**] the support of any h_0-minimizing measure is contained in the energy level $0 = \alpha(0)$. Since any $c \in F_0$ is also a subderivative to β at h_0, the support of any h-minimizing measure is contained in the energy level $\alpha(c)$ so we have

$$0 = \alpha(c) \quad \text{for all } c \in F_0.$$

In other words, the faces of α are contained in the level sets of α for an autonomous Lagrangian. In addition, since c and 0 are subderivatives to β at h_0, we have

$$\langle c, h_0 \rangle = \alpha(c) + \beta(h_0) = \beta(h_0),$$
$$\langle 0, h_0 \rangle = \alpha(0) + \beta(h_0) = \beta(h_0),$$

thus

$$\langle c, h_0 \rangle = \langle 0, h_0 \rangle = 0 \quad \text{for all } c \in F_0.$$ □

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Since $I_{\mathbb{R}}(0) = 0$, there is nothing to prove in the case $h_0 = 0$, hence by the lemma above we may assume $h_0 \neq 0$ and V_0 has codimension one.

It would help if we had an ergodic h_0-minimizing measure. Such a measure need not exist because $(h_0, \beta(h_0))$ may not be an extremal point of the epigraph of β, that is, h_0 may lie in the relative interior of a face of β. Such a face must be radial, i.e. contained in $\{th_0 : t \in [0, +\infty[\}$, because β is not differentiable at h_0 in any direction but the radial. Then the radial face of β containing h_0 has at least one non-zero extremal point th_0 with $t \in]0, +\infty[$. Furthermore if the tangent cone to β at h_0 is contains no plane, then neither does the tangent cone to β at th_0 (see [**Mt**, Lemma 17]). Since $I_{\mathbb{R}}(th_0) = I_{\mathbb{R}}(h_0)$, we may, for the purpose of proving Theorem 5, assume that h_0 itself is an extremal point of β. Then by [**Mn92**] there exists an ergodic h_0-minimizing measure μ.

Now we prove that the support of the ergodic measure μ does not meet the zero section of TM. Let c be a cohomology class such that μ is c-minimizing. By [**C95**] the support of μ is contained in the energy level $\alpha(c)$, so our hypothesis implies that $\alpha(c) > \min \alpha$.

By [**FFR**, Proposition 3.2] the intersection of the support of a minimizing measure with the zero section consists of fixed points of the Euler–Lagrange flow. So all we need to prove is that the support of μ does not contain any fixed point. Suppose that it does. The Dirac measure δ supported by a fixed point contained in the Aubry set of c is c-minimizing by [**Mn92**] (an invariant measure whose support is contained in the Aubry set is minimizing). However, the homology class of δ is zero, so we have

$$\alpha(c) + \beta(0) = \langle c, 0 \rangle = 0 \quad \text{whence } \alpha(c) = -\beta(0) = \min \alpha,$$

which is a contradiction.

Now we proceed with the proof of Theorem 5. Let $c_1, \ldots, c_{b-1}$ be a basis of V_0 and let $\omega_1, \ldots, \omega_{b-1}$ be smooth closed 1-forms on M such that $[\omega_i] = c_i$, $i = 1, \ldots, k$. Consider the functions

$$u_{1,i}(x) := h(L - \omega_i)(x_0, x),$$
$$u_0(x) := h(L)(x_0, x),$$

for $1, \ldots, b-1$, and replace each 1-form ω_i by

$$\omega_i' := \omega_i + du_{1,i} - du_0.$$

This is an almost-everywhere defined, integrable and bounded 1-form, and it vanishes identically on $\tilde{\mathcal{A}}_0$.

Complete $c_1, \ldots, c_{b-1}$ to a basis $c_1, \ldots, c_b$ of $H^1(M, \mathbb{R})$. Take ω_b to be any smooth 1-form with cohomology c_b. Pick $a_1, \ldots, a_b$ an integer basis of $H^1(M, \mathbb{R})$. Let $(\nu_{ij})_{i,j=1,\ldots,b}$ be the matrix whose ith line consists of the coordinates of a_i in the basis $c_1, \ldots, c_b$, so $a_i = \sum_{j=1}^{b} \nu_{ij} c_j$. Set, for $j = 1, \ldots, b$,

$$\eta_i := \sum_{j=1}^{b} \nu_{ij} \omega_j'.$$

This is an almost-everywhere defined, integrable and bounded 1-form with cohomology a_i.

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Now let $(\lambda_{ij})_{i,j=1,\dots,b}$ be the inverse matrix of $(v_{ij})_{i,j=1,\dots,b}$, so $c_i = \sum_{j=1}^{b} \lambda_{ij} a_j$. Let Λ be the matrix $(\lambda_{ij})_{i=1,\dots,b-1,\, j=1,\dots,b}$, that is, we just delete the last line of $(\lambda_{ij})_{i,j=1,\dots,b}$.

Endow $H_1(M, \mathbb{R})$ with the dual basis to $a_1, \dots, a_b$ and $\mathbb{R}^{b-1}$ with its canonical basis. Consider the linear map from $H_1(M, \mathbb{R})$ to $\mathbb{R}^{b-1}$ whose matrix with respect to said basis is Λ. We again call this linear map Λ for simplicity. So if $h \in H_1(M, \mathbb{R})$, we have

$$\begin{aligned}\Lambda(h) &= \left(\sum_{j=1}^{b} \lambda_{ij} \langle a_j, h\rangle\right)_{i=1,\dots,b-1} \\ &= (\langle c_i, h\rangle)_{i=1,\dots,b-1}.\end{aligned}$$

Thus, the kernel of Λ is the straight line generated by h_0.

Consider the map

$$\begin{aligned}\Phi = (\Phi_1, \dots, \Phi_b)\colon\ M &\longrightarrow \mathbb{T}^b \\ x &\longmapsto \left(\int_{x_0}^{x} \eta_i\right)_{i=1,\dots,b} \bmod \mathbb{Z}^b.\end{aligned}$$

The map Φ is Lipschitz. We assume, without loss of generality, that $\langle a_1, h_0\rangle \neq 0$. Pick an orbit $\gamma\colon\ \mathbb{R} \longrightarrow M$ contained in the support $\operatorname{supp} \mu$ of μ. We have

$$d\Phi(\dot\gamma(t)) = (\eta_i(\dot\gamma(t)))_{i=1,\dots,b} \text{ so } \Lambda(d\Phi(\dot\gamma(t))) = (\omega_i'(\dot\gamma(t)))_{i=1,\dots,b} = 0$$

and $\Phi(\gamma(\mathbb{R}))$ is a connected subset of a straight line of direction h_0 in $\mathbb{T}^b$. Note that $\Lambda \circ \Phi$ is the map

$$\begin{aligned}\overline{M} &\longrightarrow \mathbb{R}^{b-1} \\ x &\longmapsto \left(\int_{x_0}^{x} \omega_i'\right)_{i=1,\dots,b},\end{aligned}$$

so by Proposition 6 $\Lambda \circ \Phi$ is 2-Hölder on K_i.

We claim that for μ-almost every orbit γ the closure of $\Phi(\gamma(\mathbb{R}))$ in $\mathbb{T}^b$ is a subtorus of dimension $I_{\mathbb{R}}(h_0)$. It is sufficient to prove that the image of μ-almost every orbit is a whole line directed by h_0. This follows from

$$\frac{1}{t}\left(\int_0^t \eta_b(\dot\gamma(t))\right) \longrightarrow \langle a_1, h_0\rangle \neq 0$$

when $t \to \pm\infty$, by the Birkhoff ergodic theorem.

Since we have seen that the support of μ does not meet the zero section of TM, we know from Mather's graph theorem that the support of μ is a Lipschitz lamination whose leaves are the orbits. In other words, one can cover $\operatorname{supp} \mu$ with finitely many charts (U_i, f_i) such that:

- the U_i are open sets of M that cover $\operatorname{supp} \mu$;
- each f_i is a bi-Lipschitz homeomorphism from $]-1, 1[^n$ to U_i;
- the horizontal lines $\{y\} \times]-1, 1[$ are mapped into pieces of leaf;
- the image $\Phi(U_i)$ is contained in a small disk in $\mathbb{T}^b$.

As a consequence, we can lift $\Phi_{|U_i}$ to an $\mathbb{R}^b$-valued map. Let us denote by K_i the set of points y in $]-1, 1[^{n-1}$ such that $f_i(y, 0)$ belongs to the support of μ, so $f_i^{-1}(\operatorname{supp} \mu) = K_i \times]-1, 1[$. Let us define a map $\Psi_i : K_i \to \mathbb{R}^{b-1}$ by the expression

$$\Psi_i(y) = \Lambda \circ \Phi \circ f_i(y, 0).$$

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Since f_i is Lipschitz, Ψ_i is 2-Hölder on K_i. Since the images $\Phi(\operatorname{supp}\mu \cap U_i)$ cover a torus of dimension $I_{\mathbb{R}}(h_0)$, there exists i such that $\Phi(\operatorname{supp}\mu \cap U_i)$ contains a set of positive Lebesgue measure of a vector space of dimension $I_{\mathbb{R}}(h_0)$. Thus, $\Psi_i(K_i)$ contains a set of positive Lebesgue measure of a vector space of dimension $I_{\mathbb{R}}(h_0) - 1$. Since Ψ_i is 2-Hölder on K_i, and K_i is a subset of $\mathbb{R}^{\dim M-1}$, Ferry's lemma 8 implies that $1/2(\dim M - 1) > I_{\mathbb{R}}(h_0) - 1$, that is, $\dim M > 2I_{\mathbb{R}}(h_0) - 1$. □

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A. *Appendix*

A.1. *Irrationality.*

Recall that Γ is the torsion-free part of $H_1(M, \mathbb{Z})$. Let

$$J: H_1(M, \mathbb{R}) \longrightarrow H_1(M, \mathbb{R})/\Gamma$$

denote the canonical projection.

PROPOSITION A.1. *For any $h \in H_1(M, \mathbb{R})$, the irrationality $I_{\mathbb{Z}}(h)$ of h is the smallest possible dimension of an integer subspace E_h such that $E_h + \Gamma$ contains nh for some $n \in \mathbb{Z}$. Similarly, $I_{\mathbb{R}}(h)$ is the smallest possible dimension of an integer subspace that contains h. Moreover, $I_{\mathbb{R}}(h)$ is the dimension of the $\mathbb{Q}$-subspace of $\mathbb{R}$ generated by the coordinates of h in any integer basis of $H_1(M, \mathbb{R})$.*

Proof. Let $h \in H_1(M, \mathbb{R})$ be k-irrational with $k \geq 1$. By the definition of irrationality the closure of $J(\mathbb{Z}h)$ in $H_1(M, \mathbb{R})/\Gamma$ is a finite union of k-dimensional tori. Let G_h be the connected component of the identity in this finite union of k-dimensional tori. Then, for some $n \in \mathbb{Z}$, $J(nh)$ lies in G_h. The inverse image of G_h under J is $E_h + \Gamma$, where E_h is a k-dimensional vector subspace of $H_1(M, \mathbb{R})$. Now

$$G_h = (E_h + \Gamma)/\Gamma \cong E_h/(E_h \cap \Gamma)$$

is compact, so $E_h \cap \Gamma$ is cocompact in E_h. Thus, E_h is generated by elements of Γ, that is, E_h is integer. This proves that $k \geq$ the smallest possible dimension of an integer subspace E_h such that $E_h + \Gamma$ contains nh for some n.

Conversely, assume that $nh \in E_h + \Gamma$, with E_h integer of dimension d and $n \in \mathbb{Z}$. Then $n\mathbb{Z}h \subset E_h + \Gamma$, thus

$$J(n\mathbb{Z}h) \subset J(E_h) = E_h/(E_h \cap \Gamma)$$

which is a torus of dimension d. Thus, $J(\mathbb{Z}h)$ is contained in at most n tori of dimension d, thus $k \leq d$. This proves that k is the smallest possible dimension of an integer subspace E_h such that $E_h + \Gamma$ contains h.

Let us prove the rest of the proposition. Take $h \in H_1(M, \mathbb{R})$. The closure of $J(\mathbb{R}h)$ in $H_1(M, \mathbb{R})/\Gamma$ is an $I_{\mathbb{R}}(h)$-dimensional tori G_h. The inverse image of G_h under J is $E_h + \Gamma$, where E_h is an $I_{\mathbb{R}}(h)$-dimensional vector subspace of $H_1(M, \mathbb{R})$. Now

$$G_h = (E_h + \Gamma)/\Gamma \cong E_h/(E_h \cap \Gamma)$$

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is compact, so $E_h \cap \Gamma$ is cocompact in E_h, that is, E_h is integer. The connected subset $\mathbb{R}h$ contains zero and is contained in $E_h + \Gamma$, so it is contained in the connected component of zero in $E_h + \Gamma$, which is E_h. This proves that $I_{\mathbb{R}}(h) \geq$ the smallest possible dimension of an integer subspace that contains h.

Conversely, assume that h is contained in an integer subspace E_h of dimension k. Then E_h contains $\mathbb{R}h$ so $J(\mathbb{R}h)$ is contained in the k-dimensional torus $E_h/(E_h \cap \Gamma)$, thus $I(h) \leq k$. This proves that $I_{\mathbb{R}}(h)$ is the smallest possible dimension of an integer subspace that contains h .

Now let us prove the last statement of the proposition. Take $h \in H_1(M, \mathbb{R})$. Let $a_1, \ldots, a_b$ be an integer basis of $H_1(M, \mathbb{R})$, that is, a basis of $H_1(M, \mathbb{R})$ all of whose elements are in Γ. Set $h = \sum_{i=1}^{b} \lambda_i a_i$. Let

$$k := \dim \mathrm{Vect}_{\mathbb{Q}}\{\lambda_1, \ldots, \lambda_b\}.$$

Modulo a permutation of $\lambda_1, \ldots, \lambda_b$ we may assume that

$$\begin{aligned} \lambda_{k+1} &= \sum_{j=1}^{k} r_{j,k+1}\lambda_j \\ &\vdots \quad \vdots \\ \lambda_b &= \sum_{j=1}^{k} r_{j,b}\lambda_j, \end{aligned}$$

where all of the $r_{j,i}$ are rationals. Let N be the lowest common multiple of their denominators (in irreducible form). Then

$$\begin{aligned} Nh &= \sum_{i=1}^{b} N\lambda_i a_i \\ &= \sum_{i=1}^{k} N\lambda_i a_i + \sum_{i=k+1}^{b} \left(\sum_{j=1}^{k} N r_{j,i}\lambda_j \right) a_i \\ &= \sum_{i=1}^{k} \lambda_i \left[N a_i + \sum_{l=k+1}^{b} N r_{i,l} a_l \right], \end{aligned}$$

that is, h lies in the integer subspace of $H_1(M, \mathbb{R})$ generated by the k classes

$$N a_i + \sum_{l=k+1}^{b} N r_{i,l} a_l, \quad i = 1, \ldots, k$$

so $I_{\mathbb{R}}(h) \leq k$.

Conversely, let E_h be an integer subspace of $H_1(M, \mathbb{R})$ of dimension $d := I_{\mathbb{R}}(h)$ containing h. Let $b_1, \ldots, b_d$ be an integer basis of E_h and set $h = \sum_{j=1}^{d} \nu_j b_j$. Choose a basis B of Γ as a $\mathbb{Z}$-module and let A be the matrix of the coordinates of $a_1, \ldots, a_b$ in the basis B. Let D be the determinant of the matrix A. Since $b_1, \ldots, b_d$ are elements of Γ there exist integers $r_{i,j}$, $i = 1, \ldots, b$, $j = 1, \ldots, d$ such that

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$$b_j = \sum_{i=1}^{b} \frac{r_{i,j}}{D} a_i \quad \text{for all } j = 1, \ldots, d.$$

Then

$$h = \sum_{i=1}^{b} \left(\sum_{j=1}^{d} \frac{r_{i,j}}{D} v_j \right) a_i$$

so the coordinates of h in the basis $a_1, \ldots, a_b$ are linear combinations, with rational coefficients, of the v_j, $j = 1, \ldots, d$, thus the $\mathbb{Q}$-subspace of $\mathbb{R}$ they generate has dimension at most $d = I_{\mathbb{R}}(h)$. □

A.2. *A consequence of the integrality of $\tilde{V}_0$.* The following two easy propositions are the reasons why the integrality of $\tilde{V}_0$ matters in the problem we study.

PROPOSITION A.2. *Let L be a Tonelli Lagrangian on a closed manifold M with first Betti number b. Assume that $\tilde{V}_0$ is an l-dimensional integer subspace of $H^1(M \times \mathbb{T}, \mathbb{R})$. Let h be such that the zero cohomology class defines a subderivative to β at h. Then h is at most $(b-l)$-irrational.*

Proof. Take cohomology classes $c_1, \ldots, c_l$ in F_0, the largest face of α containing zero in its interior, and take $\omega_1, \ldots, \omega_l$ closed 1-forms on M such that:

- $[\omega_i] = c_i$, $i = 1, \ldots, l$;
- for all $i = 1, \ldots, l$, there exists $n_i \in \mathbb{Z}$, $n_i(c_i, \alpha(c_i)) \in H^1(M \times \mathbb{T}, \mathbb{Z})$, in particular $n_i \alpha(c_i) \in \mathbb{Z}$;
- the classes $(c_i, \alpha(c_i))$, $i = 1, \ldots, l$ form a basis of $\tilde{V}_0$ as a vector space.

Complete $n_i(c_i, \alpha(c_i))$, $i = 1, \ldots, l$ to a basis B of $H^1(M \times \mathbb{T}, \mathbb{R})$ that consists of elements of $H^1(M, \mathbb{Z}) \times \mathbb{Z}$. Let μ be a minimizing measure for L. Since the zero cohomology class lies in the interior of F_0 and $c_i \in F_0$, $i = 1, \ldots, l$, the measure μ is also $L - \omega_i$-minimizing. Thus, for $i = 1, \ldots, l$, we have

$$\int_{TM \times \mathbb{T}} (L - \omega_i)\, d\mu = -\alpha(c_i),$$
$$\int_{TM \times \mathbb{T}} L\, d\mu = 0$$

(recall that we work under the assumption that $\alpha(0) = 0$). Therefore,

$$\int_{TM \times \mathbb{T}} \omega_i\, d\mu = \alpha(c_i) \in \mathbb{Q}. \tag{1}$$

On the other hand,

$$\int_{TM \times \mathbb{T}} \omega_i\, d\mu = \langle c_i, [\mu] \rangle \tag{2}$$

so the first l coordinates of $[\mu]$ in the basis $h_1, \ldots, h_b$ of $H_1(M, \mathbb{R})$ dual to B are rational. Hence, for some $N \in \mathbb{Z}$, Nh belongs to $\Gamma + \mathrm{Vect}(h_{l+1}, \ldots, h_b)$. Since $\mathrm{Vect}(h_{l+1}, \ldots, h_b)$ is integer and of dimension $b - l$, the irrationality of $[\mu]$ is at most $b - l$. □

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PROPOSITION A.3. *Let L be a Tonelli Lagrangian on a closed manifold M with first Betti number b. Assume that for every cohomology class c, $\tilde{V}_c$ is an integer subspace of $H^1(M \times \mathbb{T}, \mathbb{R})$. Let h be a k-irrational homology class. Then β is differentiable at h in at least k directions.*

Proof. Let F be the set of subderivatives to β at h, then F is a face of α. Take c in the interior of F, so $F = F_c$. Since h is k-irrational, by the previous proposition we have

$$\dim F_c \leq \dim \tilde{V}_c \leq b - k$$

whence β is differentiable at h in at least k directions. □

A.3. *Behaviour of the Aubry set with respect to the faces of α.* Recall that the static classes of L are the classes of the equivalence relation for all $(x, t), (y, s) \in M \times \mathbb{T}$,

$$(x, t) \sim (y, s) \quad \text{if } h(L)((x, t), (y, s)) + h(L)((y, s), (x, t)) = 0.$$

The following proposition is useful.

PROPOSITION A.4. *Let:*

- *L be a Tonelli Lagrangian on a closed manifold M;*
- *c be a cohomology class in F_0;*
- *ω be a closed 1-form representing c.*

Then the Aubry set (respectively static classes) for L are contained in the Aubry set (respectively static classes) for $L - \omega$, and furthermore the curves realizing the $\liminf$ *in the Peierls barrier for L also realize the* $\liminf$ *in the Peierls barrier for $L - \omega$.*

In particular, if c lies in the interior of F_0, then the Aubry set (respectively static classes) for L and $L - \omega$ are the same, and the curves realizing the $\liminf$ *in the Peierls barrier for L are the same as the curves realizing the* $\liminf$ *in the Peierls barrier for $L - \omega$.*

REMARK A.5. This proposition is a slight improvement on [**Mt03**, Proposition 6] (see [**Be02**] for the time-periodic case) which says the Aubry sets for L and $L - \omega$ are the same.

Proof. Take (x_1, t_1) and (x_2, t_2) in the same static class for L. We have

$$h(L)((x_1, t_1), (x_2, t_2)) + h(L)((x_2, t_2), (x_1, t_1)) = 0,$$

that is to say, there exist two increasing sequence of integers k_n, l_n, $n \in \mathbb{N}$ and curves

$$\gamma_n : [t_1, t_2 + k_n] \longrightarrow M,$$
$$\delta_n : [t_2, t_1 + l_n] \longrightarrow M,$$

such that $\gamma_n(t_1) = \delta_n(t_1 + l_n) = x_1$, $\gamma_n(t_2 + k_n) = \delta_n(t_2) = x_2$ and

$$\int_{t_1}^{t_2+k_n} L(\gamma_n, \dot{\gamma}_n, t)\, dt + \int_{t_2}^{t_1+l_n} L(\delta_n, \dot{\delta}_n, t)\, dt \longrightarrow 0$$

when $n \to \infty$. Since $c \in F_0$, and F_0 contains 0 in its interior, we can find $\theta_1 > 0$ such that $c_1 := -\theta_1 c \in F_0$ so, using the notation $\theta := (1 + \theta_1)^{-1}$, we have $(1 - \theta)c + \theta c_1 = 0$. Thus, α being affine in F_0, we have

$$0 = \alpha(0) = \alpha((1 - \theta)c) + \alpha(\theta c_1) = (1 - \theta)\alpha(c) + \theta\alpha(c_1).$$

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Now if v, w are weak KAM solutions for $L-\omega$ and $L-\omega_1$ respectively (with $\omega_1 := -\theta_1\omega$), we have

$$\begin{aligned}
&\int_{t_1}^{t_2+k_n}(L-\omega+\alpha(c))(\gamma_n,\dot\gamma_n,t)\,dt+\int_{t_2}^{t_1+l_n}(L-\omega+\alpha(c))(\delta_n,\dot\delta_n,t)\,dt\\
&\quad\geq v(x_2,t_2)-v(x_1,t_1)+v(x_1,t_1)-v(x_2,t_2)=0,\\
&\int_{t_1}^{t_2+k_n}(L-\omega_1+\alpha(c_1))(\gamma_n,\dot\gamma_n,t)\,dt+\int_{t_2}^{t_1+l_n}(L-\omega_1+\alpha(c_1))(\delta_n,\dot\delta_n,t)\,dt\\
&\quad\geq w(x_2,t_2)-w(x_1,t_1)+w(x_1,t_1)-w(x_2,t_2)=0.
\end{aligned}$$

Summing $1-\theta$ times the first inequality with θ times the second we get a sum of two non-negative terms which converges to zero, so each term must converge to zero. Consequently,

$$h(L-\omega)((x_1,t_1),(x_2,t_2))+h(L-\omega)((x_2,t_2),(x_1,t_1))=0$$

which proves that (x_1,t_1) and (x_2,t_2) are in the same static class for $L-\omega$. This proves the first statement of Proposition A.4.

The second statement follows by swapping L and $L-\omega$, because if c lies in the interior of F_0, then $F_c=F_0$, in particular 0 lies in the interior of F_c. □

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