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J. A. SCOTT
1 Shiptons Lane, Great Somerford, Chippenham SN15 5EJ

102.53 The excentral triangle and a curious application to inequalities

Introduction

The celebrated Weitzenböck's inequality from 1919 is [1]

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta, \tag{1}$$

where a, b, c are sides of a triangle ABC and Δ is its area. Equality holds if, and only if, the triangle is equilateral. The inequality was given as Problem 2 at the third International Mathematical Olympiad held in 1961 in Hungary. One sharpening is the well-known Finsler-Hadwiger inequality from 1937 [2]

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta + (a - b)^2 + (b - c)^2 + (c - a)^2, \tag{2}$$

which can be rewritten as

$$a(s - a) + b(s - b) + c(s - c) \geqslant 2\sqrt{3}\Delta, \tag{3}$$

where s is the semiperimeter of ABC. Again, equality holds only for the equilateral triangle. For eleven proofs of (1), with one of them proving the generalisation (2), see [3]. For another proof of (2) together with its complementary inequality

$$a^2 + b^2 + c^2 \leqslant 4\sqrt{3}\Delta + 3\left[(a - b)^2 + (b - c)^2 + (c - a)^2\right]$$

see [4], where both inequalities are derived from the Gerretsen's inequalities. Even stronger than (2) is the recent inequality [5]

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3 + \frac{R - 2r}{R}} + (a - b)^2 + (b - c)^2 + (c - a)^2,$$

where R and r are the circumradius and inradius of ABC respectively. For the reader's convenience, we shall prove the sharpening of (1) in the opposite direction

$$ab + bc + ca \geqslant 4\sqrt{3}\Delta,$$

sometimes referred to as Gordon's inequality. First we prove

$$ab + bc + ca \geqslant 18Rr,$$

which by $R = \dfrac{abc}{4\Delta}$ and $r = \dfrac{2\Delta}{a + b + c}$ is equivalent to

$$(a + b + c)(ab + bc + ca) \geqslant 9abc.$$

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The last inequality follows from multiplying the AM-GM inequalities

$$a + b + c \geqslant 3\sqrt[3]{abc},\ ab + bc + ca \geqslant 3\sqrt[3]{a^2b^2c^2}.$$

Now by the well-known result $R \geqslant \dfrac{2s}{3\sqrt{3}}$, we get

$$ab + bc + ca \geqslant 18Rr \geqslant 4\sqrt{3}\Delta.$$

Inequality (1) has an interesting geometric interpretation [6]. Multiplying it by $\frac{1}{4}\sqrt{3}$ gives

$$\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} + \frac{c^2\sqrt{3}}{4} \geqslant 3\Delta,$$

and now it says that the combined sum of the areas of the three equilateral triangles erected on the sides of the triangle is at least three times the area of the triangle. For the proof, the authors in [6] employed the Fermat point F of the triangle, which minimizes the sum $AP + BP + CP$ for a point P in or on the triangle ABC.

The excentral triangle

We show in this Note that the stronger Finsler-Hadwiger inequality (2) can be derived from the Weitzenböck's inequality (1). Consider for that purpose the excentral triangle $I_AI_BI_C$ of ABC, which is formed by the excentres I_A, I_B, I_C of ABC. See Figure 1.

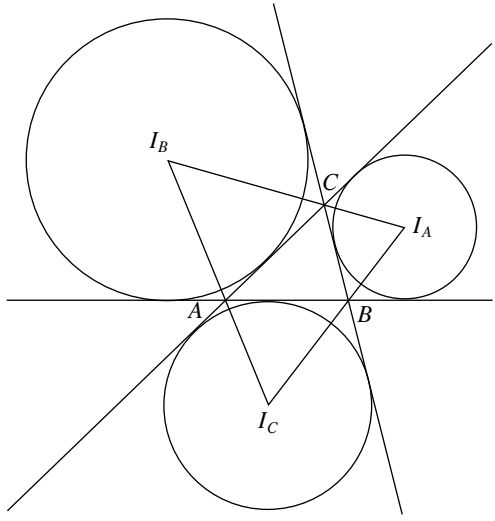


FIGURE 1

We shall see that this triangle holds a surprise: when a triangle ABC is given, (1) applied to the excentral triangle of ABC, implies (2) for ABC. It is straightforward that the angles of $I_AI_BI_C$ are $\frac{1}{2}(\pi - A)$, $\frac{1}{2}(\pi - B)$, $\frac{1}{2}(\pi - C)$

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and it is known [7, 8, 9], that its nine-point circle is the circumcircle of ABC. Hence the circumradius of $I_AI_BI_C$ is $2R$, where R is the circumradius of ABC, and by the *Sine Law* we obtain for its sides

$$I_AI_B = 4R\cos\frac{C}{2}, \qquad I_BI_C = 4R\cos\frac{A}{2}, \qquad I_CI_A = 4R\cos\frac{B}{2}.$$

Its area is

$$\Delta\left(I_AI_BI_C\right) = 2Rs.$$

Applying the Weitzenböck's inequality to $I_AI_BI_C$, we get

$$16R^2\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right) \geqslant 8\sqrt{3}Rs. \tag{4}$$

By Euler's half-angle formula $\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$,

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = \frac{s}{abc}\left(a(s-a) + b(s-b) + c(s-c)\right),$$

which, by $abc = 4Rrs$ and $\Delta = rs$ substituted in (4), yields (3), the equivalent form of Finsler-Hadwiger inequality (2).

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MARTIN LUKAREVSKI

Department of Mathematics and Statistics, University Goce Delcev - Stip, Macedonia

e-mail: *martin.lukarevski@ugd.edu.mk*

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