PRESERVATION THEOREMS FOR AE-SENTENCES

DIEGO CASTAÑO, MIGUEL CAMPERCHOLI, AND DIEGO VAGGIONE

Abstract. An *AE-sentence* is a sentence of the form $\forall x_1 \dots x_n \exists! z_1 \dots z_m \, \alpha(\mathbf{x}, \mathbf{z})$, where α is a finite conjunction of equations. Given an arbitrary quasivariety $\mathcal{Q}$, we give a purely semantical characterization of when a class $\mathcal{K} \subseteq \mathcal{Q}$ with $S(\mathcal{K}) = \mathcal{Q}$ is axiomatizable relative to $\mathcal{Q}$ by AE-sentences. Along the way, we also characterize axiomatizability by *generalized AE-sentences*, which are of the form described above, except that both the number of existential quantifiers and of equations in α are allowed to be infinite. The article concludes with an analysis of how the main theorems can be improved in the context of finitely generated quasivarieties.

§1. Introduction. In this article we explore subclasses of a quasivariety axiomatizable by a very special kind of sentence. An *AE-sentence* is an expression φ of the form

$$\forall x_1 \dots x_n \exists! \, z_1 \dots z_m \bigwedge_{i=1}^{k} r_i(\mathbf{x}, \mathbf{z}) = s_i(\mathbf{x}, \mathbf{z}),$$

where r_i's and s_i's are terms. To clarify the semantics, given a structure $\mathbf{A}$, it is the case that $\mathbf{A} \vDash \varphi$ if and only if for every $\mathbf{a} \in A^n$ there is a unique $\mathbf{b} \in A^m$ such that $r_i^{\mathbf{A}}(\mathbf{a}, \mathbf{b}) = s_i^{\mathbf{A}}(\mathbf{a}, \mathbf{b})$ for $i \in \{1, \dots, k\}$. The sentence φ induces in each of its models m new n-ary operations that are simultaneously definable by a conjunction of term-equalities. Now, given a quasivariety $\mathcal{Q}$, there is a natural expansion of the members of $\mathcal{K} := \mathcal{Q} \cap \mathrm{Mod}(\varphi)$, where new operation symbols are added to the language to represent the operations induced by φ. It turns out that this expansion, which we denote by $\mathcal{K}^{\varphi}$, is itself a quasivariety. Furthermore, since the new operations depend closely on the structure of the original algebras, there is usually a strong connection between the quasivarieties $\mathcal{Q}$ and $\mathcal{K}^{\varphi}$. An instance of this phenomenon is that $\mathcal{Q}$-morphisms are always $\mathcal{K}^{\varphi}$-morphisms. Let us take a look at some examples.

Let $\mathcal{D}$ be the class of bounded distributive lattices, and let φ_c be the AE-sentence

$$\forall x \exists! \, z \, (x \wedge z = 0, x \vee z = 1).$$

Of course, in this case, the class $\mathcal{K} := \mathcal{D} \cap \mathrm{Mod}(\varphi_c)$ is the class of complemented distributive lattices, and $\mathcal{K}^{\varphi_c}$ is the class of Boolean algebras.

A second example is provided by the quasivariety $\mathcal{C}$ of cancellative commutative monoids, that is, the class of structures $\mathbf{M} := \langle M, \cdot, 1 \rangle$, where $\cdot$ is an associative

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and commutative binary operation with neutral element 1, and the cancellative law $\forall xyz(xy = xz \to y = z)$ holds in $\mathbf{M}$. If we consider the AE-sentence

$$\varphi_i := \forall x \exists! z\, (xz = 1),$$

it is clear that, in this case, the class of expansions of models in $\mathcal{C} \cap \mathrm{Mod}(\varphi_i)$ is the variety of Abelian groups.

Let $\mathcal{Q}$ be the quasivariety generated by the integral domains (i.e., commutative rings with unit with no nontrivial zero-divisors). Take

$$\varphi_r := \forall x \exists! z\, (x = xzx,\ z = zxz).$$

The class $\mathcal{K} := \mathcal{Q} \cap \mathrm{Mod}(\varphi_r)$ is the class of commutative Von Neumann regular rings [15].

Even though in the above examples a single AE-sentence is considered, it also makes perfect sense to investigate the same construction starting from a set of AE-sentences. This motivates the following definition: a subclass $\mathcal{K}$ of the quasivariety $\mathcal{Q}$ is called an *algebraically expandable subclass* (*AE-subclass* for short) of $\mathcal{Q}$ provided there is a set Σ of AE-sentences such that $\mathcal{K} = \mathcal{Q} \cap \mathrm{Mod}(\Sigma)$. A study of the AE-subclasses of several well-known quasivarieties can be found in [5], where it is also revealed that in this type of investigations it is possible to encounter, among the expansions, not only previously known quasivarieties, but also promising new algebraic structures.

Since the original study of axiomatizability by AE-sentences, the tools and techniques developed in [5] have branched out into some other interesting mathematical inquiries. We mention two examples next.

The first is the study of the set of all functions definable by conjunctions of equations on a given algebra $\mathbf{A}$, which we call *algebraic functions* on $\mathbf{A}$. Notice that this set is exactly the set of all functions induced by AE-sentences that hold in $\mathbf{A}$, and has the desirable features of being closed under composition and containing all term-operations of $\mathbf{A}$, i.e., it is a clone. Thus, algebraic functions on $\mathbf{A}$ are a natural generalization of the term-operations of $\mathbf{A}$ and have shown to be a valuable theoretical tool serving as an intermediate step when solving problems that aim to establish the existence of certain term operations (see, e.g., [2, 7, 14]). An in-depth look into the problem of characterizing the set of algebraic functions for each member of a diversity of well-known varieties constitutes the article [6].

A second field where AE-sentences produced interesting insights is that of algebraic logic. If the algebraic counterpart of an algebraizable logic L is a quasivariety $\mathcal{Q}$, each AE-subclass of $\mathcal{Q}$ leads to a very special kind of expansion of L. As in the algebraic case, these new logics have some strong ties to the original logic L, and, in particular, turn out to be algebraizable. These expansions are called *algebraic expansions* of L and are studied in [4] and turn out to be a particular case of X. Caicedo's expansions by implicit connectives [1].

Before stating the problem we address in this note, let us return to the examples of distributive lattices, cancellative commutative monoids, and integral domains given above. All of these share the common feature that every algebra in the ambient quasivariety $\mathcal{Q}$ can be realized as a subalgebra of a member of the AE-subclass $\mathcal{K}$. (For example, every distributive lattice can be embedded in a complemented distributive lattice.) That is to say, $S(\mathcal{K}) = \mathcal{Q}$.

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Let us fix an arbitrary quasivariety $\mathcal{Q}$ for the remainder of this introduction. The main contribution of this article is a preservation theorem characterizing in purely semantical terms the AE-subclasses $\mathcal{K}$ of $\mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$ (see Theorem 16). Such classes $\mathcal{K}$ are easily seen to be closed under

(1) direct products,
(2) $\mathcal{Q}$-homomorphic images, and
(3) nonempty arbitrary intersection of subalgebras.

However, it does not take long to realize that these closure conditions do not suffice. One way to produce a counterexample is to consider classes axiomatizable by *generalized AE-sentences*, that is, sentences of the form

$$\forall x_1 \dots x_n \exists! \mathbf{z} \bigwedge_{i \in I} r_i(\mathbf{x}, \mathbf{z}) = s_i(\mathbf{x}, \mathbf{z}),$$

where both $\mathbf{z}$ and I are allowed to be infinite. (In the special case where $|I| \leq \omega$ these formulas are called AE_ω*-sentences*.) It is easy to see that generalized AE-sentences are also preserved by the three closure operators. Example 15 exhibits a class $\mathcal{K}$, with $S(\mathcal{K}) = \mathcal{Q}$, that is axiomatizable by a set of generalized AE-sentences but is not an AE-subclass of $\mathcal{Q}$. This indicates that further conditions are needed for a class $\mathcal{K}$ to be an AE-subclass of $\mathcal{Q}$. As it turns out, a finer inspection of how the intersection property behaves in $\mathcal{K}$ yields the missing pieces.

Axiomatizable classes closed under nonempty arbitrary intersection of subalgebras were originally introduced and studied by A. Robinson [12] and have been further investigated by other authors (see, e.g., [9–11]). Following M. Rabin [11] we say that such a class has the *intersection property* (*i.p.* for short).[1] Any class $\mathcal{K}$ with the i.p. induces on each algebra $\mathbf{A}$ of $\mathcal{K}$ a closure operator $C_{\mathbf{A}}$ on the universe of $\mathbf{A}$ such that for each $S \subseteq A$:

$$C_{\mathbf{A}}(S) := \bigcap \{B : \mathbf{B} \in \mathcal{K}, \mathbf{B} \leq \mathbf{A}, S \subseteq B\}.$$

We say that a class $\mathcal{K} \subseteq \mathcal{Q}$ with the i.p. is *locally small* provided that there is a cardinal κ such that for every $\mathbf{A} \in \mathcal{K}$ and every finite $S \subseteq A$ we have $|C_{\mathbf{A}}(S)| \leq \kappa$. With this definition in place we move one step further towards a characterization of AE-subclasses of $\mathcal{Q}$, which also turns out to be an interesting result in its own right.

THEOREM. *Let $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. The following are equivalent*:

1. *$\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by a set of generalized AE-sentences.*
2. *$\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by AE_ω-sentences.*
3. *$\mathcal{K}$ has the i.p., is closed under direct products and $\mathcal{Q}$-homomorphic images, and is locally small.*

In view of Example 15 there is still work to be done in order to complete our characterization of AE-subclasses of $\mathcal{Q}$. The last piece of the puzzle is yet another condition we need to impose on the closure operators induced by $\mathcal{K}$. It is a natural property that describes the interaction between $\mathcal{Q}$-morphisms and the closure of finite sets, which we call *local domination property* (see Section 5.2 for the definition). We can now state the main result of this article.

[1] A. Robinson uses the adjective *convex* for first-order theories whose models have the intersection property.

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THEOREM. *Let $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. The following are equivalent*:

1. *$\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by AE-sentences.*
2. *$\mathcal{K}$ has the i.p., is closed under direct products and $\mathcal{Q}$-homomorphic images, is locally small, and has the local domination property.*

The article concludes with an analysis of AE-subclasses of finitely generated quasivarieties, where it is shown that the hypotheses of local smallness and domination property can both be dropped and the equivalence still holds. Finally, we study the case in which $\mathcal{Q}$ is generated by a finite set $\mathcal{F}$ of finite injective algebras. We obtain a theorem reminiscent of the HSP theorem that describes the least AE-subclass of $\mathcal{Q}$ containing $\mathcal{S}$ for any given $\mathcal{S}$ with $\mathcal{F} \subseteq \mathcal{S} \subseteq \mathcal{Q}$.

§2. Generalized AE-sentences and algebraic expansions. The notation used throughout is fairly standard. To be on the safe side we specify next some of our notational choices and other conventions. All first-order languages considered in this article are algebraic, that is, they contain no relation symbols. Boldface capital letters $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots)$ denote structures, and the corresponding non-bold symbols $(A, B, C, \ldots)$ stand for their universes. Both finite and infinite sequences of objects (such as elements, variables, terms, etc.) are denoted in lowercase boldface $(\mathbf{a}, \mathbf{x}, \mathbf{t}, \ldots)$. All sequences we consider are well ordered. Given algebras $\mathbf{A}, \mathbf{B}, \mathbf{C}$ with $\mathbf{C} \leq \mathbf{B}$, and a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$, we write:

- $h[\mathbf{A}]$ for the substructure of $\mathbf{B}$ with universe $h[A] := \{h(a) : a \in A\}$;
- $h^{-1}[\mathbf{C}]$ for the substructure of $\mathbf{A}$ with universe $h^{-1}[C] := \{a \in A : h(a) \in C\}$, provided $h^{-1}[C]$ is nonempty.

Let L be a first-order language without relation symbols. Recall that an *equation* is an atomic L-formula and a *quasi-equation* is an L-formula of the form $\bigwedge_{i=1}^{n} r_i = s_i \to r = s$, where $r, s, r_1, s_1, \ldots, r_n, r_n$ are L-terms. A *(quasi-)identity* is the universal closure of a (quasi-)equation. By a *(quasi-)variety* we mean a class of algebras axiomatized by a set of (quasi-)identities.

Let us introduce next the type of sentence on which this note focuses. A *generalized AE-sentence* of L is a formal expression of the form $\forall \mathbf{x} \exists! \, \mathbf{z} \, \alpha(\mathbf{x}, \mathbf{z})$, where

- $\mathbf{x}$ is a finite sequence of distinct variables;
- $\mathbf{z}$ is a nonempty, possibly infinite sequence of distinct variables that do not occur in $\mathbf{x}$;
- α is a conjunction of a possibly infinite set of L-atomic formulas such that every variable in $\mathbf{z}$ occurs in some conjunct of α.

In the case where α has at most ω conjuncts, we say that $\forall \mathbf{x} \exists! \, \mathbf{z} \, \alpha$ is an *AE_ω-sentence*.

A generalized AE-sentence $\varphi := \forall \mathbf{x} \exists! \, \mathbf{z} \, \alpha(\mathbf{x}, \mathbf{z})$ is understood as the conjunction of the following two (possibly infinitary) formulas:

- $E(\varphi) := \forall \mathbf{x} \exists \mathbf{z} \, \alpha(\mathbf{x}, \mathbf{z})$;
- $U(\varphi) := \forall \mathbf{x} \mathbf{z} \mathbf{w} \, (\alpha(\mathbf{x}, \mathbf{z}) \wedge \alpha(\mathbf{x}, \mathbf{w}) \to \mathbf{z} = \mathbf{w})$.

As the semantics of the two formulas above is self-explanatory, it is clear now how to evaluate whether φ holds in a given first-order structure.[2]

[2]This semantics explains why we require every existential variable to occur in the matrix; otherwise, it would render the AE-sentence only satisfiable in trivial algebras.

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Let $\varphi := \forall x_1 \dots x_n \exists! \mathbf{z}\, \alpha(x_1, \dots, x_n, \mathbf{z})$ be a generalized AE-sentence, and let $\mathbf{A}$ be an algebra that satisfies φ. Note that φ induces an n-ary operation on A for each z in $\mathbf{z}$. More precisely, we define the sequence $\langle [\varphi]_z^{\mathbf{A}} : z \text{ in } \mathbf{z} \rangle$ of n-ary operations on A by:

$$\langle [\varphi]_z^{\mathbf{A}}(a_1, \dots, a_n) : z \text{ in } \mathbf{z} \rangle = \mathbf{b} \text{ if and only if } \mathbf{A} \vDash \alpha[a_1, \dots, a_n, \mathbf{b}], \tag{1}$$

for arbitrary $a_1, \dots, a_n \in A$.

Given a set Σ of generalized AE-sentences in a language L, we define the expansion L^Σ of L as follows: for each $\varphi := \forall x_1 \dots x_n \exists! \mathbf{z}\, \alpha(x_1, \dots, x_n, \mathbf{z})$ in Σ and each z in $\mathbf{z}$ we add to L a new n-ary function symbol $f_{\varphi,z}$. Now, given a model $\mathbf{A}$ of Σ we define the L^Σ-expansion $\mathbf{A}^\Sigma$ of $\mathbf{A}$ by interpreting for each $\varphi := \forall x_1 \dots x_n \exists! \mathbf{z}\, \alpha(x_1, \dots, x_n, \mathbf{z})$ in Σ the symbols $\{f_{\varphi,z} : z \text{ in } \mathbf{z}\}$ as the functions induced by φ on $\mathbf{A}$, that is, $f_{\varphi,z}^{\mathbf{A}^\Sigma} := [\varphi]_z^{\mathbf{A}}$. Also, if $\mathcal{K}$ is a class of models of Σ, we define the class of L^Σ-algebras

$$\mathcal{K}^\Sigma := \{\mathbf{A}^\Sigma : \mathbf{A} \in \mathcal{K}\}.$$

We call this class the *algebraic expansion* of $\mathcal{K}$ relative to Σ. (This definition should explain the choice for the name of AE-sentences.)

As mentioned in the introduction, in this article we investigate the subclasses $\mathcal{K}$ of a given quasivariety $\mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$ and $\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by a set of generalized AE-sentences. An important feature of this type of classes is that their algebraic expansions always turn out to be quasivarieties.

LEMMA 1. *Let $\mathcal{Q}$ be a quasivariety and Σ a set of generalized AE-sentences both in the language L. Put*

$$\mathcal{K} := \mathcal{Q} \cap \operatorname{Mod} \Sigma,$$

and assume $S(\mathcal{K}) = \mathcal{Q}$. Then, the class $\mathcal{K}^\Sigma$ is a quasivariety. Moreover, $\mathcal{K}^\Sigma$ can be axiomatized with a quasi-equational base for $\mathcal{Q}$ together with a set of L^Σ-identities.

PROOF. We prove the moreover part. Take $\sigma := \forall x_1 \dots x_n \exists! \mathbf{z}\, \alpha(x_1, \dots, x_n, \mathbf{z})$ in Σ, and let $\{f_z : z \text{ in } \mathbf{z}\}$ be the function symbols in L^Σ corresponding to σ. Define $\mathbf{f}(\mathbf{x}) := \langle f_z(\mathbf{x}) : z \text{ in } \mathbf{z} \rangle$. Note that the infinitary sentence $\forall \mathbf{x}\, \alpha(\mathbf{x}, \mathbf{f}(\mathbf{x}))$ is equivalent to a set of first-order identities; call it Φ_σ. Let Γ be a quasi-equational base for $\mathcal{Q}$. We claim that $T := \Gamma \cup \bigcup_{\sigma \in \Sigma} \Phi_\sigma$ axiomatizes $\mathcal{K}^\Sigma$. Obviously, the class $\mathcal{K}^\Sigma$ satisfies T. We prove that every model of T belongs to $\mathcal{K}^\Sigma$. Fix an L^Σ-algebra $\mathbf{A}$ satisfying T, and let $\mathbf{A}_L$ be the reduct of $\mathbf{A}$ to L. Take $\sigma \in \Sigma$, and note that, as $\mathbf{A} \vDash \Phi_\sigma$, we clearly have $\mathbf{A} \vDash E(\sigma)$; in particular, $\mathbf{A}_L \vDash E(\sigma)$. Also, since $\mathcal{K} \vDash U(\sigma)$ and $\mathcal{Q} = S(\mathcal{K})$, it follows that $\mathcal{Q} \vDash U(\sigma)$. Now, as $\mathbf{A}_L \in \mathcal{Q}$, we have $\mathbf{A}_L \vDash U(\sigma)$. Since σ is arbitrary in Σ, we have established that $\mathbf{A}_L \in \mathcal{K}$. Finally, observe that, as $\mathbf{A} \vDash \Sigma$, we necessarily have $\mathbf{A} = (\mathbf{A}_L)^\Sigma \in \mathcal{K}^\Sigma$. ⊣

Next, we take a small detour to highlight a fundamental feature of AE-subclasses of a quasivariety: their algebraic expansions do not depend on the choice of axiomatizing generalized AE-sentences. (A precursor of this result can be found in [6].) In order to precisely state this result we need to recall the notion of interpretation between quasivarieties.

Fix a countably infinite set of variables $X := \{x_1, x_2, \dots\}$; given a language L we write $Tm(L)$ for the set of L-terms over the variables in X. Let L_1 and L_2 be two

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expansions of a language L. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of algebras over L_1 and L_2, respectively. An *L-interpretation* of $\mathcal{K}_1$ in $\mathcal{K}_2$ is a function $T\colon L_1 \to Tm(L_2)$ such that:

- $T(f) = f(x_1, \dots, x_n)$ for every n-ary symbol $f \in L$;
- T maps each symbol in $L_1 \setminus L$ of arity $n \geq 1$ to an L_2-term in the variables $x_1, \dots, x_n$;
- T maps each nullary symbol c in $L_1 \setminus L$ to a unary L_2-term $t(x_1)$ that is constant valued in $\mathcal{K}_2$;
- for every member $\mathbf{A} := \langle A, \{g^{\mathbf{A}} : g \in L_2\}\rangle$ in $\mathcal{K}_2$, the algebra
$$\mathbf{A}^T := \langle A, \{T(f)^{\mathbf{A}} : f \in L_1\}\rangle$$
belongs to $\mathcal{K}_1$, where we interpret the images of constant symbols in $L_1 \setminus L$ as constants instead of unary constant term-operations.

If T and S are L-interpretations of $\mathcal{K}_1$ in $\mathcal{K}_2$ and $\mathcal{K}_2$ in $\mathcal{K}_1$, respectively, such that the maps $\mathbf{A} \mapsto \mathbf{A}^T$ and $\mathbf{A} \mapsto \mathbf{A}^S$ are mutually inverse, we say that $\mathcal{K}_1$ and $\mathcal{K}_2$ are *L-term-equivalent*.

THEOREM 2. *Let $\mathcal{Q}$ be a quasivariety in the language L and let Σ_1 and Σ_2 be sets of generalized AE-sentences in L. Put*

$$\mathcal{K}_1 := \mathcal{Q} \cap \operatorname{Mod}\Sigma_1 \text{ and } \mathcal{K}_2 := \mathcal{Q} \cap \operatorname{Mod}\Sigma_2.$$

Suppose $\mathcal{K}_2 \subseteq \mathcal{K}_1$ and $S(\mathcal{K}_2) = \mathcal{Q}$. Then, there is an L-interpretation T of $\mathcal{K}_1^{\Sigma_1}$ in $\mathcal{K}_2^{\Sigma_2}$ such that:

1. *For all $\mathbf{A} \in \mathcal{K}_2$ we have that $(\mathbf{A}^{\Sigma_2})^T = \mathbf{A}^{\Sigma_1}$.*
2. *$S(\mathcal{K}_2^{\Sigma_1}) = \mathcal{K}_1^{\Sigma_1}$.*

PROOF. We provide the proof for the special case where every AE-sentence in Σ_1 has at least one universal quantifier. The general case requires treating AE-sentences with no universal quantifiers separately, but, in essence, the proof runs the same way.

We start by defining the L-interpretation T. Fix $\varphi \in \Sigma_1$, say

$$\varphi := \forall x_1 \dots x_n \exists! \mathbf{z}\, \alpha(\mathbf{x}, \mathbf{z}),$$

and let $\mathbf{F}$ be the n-generated free algebra in $\mathcal{K}_2^{\Sigma_2}$ freely generated by $x_1, \dots, x_n$. Note that $\mathbf{F} \vDash \varphi$, and hence, there is a sequence $\langle t_z(x_1, \dots, x_n) : z \text{ in } \mathbf{z}\rangle$ of L^{Σ_2} terms such that

$$\mathbf{F} \vDash \alpha[\mathbf{x}, \langle t_z^{\mathbf{F}}(\mathbf{x}) : z \text{ in } \mathbf{z}\rangle]. \tag{2}$$

Now, for each z in $\mathbf{z}$ we define $T(f_{\varphi,z}) := t_z$. Repeating this process for each sentence in Σ_1 we have a complete definition of T.

Next, we establish 1. Take $\mathbf{A} \in \mathcal{K}_2$, and, for readability, put $\mathbf{A}_1 := \mathbf{A}^{\Sigma_1}$ and $\mathbf{A}_2 := \mathbf{A}^{\Sigma_2}$. Fix $\varphi := \forall x_1 \dots x_n \exists! \mathbf{z}\, \alpha(\mathbf{x}, \mathbf{z})$ in Σ_1 and $a_1, \dots, a_n \in A$. Let $h\colon \mathbf{F} \to \mathbf{A}_2$ be the homomorphism mapping $x_1, \dots, x_n \mapsto a_1, \dots, a_n$. From (2) it follows that

$$\mathbf{A}_2 \vDash \alpha[\mathbf{a}, \langle t_z^{\mathbf{A}_2}(\mathbf{a}) : z \text{ in } \mathbf{z}\rangle].$$

Clearly, this says that $f_{\varphi,z}^{\mathbf{A}_1} = [\varphi]_z^{\mathbf{A}} = t_z^{\mathbf{A}_2} = T(f_{\varphi,z})^{\mathbf{A}_2} = f_{\varphi,z}^{\mathbf{A}_2^T}$ for every z in $\mathbf{z}$. This completes the proof of 1.

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We now prove 2. The forward inclusion is immediate since $\mathcal{K}_1^{\Sigma_1}$ is a quasivariety, and thus closed under substructures. We turn to the remaining direction. Fix $\mathbf{A} \in \mathcal{K}_1$. By assumption, there is $\mathbf{B} \in \mathcal{K}_2$ such that $\mathbf{A} \leq \mathbf{B}$. Observe that $\mathbf{B} \vDash \Sigma_1$. Finally, it is straightforward to check that $\mathbf{A}^{\Sigma_1} \leq \mathbf{B}^{\Sigma_1}$. ⊣

From Theorem 2 independence of the choice of axioms follows easily.

COROLLARY 3 (cf. [5, Theorem 5]). *Let $\mathcal{Q}$ be a quasivariety in the language L and let Σ_1 and Σ_2 be sets of generalized AE-sentences in L such that*

$$\mathcal{K} := \mathcal{Q} \cap \operatorname{Mod} \Sigma_1 = \mathcal{Q} \cap \operatorname{Mod} \Sigma_2.$$

Suppose $S(\mathcal{K}) = \mathcal{Q}$. Then, $\mathcal{K}^{\Sigma_1}$ and $\mathcal{K}^{\Sigma_2}$ are L-term-equivalent.

PROOF. By Theorem 2 there are L-interpretations T of $\mathcal{K}^{\Sigma_1}$ in $\mathcal{K}^{\Sigma_2}$ and S of $\mathcal{K}^{\Sigma_2}$ in $\mathcal{K}^{\Sigma_1}$ satisfying for all $\mathbf{A} \in \mathcal{K}$ that:

- $(\mathbf{A}^{\Sigma_2})^T = \mathbf{A}^{\Sigma_1}$;
- $(\mathbf{A}^{\Sigma_1})^S = \mathbf{A}^{\Sigma_2}$.

Then, given $\mathbf{A} \in \mathcal{K}$ we have $(\mathbf{A}^{\Sigma_2})^{TS} = (\mathbf{A}^{\Sigma_1})^S = \mathbf{A}^{\Sigma_2}$ and $(\mathbf{A}^{\Sigma_1})^{ST} = (\mathbf{A}^{\Sigma_2})^T = \mathbf{A}^{\Sigma_1}$. ⊣

§3. Preservation properties of generalized AE-sentences. In this section we collect some fundamental preservations of generalized AE-sentences by certain model-theoretic constructions. We start with some easy observations whose proofs we leave to the reader. Recall that, given homomorphisms $h, g\colon \mathbf{A} \to \mathbf{B}$, the set $E(h,g) := \{a \in A : h(a) = g(a)\}$ is called the *equalizer* of h and g. Observe that $E(h,g)$ is always a subuniverse of $\mathbf{A}$, and when $E(h,g) \neq \emptyset$, we write $\mathbf{E}(h,g)$ for the corresponding substructure of $\mathbf{A}$.

LEMMA 4. *Let φ be a generalized AE-sentence, and suppose $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \cup \{\mathbf{A}_i : i \in I\} \vDash \varphi$.*

(1) *If $\mathbf{A}, \mathbf{B} \leq \mathbf{C}$ and $\mathbf{A} \cap \mathbf{B} \neq \emptyset$, then $\mathbf{A} \cap \mathbf{B} \vDash \varphi$.*
(2) *If $h : \mathbf{A} \to \mathbf{B}$, then $h[\mathbf{A}] \vDash \varphi$.*
(3) *If $\mathbf{C} \leq \mathbf{B}$, $h : \mathbf{A} \to \mathbf{B}$ and $h^{-1}[C] \neq \emptyset$, then $h^{-1}[\mathbf{C}] \vDash \varphi$.*
(4) *If $h, g : \mathbf{A} \to \mathbf{B}$ and $E(h,g) \neq \emptyset$, then $\mathbf{E}(h,g) \vDash \varphi$.*
(5) *$\prod_{i \in I} \mathbf{A}_i \vDash \varphi$.*

The previous lemma motivates the introduction of the following class operators. Let $\mathcal{Q}$ be a quasivariety and $\mathcal{K} \subseteq \mathcal{Q}$:

- $H_{\mathcal{Q}}(\mathcal{K}) := \{h[\mathbf{A}] : \mathbf{A} \in \mathcal{K}, \mathbf{B} \in \mathcal{Q} \text{ and } h\colon \mathbf{A} \to \mathbf{B}\}$.
- $S_{\cap}(\mathcal{K}) := \{\mathbf{A} \cap \mathbf{B} : \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K} \text{ with } \mathbf{A} \leq \mathbf{C}, \mathbf{B} \leq \mathbf{C}, A \cap B \neq \emptyset\}$.
- $H^{-1}(\mathcal{K}) := \{h^{-1}[\mathbf{C}] : \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}, h\colon \mathbf{A} \to \mathbf{B}, \mathbf{C} \leq \mathbf{B}, \text{ and } h^{-1}[C] \neq \emptyset\}$.
- $S_E(\mathcal{K}) := \{\mathbf{E}(h,g) : \mathbf{A}, \mathbf{B} \in \mathcal{K}, h, g\colon \mathbf{A} \to \mathbf{B} \text{ and } E(h,g) \neq \emptyset\}$.

Since the above introduced operators are rather uncommon, let us record some of their interactions and basic properties.

LEMMA 5. *Let $\mathcal{Q}$ be a quasivariety and $\mathcal{K} \subseteq \mathcal{Q}$. The following hold*:

(1) *$H_{\mathcal{Q}} H_{\mathcal{Q}}(\mathcal{K}) = H_{\mathcal{Q}}(\mathcal{K})$.*
(2) *$IS_E(\mathcal{K}) = S_E I(\mathcal{K})$ and $IS_{\cap}(\mathcal{K}) \subseteq S_{\cap} I(\mathcal{K})$.*

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(3) $PS_E(\mathcal{K}) \subseteq S_E P(\mathcal{K})$ *and* $PH_{\mathcal{Q}}(\mathcal{K}) \subseteq H_{\mathcal{Q}} P(\mathcal{K})$.
(4) $S_\cap(\mathcal{K}) \subseteq H_{\mathcal{Q}} S_E P(\mathcal{K})$.
(5) $H^{-1}(\mathcal{K}) \subseteq S_\cap IP(\mathcal{K})$.
(6) $S_E(\mathcal{K}) \subseteq S_\cap IP(\mathcal{K})$.

PROOF. (1)–(3) are left to the reader.

(4) Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ with $\mathbf{A}, \mathbf{B} \leq \mathbf{C}$. Take $h, g \colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ defined by $h(a, b) := a$ and $g(a, b) := b$. Now $\mathbf{A} \cap \mathbf{B} = h[\mathbf{E}(h, g)] \in \mathcal{K}$.

(5) Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ with $\mathbf{C} \leq \mathbf{B}$, and suppose $h \colon \mathbf{A} \to \mathbf{B}$ is such that $h^{-1}[C] \neq \emptyset$. Write $\mathbf{h}$ for the subalgebra of $\mathbf{A} \times \mathbf{B}$ with universe $\{\langle a, h(a)\rangle : a \in A\}$. Now observe that $h^{-1}[\mathbf{C}] \cong \mathbf{h} \cap (\mathbf{A} \times \mathbf{C})$, thus $h^{-1}[\mathbf{C}] \in IS_\cap IP(\mathcal{K}) \subseteq S_\cap IP(\mathcal{K})$.

(6) Let $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and $h, g \colon \mathbf{A} \to \mathbf{B}$ such that $E(h, g) \neq \emptyset$. Write $\mathbf{h}$ and $\mathbf{g}$ for the subalgebras of $\mathbf{A} \times \mathbf{B}$ induced by h and g, respectively. Note that $\mathbf{E}(h, g) \cong \mathbf{h} \cap \mathbf{g}$, so $\mathbf{E}(h, g) \in IS_\cap IP(\mathcal{K}) \subseteq S_\cap IP(\mathcal{K})$. ⊣

So far, we have introduced some class operators that are relevant to our investigations and seen that there are some interesting interactions among them. Our next result shows that for a class $\mathcal{K}$ contained in a quasivariety $\mathcal{Q}$ with $S(\mathcal{K}) = \mathcal{Q}$ the closure under $H_{\mathcal{Q}}$, $S_\cap$, and P ensures several other strong closure conditions. Among these, one that is particularly relevant for our purposes is that $\mathcal{K}$ is closed under arbitrary nonempty intersections of subalgebras.

Recall that a class $\mathcal{K}$ of algebras has the *intersection property* (*i.p.*) provided that for all $\mathbf{A} \in \mathcal{K}$ and all $\mathcal{F} \subseteq S(\mathbf{A})$ with $\bigcap_{\mathbf{B} \in \mathcal{F}} B \neq \emptyset$ we have $\bigcap \mathcal{F} \in \mathcal{K}$.

PROPOSITION 6. *Let $\mathcal{Q}$ be a quasivariety, $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. Suppose $\mathcal{K}$ is closed under $H_{\mathcal{Q}}$, $S_\cap$, and P. Then, the following hold*:

(1) *$\mathcal{K}$ is closed under H^{-1}.*
(2) *$\mathcal{K}$ is closed under S_E.*
(3) *$\mathcal{K}$ is closed under P_U.*
(4) *$\mathcal{K}$ has the i.p.*
(5) *$\mathcal{K}$ is closed under directed unions.*

PROOF. (1) and (2) follow from (5) and (6) of Lemma 5, respectively. We leave the easy proof of (3) to the reader.

Let us prove (4). Take $\{\mathbf{A}\} \cup \{\mathbf{A}_i : i \in I\} \subseteq \mathcal{K}$ with each $\mathbf{A}_i \leq \mathbf{A}$ and satisfying $\bigcap_{i \in I} A_i \neq \emptyset$. Let $\imath$ be the canonical embedding of $\mathbf{A}$ into $\mathbf{A}^I$. Observe that $\imath[\mathbf{A}]$, $\prod_{i \in I} \mathbf{A}_i$ and $\mathbf{A}^I$ are all in $\mathcal{K}$, so $\imath[\mathbf{A}] \cap \prod_{i \in I} \mathbf{A}_i \in \mathcal{K}$. It is easy to check that $\imath[\bigcap_{i \in I} \mathbf{A}_i] = \imath[\mathbf{A}] \cap \prod_{i \in I} \mathbf{A}_i \in \mathcal{K}$, and thus $\bigcap_{i \in I} \mathbf{A}_i \in \mathcal{K}$.

Next, we prove (5). Let $\langle I, \leq\rangle$ be an upward directed poset, and for each $i \in I$ let $\mathbf{A}_i \in \mathcal{K}$ such that $\langle \mathbf{A}_i : i \in I\rangle$ is a directed system, that is, $i \leq j$ implies $\mathbf{A}_i \leq \mathbf{A}_j$. We prove that $\mathbf{A} := \bigcup_{i \in I} \mathbf{A}_i$ is in $\mathcal{K}$. Since $\mathbf{A} \in \mathcal{Q}$, by assumption, there is $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A} \leq \mathbf{B}$. Now, observe that the set $\{\uparrow i : i \in I\}$, where $\uparrow i := \{j \in I : i \leq j\}$, has the finite intersection property. Hence, there is an ultrafilter U over I containing $\uparrow i$ for all $i \in I$. Next, consider $C := \{c/U \in B^I/U : c \in \prod_{i \in I} A_i\}$, and note that $\mathbf{C} \cong \prod_{i \in I} \mathbf{A}_i/U \in \mathcal{K}$. Let $\imath$ be the canonical embedding of $\mathbf{B}$ into $\mathbf{B}^I/U$. Note that $\imath[\mathbf{B}]$, $\mathbf{C}$, and $\mathbf{B}^I/U$ are in $\mathcal{K}$, and thus $\imath[\mathbf{B}] \cap \mathbf{C} \in \mathcal{K}$. The final step to establish that $\mathbf{A}$ is in $\mathcal{K}$ is to show that $\imath[\mathbf{A}] = \imath[\mathbf{B}] \cap \mathbf{C}$. Take $b \in B$ such that $\imath(b) \in C$, that is, $\imath(b) = c/U$ for some $c \in \prod_{i \in I} A_i$. In particular, there is $i \in I$ such that $b = c_i \in A$. This takes care of the right to left inclusion. For the remaining inclusion,

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fix $a \in A$, say $a \in A_{i_0}$. For each $i \in I \setminus \uparrow i_0$ choose an element $a_i \in A_i$. Now, define $c \in \prod_{i \in I} A_i$ by

$$c_i := \begin{cases} a & \text{if } i \in \uparrow i_0, \\ a_i & \text{otherwise.} \end{cases}$$

Finally, observe that, since $\uparrow i_0 \in U$, we have that $\iota(a) = c/U$. ⊣

It is worth mentioning that items (4) and (5) are inspired by A. Robinson's work on first-order classes with the i.p. [13].

REMARK 7. In what follows, if a class $\mathcal{K}$ satisfies the assumptions of Proposition 6, in the light of item (4) of said lemma, we shall make free use of the fact that $\mathcal{K}$ has the i.p. without explicitly mentioning it.

Let us summarize in one statement the conditions we have found so far to be necessary for a subclass $\mathcal{K}$ of a quasivariety $\mathcal{Q}$ with $S(\mathcal{K}) = \mathcal{Q}$ to be axiomatizable relative to $\mathcal{Q}$ by generalized AE-sentences.

LEMMA 8. *Let $\mathcal{Q}$ be a quasivariety, $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. Suppose $\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by a set of generalized AE-sentences. Then, $\mathcal{K}$ has the i.p. is closed under directed unions and under $H_{\mathcal{Q}}$, $S_{\cap}$, H^{-1}, S_E, P, and P_U.*

As we shall shortly see, the closure conditions listed in Lemma 8 do not suffice to characterize axiomatizability by generalized AE-sentences. In our next section we take a closer look at how the i.p. behaves in our classes of interest.

§4. The closure operator induced by a class with the i.p. Let $\mathcal{K}$ be a class with the i.p. For each $\mathbf{A} \in \mathcal{K}$ we have a closure operator on A defined by $C_{\mathbf{A}}(S) := \bigcap\{B : S \subseteq B, \mathbf{B} \leq \mathbf{A}, \mathbf{B} \in \mathcal{K}\}$. Of course, $C_{\mathbf{A}}(S)$ is always a subuniverse of $\mathbf{A}$. For notational convenience, depending on the context, $C_{\mathbf{A}}(S)$ may also stand for the algebra with universe $C_{\mathbf{A}}(S)$.

As we shall expose in our next two sections, the closure operators induced by our classes of interest enjoy some very particular properties.

4.1. Local smallness. The closure operators induced by classes axiomatizable by generalized AE-sentences have a convenient algebraic description.

LEMMA 9. *Let $\mathcal{Q}$ be a quasivariety, $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. Suppose $\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by a set Σ of generalized AE-sentences. Then, for every $\mathbf{A} \in \mathcal{K}$ and $S \subseteq A$ we have $C_{\mathbf{A}}(S) = \langle S \rangle^{\mathbf{A}^{\Sigma}}$.*

PROOF. Fix $\mathbf{A} \in \mathcal{K}$ and $S \subseteq A$. Observe first that, if $\mathbf{B} \leq \mathbf{A}$ with $\mathbf{B} \in \mathcal{K}$, then $\mathbf{B}^{\Sigma} \leq \mathbf{A}^{\Sigma}$. This entails $\langle S \rangle^{\mathbf{A}^{\Sigma}} \subseteq C_{\mathbf{A}}(S)$. Next, if $\mathbf{K} \leq \mathbf{A}^{\Sigma}$, then its reduct $\mathbf{K}_0$ to the language of $\mathcal{Q}$ belongs to $\mathcal{K}$. This is enough to establish that $C_{\mathbf{A}}(S) \subseteq \langle S \rangle^{\mathbf{A}^{\Sigma}}$. ⊣

A class $\mathcal{K}$ with the i.p. is said to be *locally small* provided that there is a cardinal κ such that for every $\mathbf{A} \in \mathcal{K}$ and each finite subset $S \subseteq A$ we have $|C_{\mathbf{A}}(S)| \leq \kappa$.

LEMMA 10. *Let $\mathcal{Q}$ be a quasivariety in the language L, and let $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. If $\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by a set of generalized AE-sentences, then $\mathcal{K}$ is locally small.*

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PROOF. Let Σ be a set of generalized AE-sentences that axiomatizes $\mathcal{K}$ relative to $\mathcal{Q}$. We can assume there are disjoint sets X, Z of variables, with $|X| = \omega$ and $|Z| = \kappa \geq \omega$, such that every sentence in Σ has its universal variables in X and its existential variables in Z. Let λ be the cardinality of the κ-generated $\mathcal{Q}$-free algebra. Since κ is infinite, it is a bound to the cardinality of the ω-generated $\mathcal{Q}$-free algebra, and thus, we may assume without loss that $|L| \leq \lambda$. Furthermore, the number of atomic formulas in L with variables from $X \cup Z$ is also bounded by λ. Hence, modulo equivalence over $\mathcal{Q}$, the number of generalized AE-sentences with existential variables in Z is at most 2^λ. In particular, we may assume that $|\Sigma| \leq 2^\lambda$.

Next, let L^Σ and $\mathcal{K}^\Sigma$ be as defined in Section 2. Observe that $|L^\Sigma| \leq \kappa \cdot 2^\lambda = 2^\lambda$, and so the ω-generated $\mathcal{K}^\Sigma$-free algebra has cardinality at most 2^λ. We can now apply Lemma 9 to conclude the proof. ⊣

4.2. Domination. Let $\mathcal{Q}$ be a quasivariety, $\mathbf{V}$ an algebra and $S \subseteq V$. Recall that S *dominates* $\mathbf{V}$ with respect to $\mathcal{Q}$ if for every pair of homomorphisms $h, g \colon \mathbf{V} \to \mathbf{B}$, with $\mathbf{B} \in \mathcal{Q}$, we have that $h|_S = g|_S$ implies $h = g$. The concept of domination, introduced by Isbell [8] in his investigation on epimorphisms, turns out to provide one of the key ingredients in our main theorems. Our next result shows that there is a domination phenomenon that manifests on the behavior of the closure operators we are studying.

LEMMA 11. *Let $\mathcal{Q}$ be a quasivariety, $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. Suppose $\mathcal{K}$ is closed under $H_\mathcal{Q}$, $S_\cap$, and P. Let $\mathbf{A} \in \mathcal{K}$ and $S \subseteq A$. Then, S dominates $C_\mathbf{A}(S)$ with respect to $\mathcal{Q}$.*

PROOF. Let $h, g \colon C_\mathbf{A}(S) \to \mathbf{B}$ with $\mathbf{B} \in \mathcal{Q}$ such that $h|_S = g|_S$. Put $\mathbf{C} := h[C_\mathbf{A}(S)]$ and $\mathbf{D} := g[C_\mathbf{A}(S)]$. Observe that $h[S] = g[S] \subseteq C \cap D$. Since $\mathcal{K}$ is closed under $H_\mathcal{Q}$, we have that $\mathbf{C}, \mathbf{D} \in \mathcal{K}$. Moreover, as $S(\mathcal{K}) = \mathcal{Q}$ and $\mathcal{K}$ is closed under $S_\cap$, it follows that $\mathbf{C} \cap \mathbf{D} \in \mathcal{K}$. Item (1) in Proposition 6 yields $h^{-1}[\mathbf{C} \cap \mathbf{D}] \in \mathcal{K}$, and now, since $S \subseteq h^{-1}[C \cap D]$, we conclude that $h^{-1}[\mathbf{C} \cap \mathbf{D}] = C_\mathbf{A}(S)$. This shows that $C = C \cap D$, so $C \subseteq D$. Analogously, $D \subseteq C$, hence $\mathbf{C} = \mathbf{D}$. Thus, $h, g \colon C_\mathbf{A}(S) \to \mathbf{C}$. Using item 2 in Proposition 6, we get that $\mathbf{E}(h, g) \in \mathcal{K}$. Finally, as $S \subseteq E(h, g)$, we conclude that $\mathbf{E}(h, g) = C_\mathbf{A}(S)$, that is, $h = g$. ⊣

§5. Main theorems.

5.1. Axiomatizability by generalized AE-sentences. With the information collected so far, we are able to state and prove a characterization of axiomatizability by generalized AE-sentences. However, the following general result about domination in quasivarieties allows us to show the surprising fact that any such axiomatizability is equivalent to an axiomatizability by AE_ω-sentences.

LEMMA 12. *Let $\mathcal{Q}$ be a quasivariety, $\mathbf{A} \in \mathcal{Q}$ and $S \subseteq A$ such that S dominates $\mathbf{A}$ with respect to $\mathcal{Q}$. Then, for each $a \in A \setminus S$ there is a set $W \subseteq A$ such that*:

- $a \in W$;
- $|W| \leq \omega$;
- *S dominates $\langle S \cup W \rangle^\mathbf{A}$ with respect to $\mathcal{Q}$.*

PROOF. We start with an auxiliary result.

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(†) For each $c \in A \setminus S$ there is a finite subset $D_c \subseteq A$ such that for every $\mathbf{V} \leq \mathbf{A}$ with $S \cup \{c\} \cup D_c \subseteq V$, every $\mathbf{B} \in \mathcal{Q}$ and every pair of homomorphisms $h, g \colon \mathbf{V} \to \mathbf{B}$ with $h|_S = g|_S$ we have that $h(c) = g(c)$.

Under the assumptions of the claim, Theorem 3.2 of [3] provides the existence of a first-order formula $\exists \mathbf{z}\, \alpha(\mathbf{x}, y, \mathbf{z})$, where α is a conjunction of equations, with the following two properties:

- $\mathcal{Q} \vDash \exists \mathbf{z}\, \alpha(\mathbf{x}, y_1, \mathbf{z}) \wedge \exists \mathbf{z}\, \alpha(\mathbf{x}, y_2, \mathbf{z}) \to y_1 = y_2$;
- there are tuples $\mathbf{s}$ from S and $\mathbf{d}$ from A such that $\mathbf{A} \vDash \alpha[\mathbf{s}, c, \mathbf{d}]$.

Next, we recursively define a sequence of sets W_i for $i \in \omega$ as follows. Set $W_0 := \{a\}$ and $W_{i+1} := \bigcup_{w \in W_i} D_w$; in the recursive step, D_w is provided by (†). Now define $W := \bigcup_{i \in \omega} W_i$. Clearly, W satisfies that $a \in W$ and $|W| \leq \omega$. To verify the third condition, take $\mathbf{B} \in \mathcal{Q}$ and homomorphisms $h, g \colon \langle S \cup W \rangle^{\mathbf{A}} \to \mathbf{B}$ with $h|_S = g|_S$. Observe that all we need to show is that $h|_W = g|_W$. Indeed, given $c \in W$, since $D_c \subseteq W$, applying (†) with $\mathbf{V} = \langle S \cup W \rangle^{\mathbf{A}}$ immediately produces $h(c) = g(c)$. ⊣

Now we are ready to present our preservation result for generalized AE-sentences.

THEOREM 13. *Let $\mathcal{Q}$ be a quasivariety, $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. Let $* \in \{\cap, E\}$. The following are equivalent*:

1. *$\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by a set of generalized AE-sentences.*
2. *$\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by AE_ω-sentences.*
3. *$\mathcal{K}$ is closed under $H_{\mathcal{Q}}$, S_*, P and locally small.*

PROOF. 2.⇒1. Trivial.

1.⇒3. This follows from Lemmas 8 and 10.

3.⇒2. We start by observing that, under the remaining assumptions, it is equivalent for $\mathcal{K}$ to be closed under $S_\cap$ or S_E (see Lemma 5, items (4) and (6)). Consequently, by Proposition 6, $\mathcal{K}$ has the i.p.

We proceed next to construct axioms for $\mathcal{K}$. Fix a positive integer n, and let η be a cardinal witnessing the local smallness of $\mathcal{K}$. Define the collection

$$X := \{\langle \mathbf{K}, k_1, \dots, k_n \rangle : \mathbf{K} \in \mathcal{K}, k_1, \dots, k_n \in K, C_{\mathbf{K}}(\mathbf{k}) = K, K \subseteq \eta\};$$

observe that X is a set. Next, let

$$\langle \mathbf{P}, s_1, \dots, s_n \rangle := \prod X,$$

and $S := \{s_1, \dots, s_n\}$. Since $\mathbf{P} \in \mathcal{K}$, we can define $\mathbf{F}$ as the subalgebra of $\mathbf{P}$ with universe $C_{\mathbf{P}}(S)$. The freeness of $\mathbf{F}$ for $\mathcal{K}$, established in the next two assertions, serves a fundamental role in the remainder of the proof. First note that

(i) for each $\langle \mathbf{K}, k_1, \dots, k_n \rangle \in X$ the projection map $\pi_{\mathbf{K}}$ from F into K maps $\mathbf{s} \mapsto \mathbf{k}$ and is onto.

In fact, if $\mathbf{K}' := \pi_{\mathbf{K}}[\mathbf{F}]$, then $\mathbf{K}' \in \mathcal{K}$, and, as $k_1, \dots, k_n \in K'$, we must have $\mathbf{K}' = \mathbf{K}$.

Moreover, it follows from (i) that

(ii) for each $\mathbf{A} \in \mathcal{K}$ and elements $a_1, \dots, a_n \in A$ such that $C_{\mathbf{A}}(\mathbf{a}) = A$, there is a surjective homomorphism $h \colon \mathbf{F} \to \mathbf{A}$ mapping $\mathbf{s} \mapsto \mathbf{a}$.

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Indeed, since $|A| \leq \eta$, there is $\langle \mathbf{K}, k_1, \dots, k_n \rangle \in X$ and an isomorphism $\gamma \colon \mathbf{K} \to \mathbf{A}$ mapping $\mathbf{k} \mapsto \mathbf{a}$. Thus, $\gamma \circ \pi_{\mathbf{K}} \colon \mathbf{F} \to \mathbf{A}$ is the desired surjection.

We proceed now to populate Φ_n with an AE_ω-sentence φ_v for each $v \in F \setminus S$. Fix an arbitrary $v \in F \setminus S$. By Lemmas 11 and 12, there is a sequence $\mathbf{v}$ starting with v, having length at most ω and such that $\mathbf{V} := \langle \mathbf{v} \rangle^{\mathbf{F}}$ satisfies the following:

(iii) $S \cup \{v\} \subseteq V \subseteq F$,
(iv) for every pair of homomorphisms $h, g \colon \mathbf{V} \to \mathbf{B}$, with $\mathbf{B} \in \mathcal{Q}$, we have that $h|_S = g|_S$ implies $h = g$.

Let $\Delta(x_1, \dots, x_n, \mathbf{z})$ be the positive atomic diagram of the tuple $\langle s_1, \dots, s_n, \mathbf{v} \rangle$ in $\mathbf{F}$. Observe that, by (iv), we have

$$\mathcal{Q} \vDash \Delta(\mathbf{x}, \mathbf{z}) \cup \Delta(\mathbf{x}, \mathbf{z}') \to \mathbf{z} = \mathbf{z}'.$$

For each variable z in the sequence $\mathbf{z}$, by compactness, there is a finite subset δ_z of Δ such that

$$\mathcal{Q} \vDash \delta_z(\mathbf{x}, \mathbf{z}) \cup \delta_z(\mathbf{x}, \mathbf{z}') \to z = z'.$$

Now, let $\alpha_v(\mathbf{x}, \mathbf{z})$ be the conjunction of the atomic formulas in $\bigcup_{z \text{ in } \mathbf{z}} \delta_z(\mathbf{x}, \mathbf{z})$. Of course, as $|\mathbf{z}| \leq \omega$, there are at most ω conjuncts in α_v. Also, by construction, we clearly have

(v) $\mathcal{Q} \vDash \alpha_v(\mathbf{x}, \mathbf{z}) \wedge \alpha_v(\mathbf{x}, \mathbf{z}') \to \mathbf{z} = \mathbf{z}'$.

Let us define the AE_ω-sentence

$$\varphi_v := \forall \mathbf{x} \exists! \, \mathbf{z} \, \alpha_v(\mathbf{x}, \mathbf{z}),$$

and put

$$\Phi_n := \{\varphi_v : v \in F \setminus S\}.$$

Since Φ_n will form part of the axiomatization of $\mathcal{K}$, we establish next that $\mathcal{K} \vDash \Phi_n$. So, fix $\varphi_v \in \Phi_n$, an algebra $\mathbf{A} \in \mathcal{K}$, and $a_1, \dots, a_n \in A$. Let $\mathbf{A}'$ be the subalgebra of $\mathbf{A}$ with universe $C_{\mathbf{A}}(a_1, \dots, a_n)$. By (ii), there is a homomorphism $h \colon \mathbf{F} \to \mathbf{A}'$ mapping $\mathbf{s} \to \mathbf{a}$. Now, since $\mathbf{F} \vDash \alpha_v[\mathbf{s}, \mathbf{v}]$, we have $\mathbf{A} \vDash \alpha_v[\mathbf{a}, h(\mathbf{v})]$. Finally, observe that, by (v), these are the unique solutions to $\alpha_v(\mathbf{a}, \mathbf{z})$ in $\mathbf{A}$.

We now prove that

(vi) for all $\mathbf{A} \in \mathcal{Q}$ such that $\mathbf{A} \vDash \Phi_n$ and all $a_1, \dots, a_n \in A$ there is $\mathbf{K} \leq \mathbf{A}$ such that $\mathbf{K} \in \mathcal{K}$ with $a_1, \dots, a_n \in K$ and $C_{\mathbf{K}}(\mathbf{a}) = K$.

Consider $\mathbf{A} \in \mathcal{Q}$ such that $\mathbf{A} \vDash \Phi_n$, and take $a_1, \dots, a_n \in A$. Since $S(\mathcal{K}) = \mathcal{Q}$, there is $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A} \leq \mathbf{B}$. Put $\mathbf{K} := C_{\mathbf{B}}(\mathbf{a})$, and take $k \in K$. We aim to prove that $k \in A$. By (ii), there is a surjective homomorphism $h \colon \mathbf{F} \to \mathbf{K}$ mapping $\mathbf{s} \mapsto \mathbf{a}$. Let $v \in F$ be such that $h(v) = k$. If $v \in S$, we are done. So, assume $v \notin S$. By the construction of α_v, there is a sequence $\mathbf{v}$ from $\mathbf{F}$ starting with v such that $\mathbf{F} \vDash \alpha_v[\mathbf{s}, \mathbf{v}]$. Then, $\mathbf{K} \vDash \alpha_v[\mathbf{a}, h(\mathbf{v})]$. On the other hand, since $\mathbf{A} \vDash \varphi_v$, there is a sequence $\mathbf{b}$ from A such that $\mathbf{A} \vDash \alpha_v[\mathbf{a}, \mathbf{b}]$. But, (v) says that $h(\mathbf{v}) = \mathbf{b}$, and thus, $k \in A$. This proves that $K \subseteq A$, and the proof of (vi) concludes by observing that $\mathbf{K} = C_{\mathbf{K}}(\mathbf{a})$.

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Next, let us define

$$\Phi := \bigcup_{n\in\omega} \Phi_n.$$

The last step is to show that Φ axiomatizes $\mathcal{K}$ relative to $\mathcal{Q}$.

Assume $\mathbf{A} \in \mathcal{Q}$ is such that $\mathbf{A} \models \Phi$. For each finite subset E of A, by (vi), there is $\mathbf{K}_E \in \mathcal{K}$ with $E \subseteq K_E \subseteq A$ and $K_E = C_{\mathbf{K}_E}(E)$. Finally, observe that $\mathbf{A}$ is the union of the directed system $\langle \mathbf{K}_E : \text{finite } E \subseteq A\rangle$, and hence, we are done by (5) of Proposition 6. ⊣

5.2. Axiomatizability by AE-sentences. A natural question to ask at this point is whether there are classes $\mathcal{K}$ which fall under the assumptions of Theorem 13, and thus are axiomatizable by AE_ω-sentences, but which are not axiomatizable by finitary AE-sentences. A positive answer is provided by Example 15 below.

It is again in the behavior of the closure operator induced by $\mathcal{K}$ where the property distinguishing the finitary case manifests.

We say that $\mathcal{K}$ has the *local domination property* with respect to $\mathcal{Q}$ if for every $\mathbf{A} \in \mathcal{K}$, every finite subset $S \subseteq A$, and each $a \in C_{\mathbf{A}}(S) \setminus S$ there is a subalgebra $\mathbf{V}$ of $\mathbf{A}$ such that:

- $S \cup \{a\} \subseteq V \subseteq C_{\mathbf{A}}(S)$;
- $\mathbf{V}$ is finitely generated;
- S dominates $\mathbf{V}$ with respect to $\mathcal{Q}$.

LEMMA 14. *Let $\mathcal{Q}$ be a quasivariety in the language L, and let $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. If $\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by AE-sentences, then $\mathcal{K}$ has the local domination property.*

PROOF. Let Σ be a set of AE-sentences axiomatizing $\mathcal{K}$ relative to $\mathcal{Q}$. Fix $\mathbf{A} \in \mathcal{K}$, a finite subset S of A and $a \in C_{\mathbf{A}}(S)$. By Lemma 9, there is an L^Σ-term $t(x_1, \dots, x_n)$ and $s_1, \dots, s_n \in S$ such that $a = t^{\mathbf{A}^\Sigma}(s_1, \dots, s_n)$. Let $\sigma_1, \dots, \sigma_n \in \Sigma$ be such that every new function symbol occurring in t comes from some σ_i. It is not hard to show that there is an AE-sentence $\delta := \forall x_1 \dots x_n \exists! z_1 \dots z_m\, \alpha$ that holds in $\mathcal{K}$ such that $t^{\mathbf{K}^\Sigma} = [\delta]^{\mathbf{K}}_{z_1}$ for every $\mathbf{K} \in \mathcal{K}$. Now, since $C_{\mathbf{A}}(S) \models \delta$, there are $b_1, \dots, b_m \in C_{\mathbf{A}}(S)$ such that $b_1 = a$ and $C_{\mathbf{A}}(S) \models \alpha[s_1, \dots, s_n, b_1, \dots, b_m]$. Let $\mathbf{V}$ be the subalgebra of $\mathbf{A}$ generated by $S \cup \{b_1, \dots, b_m\}$. Using the fact that $\mathcal{Q} \models U(\delta)$, it is straightforward to prove that S dominates $\mathbf{V}$. ⊣

Our next example shows that it is possible for a class $\mathcal{K}$ to be axiomatizable relative to the quasivariety $\mathcal{Q} = S(\mathcal{K})$ by an AE_ω-sentence but fail to be axiomatizable relative to $\mathcal{Q}$ by AE-sentences.

EXAMPLE 15. Let L be the language $\{0, f\}$, where 0 is nullary and f is binary. We say that an element $a \in A$ is *n-reachable* if there is a sequence $a_0, \dots, a_n$ from A such that $a_0 = 0^{\mathbf{A}}$, $a_n = a$ and $f^{\mathbf{A}}(a_{i-1}, a_i) = 0^{\mathbf{A}}$ for $i \in \{1, \dots, n\}$. We say that an element $a \in A$ has a *successor* if there is $b \in A$ such that $f^{\mathbf{A}}(a, b) = 0^{\mathbf{A}}$.

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For each positive integer n define α_n as the formula

$$f(0,z_1)=0\wedge\bigwedge_{i=1}^{n}f(z_i,z_{i+1})=0\wedge f(0,w_1)=0\wedge\bigwedge_{i=1}^{n}f(w_i,w_{i+1})=0\to z_n=w_n,$$

and let q_n be the universal closure of α_n. Observe that an $\mathbf{A}$ satisfies q_n if and only if there is at most one element in A that is n-reachable and has a successor. Define

$$\mathcal{Q}:=\mathrm{Mod}(\{q_n:n\geq 1\}).$$

Next, consider the AE_ω-sentence φ given by

$$\exists!\, z_1 z_2 \ldots\ f(0,z_1)\wedge\bigwedge_{i\geq 1}f(z_i,z_{i+1})=0.$$

Observe that $\mathcal{Q}\vDash U(\varphi)$. Also, given $\mathbf{A}\in\mathcal{Q}$ define

$$R_{\mathbf{A}}:=\{n:\ \text{there is an } n\text{-reachable element in } \mathbf{A}\}.$$

Clearly, $R_{\mathbf{A}}$ is an initial segment of the positive integers. Furthermore, it is not hard to see that $\mathbf{A}\vDash\varphi$ iff $R_{\mathbf{A}}$ is the set of all positive integers.

Let $\mathcal{K}:=\mathcal{Q}\cap\mathrm{Mod}(\varphi)$; we prove that $S(\mathcal{K})=\mathcal{Q}$. Fix an arbitrary $\mathbf{A}\in\mathcal{Q}\setminus\mathcal{K}$. Then, $R_{\mathbf{A}}$ has a greatest element n. Let $a\in A$ be n-reachable. We construct an extension $\bar{\mathbf{A}}$ of $\mathbf{A}$ by adding ω new distinct elements $b_{n+1},b_{n+2},\ldots$ to its universe and defining $f^{\bar{\mathbf{A}}}$ as an extension of $f^{\mathbf{A}}$ such that:

- $f^{\bar{\mathbf{A}}}(a,b_{n+1})=0^{\mathbf{A}}$;
- $f^{\bar{\mathbf{A}}}(b_{n+i},b_{n+i+1})=0^{\mathbf{A}}$ for $i\geq 1$;
- $f^{\bar{\mathbf{A}}}(x,y)\neq 0^{\mathbf{A}}$ for all other new cases.

It is not hard to see that $\bar{\mathbf{A}}\in\mathcal{K}$.

Finally, we claim that $\mathcal{K}$ does not have the local domination property, and thus is not axiomatizable relative to $\mathcal{Q}$ by AE-sentences. Let $\mathbf{K}\in\mathcal{K}$ be the algebra with universe ω, with $0^{\mathbf{K}}:=0$ and

$$f^{\mathbf{K}}(x,y):=\begin{cases}0 & \text{if } y=x+1,\\ 1 & \text{otherwise.}\end{cases}$$

Clearly, $C_{\mathbf{K}}(\{0\})=K$. We show that there is no finitely generated subalgebra $\mathbf{V}$ of $\mathcal{K}$ containing 1 and dominated by $\{0\}$. Fix a finitely generated subalgebra $\mathbf{V}$ of $\mathcal{K}$. By the definition of $\mathcal{K}$ it is clear that $\mathbf{V}$ is finite, and hence there is a greatest positive integer m that belongs to V. Let $\mathbf{A}$ be the algebra with universe $\{0,1,\ldots,m\}\cup\{m'\}$, where m' is a new element, $0^{\mathbf{A}}:=0$ and

$$f^{\mathbf{A}}(x,y):=\begin{cases}0 & \text{if } y=x+1,\\ 0 & \text{if } x=m-1,\ y=m',\\ 1 & \text{otherwise.}\end{cases}$$

The reader may easily check that $\mathbf{A}\in\mathcal{Q}$. Now, define $h,g\colon\mathbf{V}\to\mathbf{A}$ by $h(x):=x$ for $x\in V$, $g(x):=x$ for $x\in V\setminus\{m\}$, $g(m):=m'$. So, h,g agree on $\{0\}$; however, they do not agree on V. Hence, $\mathcal{K}$ does not have the local domination property.

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Having established the necessity of the local domination property, we are now ready to state the main theorem of this note.

THEOREM 16. *Let $\mathcal{Q}$ be a quasivariety, $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. The following are equivalent*:

1. *$\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by AE-sentences.*
2. *$\mathcal{K}$ is closed under $H_{\mathcal{Q}}$, $S_{\cap}$, and P, is locally small, and has the local domination property.*
3. *$\mathcal{K}$ is closed under $H_{\mathcal{Q}}$, $S_{\cap}$, and P, is a first-order class, and has the local domination property.*

PROOF. 1.⇒3. is a consequence of Lemmas 8 and 14.

To prove 3.⇒2. it suffices to show that the first-order axiomatizability of $\mathcal{K}$ yields local smallness. Let $\eta := \max\{\omega, |L|\}$, where L is the language of $\mathcal{Q}$. Fix $\mathbf{A} \in \mathcal{K}$ and a finite $S \subseteq A$. If $\mathbf{K}$ is the subalgebra of $\mathbf{A}$ with universe $C_{\mathbf{A}}(S)$, by the downward Löwenheim–Skolem Theorem, there is an elementary substructure $\mathbf{K}_0$ of $\mathbf{K}$ containing S with $|K| \leq \eta$. As $\mathcal{K}$ is axiomatizable, we know that $\mathbf{K}_0 \in \mathcal{K}$, and thus, $\mathbf{K}_0 = \mathbf{K}$. It follows that η witnesses the local smallness of $\mathcal{K}$.

The proof of 2.⇒1. is the same as that of 3.⇒2. of Theorem 13 introducing the following change in the construction of φ_v. At the step where the sequence $\mathbf{v}$ is chosen of length at most ω (using Lemmas 11 and 12), observe that, due to the local domination property, we can actually choose $\mathbf{v}$ to be of finite length. ⊣

In item 3. of the theorem above we can replace the condition of $\mathcal{K}$ being first-order axiomatizable with $\mathcal{K}$ being closed under elementary substructures. This is due to the fact that $\mathcal{K}$ is already closed under isomorphisms and ultraproducts.

§6. The finitely generated case. Our main theorem takes a particularly neat form in the finitely generated case. First we need the following auxiliary result.

LEMMA 17. *Let $\mathcal{Q}$ be a quasivariety, $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. Suppose $\mathcal{K}$ is closed under $H_{\mathcal{Q}}$, $S_{\cap}$, and P. Suppose there is a finite set $\mathcal{F}$ of finite algebras such that $\mathcal{Q} = Q(\mathcal{F})$ and $\mathcal{F} \subseteq \mathcal{K}$. Then, for every $\mathbf{A} \in \mathcal{K}$ and each finite subset $S \subseteq A$ we have $|C_{\mathbf{A}}(S)| < \omega$.*

PROOF. Fix $\mathbf{A} \in \mathcal{K}$ and a finite $S = \{s_1, \ldots, s_n\} \subseteq A$. We can assume without loss that $\mathbf{A}$ is a subalgebra of $\mathbf{P} := \prod_{i \in I} \mathbf{A}_i$ with each $\mathbf{A}_i \in \mathcal{F}$. Define the equivalence relation $\sim$ on I by $i \sim j$ iff $\mathbf{A}_i = \mathbf{A}_j$ and $\pi_i(s_\ell) = \pi_j(s_\ell)$ for all $\ell \in \{1, \ldots, n\}$. Observe that, since $\mathcal{F}$ and S are finite, there is a finite amount of $\sim$-classes. Let I_0 be a transversal of $\sim$, and let S_0 be the projection of S into $\mathbf{P}_0 := \prod_{i \in I_0} \mathbf{A}_i$; clearly, P_0 is finite. Now, take $\mathbf{K}$ to be the subalgebra of $\mathbf{P}_0$ with universe $C_{\mathbf{P}_0}(S_0)$. Next, define for each $k \in K$ the element $\widetilde{k} \in P$ with $\widetilde{k}(i) := k(j)$, where j is the unique $j \in I_0$ with $j \sim i$. Let $\widetilde{K} := \{\widetilde{k} : k \in K\}$, and note that the following hold:

- $S \subseteq \widetilde{K}$;
- $\widetilde{K}$ is finite;
- $\widetilde{\mathbf{K}} \cong \mathbf{K}$, and thus, $\widetilde{\mathbf{K}} \in \mathcal{K}$.

Finally, observe that $S \subseteq A \cap \widetilde{K}$ and $\mathbf{A} \cap \widetilde{\mathbf{K}} \in \mathcal{K}$. So, $C_{\mathbf{A}}(S)$ is finite. ⊣

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THEOREM 18. *Let $\mathcal{Q}$ be a quasivariety, $\mathcal{K} \subseteq \mathcal{Q}$ such that $S(\mathcal{K}) = \mathcal{Q}$. Further, suppose that there is a finite set $\mathcal{F}$ of finite algebras such that $\mathcal{Q} = Q(\mathcal{F})$ and $\mathcal{F} \subseteq \mathcal{K}$. The following are equivalent*:

1. *$\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by a set of generalized AE-sentences.*
2. *$\mathcal{K}$ is axiomatizable relative to $\mathcal{Q}$ by AE-sentences.*
3. *$\mathcal{K}$ is closed under $H_{\mathcal{Q}}$, $S_{\cap}$, and P.*

PROOF. 1.⇒3. From Lemma 8.

3.⇒2. We aim to apply Theorem 16. By Lemma 17 we know that the closures of finite sets are finite. Of course, this implies the local smallness of $\mathcal{K}$, but also, in view of Lemma 11, it also says that $\mathcal{K}$ has the local domination property.

2.⇒1. Trivial. ⊣

6.1. Injectivity and class operators. In this final section of the article we take a look at the very particular case where $\mathcal{Q}$ is generated as a quasivariety by a finite set $\mathcal{F}$ of finite injective algebras. These strong assumptions produce a simplification of the interaction of the class operators involved. This, in turn, allows us to give for each $\mathcal{S}$ with $\mathcal{F} \subseteq \mathcal{S} \subseteq \mathcal{Q}$ a tight description of the smallest AE-subclass of $\mathcal{Q}$ that contains $\mathcal{S}$.

Let $\mathcal{Q}$ be a quasivariety and $\mathbf{D} \in \mathcal{Q}$. Recall that $\mathbf{D}$ is *injective* in $\mathcal{Q}$ provided that for all $\mathbf{A}, \mathbf{B} \in \mathcal{Q}$ with $\mathbf{A} \leq \mathbf{B}$ and all $h\colon \mathbf{A} \to \mathbf{D}$ there is $\bar{h}\colon \mathbf{B} \to \mathbf{D}$ such that $\bar{h}|_A = h$. Note that any product of algebras that are injective in $\mathcal{Q}$ is once again injective in $\mathcal{Q}$.

In the case that we have access to an injective finite set of finite generators the interaction between our class operators simplifies significantly.

LEMMA 19. *Let $\mathcal{Q} := Q(\mathcal{F})$, where $\mathcal{F}$ is a finite set of finite algebras. Suppose $\mathcal{F} \subseteq \mathcal{S} \subseteq \mathcal{Q}$. Then*:

(1) *If $P(\mathcal{F}) \subseteq \mathcal{S}$, then $S_E H_{\mathcal{Q}}(\mathcal{S}) \subseteq H_{\mathcal{Q}} S_E(\mathcal{S})$.*

In addition, if each algebra in $\mathcal{F}$ is injective in $\mathcal{Q}$, then the following hold:

(2) *If $P(\mathcal{F}) \subseteq \mathcal{S}$, then $S_E S_E(\mathcal{S}) \subseteq S_E(\mathcal{S})$.*

(3) *$S_E S_E P(\mathcal{S}) \subseteq S_E P(\mathcal{S})$.*

(4) *$H_{\mathcal{Q}} S_E P(\mathcal{S})$ is closed under $H_{\mathcal{Q}}$, S_E, and P.*

PROOF. (1) Since $P(\mathcal{F}) \subseteq \mathcal{S}$, every equalizer can be obtained from morphisms having a product of algebras in $\mathcal{F}$ as codomain. Let $\mathbf{E} \in S_E H_{\mathcal{Q}}(\mathcal{S})$. So there are $\mathbf{A} \in H_{\mathcal{Q}}(\mathcal{S})$, an algebra $\mathbf{D}$ in $P(\mathcal{F})$ and $h, g\colon \mathbf{A} \to \mathbf{D}$ such that $\mathbf{E} = \mathbf{E}(h, g)$. Suppose $p\colon \mathbf{B} \to \mathbf{A}$, with $\mathbf{B} \in \mathcal{S}$, is an onto morphism. Now $hp, gp\colon \mathbf{B} \to \mathbf{D}$, and thus $\mathbf{E}(hp, gp) \in S_E(\mathcal{S})$. Finally, observe that $E(h, g) = \{p(b) : b \in B \text{ and } hp(b) = gp(b)\} = p[E(hp, gp)]$, so $\mathbf{E}(h, g) \in H_{\mathcal{Q}} S_E(\mathcal{S})$.

(2) Since $P(\mathcal{F}) \subseteq \mathcal{S}$, every equalizer in $S_E(\mathcal{S})$ can be obtained from morphisms having a product of algebras in $\mathcal{F}$ as codomain. Fix $\mathbf{E} \in S_E S_E(\mathcal{S})$, there are $\mathbf{A} \in S_E(\mathcal{S})$, an algebra $\mathbf{D} \in P(\mathcal{F})$ and $h, g\colon \mathbf{A} \to \mathbf{D}$ with $\mathbf{E} = \mathbf{E}(h, g)$. We also have $\mathbf{A}' \in \mathcal{S}$, an algebra $\mathbf{D}' \in P(\mathcal{F})$ and $h', g'\colon \mathbf{A}' \to \mathbf{D}'$ such that $\mathbf{A} = \mathbf{E}(h', g')$. In particular $\mathbf{A} \leq \mathbf{A}'$, and since $\mathbf{D}$ is injective in $\mathcal{Q}$, there are $\bar{h}, \bar{g}\colon \mathbf{A}' \to \mathbf{D}$ extending h and g. Define $H, G\colon \mathbf{A}' \to \mathbf{D} \times \mathbf{D}'$ by $H(a) := \langle \bar{h}(a), h'(a) \rangle$ and $G(a) := \langle \bar{g}(a), g'(a) \rangle$. Finally, note that $\mathbf{E}(h, g) = \mathbf{E}(H, G) \in S_E(\mathcal{S})$.

(3) Straightforward from (2).

(4) Clearly, $\mathcal{K} := H_{\mathcal{Q}} S_E P(\mathcal{S})$ is closed under $H_{\mathcal{Q}}$. Now, observe that $\mathcal{K}$ is closed under P since, by Lemma 5, we have $P H_{\mathcal{Q}} S_E P(\mathcal{S}) \subseteq H_{\mathcal{Q}} S_E P P(\mathcal{S}) \subseteq$

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$H_{\mathcal{Q}}S_E IPIP(\mathcal{S}) \subseteq H_{\mathcal{Q}}S_E IP(\mathcal{S}) = H_{\mathcal{Q}}IS_E P(\mathcal{S}) = H_{\mathcal{Q}}S_E P(\mathcal{S}) = \mathcal{K}$. To see that $\mathcal{K}$ is closed under S_E note that (1) and (3) say that $S_E H_{\mathcal{Q}}S_E P(\mathcal{S}) \subseteq H_{\mathcal{Q}}S_E S_E P(\mathcal{S}) \subseteq \mathcal{K}$. ⊣

We present next the final result of this note, which is pleasantly reminiscent of the HSP theorem for varieties.

THEOREM 20. *Let $\mathcal{Q} := Q(\mathcal{F})$, where $\mathcal{F}$ is a finite set of finite algebras, and assume that each algebra in $\mathcal{F}$ is injective in $\mathcal{Q}$. Suppose $\mathcal{F} \subseteq \mathcal{S} \subseteq \mathcal{Q}$. Then, the class $H_{\mathcal{Q}}S_E P(\mathcal{S})$ is the least AE-subclass of $\mathcal{Q}$ containing $\mathcal{S}$.*

PROOF. Put $\mathcal{K} := H_{\mathcal{Q}}S_E P(\mathcal{S})$. Clearly, $S(\mathcal{K}) = \mathcal{Q}$ and, by item (4) in Lemma 19, $\mathcal{K}$ is closed under $H_{\mathcal{Q}}$, S_E, and P. Hence, by Theorem 18, we are done. ⊣

REFERENCES

[1] X. CAICEDO, *Implicit connectives of algebraizable logics*. ***Studia Logica***, vol. 78 (2004), pp. 155–170.

[2] X. CAICEDO, M. CAMPERCHOLI, K. A. KEARNES, P. S. TERRAF, A. SZENDREI, and D. VAGGIONE, *Every minimal dual discriminator variety is minimal as a quasivariety*. ***Algebra Universalis***, vol. 82 (2021), no. 2, p. 36.

[3] M. CAMPERCHOLI, *Dominions and primitive positive functions*. ***The Journal of Symbolic Logic***, vol. 83 (2018), no. 1, pp. 40–54.

[4] M. CAMPERCHOLI, D. N. CASTAÑO, J. P. D. VARELA, and J. GISPERT, *Algebraic expansions of logics*. ***The Journal of Symbolic Logic***, vol. 88 (2023), no. 1, pp. 74–92.

[5] M. CAMPERCHOLI and D. J. VAGGIONE, *Algebraically expandable classes*. ***Algebra Universalis***, vol. 61 (2009), no. 2, pp. 151–186.

[6] ———, *Algebraic functions*. ***Studia Logica***, vol. 98 (2011), nos. 1–2, pp. 285–306.

[7] ———, *Implicit definition of the quaternary discriminator*. ***Algebra Universalis***, vol. 68 (2012), no. 1, pp. 1–16.

[8] J. R. ISBELL, *Epimorphisms and dominions*, ***Proceedings of the Conference on Categorical Algebra*** (S. Eilenberg, D. K. Harrison, S. MacLane, H. Röhrl, editors), Springer, Berlin, Heidelberg, 1966, pp. 232–246.

[9] D. KUEKER, *Core structures for theories*. ***Fundamenta Mathematicae***, vol. 89 (1975), no. 2, pp. 155–171.

[10] M. RABIN, *Characterization of convex systems of axioms*. ***Notices of the American Mathematical Society***, vol. 7 (1960), p. 503.

[11] M. O. RABIN, *Classes of models and sets of sentences with the intersection property*. ***Annales de la faculté des Sciences de l'université de Clermont. Mathématiques***, vol. 7 (1962), no. 1, pp. 39–53.

[12] A. ROBINSON, ***On the Metamathematics of Algebra***, Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, 1951.

[13] ———, ***Complete Theories***, North-Holland Publishing Co., Amsterdam, 1956.

[14] D. J. VAGGIONE, *Infinitary baker-Pixley theorem*. ***Algebra Universalis***, vol. 79 (2018), no. 3, Article no. 67, 14 pp.

[15] J. VON NEUMANN, *On regular rings*. ***Proceedings of the National Academy of Sciences***, vol. 22 (1936), no. 12, pp. 707–713.

DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDAD NACIONAL DEL SUR
BAHÍA BLANCA B8000
ARGENTINA
E-mail: diego.castano@uns.edu.ar

FACULTAD DE MATEMÁTICA
ASTRONOMÍA Y FÍSICA (FA.M.A.F.)
UNIVERSIDAD NACIONAL DE CÓRDOBA
CÓRDOBA X5000
ARGENTINA
E-mail: miguel.campercholi@unc.edu.ar
E-mail: dvaggione@gmail.com

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