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The Noncommutative Infinitesimal Equivariant Index Formula

by

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Abstract

In this paper, we establish an infinitesimal equivariant index formula in the noncommutative geometry framework using Greiner's approach to heat kernel asymptotics. An infinitesimal equivariant index formula for odd dimensional manifolds is also given. We define infinitesimal equivariant eta cochains, prove their regularity and give an explicit formula for them. We also establish an infinitesimal equivariant family index formula and introduce the infinitesimal equivariant eta forms as well as compare them with the equivariant eta forms.

Key Words: Infinitesimal equivariant Chern-Connes characters; infinitesimal equivariant eta cochains; infinitesimal equivariant family index; infinitesimal equivariant eta forms.

Mathematics Subject Classification 2010: 58J20, 19K56.

1. Introduction

The Atiyah-Bott-Segal-Singer index formula is a generalization of the Atiyah-Singer index theorem for manifolds admitting group actions. In [BV1], Berline and Vergne gave a heat kernel proof of the Atiyah-Bott-Segal-Singer index formula. In [LYZ], Lafferty, Yu and Zhang gave a very simple and direct geometric proof of the equivariant index formula of Dirac operators. In [PW], Ponge and Wang gave another proof of the equivariant index formula using Greiner's approach to heat kernel asymptotics. For manifolds with boundary, Donnelly [Do] introduced the equivariant eta invariant and generalized the Atiyah-Patodi-Singer index theorem to the equivariant setting. Zhang proved the regularity of the equivariant eta invariant in [Zh]. In [Fa], Fang established an equivariant index formula for odd dimensional manifolds.

The equivariant index formula has an infinitesimal version, which is called the Kirillov formula. Berline and Vergne [BV2] established the Kirillov formula using the equivariant index formula and the localization formula. Bismut introduced the Bismut Laplacian and gave a direct heat kernel proof of the Kirillov formula in [Bi]. The infinitesimal equivariant index formula for manifolds with boundary was established in [Go], where Goette introduced infinitesimal equivariant eta invariants

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and compared equivariant eta invariants with infinitesimal equivariant eta invariants.

On the noncommutative geometry side, Connes [Co] defined the Chern-Connes character of a θ-summable Fredholm module (H, D) over a unital C^*-algebra A, which takes value in the entire cyclic cohomology of A. In [JLO], Jaffe, Lesniewski and Osterwalder introduced an equivariant but convenient version of the Chern-Connes character, which is known as the JLO character. The JLO character was computed in [CM1] and [BlF]. An explicit formula of the equivariant JLO character associated to the invariant Dirac operator, in the presence of a countable discrete group action on a smooth compact spin Riemannian manifold, was given by Azmi [Az] and moreover it was shown that this equivariant cocycle is an element of the delocalized cohomology, paired with an equivariant K-theory idempotent. When G is a compact Lie group, Chern and Hu [CH] gave an explicit formula of the equivariant Chern-Connes character associated to a G-equivariant θ-summable Fredholm module. In [Ge1], for odd dimensional manifolds, the spectral flow was written as pairing of the JLO character with the odd Chern character of an idempotent matrix.

In the framework of noncommutative geometry, Wu established an Atiyah-Patodi-Singer index theorem in [Wu]. To do so, he introduced the total eta invariant (called the higher eta invariant in [Wu]), which is a generalization of the classical Atiyah-Patodi-Singer eta invariants. Wu then proved its regularity using the Getzler symbol calculus as adopted in [BlF] and computed its radius of convergence. Subsequently, he established a variation formula of eta cochains, which he used to obtain the noncommutative Atiyah-Patodi-Singer index theorem. In [Ge2], using superconnection, Getzler gave another proof to the noncommutative Atiyah-Patodi-Singer index theorem, which is more difficult but avoided mention of the operators b and B in cyclic cohomology. In [Wa1], we introduced equivariant eta chains and established an equivariant noncommutative Atiyah-Patodi-Singer index formula which generalized Wu's theorem to the equivariant setting.

This paper is devoted to establishing an infinitesimal equivariant index formula in the noncommutative geometry framework using Greiner's approach to heat kernel asymptotics as well as establish an infinitesimal equivariant index formula for odd dimensional manifolds. In the same framework, we also give an infinitesimal equivariant index formula for manifolds with boundary.

Let D be a differential operator acting on a fiber bundle M over a compact space B. If D is elliptic along the fibers, then D can be viewed as a family of elliptic operators parameterized by B. Atiyah and Singer defined a more general index for D which is an element in the K group $K(B)$. This index is called family index. Atiyah and Singer proved that the analytic and topological indices coincide in $K(B)$. As a consequence, they could determine the Chern character of the difference

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bundle $\mathrm{Ker}D - \mathrm{Coker}D$ and gave a cohomology expression of the Chern character of the difference bundle in terms of certain characteristic classes using Chern-Weil theory. A nice exposition of family index theory can be found in [BGV, Ch.10]. In order to prove family rigidity theorems for certain elliptic operators, Liu and Ma established an equivariant family index formula [LM]. In [Wa2], using Greiner's approach to heat kernel asymptotics, we gave another proof of the local equivariant index theorem for a family of Dirac operators. We also introduced the equivariant eta forms and proved their regularity in [Wa2]. The current paper will study the infinitesimal versions too, including an infinitesimal equivariant family index formula, the definition of infinitesimal equivariant eta forms and the comparison of them with equivariant eta forms.

The rest of the paper is organized as follows: In Section 2, we establish an infinitesimal equivariant index formula in the noncommutative geometry framework using Greiner's approach to heat kernel asymptotics. An infinitesimal equivariant index formula for odd dimensional manifolds is also established. In Section 3, we define truncated infinitesimal equivariant eta cochains and prove their regularity as well as give a formula for them. In Section 4, a proof of the infinitesimal equivariant family index formula is given. We also introduce infinitesimal equivariant eta forms and compare them with equivariant eta forms.

2. The noncommutative infinitesimal equivariant index formula

2.1. The infinitesimal equivariant JLO cocycle

Let M be a compact oriented even dimensional Riemannian manifold without boundary with a fixed spin structure and S be the bundle of spinors on M. Denote by D the associated Dirac operator on $H = L^2(M;S)$, the Hilbert space of L^2-sections of the bundle S. Let $c(df) : S \to S$ denote the Clifford action with $f \in C^\infty(M)$. Suppose that G is a compact connected Lie group acting on M by orientation-preserving isometries preserving the spin structure and $\mathfrak{g}$ is the Lie algebra of G. Then G commutes with the Dirac operator. For $X \in \mathfrak{g}$, let $X_M(p) = \frac{d}{dt}|_{t=0} e^{-tX} p$ be the Killing field induced by X. Let $c(X)$ denote the Clifford action by X_M, and $\mathfrak{L}_X$ denote the Lie derivative. Define $\mathfrak{g}$-equivariant modifications of D and D^2 for $X \in \mathfrak{g}$ as follows:

$$D_X := D - \frac{1}{4}c(X);\ \ H_X := D^2_{-X} + \mathfrak{L}_X = (D + \frac{1}{4}c(X))^2 + \mathfrak{L}_X. \tag{2.1}$$

Then H_X is the equivariant Bismut Laplacian. Let $\mathbb{C}[\mathfrak{g}^*]$ denote the space of formal power series in $X \in \mathfrak{g}$ and ψ_t be the rescaling operator on $\mathbb{C}[\mathfrak{g}^*]$ defined by $X \to \frac{X}{t}$ for $t > 0$.

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Let

$$A = C_G^{\infty}(M) = \{f \in C^{\infty}(M) | f(g \cdot x) = f(x), g \in G, x \in M\}.$$

Then the data $(A, H, D + \frac{1}{4}c(X), G)$ defines a non selfadjoint perturbation of the finitely summable (hence θ-summable) equivariant unbounded Fredholm module (A, H, D, G) in the sense of [KL] (for details, see [CH] and [KL]). For $(A, H, D + \frac{1}{4}c(X), G)$, **the truncated infinitesimal equivariant JLO cochain** $\mathrm{ch}_{2k}(\sqrt{t}D, X)_J$ can be defined by the formula:

$$\mathrm{ch}_{2k}(\sqrt{t}D, X)(f^0, \cdots, f^{2k})_J := t^k \int_{\triangle_{2k}} \mathrm{Str}\Big[\psi_t e^{-t\Omega_X} f^0 e^{-\sigma_0 t(D+\frac{1}{4}c(X))^2} c(df^1)$$

$$\cdot e^{-\sigma_1 t(D+\frac{1}{4}c(X))^2} \cdots c(df^{2k}) e^{-\sigma_{2k} t(D+\frac{1}{4}c(X))^2}\Big]_J d\mathrm{Vol}_{\triangle_{2k}}, \tag{2.2}$$

where $\triangle_{2k} = \{(\sigma_0, \cdots, \sigma_{2k}) | \ \sigma_0 + \cdots \sigma_{2k} = 1\}$ is the $2k$-simplex. For an integer $J \geq 0$, denote by $\mathbb{C}[\mathfrak{g}^*]_J$ the space of polynomials in $X \in \mathfrak{g}$ of degree $\leq J$ and let $(\cdot)_J : \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\mathfrak{g}^*]_J$ be the natural projection. Then $\mathrm{ch}_{2k}(\sqrt{t}D, X)(f^0, \cdots, f^{2k})_J$ is controlled by $\psi_t(C(X))_J t^k \mathrm{Tr}(e^{-\frac{t}{2}D^2})$ via following Lemma 2.2 below (similar to Lemma 2.1 in [GS]), so it is well-defined for $t \in (0, +\infty)$. We will compute the limit of the J-jet of the infinitesimal equivariant JLO cochain

$$\lim_{t \to 0} \mathrm{ch}_{2k}(\sqrt{t}D, X)(f^0, \cdots, f^{2k})_J.$$

In the following, we give some estimates of $\mathrm{ch}_{2k}(\sqrt{t}D, X)(f^0, \cdots, f^{2k})_J$. Let H be a Hilbert space. For $q \geq 0$, denote by $||.||_q$ the Schatten p-norm on the Schatten ideal L^p. Let $L(H)$ denote the Banach algebra of bounded operators on H.

Lemma 2.1 *([CH],[Fe]) (i)* $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$*, for* $A, B \in L(H)$ *and* $AB, BA \in L^1$.
(ii) For $A \in L^1$*, we have* $|\mathrm{Tr}(A)| \leq ||A||_1$, $||A|| \leq ||A||_1$.
(iii) For $A \in L^q$ *and* $B \in L(H)$*, we have:* $||AB||_q \leq ||B||||A||_q$, $||BA||_q \leq ||B||||A||_q$.
(iv) (Hölder Inequality) If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $p, q, r > 0$, $A \in L^p$, $B \in L^q$*, then* $AB \in L^r$ *and* $||AB||_r \leq ||A||_p ||B||_q$.

Fix basis $e_1, \cdots, e_n$ of $\mathfrak{g}$ and let $X = x_1 e_1 + \cdots x_n e_n$. A J-degree polynomial on X means a J-degree polynomial on $x_1, \cdots, x_n$.

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Lemma 2.2 *For any* $1 \geq u > 0$, $t > 0$ *and* t *small,* $X \in \mathfrak{g}$ *and any order* l *differential operator* B*, we have:*

$$||e_J^{-utH_X} B||_{u^{-1}} \leq C(X)_J u^{-\frac{l}{2}} t^{-\frac{l}{2}} (\mathrm{tr}[e^{-\frac{tD^2}{2}}])^u, \tag{2.3}$$

where $C(X)_J$ *is a* J*-degree polynomial with constant coefficients on* X*.*

Proof: Let $H_X = D^2 + F_X$, where F_X is a first order differential operator with degree ≥ 1 coefficients depending on X. By the Duhamel principle, it follows that

$$||e_J^{-utH_X} B||_{u^{-1}} = ||\sum_{m\geq 0}^{J} (-ut)^m \int_{\triangle_m} e^{-v_0 utD^2} F_X e^{-v_1 utD^2}$$

$$\cdot F_X \cdots e^{-v_{m-1} utD^2} F_X e^{-v_m utD^2} B dv||_{u^{-1}}. \tag{2.4}$$

We estimate the term for $m = 2$ on the right hand side of (2.4). Other terms can be estimated similarly. We split $\triangle_2 = J_0 \cup J_1 \cup J_2$ where $J_i = \{(v_0, v_1, v_2) \in \Delta_2 | v_i \geq \frac{1}{3}\}$. Then,

$$\begin{aligned}
&(ut)^2 || \textstyle\int_{J_0} e^{-v_0 utD^2} F_X e^{-v_1 utD^2} F_X e^{-v_2 utD^2} B dv||_{u^{-1}} \\
&\leq (ut)^2 \textstyle\int_{J_0} ||e^{-\frac{v_0 ut}{2} D^2}||_{(uv_0)^{-1}} ||e^{-\frac{v_0 ut}{2} D^2} (1+D^2)^{\frac{l+2}{2}}|| ||(1+D^2)^{-\frac{l+2}{2}} F_X (1+D^2)^{\frac{l+1}{2}}|| \\
&\cdot ||e^{-v_1 utD^2}||_{(uv_1)^{-1}} ||(1+D^2)^{-\frac{l+1}{2}} F_X (1+D^2)^{\frac{l}{2}}|| ||e^{-v_2 utD^2}||_{(uv_2)^{-1}} ||1+D^2)^{-\frac{l}{2}} B|| dv \\
&\leq (ut)^2 \textstyle\int_{J_0} \left(\mathrm{Tr} e^{-\frac{t}{2} D^2}\right)^{uv_0} \left(\mathrm{Tr} e^{-tD^2}\right)^{u(v_1+v_2)} (uv_0 t)^{-\frac{l+2}{2}} \\
&\cdot ||(1+D^2)^{-\frac{l+2}{2}} F_X (1+D^2)^{\frac{l+1}{2}}|| ||(1+D^2)^{-\frac{l+1}{2}} F_X (1+D^2)^{\frac{l}{2}}|| ||1+D^2)^{-\frac{l}{2}} B|| dv \\
&\leq C(X)_2 \left(\mathrm{Tr} e^{-\frac{t}{2} D^2}\right)^u (ut)^{-\frac{l}{2}+1}
\end{aligned} \tag{2.5}$$

where we use that F_X is a first order differential operator and the equality

$$\sup\{(1+x)^{\frac{l}{2}} e^{-\frac{utx}{2}}\} = (ut)^{-\frac{l}{2}} e^{-\frac{l-ut}{2}}. \tag{2.6}$$

J_1 and J_2 can be estimated similarly. For general m, we get

$$||(-ut)^m \int_{\triangle_m} e^{-v_0 utD^2} F_X e^{-v_1 utD^2} F_X \cdots e^{-v_{m-1} utD^2}$$

$$\cdot F_X e^{-v_m utD^2} B dv||_{u^{-1}} \leq C_2 \left(\mathrm{Tr} e^{-\frac{t}{2} D^2}\right)^u (ut)^{-\frac{l}{2}+\frac{m}{2}}. \tag{2.7}$$

By (2.4) and (2.7), (2.3) is obtained. □

Similarly to Lemmas 4.3 and 4.4 in [Wa2], we have

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Lemma 2.3 *Let B_1, B_2 be pseudodifferential operators of positive order p, q respectively. Then for any s, $t > 0$, $0 \le u \le 1$ and t small, $X \in \mathfrak{g}$, we have the following estimate:*

$$||[B_1 e^{-ustH_X} B_2 e^{-(1-u)stH_X}]_J||_{s^{-1}} \le C(X)_J s^{-\frac{p+q}{2}} t^{-\frac{p+q}{2}} (\mathrm{tr}[e^{-\frac{tD^2}{4}}])^s. \tag{2.8}$$

Let B be an operator and l be a positive interger. Write

$$B^{[l]} = [H_X, B^{[l-1]}],\ B^{[0]} = B.$$

Lemma 2.4 *Let B be a finite order differential operator with coefficients on X. Then for any $s > 0$, we have:*

$$[e^{-sH_X} B]_J = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l [B^{[l]} e^{-sH_X}]_J + (-1)^N s^N (B^{[N]}(s))_J, \tag{2.9}$$

where $B^{[N]}(s)$ is given by

$$B^{[N]}(s) = \int_{\triangle_N} e^{-u_1 sH_X} B^{[N]} e^{-(1-u_1)sH_X} du_1 du_2 \cdots du_N. \tag{2.10}$$

Lemma 2.5 *Let B be a finite order differential operator with coefficients on X. Then for any $s > 0$, we have:*

$$[Be^{-sH_X}]_J = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l [e^{-sH_X} B^{[l]}]_J + (-1)^N s^N (B_1^{[N]}(s))_J, \tag{2.11}$$

where $B_1^{[N]}(s)$ is given by

$$B_1^{[N]}(s) = \int_{\triangle_N} e^{-(1-u_1)sH_X} B^{[N]} e^{-u_1 sH_X} du_1 du_2 \cdots du_N. \tag{2.12}$$

Since $\mathfrak{L}_X$ commutes with D and $c(X)$ and $f \in C_G^\infty(M)$, we have by Lemma 2.4:

$$\begin{aligned}
&[e^{-t\mathfrak{L}_X} f^0 e^{-s_1 t(D+\frac{1}{4}c(X))^2} c(df^1) e^{-(s_2-s_1)t(D+\frac{1}{4}c(X))^2} \cdots c(df^{2k}) e^{-(1-s_{2k})t(D+\frac{1}{4}c(X))^2}]_J \\
&= \sum_{\lambda_1,\cdots,\lambda_{2k}=0}^{N-1} \frac{(-1)^{\lambda_1+\cdots+\lambda_{2k}} s_1^{\lambda_1} \cdots s_{2k}^{\lambda_{2k}} t^{\lambda_1+\cdots+\lambda_{2k}}}{\lambda_1! \cdots \lambda_{2k}!} [f^0 [c(df^1)]^{[\lambda_1]} \cdots [c(df^{2k})]^{[\lambda_{2k}]} e^{-tH_X}]_J \\
&+ \sum_{1 \le q \le 2k} \sum_{\lambda_1,\cdots,\lambda_{q-1}=0}^{N-1} \frac{(-1)^{\lambda_1+\cdots+\lambda_{q-1}+N} s_1^{\lambda_1} \cdots s_{q-1}^{\lambda_{q-1}} s_q^N t^{\lambda_1+\cdots+\lambda_{q-1}+N}}{\lambda_1! \cdots \lambda_{q-1}!} [f^0 [c(df^1)]^{[\lambda_1]} \\
&\cdots [c(df^{q-1})]^{[\lambda_{q-1}]} \{[c(df^q)]^{[N]}(s_q t)\} e^{-(s_{q+1}-s_q)tH_X} \cdots c(df^{2k}) e^{-(1-s_{2k})tH_X}]_J.
\end{aligned} \tag{2.13}$$

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Since $f^0[c(df^1)]^{[\lambda_1]}\cdots[c(df^{q-1})]^{[\lambda_{q-1}]}$ is a $\lambda_1+\cdots+\lambda_{q-1}$ order differential operator, we get by Lemma 2.2 and Lemma 2.3 (see pp. 61-62 in [Fe]) that

$$\begin{aligned}&\Big|\psi_t\int_{\Delta_{2k}}t^k\sum_{1\le q\le 2k}\sum_{\lambda_1,\cdots,\lambda_{q-1}=0}^{N-1}\frac{(-1)^{\lambda_1+\cdots+\lambda_{q-1}+N}s_1^{\lambda_1}\cdots s_{q-1}^{\lambda_{q-1}}s_q^N t^{\lambda_1+\cdots+\lambda_{q-1}+N}}{\lambda_1!\cdots\lambda_{q-1}!}\mathrm{Str}[f^0[c(df^1)]^{[\lambda_1]}\\&\cdots[c(df^{q-1})]^{[\lambda_{q-1}]}\{[c(df^q)]^{[N]}(s_qt)\}e^{-(s_{q+1}-s_q)tH_X}\cdots c(df^{2k})e^{-(1-s_{2k})tH_X}]_J\,dv\Big|\\&\qquad\sim O(t^{\frac{2k-2J+\lambda_1+\cdots+\lambda_{q-1}+N-\dim M}{2}}).\end{aligned}\tag{2.14}$$

Therefore,

Theorem 2.6 *(1) if* $2k\le 2J+\dim M$, *then*

$$\begin{aligned}&\mathrm{ch}_{2k}(\sqrt{t}D,X)(f^0,\cdots,f^{2k})_J\\&=\psi_t\textstyle\sum_{\lambda_1,\ldots,\lambda_{2k}=0}^{\dim M+2J-2k}\frac{(-1)^{\lambda_1+\cdots+\lambda_{2k}}}{\lambda_1!\cdots\lambda_{2k}!}Ct^{|\lambda|+k}\mathrm{Str}[f^0[c(df^1)]^{[\lambda_1]}\cdots[c(df^{2k})]^{[\lambda_{2k}]}e^{-tH_X}]_J+O(\sqrt{t}),\end{aligned}\tag{2.15}$$

with the constant

$$C=\frac{1}{\lambda_1+1}\frac{1}{\lambda_1+\lambda_2+2}\cdots\frac{1}{\lambda_1+\cdots+\lambda_{2k}+2k}.\tag{2.16}$$

(2) if $2k>2J+\dim M$, *then*

$$\mathrm{ch}_{2k}(\sqrt{t}D,X)(f^0,\cdots,f^{2k})_J=O(\sqrt{t}).\tag{2.17}$$

2.2. Computations of infinitesimal equivariant Chern-Connes characters

Since H_X is a generalized Laplacian, the heat operator e^{-tH_X} exists and

$$(\frac{\partial}{\partial t}+H_X)e^{-tH_X}=0,\ \ H_Xe^{-tH_X}=e^{-tH_X}H_X.\tag{2.18}$$

It is easy to extend the notation of the Volterra pseudodifferential operator to the case with coefficients in $\mathbb{C}[\mathfrak{g}^*]$ (see [BGS],[Gr],[Po]). Let $Q=(H_X+\frac{\partial}{\partial t})^{-1}$ be the Volterra inverse of $H_X+\frac{\partial}{\partial t}$ as in [BGS]. Let $K_Q(x,y,X,t)$ and $k(x,y,X,t)$ be the distribution kernel of Q and the heat kernel of e^{-tH_X} respectively. Then for $t>0$ (see [BGS])

$$k(x,y,X,t)=K_Q(x,y,X,t)+O(t^\infty)\qquad\text{as }\ t\to 0^+.\tag{2.19}$$

For the definition 2.4 in [Wa2], we replace $\wedge T_z^*B$ by $\mathbb{C}[\mathfrak{g}^*]$ so that we can define Volterra symbols with coefficients in $\mathbb{C}[\mathfrak{g}^*]$ and Volterra pseudodifferential operators with coefficients in $\mathbb{C}[\mathfrak{g}^*]$. We denote the space of Volterra pseudodifferential operators with coefficients in $\mathbb{C}[\mathfrak{g}^*]$ by $\Psi_V^*(\mathbb{R}^n\times\mathbb{R},S(TM)\otimes\mathbb{C}[\mathfrak{g}^*])$.

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Recall that the quantization map $c : \wedge T^*_{\mathbb{C}}(M) \to \mathrm{Cl}(M)$ and the symbol map $\sigma = c^{-1}$ satisfy

$$\sigma(c(\xi)c(\eta)) = \xi \wedge \eta - \langle \xi, \eta \rangle. \tag{2.20}$$

Thus, for ξ and η in $\wedge T^*_{\mathbb{C}}(M)$ we have

$$\sigma(c(\xi^{(i)})c(\eta^{(j)})) = \xi^{(i)} \wedge \eta^{(j)} \mod \wedge^{i+j-2} T^*_{\mathbb{C}}(M), \tag{2.21}$$

where $\xi^{(l)}$ denotes the component in $\wedge^l T^*_{\mathbb{C}}(M)$ of $\xi \in \wedge T^*_{\mathbb{C}}(M)$. Recall that if $e_1, \cdots, e_n$ is an orthonormal frame of $T_x M$, then

$$\mathrm{Str}[c(e^{i_1}) \cdots c(e^{i_k})] = \begin{cases} 0 & \text{when } k < n, \\ (-2i)^{\frac{n}{2}} & \text{when } k = n. \end{cases} \tag{2.22}$$

We compute the Chern-Connes character at a fixed point $x_0 \in M$. Using normal coordinates centered at x_0 in M and paralleling ∂_i at x_0 along geodesics through x_0, we get the orthonormal frame $e_1, \cdots, e_n$. We define the Getzler order as follows:

$$\deg \partial_j = \frac{1}{2} \deg \partial_t = \deg c(dx_j) = \frac{1}{2} \deg(X) = -\deg x^j = 1. \tag{2.23}$$

Let $Q \in \Psi^*_V(\mathbb{R}^n \times \mathbb{R}, S(TM) \otimes \mathbb{C}[\mathfrak{g}^*])$ have the symbol

$$q(x, X, \xi, \tau) \sim \sum_{k \leq m'} q_k(x, X, \xi, \tau), \tag{2.24}$$

where $q_k(x, X, \xi, \tau)$ is an order k symbol. Then using Taylor expansions at $x = 0$ as well as at $X = 0$, we get that

$$\sigma[q(x, X, \xi, \tau)] \sim \sum_{j,k,\alpha,\beta} \frac{x^\alpha}{\alpha!} \frac{X^\beta}{\beta!} \sigma[\partial_x^\alpha \partial_X^\beta q_k(0, 0, \xi, \tau)]^{(j)}. \tag{2.25}$$

The symbol $\frac{x^\alpha}{\alpha!} \frac{X^\beta}{\beta!} \sigma[\partial_x^\alpha \partial_X^\beta q_k(0, 0, \xi, \tau)]^{(j)}$ is the Getzler homogeneous symbol of $k + j - |\alpha| + 2|\beta|$.

Definition 2.7 The J-truncated symbol of q is defined by

$$\sigma[q(x, X, \xi, \tau)]_J := \sum_{j,k,\alpha,|\beta| \leq J} \frac{x^\alpha}{\alpha!} \frac{X^\beta}{\beta!} \sigma[\partial_x^\alpha \partial_X^\beta q_k(0, 0, \xi, \tau)]^{(j)}. \tag{2.26}$$

Then $\sigma[q(x, X, \xi, \tau)]_J$ can be written as

$$\sigma[q(x, X, \xi, \tau)]_J \sim \sum_{l \geq 0} q_{(m-l)}(x, X, \xi, \tau)_J, \qquad q_{(m)} \neq 0, \tag{2.27}$$

where $q_{(m-l),J}$ is a Getzler homogeneous symbol of degree $m - l$, and the degree of X is $\leq J$.

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Definition 2.8 The integer m is called the Getzler order of Q. The symbol $q_{(m),J}$ is the truncated principle Getzler homogeneous symbol of Q. The operator $Q_{(m),J} = q_{(m)}(x, D_x, D_t)_J$ (see [BGS], [Po]) is called the truncated model operator of Q.

Lemma 2.9 *Let $Q \in \Psi_V^*(\mathbb{R}^n \times \mathbb{R}, S(TM) \otimes \mathbb{C}[\mathfrak{g}]^*)$, and Q_J has the Getzler order m and the model operator $Q_{(m),J}$. Then as $t \to 0^+$, we have:*

1) $\sigma[K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)} = O(t^{\frac{j-n-m-1}{2}})$, if $m-j$ is odd;

2) $\sigma[K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)} = t^{\frac{j-n-m-2}{2}} K_{Q_{(m),J}}(0,0,X,1)^{(j)} + O(t^{\frac{j-n-m}{2}})$, *if $m-j$ is even,*

where $[K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)}$ denotes taking the j degree form component in $\wedge^* T^*M$. In particular, when $m=-2$ and $j=n$ is even, we get

$$\sigma[K_{Q_J}(0,0,\frac{X}{t},t)]^{(n)} = K_{Q_{(-2),J}}(0,0,X,1)^{(n)} + O(t). \tag{2.28}$$

Proof: By (1.7) in [Po], we have

$$K_{Q_J}(0,0,\frac{X}{t},t) \sim \sum_{m_0-j_0 \text{ even}} t^{\frac{j_0-n-m_0-2}{2}} \breve{q}_{m_0-j_0}(0,0,\frac{X}{t},1)_J, \tag{2.29}$$

where m_0 is the operator order of Q_J. Then

$$\sigma[K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)} \sim \sum_{m_0-j_0 \text{ even}} \sum_{|\beta|\le J} t^{\frac{j_0-n-m_0-2|\beta|-2}{2}} \sigma[\frac{X^\beta}{\beta!}\frac{\partial}{\partial X^\beta}\breve{q}_{m_0-j_0}(0,0,0,1)]_J^{(j)}. \tag{2.30}$$

Let $L = m_0 - j_0 + j + 2|\beta|$. Since Q_J has Getzler order m, it follows that $L \le m$. Thus

$$\sigma[K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)} \sim \sum_{m_0-j_0 \text{ even}} \sum_{|\beta|\le J} t^{\frac{j-n-L-2}{2}} \sigma[\frac{X^\beta}{\beta!}\frac{\partial}{\partial X^\beta}\breve{q}_{m_0-j_0}(0,0,0,1)]_J^{(j)}. \tag{2.31}$$

Note that the degree of the leading term is $L = m$ and $m_0 - j_0 = m - j - 2|\beta|$. When $m-j$ is odd, as $m_0 - j_0$ is even, this is impossible. Therefore,

$$\sigma[\psi_t K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)} = O(t^{\frac{j-n-m-1}{2}}). \tag{2.32}$$

When $L = m$ and $m-j$ is even, the leading coefficient is

$$\sigma[\breve{q}_{(m)}(0,0,X,1)]_J^{(j)} = \sum_{|\beta|\le J} \sigma[\frac{X^\beta}{\beta!}\frac{\partial}{\partial X^\beta}\breve{q}_{m-j}(0,0,0,1)]^{(j)} = K_{Q_{(m),J}}(0,0,X,1)^{(j)}. \tag{2.33}$$

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For the next term, we have that $L = m-1$, $m-j$ is even, $m_0 - j_0 + j = m-1$, which is impossible, so that the next term is $O(t^{\frac{j-n-m}{2}})$. □

Let θ_X be the one-form associated with X_M which is defined by $\theta_X(Y) = g(X,Y)$ for the vector field Y. Let $\nabla^{S,X}$ be the Clifford connection $\nabla^S - \frac{1}{4}\theta_X$ on the spinor bundle and $\triangle_X$ be the Laplacian on $S(TM)$ associated with $\nabla^{S,X}$. Let $\mu(X)(\cdot) = \nabla^{TM}_{\cdot} X_M$. Define $\alpha : U \times \mathfrak{g} \to \mathbb{C}$ via the formula

$$\alpha_X(x) := -\frac{1}{4}\int_0^1 (\iota(\mathcal{R})\theta_X)(tx)t^{-1}dt, \;\; \rho(X,x) = e^{\alpha_X(x)}, \tag{2.34}$$

where $\mathcal{R} = \sum_{i=1}^n x_i \frac{\partial}{\partial_{x_i}}$. Recall

Lemma 2.10 *([BGV Lemma 8.13]) The following identity holds*

$$H_X = \Delta_X + \frac{1}{4}r_M, \tag{2.35}$$

where r_M is the scalar curvature. In the trivialization of $S(TM)$ over U, the conjugate $\rho(X,x)(\nabla^{S,X}_{\partial_i})\rho(X,x)^{-1}$ is given by

$$\begin{aligned}&\rho(X,x)(\nabla^{S,X}_{\partial_i})\rho(X,x)^{-1}\\ &= \partial_i - \frac{1}{4}\sum_{j,a<b}\big\langle R(\partial_i,\partial_j)e_a,e_b\big\rangle c(e^a)c(e^b)x^j - \frac{1}{4}\mu^M_{ij}(X)x^j + O_G(0),\end{aligned} \tag{2.36}$$

where $O_G(0)$ is the Getzler order 0 operator.

By Lemma 2.10, we get

Proposition 2.11 *In the trivialization of $S(TM)$ over U and the normal coordinate, the model operator of $\rho(X,x)H_X\rho(X,x)^{-1}$ is*

$$\begin{aligned}(\rho(X,x)H_X\rho(X,x)^{-1})_{(2)} &= -\sum_{i=1}^n(\partial_i - \frac{1}{4}\sum_{j=1}^n a_{ij}x_j)^2,\\ a_{ij} &= \Big\langle R^{TM}\partial_i,\partial_j\Big\rangle + \big\langle\mu(X)\partial_i,\partial_j\big\rangle.\end{aligned} \tag{2.37}$$

Let

$$\widetilde{[c(df^j)]}^{[\lambda_j]} = [\rho H_X\rho^{-1},\widetilde{[c(df^j)]}^{[\lambda_j-1]}];\;\; \widetilde{[c(df^j)]}^{[0]} = c(df^j).$$

Then

$$\rho[c(df^j)]^{[\lambda_j]}\rho^{-1} = \widetilde{[c(df^j)]}^{[\lambda_j]};\;\; O_G(\rho[c(df^j)]^{[\lambda_j]}\rho^{-1}) = 2\lambda_j, \;\text{ for }\; \lambda_j > 0. \tag{2.38}$$

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We will compute

$$\lim_{t\to 0} t^{|\lambda|+k}\psi_t \mathrm{Str}[f^0[c(df^1)]^{[\lambda_1]}\cdots[c(df^{2k})]^{[\lambda_{2k}]}e^{-tH_X}]_J.$$

Since $\rho e^{-tH_X}\rho^{-1} = e^{-t\rho H_X\rho^{-1}}$ and by (2.38), for a fixed point x_0, we have

$$\begin{aligned}&\lim_{t\to 0} t^{|\lambda|+k}\psi_t \mathrm{Str}[f^0[c(df^1)]^{[\lambda_1]}\cdots[c(df^{2k})]^{[\lambda_{2k}]}e^{-tH_X}]_J\\ &= \lim_{t\to 0} t^{|\lambda|+k}\psi_t \mathrm{Str}[f^0\widetilde{[c(df^1)]}^{[\lambda_1]}\cdots\widetilde{[c(df^{2k})]}^{[\lambda_{2k}]}e^{-t\rho H_X\rho^{-1}}]_J. \quad (2.39)\end{aligned}$$

By (2.38), when $(\lambda_1,\cdots,\lambda_{2k}) \neq (0,\cdots,0)$, we get

$$O_G(f^0\widetilde{[c(df^1)]}^{[\lambda_1]}\cdots\widetilde{[c(df^{2k})]}^{[\lambda_{2k}]}) = O_G(2|\lambda|+2k-1);$$

$$O_G(f^0\widetilde{[c(df^1)]}^{[\lambda_1]}\cdots\widetilde{[c(df^{2k})]}^{[\lambda_{2k}]}(\rho H_X\rho^{-1}+\partial_t)^{-1}) = O_G(2|\lambda|+2k-3). \quad (2.40)$$

By (2.40),(2.22) and Lemma 2.9,

$$\lim_{t\to 0} t^{|\lambda|+k}\psi_t \mathrm{Str}[f^0[c(df^1)]^{[\lambda_1]}\cdots[c(df^{2k})]^{[\lambda_{2k}]}e^{-tH_X}]_J = 0. \quad (2.41)$$

When $(\lambda_1,\cdots,\lambda_{2k}) = (0,\cdots,0)$, then $O_G(f^0c(df^1)\cdots c(df^{2k})) = 2k$ and

$$O_G(f^0c(df^1)\cdots c(df^{2k})(\rho H_X\rho^{-1}+\partial_t)^{-1}) = O_G(2k-2).$$

The model operator of $f^0c(df^1)\cdots c(df^{2k})(\rho H_X\rho^{-1}+\partial_t)^{-1}$ is

$$f^0\wedge df^1\wedge\cdots\wedge df^{2k}((\rho H_X\rho^{-1})_{(2)}+\partial_t)^{-1}.$$

By Lemma 2.9 and Proposition 2.11 in connection with the Mehler formula, we get

$$\begin{aligned}&\lim_{t\to 0^+} t^k\sigma[\psi_t f^0c(df^1)\cdots c(df^{2k})e^{-tH_X}]_J^{(n)}\\ &= (2\pi\sqrt{-1})^{-n/2}[f^0\wedge df^1\wedge\cdots\wedge df^{2k}\widehat{A}(F_{\mathfrak{g}}^M(X))]_J^{(n)}, \quad (2.42)\end{aligned}$$

where $\widehat{A}(F_{\mathfrak{g}}^M(X))$ is the equivariant $\widehat{A}$-genus. By (2.41), (2.42) and Theorem 2.6, we get when $J\to+\infty$ that

Theorem 2.12 *When $2k \leq \dim M$ and X is small which means that $||X_M||$ is sufficient small, then for $f^j \in C_G^\infty(M)$,*

$$\begin{aligned}&\lim_{J\to+\infty}\lim_{t\to 0}\mathrm{ch}_{2k}(\sqrt{t}D,X)(f^0,\cdots,f^{2k})_J\\ &= \frac{1}{(2k)!}(2\pi\sqrt{-1})^{-n/2}\int_M f^0\wedge df^1\wedge\cdots\wedge df^{2k}\widehat{A}(F_{\mathfrak{g}}^M(X))d\mathrm{Vol}_M. \quad (2.43)\end{aligned}$$

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Remark Theorem 2.12 is not a direct consequence of the equivariant Chern-Connes character formula due to Chern-Hu in [CH] and the localization formula because $f^0 \wedge df^1 \wedge \cdots \wedge df^{2k} \widehat{A}(F_{\mathfrak{g}}^M(X))$ is not an equivariant closed form.

Let $p \in M_r(\mathbb{C}^\infty(M))$ be a selfadjoint idempotent, and

$$\mathrm{Ch}(\mathrm{Im}(p)) = \sum_{k=0}^{\infty} (-\frac{1}{2\pi\sqrt{-1}})^k \frac{1}{k!} \mathrm{Tr}[p(dp)^{2k}]. \tag{2.44}$$

Let $D_{\mathrm{Im}p}$ be the Dirac operator with coefficients from $\mathrm{Im}p$. Let $S(TM) = S^+(TM) \oplus S^-(TM)$ and $D_{\mathrm{Im}p,+}$ be the restriction on $S^+(TM) \otimes \mathrm{Im}p$. Then, by the infinitesimal equivariant index formula and Theorem 2.12, we get

Corollary 2.13 *When X is small, we have*

$$\mathrm{Ind}_{e^{-X}}(D_{\mathrm{Im}p,+}) = \lim_{J\to+\infty} \lim_{t\to 0} \left\langle \mathrm{ch}_{\mathrm{even}}(\sqrt{t}D, X)_J, \mathrm{ch}(p) \right\rangle. \tag{2.45}$$

Next, we shall give an infinitesimal equivariant index formula for odd dimensional manifolds. Let M be a compact oriented odd dimensional Riemannian manifold without boundary with a fixed spin structure and S be the bundle of spinors on M. The fundamental setup is consistent with that in Section 2.1. Let $g \in GL_r(C^\infty(M))$, $g(hx) = g(x)$ for $h \in G$ and $x \in M$. For $0 \le u \le 1$, on the bundle $S(TM) \otimes C^r$, let

$$D_{-X,u} = (1-u)D_{-X} + ug^{-1}D_{-X}g = D_{-X} + ug^{-1}dg, \quad A = g^{-1}dg, \tag{2.46}$$

$$H_{X,u} = D_{-X,u}^2 + uA(X_M) + L_X. \tag{2.47}$$

We will compute

$$\lim_{t\to 0} \int_0^1 \sqrt{t}\,\mathrm{Tr}\left[Ae^{-tH_{\frac{X}{t},u}} \right] du.$$

By Lemma 2.10, we have

Proposition 2.14 *The following identity holds*

$$H_{X,u} = \triangle_X + \frac{1}{4} r_M + u^2 c(A)^2 + u(D^{cl}(c(A)) - 2\nabla_A^{S(TM),X}), \tag{2.48}$$

where D^{cl} is the Dirac operator on the Clifford bundle.

By Lemma 2.10 and Proposition 2.11, we get

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Proposition 2.15 *In the trivialization of $S(TM)$ over U and the normal coordinate, the model operator of $\rho(X,x)H_{X,u}\rho(X,x)^{-1}$ is*

$$(\rho(X,x)H_{X,u}\rho(X,x)^{-1})_{(2)} = -\sum_{i=1}^{n}(\partial_i - \frac{1}{4}\sum_{j=1}^{n}a_{ij}x_j)^2 + u^2A^2 + udA, \quad (2.49)$$

$$\text{where} \qquad a_{ij} = \left\langle R^{TM}\partial_i,\partial_j\right\rangle + \left\langle \mu(X)\partial_i,\partial_j\right\rangle. \quad (2.50)$$

By Lemma 2.9 2), similarly to Theorem 2.12, we get

Theorem 2.16 *When X is small, then*

$$\lim_{t\to 0}\int_0^1 \sqrt{t}\,\mathrm{Tr}\Big[Ae^{-tH_{\frac{X}{t},u}}\Big]du = (2\pi\sqrt{-1})^{-n/2}\int_M \widehat{A}(F_{\mathfrak{g}}^M(X))\mathrm{ch}(g)d\,\mathrm{Vol}_M, \quad (2.51)$$

where the odd Chern character is defined by

$$\mathrm{ch}(g) = \sum_{k=0}^{+\infty}(-1)^k\frac{k!}{(2k+1)!}\mathrm{Tr}[(g^{-1}dg)^{2k+1}]. \quad (2.52)$$

By Lemma 2.9, we know that Theorem 2.12 also holds for odd dimensional manifolds. So by Theorem 2.16, we get

Corollary 2.17 *When X is small, we have*

$$\lim_{t\to 0}\int_0^1 \sqrt{t}\,\mathrm{Tr}\Big[Ae^{-tH_{\frac{X}{t},u}}\Big]du = \lim_{J\to+\infty}\lim_{t\to 0}\left\langle \mathrm{ch}_{\mathrm{odd}}(\sqrt{t}D,X)_J,\mathrm{ch}(g)\right\rangle. \quad (2.53)$$

3. Infinitesimal equivariant eta cochains

Let N be a compact oriented odd dimensional Riemannian manifold without boundary with a fixed spin structure and S be the bundle of spinors on N. The fundamental setup is consistent with that in Section 2.1. Define

$$\begin{aligned}\mathrm{ch}_k(\sqrt{t}D_{-X},D_X)(f^0,\cdots,f^k)_J := t^{k/2}\sum_{j=0}^{k}\int_{\Delta_{k+1}}\mathrm{Tr}\Big[\psi_t e^{-t\Omega_X}f^0e^{-s_1t(D+\frac{1}{4}c(X))^2}c(df^1)\\ \cdot e^{-(s_2-s_1)t(D+\frac{1}{4}c(X))^2}\cdots c(df^j)e^{-(s_{j+1}-s_j)t(D+\frac{1}{4}c(X))^2}\\ \cdot D_X e^{-(s_{j+2}-s_{j+1})t(D+\frac{1}{4}c(X))^2}c(df^{j+1})\cdots c(df^k)e^{-(1-s_{k+1})t(D+\frac{1}{4}c(X))^2}\Big]_J d\,\mathrm{Vol}_{\Delta_{k+1}},\end{aligned} \quad (3.1)$$

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where $\triangle_{k+1} = \{(s_1,\cdots,s_{k+1})|0 \le s_1 \le s_2 \le \cdots \le s_{k+1} \le 1\}$ is the $k+1$-simplex. Formally, **truncated infinitesimal equivariant η cochains** on $C_G^\infty(N)$ are defined by the formulas:

$$\widetilde{\eta}_{X,k}(D)_J = \frac{1}{\Gamma(\frac{1}{2})}\int_\varepsilon^\infty \frac{1}{2\sqrt{t}}\mathrm{ch}_k(\sqrt{t}D_{-X}, D_X)_J\, dt, \tag{3.2}$$

$$\eta_{X,k}(D)_J = \frac{1}{\Gamma(\frac{1}{2})}\int_\varepsilon^\infty \frac{1}{2\sqrt{t}}\mathrm{ch}_k(\sqrt{t}D_{-X}, D_{-X})_J\, dt, \tag{3.3}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and ε is a small positive number. Then $\widetilde{\eta}_{X,0}(D)(1)$ is half of the truncated infinitesimal equivariant eta invariant defined by Goette in [Go]. In order to prove that the above expression is well defined, it is necessary to check integrality near infinity of the integration. In fact, when $k > \dim N + 1 + 2J$, we can take $\varepsilon = 0$. First, we prove regularity at zero.

Lemma 3.1 *When $t \to 0^+$, then for small X and $f^0,\cdots,f^k \in C_G^\infty(N)$, we have*

$$\mathrm{ch}_k(\sqrt{t}D_{-X}, D_X)_J(f^0,\cdots,f^k) = O(t^{-\frac{1}{2}}). \tag{3.4}$$

When $k > \dim N + 1 + 2J$,

$$\mathrm{ch}_k(\sqrt{t}D_{-X}, D_X)_J(f^0,\cdots,f^k) = O(t^{\frac{1}{2}}). \tag{3.5}$$

In (3.1), the difference between infinitesimal equivariant eta cochains and equivariant eta cochains is that D_{-X} does not commute with D_X. So we cannot apply the trick in [Wa1] directly. By $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, we have

$$\begin{aligned}
&\mathrm{Tr}\Big[\psi_t e^{-t\mathfrak{L}_X} f^0 e^{-s_1 t(D+\frac{1}{4}c(X))^2} c(df^1) e^{-(s_2-s_1)t(D+\frac{1}{4}c(X))^2}\cdots c(df^j) e^{-(s_{j+1}-s_j)t(D+\frac{1}{4}c(X))^2}\\
&\quad\cdot D_X e^{-(s_{j+2}-s_{j+1})t(D+\frac{1}{4}c(X))^2} c(df^{j+1})\cdots c(df^k) e^{-(1-s_{k+1})t(D+\frac{1}{4}c(X))^2}\Big]_J\\
&\quad= \mathrm{Tr}\big[\psi_t D_X e^{-(s_{j+2}-s_{j+1})tH_X} c(df^{j+1})\cdots c(df^k) e^{-(1-s_{k+1})tH_X}\\
&\quad\cdot f^0 e^{-s_1 tH_X} c(df^1)\cdot e^{-(s_2-s_1)tH_X}\cdots c(df^j) e^{-(s_{j+1}-s_j)tH_X}\big]_J.
\end{aligned} \tag{3.6}$$

By Lemma 2.5, we commute $e^{-(s_{j+1}-s_j)tH_X}$ with $c(df^j)$ and then commute heat operators from the right to the left. We write the result for the case that $k = 2$, $j = 1$.

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For the general case the result is similar.

$$\begin{aligned}
&D_X e^{-(s_3-s_2)tH_X} c(df^{j+1}) c(df^2) e^{-(1-s_3)tH_X} f^0 e^{-s_1 tH_X} c(df^1) \cdot e^{-(s_2-s_1)tH_X} \\
&= \sum_{\lambda_1,\lambda_2,\lambda_3=0}^{N-1} \frac{t^{\lambda_1+\lambda_2+\lambda_3}}{\lambda_1!\lambda_2!\lambda_3!} (s_2-s_1)^{\lambda_1} s_2^{\lambda_2} (1-s_3+s_2)^{\lambda_3} D_X e^{-tH_X} c(df^2)^{[\lambda_3]} (f^0)^{[\lambda_2]} c(df^1)^{[\lambda_1]} \\
&\qquad + \sum_{\lambda_1,\lambda_2=0}^{N-1} \frac{t^{\lambda_1+\lambda_2+N}}{\lambda_1!\lambda_2!} (s_2-s_1)^{\lambda_1} s_2^{\lambda_2} (1-s_3+s_2)^N D_X \\
&\qquad \cdot e^{-t(s_3-s_2)H_X} \{c(df^2)_1^{[N]}[(1-s_3+s_2)t]\} (f^0)^{[\lambda_2]} c(df^1)^{[\lambda_1]} \\
&\quad + \sum_{\lambda_1=0}^{N-1} \frac{t^{\lambda_1+N}}{\lambda_1!} (s_2-s_1)^{\lambda_1} s_2^N D_X e^{-t(s_3-s_2)H_X} c(df^2) e^{-t(1-s_3)H_X} [(f^0)_1^{[N]}(t_1 s_2)] c(df^1)^{[\lambda_1]} \\
&\quad + t^N (s_2-s_1)^N D_X e^{-t(s_3-s_2)H_X} c(df^2) e^{-t(1-s_3)H_X} f^0 e^{-ts_1 H_X} [c(df^1)_1^{[N]}[(s_2-s_1)t]].
\end{aligned} \tag{3.7}$$

For the second term on the right hand side of (3.7), we have

$$\begin{aligned}
&A := t^{k/2} \int_{\Delta_{k+1}} \Big| \mathrm{Tr}\Big[\psi_t \textstyle\sum_{\lambda_1,\lambda_2=0}^{N-1} \frac{t^{\lambda_1+\lambda_2+N}}{\lambda_1!\lambda_2!} (s_2-s_1)^{\lambda_1} s_2^{\lambda_2} (1-s_3+s_2)^N D_X e^{-t(s_3-s_2)H_X} \\
&\cdot \int_{\Delta_N} e^{-t(1-u_1)(1-s_3+s_2)H_X} c(df^2)^{[N]} e^{-tu_1(1-s_3+s_2)H_X} (f^0)^{[\lambda_2]} c(df^1)^{[\lambda_1]} \Big]_J \Big| du_1 \cdots du_n ds_1 ds_2 ds_3 \\
&= \sum_{\lambda_1,\lambda_2=0}^{N-1} \frac{t^{\lambda_1+\lambda_2+N+k/2-J}}{\lambda_1!\lambda_2!} \int_{\Delta_{k+1}} \int_{\Delta_N} (s_2-s_1)^{\lambda_1} s_2^{\lambda_2} (1-s_3+s_2)^N \\
&\cdot \Big| \mathrm{Tr}\Big[D_X e^{-t\sigma_1 H_X} e^{-t\sigma_2 H_X} c(df^2)^{[N]} e^{-t\sigma_3 H_X} (f^0)^{[\lambda_2]} c(df^1)^{[\lambda_1]} \Big]_J \Big| du_1 \cdots du_n ds_1 ds_2 ds_3,
\end{aligned} \tag{3.8}$$

where $\sigma_1+\sigma_2+\sigma_3 = 1$, $\sigma_1,\sigma_2,\sigma_3 \geq 0$ and

$$\sigma_1 = s_3 - s_2;\ \ \sigma_2 = (1-u_1)(1-s_3+s_2),\ \ \sigma_3 = u_1(1-s_3+s_2). \tag{3.9}$$

We divide the region into three parts as shown in Lemma 2.2. By the Weyl theorem, we get when $N \geq n+2-k+2J$, that

$$A \sim O(t^{\frac{N+k+|\lambda|-n-1-2J}{2}}) \sim O(t^{\frac{1}{2}}). \tag{3.10}$$

Similarly, we get when $N \geq n+2-k+2J$, that the third and fourth terms on the right hand side of (3.7) are also $O(t^{\frac{1}{2}})$. When $k \geq n+2+2J$, then $N \geq n+2-k+2J$. So we get

Theorem 3.2 1) *If $k \leq n+1+2J$ and X is small, then when $t \to 0^+$, we have:*

$$\begin{aligned}
&\mathrm{ch}_k(\sqrt{t}D_{-X}, D_X)(f^0,\cdots,f^k)_J = \sum_{j=0}^{k} (-1)^j \sum_{0\leq\lambda_1,\cdots,\lambda_k\leq N-1} \frac{(-1)^{|\lambda|} C' t^{|\lambda|+\frac{k}{2}}}{\lambda!} \\
&\cdot \mathrm{Tr}\Big[\psi_t c(df^{j+1})^{[\lambda_{k+1}]} \cdots c(df^k)^{[\lambda_{j+2}]} (f^0)^{[\lambda_{j+1}]} c(df^1)^{[\lambda_j]} \cdots c(df^j)^{[\lambda_1]} D_X e^{-tH_X} \Big]_J + O(t^{\frac{1}{2}}),
\end{aligned} \tag{3.11}$$

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where C' is a constant.

2) *If $k > n+1+2J$, then when $t \to 0^+$, we have:*

$$\mathrm{ch}_k(\sqrt{t}D_{-X}, D_X)(f^0,\cdots,f^k)_J \sim O(t^{\frac{1}{2}}). \tag{3.12}$$

1),2) also hold for $\mathrm{ch}_k(\sqrt{t}D_{-X}, D_{-X})(f^0,\cdots,f^k)_J$.

Lemma 3.3 *When $t \to 0^+$, we have:*

$$t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}\Big[\psi_t c(df^{j+1})^{[\lambda_{k+1}]}\cdots c(df^k)^{[\lambda_{j+2}]}$$

$$\cdot(f^0)^{[\lambda_{j+1}]}c(df^1)^{[\lambda_j]}\cdots c(df^j)^{[\lambda_1]}D_X e^{-tH_X}\Big]_J \sim O(t^{-\frac{1}{2}}); \tag{3.13}$$

$$t^{|\lambda|+\frac{k}{2}}\int_0^1 \mathrm{Tr}\Big[\psi_t c(df^{j+1})^{[\lambda_{k+1}]}\cdots c(df^k)^{[\lambda_{j+2}]}$$

$$\cdot(f^0)^{[\lambda_{j+1}]}c(df^1)^{[\lambda_j]}\cdots c(df^j)^{[\lambda_1]}e^{-t\sigma_0 H_X}D_X e^{-t(1-\sigma_0)H_X}\Big]_J d\sigma_0 \sim O(t^{\frac{1}{2}}). \tag{3.14}$$

Proof: We introduce an auxiliary Grassmann variable z as shown in [BF]. Let

$$\widetilde{H_X} = H_X - zD_X;\ h(x) = 1 + \frac{1}{2}z\sum_{j=1}^n x_i c(e_i). \tag{3.15}$$

Then we have by Lemma 2.10 that

$$\widetilde{H_X} = -\sum_{j=1}^n(\nabla^{S,X}_{e_j} - \frac{1}{2}c(e_j)z)^2 + \sum_{j=1}^n(\nabla^{S,X}_{\nabla^{TM}_{e_j}e_j} - \frac{1}{2}c(\nabla^{TM}_{e_j}e_j)z) + \frac{1}{4}r_M. \tag{3.16}$$

Using Lemma 8.13 in [BGV], we have

$$\rho\widetilde{H_X}\rho^{-1} = -\sum_{j=1}^n(\nabla^S_{e_j} - \frac{1}{4}\sum_i\langle\mu_X(e_j),\partial_i\rangle x^i + \langle h_j(x), X\rangle - \frac{1}{2}c(e_j)z)^2$$

$$+\sum_{j=1}^n(\nabla^S_{\nabla^{TM}_{e_j}e_j} - \frac{1}{2}c(\nabla^{TM}_{e_j}e_j)z - \frac{1}{4}\sum_i\Big\langle\mu_X(\nabla^{TM}_{e_j}e_j),\partial_i\Big\rangle x^i + \Big\langle\overline{h_j}(x), X\Big\rangle) + \frac{1}{4}r_M, \tag{3.17}$$

where $h_j(x)$, $\overline{h_j}(x) = O(|x|^2)$. Thus

$$(h\rho)\widetilde{H_X}(h\rho)^{-1} = \rho H_X\rho^{-1} + zu, \text{ where } O_G(u) \le 0 \text{ has no } z. \tag{3.18}$$

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By the Duhamel principle, we have

$$\exp(-t\widetilde{H}_X) = \exp(-tH_X) + tz\int_0^1 e^{-t\sigma_0 H_X} D_X e^{-t(1-\sigma_0)H_X} d\sigma_0. \tag{3.19}$$

By (3.18) and (3.19), it follows that

$$(h\rho)^{-1}\exp(-t(\rho H_X\rho^{-1}+zu))(h\rho) = \rho^{-1}\exp(-t\rho H_X\rho^{-1})\rho$$

$$+tz\int_0^1 e^{-t\sigma_0 H_X} D_X e^{-t(1-\sigma_0)H_X} d\sigma_0. \tag{3.20}$$

Let

$$A_0 := \widetilde{c(df^{j+1})}^{[\lambda_{k+1}]}\cdots\widetilde{c(df^k)}^{[\lambda_{j+2}]}\widetilde{(f^0)}^{[\lambda_{j+1}]}\widetilde{c(df^1)}^{[\lambda_j]}\cdots\widetilde{c(df^j)}^{[\lambda_1]},$$

$$A_1 = c(df^{j+1})^{[\lambda_{k+1}]}\cdots c(df^k)^{[\lambda_{j+2}]}(f^0)^{[\lambda_{j+1}]}c(df^1)^{[\lambda_j]}\cdots c(df^j)^{[\lambda_1]}.$$

$$\mathrm{Tr}[\psi_t A_0 h^{-1}\exp(-t(\rho H_X\rho^{-1}+zu))h] = \mathrm{Tr}[\psi_t A_0\exp(-t\rho H_X\rho^{-1})]$$

$$+tz\int_0^1 \mathrm{Tr}[\psi_t A_1 e^{-t\sigma_0 H_X} D_X e^{-t(1-\sigma_0)H_X}]d\sigma_0. \tag{3.21}$$

$$\mathrm{Tr}[\psi_t A_0 h^{-1}\exp(-t(\rho H_X\rho^{-1}+zu))h] = \mathrm{Tr}[\psi_t h^{-1}A_0\exp(-t(\rho H_X\rho^{-1}+zu))h]$$

$$+\mathrm{Tr}[\psi_t[A_0,h^{-1}]\exp(-t(\rho H_X\rho^{-1}+zu))h]. \tag{3.22}$$

Now

$$t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}[\psi_t[A_0,h^{-1}]\exp(-t(\rho H_X\rho^{-1}+zu))h]_J = O(t^{3/2}). \tag{3.23}$$

In fact, by direct computations when $\lambda \neq (0,\cdots,0)$, we have $O_G([A_0,h]) = 2|\lambda| + k - 2$ up to terms $x_j L_j z$, where L_j is an operator. When $\lambda = (0,\cdots,0)$, $[A_0,h] = \sum_j x_j L_j$ and we fix a point x_0, so in this case (3.23) is zero. By Lemma 2.9 1), we get (3.23). Since

$$(\partial_t+\rho H_X\rho^{-1}+zu)^{-1} = (\partial_t+\rho H_X\rho^{-1})^{-1} - z(\partial_t+\rho H_X\rho^{-1})^{-1}u(\partial_t+\rho H_X\rho^{-1})^{-1}, \tag{3.24}$$

we have

$$t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}[\psi_t A_0\exp(-t(\rho H_X\rho^{-1}+zu))]_J - t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}[\psi_t A_0\exp(-t\rho H_X\rho^{-1})]_J = O(t^{3/2}). \tag{3.25}$$

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By (3.21)-(3.25), we get (3.14). Letting $D_{-X}e^{-t\sigma_0 H_X} = e^{-t\sigma_0 H_X}D_{-X}$, we get

$$e^{-t\sigma_0 H_X}D_X e^{-t(1-\sigma_0)H_X} = D_X e^{-tH_X} + \frac{1}{2}c(X)e^{-tH_X} - \frac{1}{2}e^{-t\sigma_0 H_X}c(X)e^{-t(1-\sigma_0)H_X}. \tag{3.26}$$

Using Lemma 2.4, similarly to Theorem 2.6, we get

$$t^{|\lambda|+\frac{k}{2}}\int_0^1 \mathrm{Tr}[\psi_t A_1 e^{-t\sigma_0 H_X}D_X e^{-t(1-\sigma_0)H_X}]d\sigma_0$$

$$= t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}[\psi_t A_1 D_X e^{-tH_X}] + \sum_{1\le l\le K_0} t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}[\psi_t A_1 t^l c(X)^{[l]}e^{-tH_X}] + O(t^{1/2}). \tag{3.27}$$

Letting $O_G(X) = 2$ and n is odd, we get

$$\sum_{1\le l\le K_0} t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}[\psi_t A_1 t^l c(X)^{[l]}e^{-tH_X}] = O(t^{-1/2}). \tag{3.28}$$

By (3.14),(3.27) and (3.28), we get (3.13). □

Remark Lemma 2.12 in [Wa1] is not correct. But using the trick in (3.23), we can prove the regularity of equivariant eta chains in [Wa1].

Next, we prove the regularity at infinity. Let $\mathcal{M}$ be the algebra generated by pseudodifferential operators and smoothing operators. Let $\mathcal{N}$ be the ideal of all smooth operators in $\mathcal{M}$. The algebra $\mathbb{C}[\mathfrak{g}^*]_J$ possesses a natural filtration

$$\mathbb{C}[\mathfrak{g}^*]_{J,j} := \frac{(\mathfrak{g}^*)^j\mathbb{C}[\mathfrak{g}^*]}{(\mathfrak{g}^*)^{J+1}\mathbb{C}[\mathfrak{g}^*]}.$$

Let $\mathcal{M}_j$ be the algebra generated by differential operators and smoothing operators acting on $\Gamma(S(TN))$ with coefficients in $\mathbb{C}[\mathfrak{g}^*]_{J,j}$. Let $\mathcal{N}_j$ denote the algebra generated by smoothing operators acting on $\Gamma(S(TN))$ with coefficients in $\mathbb{C}[\mathfrak{g}^*]_{J,j}$. The elements of $\mathbb{C}[\mathfrak{g}^*]_{J,j}$ are nilpotent of order $\le J+1$ in $\mathbb{C}[\mathfrak{g}^*]_J$ for $j \ge 1$, so the elements of $\mathcal{M}_j$ and $\mathcal{N}_j$ are also nilpotent of the same order. Note that the subspace $1+\mathcal{N}_j$ of $\mathcal{M}$ forms a group with inverse $(1+K_X)^{-1} = \sum_{j=0}^{J}(-K_X)^j$. Let $P_0 \in \mathcal{N}$ be the projection onto $\ker(D)$ and set $P_1 := 1 - P_0 \in \mathcal{M}$. For any $A_X \in \mathrm{End}(\Gamma(S(TN))) \otimes \mathbb{C}[\mathfrak{g}^*]_J$ we shall write

$$A_X = \begin{vmatrix} P_0A_XP_0 & P_0A_XP_1 \\ P_1A_XP_0 & P_1A_XP_1 \end{vmatrix} \in \begin{vmatrix} \mathcal{N} & \mathcal{N} \\ \mathcal{N} & \mathcal{M} \end{vmatrix}.$$

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Lemma 3.4 *(Lemma 2.33 [Go]) There exists $\gamma_X \in 1+\mathcal{N}_1$ which commutes with $\mathfrak{L}_X$, such that*

$$\gamma_X D_{-X}^2 \gamma_X^{-1} = \begin{vmatrix} U_X & 0 \\ 0 & V_X \end{vmatrix} \in \begin{vmatrix} \mathcal{N}_2 & 0 \\ 0 & T+\mathcal{N}_2 \end{vmatrix},$$

where $T - D^2 \in \mathcal{M}_1$ and U_X has the form $P_0 U_X' P_0$.

By Lemma 3.4, we have

$$t\gamma_{\frac{X}{t}} H_{\frac{X}{t}} \gamma_{\frac{X}{t}}^{-1} = \begin{vmatrix} tU_{\frac{X}{t}} + P_0 \mathfrak{L}_X P_0 & 0 \\ 0 & tV_{\frac{X}{t}} + P_1 \mathfrak{L}_X P_1 \end{vmatrix}, \tag{3.29}$$

$$\gamma_{\frac{X}{t}} D_{\frac{X}{t}} \gamma_{\frac{X}{t}}^{-1} = \begin{vmatrix} 0 & 0 \\ 0 & D \end{vmatrix} + O(t^{-1}). \tag{3.30}$$

$$e^{-tH_{\frac{X}{t}}} = \gamma_{\frac{X}{t}}^{-1} \begin{vmatrix} P_0 e^{-tU_{\frac{X}{t}} - \mathfrak{L}_X} P_0 & 0 \\ 0 & P_1 e^{-tV_{\frac{X}{t}} - \mathfrak{L}_X} P_1 \end{vmatrix} \gamma_{\frac{X}{t}}, \tag{3.31}$$

$$\begin{aligned} e^{-t\sigma_l H_{\frac{X}{t}}} D_{\frac{X}{t}} e^{-t\sigma_{l+1} H_{\frac{X}{t}}} &= \gamma_{\frac{X}{t}}^{-1} \begin{vmatrix} 0 & 0 \\ 0 & P_1 e^{-t\sigma_l [V_{\frac{X}{t}} + \mathfrak{L}_{\frac{X}{t}}]} D e^{-t\sigma_{l+1}[V_{\frac{X}{t}} + \mathfrak{L}_{\frac{X}{t}}]} P_1 \end{vmatrix} \gamma_{\frac{X}{t}} \\ &+ \gamma_{\frac{X}{t}}^{-1} \begin{vmatrix} P_0 e^{-t\sigma_l [U_{\frac{X}{t}} + \mathfrak{L}_{\frac{X}{t}}]} P_0 & 0 \\ 0 & P_1 e^{-t\sigma_l [V_{\frac{X}{t}} + \mathfrak{L}_{\frac{X}{t}}]} P_1 \end{vmatrix} L \\ &\cdot \begin{vmatrix} P_0 e^{-t\sigma_{l+1} [U_{\frac{X}{t}} + \mathfrak{L}_{\frac{X}{t}}]} P_0 & 0 \\ 0 & P_1 e^{-t\sigma_{l+1} [V_{\frac{X}{t}} + \mathfrak{L}_{\frac{X}{t}}]} P_1 \end{vmatrix} \gamma_{\frac{X}{t}} O(t^{-1}) + O(t^{-1}), \end{aligned} \tag{3.32}$$

where L is a zero order operator. As $\gamma_{\frac{X}{t}} = 1 + O(t^{-1}) S_0$ where S_0 is a smoothing operator, and we do not commute operators during the process of proving regularity at infinity, we can assume that $\gamma_{\frac{X}{t}} = 1$.

Lemma 3.5 *When $t \to +\infty$, we have:*

$$\mathrm{ch}_k(\sqrt{t} D_{-X}, D_X)(f^0, \cdots, f^k)_J \sim O(t^{-1}). \tag{3.33}$$

This also holds for $\mathrm{ch}_k(\sqrt{t} D_{-X}, D_{-X})(f^0, \cdots, f^k)_J$.

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Proof: Recall that $\mathrm{ch}_k(\sqrt{t}D_{-X}, D_X)(f^0,\cdots,f^k)_J = \sum_{0\le j\le k}(-1)^j T_j$, where

$$T_j = t^{k/2}\int_{\Delta_{k+1}} \mathrm{Tr}\Big[f^0 e^{-\sigma_0 t H_{\frac{X}{t}}} c(df^1) e^{-\sigma_1 t H_{\frac{X}{t}}}\cdots c(df^j) e^{-\sigma_j t H_{\frac{X}{t}}} D_{\frac{X}{t}}$$

$$e^{-\sigma_{j+1} t H_{\frac{X}{t}}} c(df^{j+1})\cdots c(df^k) e^{-\sigma_{k+1} t H_{\frac{X}{t}}}\Big]_J d\mathrm{Vol}_{\Delta_{k+1}}. \tag{3.34}$$

By (3.31) and (3.32), we have

$$\begin{aligned} T_j = t^{k/2}\textstyle\int_{\Delta_{k+1}} \mathrm{Tr}\Big[& f^0[P_0 e^{-\sigma_0(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_0(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]c(df^1) \\ & \cdot[P_0 e^{-\sigma_1(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_1(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]\cdots c(df^j)\Big[P_1 e^{-t\sigma_j[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]} D e^{-t\sigma_{j+1}[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_1 \\ & +(P_0 e^{-t\sigma_j[U_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_0 + P_1 e^{-t\sigma_j[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_1) \\ & \cdot L(P_0 e^{-t\sigma_{j+1}[U_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_0 + P_1 e^{-t\sigma_{j+1}[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_1)O(t^{-1})\Big] \\ & \cdot c(df^{j+1})\cdots c(df^k)[P_0 e^{-\sigma_{k+1}(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_{k+1}(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]\Big]_J d\mathrm{Vol}_{\Delta_{k+1}}, \end{aligned} \tag{3.35}$$

where $P_0 e^{-t\sigma_{l+1}[U_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_0$ and $P_1 e^{-t\sigma_{l+1}[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_1$ stand for respectively

$$\begin{vmatrix} P_0 e^{-t\sigma_{l+1}[U_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_0 & 0 \\ 0 & 0 \end{vmatrix} \text{ and } \begin{vmatrix} 0 & 0 \\ 0 & P_1 e^{-t\sigma_{l+1}[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_1 \end{vmatrix}.$$

Let

$$\begin{aligned} T'_j = t^{k/2}\int_{\Delta_{k+1}} \mathrm{Tr}\Big[& f^0[P_0 e^{-\sigma_0(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_0(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]c(df^1) \\ & \cdot[P_0 e^{-\sigma_1(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_1(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]\cdots c(df^j)\Big[P_1 e^{-t\sigma_j[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]} D e^{-t\sigma_{j+1}[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_1\Big] \\ & \cdot c(df^{j+1})\cdots c(df^k)[P_0 e^{-\sigma_{k+1}(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_{k+1}(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]\Big]_J d\mathrm{Vol}_{\Delta_{k+1}}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} T''_j = t^{k/2-1}\textstyle\int_{\Delta_{k+1}} \mathrm{Tr}\Big[& f^0[P_0 e^{-\sigma_0(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_0(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]c(df^1) \\ & \cdot[P_0 e^{-\sigma_1(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_1(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]\cdots c(df^j) \\ & \Big[(P_0 e^{-t\sigma_j[U_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_0 + P_1 e^{-t\sigma_j[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_1)L(P_0 e^{-t\sigma_{j+1}[U_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_0 + P_1 e^{-t\sigma_{j+1}[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]}P_1)\Big] \\ & \cdot c(df^{j+1})\cdots c(df^k)[P_0 e^{-\sigma_{k+1}(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0 + P_1 e^{-\sigma_{k+1}(tV_{\frac{X}{t}}+\mathfrak{L}_X)}P_1]\Big]_J d\mathrm{Vol}_{\Delta_{k+1}}. \end{aligned} \tag{3.37}$$

We estimate (3.36) first. Since $P_0 c(df^j)P_0 = 0$, only the terms containing no more than $\frac{k}{2}+1$ copies of $P_0 e^{-\sigma_l(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0$ give a non-zero contribution. In fact, the term containing no copy of $P_0 e^{-\sigma_l(tU_{\frac{X}{t}}+\mathfrak{L}_X)}P_0$ has exponential decay. Note that

$$V_{\frac{X}{t}} + \mathfrak{L}_{\frac{X}{t}} = P_1 D^2 P_1 + F_{\frac{X}{t}}, \tag{3.38}$$

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where $F_{\frac{X}{t}} \in \mathcal{M}_1$. Similarly to Lemma 2.2, by (3.38) we have that when X is small and $t \to \infty$,

$$||P_1 e_J^{-ut[V_{\frac{X}{t}}+\mathfrak{L}_{\frac{X}{t}}]} P_1 B||_{u^{-1}} \leq C(X)_J u^{-\frac{l}{2}} t^{-\frac{l}{2}} (\mathrm{tr}[P_1 e^{-\frac{tD^2}{2}} P_1])^u, \tag{3.39}$$

where B is a l-order operator. Using $||\mathrm{Tr}(P_1 e^{-sD^2} P_1)|| \leq C_0 e^{-s\lambda^2}$, for $s \geq 1$ and (3.39), similarly to Lemma 1.1 in [Wu], we get the exponential decay. Thus, it remains to deal with the terms containing at least $\frac{k}{2}+1$ copies of $P_1 e^{-\sigma_l(tV_{\frac{X}{t}}+\mathfrak{L}_X)} P_1$, as well as at least one copy of $P_0 e^{-\sigma_l(tU_{\frac{X}{t}}+\mathfrak{L}_X)} P_0$. We may use the trick in Lemma 2 in [CM2] to prove $T_j' = O(t^{-1})$. Similarly, we can prove $T_j'' = O(t^{-2})$ and then get Lemma 3.5. □

Now we shall show the convergence of the total truncated infinitesimal equivariant eta cochain. Let $C_G^1(N)$ be the Banach algebra of once differentiable functions on N with the norm defined by

$$||f||_1 := \sup_{x\in N}|f(x)| + \sup_{x\in N}||df(x)||.$$

Let

$$\phi_{X,J} = \{\phi_{X,J,0}, \cdots, \phi_{X,J,2q}, \cdots\}$$

be a truncated infinitesimal equivariant even cochain sequence in the bar complex of $C^1(N)$. Define

$$||\phi_{X,J,2q}|| = \sup_{||f_i||_1\leq 1;\ 0\leq i\leq 2q}\{|\phi_{X,J,2q}(f_0, \cdots, f_{2q})|\}.$$

Definition 3.6 The radius of convergence of $\phi_{X,J}$ is defined to be that of the power series $\sum q!||\phi_{X,J,2q}||z^q$. The space of cochain sequences with radius of convergence r at least larger than zero is denoted by $C_r^{\mathrm{even},X,J}(C_G^1(N))$ (define $C_r^{\mathrm{odd},X,J}(C_G^1(N))$ similarly).

In general, the sequence

$$\eta_X(D)_J = \{\cdots, \eta_{X,2q}(D)_J, \eta_{X,2q+2}(D)_J, \cdots\},$$

which is called a total truncated infinitesimal equivariant eta cochain, is not an entire cochain. Similarly to Proposition 2.16 in [Wa1], we have

Proposition 3.7 *Suppose that D is invertible with λ the smallest positive eigenvalue of $|D|$ and X is small. Then the truncated infinitesimal equivariant total eta cochain $\eta_X(D)_J$ has radius of convergence r satisfying the inequality: $r \geq 4\lambda^2 > 0$ i.e. $\eta_X(D)_J \in C_{4\lambda^2}^{\mathrm{even},X,J}(C_G^1(N))$.*

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For the idempotent $p \in \mathcal{M}_r(C^\infty(N))$, let $||dp|| = |||[D,p]||| = \sum_{i,j} ||dp_{i,j}||$ where $p_{i,j}$ $(1 \le i,j \le r)$ is the entry of p. Similarly to Proposition 2.17 in [Wa1], we have

Proposition 3.8 *Assume that D is invertible with λ the smallest positive eigenvalue of $|D|$, $||dp|| < \lambda$ and X is small. Then the pairing $\langle \eta_X(D)_J, Ch(p)\rangle$ is well-defined.*

Next we establish the main theorem in this section. Suppose D is invertible with λ the smallest eigenvalue of $|D|$, and $p = p^* = p^2 \in \mathcal{M}_r(C^\infty_G(N))$ is an idempotent which satisfies $||dp|| < \lambda$. Let

$$p(D \otimes I_r)p : \ p(H \otimes \mathbf{C^r}) = L^2(N, S \otimes p(\mathbf{C^r})) \to L^2(N, S \otimes p(\mathbf{C^r}))$$

be the Dirac operator with coefficients from $F = p(\mathbf{C^r})$. Since $p \in \mathcal{M}_r(C^\infty_G(N))$, we have

$$e^{-X}[p(D \otimes I_r)p] = [p(D \otimes I_r)p]e^{-X}.$$

Let

$$\mathbf{D_{-X}} = \begin{bmatrix} 0 & -D_{-X} \otimes I_r \\ D_{-X} \otimes I_r & 0 \end{bmatrix}; \ p = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix};$$

$$\sigma = i\begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix}; \ e^{-X} = \begin{bmatrix} e^{-X} & 0 \\ 0 & e^{-X} \end{bmatrix},$$

be operators from $H \otimes \mathbf{C^r} \oplus H \otimes \mathbf{C^r}$ to itself. Then

$$\mathbf{D_{-X}}\sigma = -\sigma \mathbf{D_{-X}}; \ \sigma p = p\sigma.$$

Moreover $\mathbf{D_{-X}} e^{t\mathbf{D_{-X}}^2}$ and $e^{t\mathbf{D_{-X}}^2}$ $(t > 0)$ are traceclass. For $u \in [0,1]$, let

$$D_{-X,u} = (1-u)D_{-X} + u[pD_{-X}p + (1-p)D_{-X}(1-p)] = D_{-X} + u(2p-1)[D,p].$$

Then

$$\mathbf{D}_{-X,u} = \begin{bmatrix} 0 & -D_{-X,u} \\ D_{-X,u} & 0 \end{bmatrix} = \mathbf{D_{-X}} + u(2p-1)[\mathbf{D_{-X}}, p].$$

Consider a family of Dirac Operators parameterized by (u,s,t), which is given by

$$\widetilde{\mathbf{D}_{-\mathbf{X}}} = t^{\frac{1}{2}}\mathbf{D}_{-X,u} + s\sigma(p - \frac{1}{2}).$$

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Let $A = d + \widetilde{\mathbf{D}_{-\mathbf{X}}}$ be a superconnection on the trivial infinite dimensional superbundle with base $[0,1]\times\mathbf{R}$ and fibre $H\otimes\mathbf{C^r}\oplus H\otimes\mathbf{C^r}$. Then we have

$$\begin{aligned} B_{X,u,s,t} &:= (d+\widetilde{\mathbf{D}_{-\mathbf{X}}})^2 \\ &= t\mathbf{D}^2_{-X,u} - s^2/4 - (1-u)t^{\frac{1}{2}}s\sigma[\mathbf{D},p] + ds\sigma(p-\frac{1}{2}) + t^{\frac{1}{2}}du(2p-1)[\mathbf{D},p]. \end{aligned} \tag{3.40}$$

Consider the differential form $\int_\varepsilon^{+\infty}\frac{1}{2\sqrt{t}}\mathrm{Str}[\psi_t e^{-tX}\mathbf{D}_{-X,u}e^{B_{X,u,s,t}}]_J$ on $[0,1]\times\mathbf{R}$. By Lemma 9.15 in [BGV] and the fact that D_u is invertible, we have

$$\begin{aligned} &d\int_\varepsilon^{+\infty}\frac{1}{2\sqrt{t}}\mathrm{Str}[\psi_t e^{-tX}\mathbf{D}_{-X,u}e^{B_{X,u,s,t}}]_J \\ &= -\int_\varepsilon^{+\infty}\frac{\partial}{\partial t}\mathrm{Str}[\psi_t e^{-tX}e^{B_{X,u,s,t}}]_J = \mathrm{Str}[e^{-X}e^{B_{X/\varepsilon,u,s,\varepsilon}}]_J. \end{aligned} \tag{3.41}$$

Let $\Gamma_u = \{u\}\times\mathbf{R}\subset[0,1]\times\mathbf{R}$ be a contour oriented in the direction of increasing s and $\gamma_s = [0,1]\times\{s\}$ be a contour oriented in the direction of increasing u . By Stokes' theorem,

$$\begin{aligned} &\int_{[0,1]\times\mathbf{R}} d\int_\varepsilon^{+\infty}\frac{1}{2\sqrt{t}}\mathrm{Str}[\psi_t e^{-tX}\mathbf{D}_{-X,u}e^{B_{X,u,s,t}}]_J \\ &= \left(\int_{\Gamma_1}-\int_{\Gamma_0}-\int_{\gamma_{+\infty}}+\int_{\gamma_{-\infty}}\right)\left[\int_\varepsilon^{+\infty}\frac{1}{2\sqrt{t}}\mathrm{Str}[\psi_t e^{-tX}\mathbf{D}_{-X,u}e^{B_{X,u,s,t}}]_J\right]. \end{aligned} \tag{3.42}$$

We have for some constant $C>0$ that,

$$\int_{\gamma_s}\left[\int_\varepsilon^{+\infty}\frac{1}{2\sqrt{t}}\mathrm{Str}[\psi_t e^{-tX}\mathbf{D}_{-X,u}e^{B_{X,u,s,t}}]_J\right]\sim O(e^{-cs^2}). \tag{3.43}$$

As shown in [Ge2] or [Wa1]:

$$\begin{aligned} &\int_{\Gamma_0}\left[\int_\varepsilon^{+\infty}\frac{1}{2\sqrt{t}}\mathrm{Str}[\psi_t e^{-tX}\mathbf{D}_{-X,u}e^{B_{X,u,s,t}}]_J\right] \\ &= -4\sqrt{-1}\pi[\langle\eta_X(D)_J,\mathrm{Ch}(p)\rangle - \frac{1}{2}\langle\eta_X(D)_J,\mathrm{rk}(p)\mathrm{Ch}_*(1)\rangle]. \end{aligned} \tag{3.44}$$

$$\begin{aligned} &\int_{\Gamma_1}\left[\int_\varepsilon^{+\infty}\frac{1}{2\sqrt{t}}\mathrm{Str}[\psi_t e^{-tX}\mathbf{D}_{-X,u}e^{B_{X,u,s,t}}]_J\right] \\ &\quad= -2\sqrt{-1}\pi\Big[\eta_X(D_p)_J - \langle\eta_X(D)_J,\mathrm{rk}(p)\mathrm{Ch}_*(1)\rangle \\ &\qquad + \frac{1}{2}\int_0^1\mathrm{Tr}[\varepsilon^{1/2}(2p-1)dpe^{-X}e^{-\varepsilon D^2_{-\frac{X}{\varepsilon},u}}]_J du\Big]. \end{aligned} \tag{3.45}$$

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By (3.41)-(3.45), we get

Theorem 3.9 *Assume D is invertible and $||dp|| < \lambda$ where λ is the smallest eigenvalue of $|D|$ and X is small. Then we have*

$$\frac{1}{2}\eta_X(p(D\otimes I_r)p)_J = \langle \eta_X(D)_J, \mathrm{Ch}(p)\rangle$$
$$+\pi\sqrt{-1}\int_0^1 \mathrm{Tr}[\varepsilon^{1/2}(2p-1)dpe^{-X}e^{-\varepsilon D^2-\frac{X}{\varepsilon},u}]_J\,du - \frac{1}{4\sqrt{-1}\pi}\int_{[0,1]\times\mathbf{R}} \mathrm{Str}[e^{-X}e^{B_{X/\varepsilon,u,s,\varepsilon}}]_J. \tag{3.46}$$

4. A family infinitesimal equivariant index formula

In this section, we give a proof of a family infinitesimal equivariant index formula. Let M be an $n+q$ dimensional compact connected manifold and B be a q dimensional compact connected manifold. Assume that $\pi : M \to B$ is a submersion of M onto B, which defines a fibration of M with fibre Z. For $y \in B$, $\pi^{-1}(y)$ is then a submanifold Z_y of M. Let TZ denote the n-dimensional vector bundle on M whose fibre T_xZ is the tangent space at x to the fibre $Z_{\pi x}$. Assume that M and B are oriented. Taking the orthogonal bundle of TZ in TM with respect to any Riemannian metric determines a smooth horizontal subbundle T^HM, i.e. $TM = T^HM \oplus TZ$. A vector field $Y \in TB$ will be identified with its horizontal lift $Y \in T^HM$, moreover T^H_xM is isomorphic to $T_{\pi(x)}B$ via π_*. Recall that B is Riemannian, so we can lift the Euclidean scalar product g_B of TB to T^HM. And we assume that TZ is endowed with a scalar product g_Z. Thus we can introduce a new scalar product $g_B \oplus g_Z$ in TM. Denote by ∇^L the Levi-Civita connection on TM with respect to this metric. Let ∇^B denote the Levi-Civita connection on TB and still denote by ∇^B the pullback connection on T^HM. Let $\nabla^Z = P_Z(\nabla^L)$, where P_Z denotes the projection to TZ. Let $\nabla^\oplus = \nabla^B \oplus \nabla^Z$ and $\omega = \nabla^L - \nabla^\oplus$ and let T be the torsion tensor of $\nabla^\oplus$. Let $SO(TZ)$ be the $SO(n)$ bundle of oriented orthonormal frames in TZ. Now we assume that the bundle TZ is spin. Let $S(TZ)$ be the associated spinor bundle. ∇^Z can be lifted to give a connection on $S(TZ)$. Let D be the tangent Dirac operator.

Let G be a compact Lie group which acts fiberwise on M. We will assume that G acts as identity on B. Without loss of generality assume G acts on (TZ, h_{TZ}) isometrically. Also assume that the action of G lifts to $S(TZ)$ and that the G-action commutes with D. Let E be the vector bundle $\pi^*(\wedge T^*B)\otimes S(TZ)$. This bundle carries a natural action m_0 of the degenerate Clifford module $C_0(M)$. The Clifford action of a horizontal cotangent vector $\alpha \in \Gamma(M, T^*_HM)$ is given by exterior multiplication $m_0(\alpha) = \varepsilon(\alpha)$ acting on the first factor $\bigwedge T^*_HM$ in E, while the Clifford action of a vertical cotangent vector simply equals its Clifford action on

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$S(TZ)$. Define the connection for $X \in \mathfrak{g}$ whose Killing vector field is in TZ,

$$\nabla^{E,-X,\oplus} := \pi^*\nabla^B \otimes 1 + 1 \otimes \nabla^{S,-X}, \tag{4.1}$$

$$\omega(Y)(U,V) := g(\nabla^L_Y U, V) - g(\nabla^{\oplus}_Y U, V), \tag{4.2}$$

$$\nabla^{E,-X,0}_Y := \nabla^{E,-X,\oplus}_Y + \frac{1}{2} m_0(\omega(Y)), \tag{4.3}$$

for $Y,U,V \in TM$. Then the Bismut connection acting on $\Gamma(M,\pi^* \wedge (T^*B) \otimes S(TZ))$ is defined by

$$\mathcal{B}^{-X} = \sum_{i=1}^{n} c(e_i^*)\nabla^{E,-X,0}_{e_i} + \sum_{j=1}^{q} c(f_j^*)\nabla^{E,-X,0}_{f_j}, \tag{4.4}$$

where $e_1,\cdots,e_n$ and $f_1,\cdots,f_q$ are orthonormal basis of TZ and TB respectively. Define the family Bismut Laplacain as follows:

$$H_X = (\mathcal{B}^{-X})^2 + \mathfrak{L}^E_X. \tag{4.5}$$

Let $\triangle^{Z,X}$ be the Laplacian on $\pi^*(\wedge T^*B) \otimes S(TZ)$ associated with $\nabla^{E,X,0}$. Similarly to Proposition 8.12 and Theorem 10.17 in [BGV], we have

Proposition 4.1 *The following identity holds*

$$H_X = \triangle^{G,X} + \frac{1}{4} r_M. \tag{4.6}$$

In this section, we establish an index theorem in the untwisted case and it is easy to extend it to the twisted case. Then by Proposition 10.15 in [BGV],

$$\mathcal{B}^{-X} = \mathcal{B} + \frac{1}{4} c(X) = D_{-X} + \mathrm{A}_{[+]}, \tag{4.7}$$

where $\mathrm{A}_{[+]}$ is an operator with coefficients in $\Omega_{\geq 1}(B)$ and $\mathcal{B}$ is the Bismut superconnection. And $D_{-X} = D + \frac{1}{4}c(X)$. Let $H_X = D^2_{-X} + F_{[+]}$, where $\mathrm{F}_{[+]}$ is an operator with coefficients in $\Omega_{\geq 1}(B)$. We define the operator e^{-tH_X} which is given by

$$e^{-tH_X} = e^{-t(D^2_{-X}+L_X)} + \sum_{k>0} (-t)^k I_k, \tag{4.8}$$

where

$$I_k = \int_{\triangle_k} e^{-\sigma_0 t(D^2_{-X}+L_X)} F_{[+]} e^{-\sigma_1 t(D^2_{-X}+L_X)} F_{[+]}$$

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$$\cdots e^{-\sigma_{k-1}t(D^2_{-X}+L_X)}F_{[+]}e^{-\sigma_k t(D^2_{-X}+L_X)}d\sigma, \tag{4.9}$$

and the sum is finite. By Theorem 2.1 in [LM], similarly to Proposition 8.11 in [BGV], we get

Proposition 4.2 *We have in the cohomology class of B,*

$$\mathrm{Ch}(\mathrm{ind}_G(e^{-X},D)) = \mathrm{Str}(\phi_t e^{-tH_{\frac{X}{t}}}), \tag{4.10}$$

which does not depend on t, and $\phi_t(dy_j) = \frac{1}{\sqrt{t}}dy_j$.

We define the operator

$$Q := (H_X + \frac{\partial}{\partial t})^{-1} = (D^2_{-X} + L_X + \frac{\partial}{\partial t})^{-1}$$

$$+\sum_{k>0}(-1)^k(D^2_{-X} + L_X + \frac{\partial}{\partial t})^{-1}[F_{[+]}(D^2_{-X} + L_X + \frac{\partial}{\partial t})^{-1}]^k, \tag{4.11}$$

where $(D^2_{-X} + L_X + \frac{\partial}{\partial t})^{-1}$ is the Volterra inverse of $D^2_{-X} + L_X + \frac{\partial}{\partial t}$ as shown in Section 2. We can define Volterra symbols with coefficients in $\mathbb{C}[\mathfrak{g}^*]\otimes\wedge T^*_z B$ and Volterra pseudodifferential operators with coefficients in $\mathbb{C}[\mathfrak{g}^*]\otimes\wedge T^*_z B$. Write the space of Volterra pseudodifferential operators with coefficients in $\mathbb{C}[\mathfrak{g}^*]\otimes\wedge T^*_z B$ by $\Psi^*_V(\mathbb{R}^n\times\mathbb{R}, S(TM)\otimes\mathbb{C}[\mathfrak{g}^*]\otimes\wedge T^*_z B)$. We define the Getzler order $O_G(dy^j) = 1$. Let $Q\in\Psi^*_V(\mathbb{R}^n\times\mathbb{R}, S(TM)\otimes\mathbb{C}[\mathfrak{g}^*]\otimes\wedge T^*_z B)$ have symbol

$$q(x,X,\xi,\tau)\sim\sum_{k\le m'}\sum_{l_0=0}^{2^{\dim B}} q_{k,l_0}(x,X,\xi,\tau)\omega_{[l_0]}, \tag{4.12}$$

where $q_{k,l_0}(x,X,\xi,\tau)$ is an order k symbol and $\omega_{[l_0]}$ is a l_0-degree differential form on B. Then using Taylor expansions at $x=0$ and at $X=0$, we get that

$$\sigma[q(x,X,\xi,\tau)]\sim\sum_{j,k,\alpha,\beta}\sum_{l_0}^{2^{\dim B}}\frac{x^\alpha}{\alpha!}\frac{X^\beta}{\beta!}\sigma[\partial_x^\alpha\partial_X^\beta q_{k,l_0}(0,0,\xi,\tau)]^{(j)}\omega_{[l_0]}. \tag{4.13}$$

The symbol $\frac{x^\alpha}{\alpha!}\frac{X^\beta}{\beta!}\sigma[\partial_x^\alpha\partial_X^\beta q_{k,l_0}(0,0,\xi,\tau)]^{(j)}\omega_{[l_0]}$ is the Getzler homogeneous of $k + j + l_0 - |\alpha| + 2|\beta|$. Similarly to the definition 2.7, we can define the J-truncated symbol of q denoted by $\sigma[q(x,X,\xi,\tau)]_J$. Also, we may define the truncated model operator of Q. Similarly to Lemma 2.9, we have

Lemma 4.3 *Let $Q\in\Psi^*_V(\mathbb{R}^n\times\mathbb{R}, S(TM)\otimes\mathbb{C}[\mathfrak{g}^*]\otimes\wedge T^*_z B)$, and assume Q_J has Getzler order m and model operator $Q_{(m),J}$. Then as $t\to 0^+$ we have:*

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1) $\sigma[\phi_t K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)} = \omega O(t^{\frac{j-n-m-2}{2}}) + O(t^{\frac{j-n-m-1}{2}})$, *where* $\omega \in \Omega^{\text{odd}}(T^*B)$, *if* $m-j$ *is odd;*

2) $\sigma[\phi_t K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)} = t^{\frac{j-n-m-2}{2}} K_{Q_{(m),J}}(0,0,X,1)^{(j)} + O(t^{\frac{j-n-m}{2}})$, *if* $m-j$ *is even,*

where $[K_{Q_J}(0,0,\frac{X}{t},t)]^{(j)}$ *denotes taking the* j *degree form component in* $\wedge^* T(M_z)$. *In particular, when* $m=-2$ *and* $j=n$ *is even, we get*

$$\sigma[\phi_t K_{Q_J}(0,0,\frac{X}{t},t)]^{(n)} = K_{Q_{(-2),J}}(0,0,X,1)^{(n)} + O(t). \tag{4.14}$$

By Lemma 4.3, similarly to the proof of Proposition 2.11 and Theorem 2.12, we have

Theorem 4.4 *In the cohomology class of* B,

$$\text{Ch}(\text{ind}_G(e^{-X},D)) = (2\pi\sqrt{-1})^{-n/2}\int_{M/B}\widehat{A}(F^Z_{\mathfrak{g}}(X))d\,\text{Vol}_{M/B}. \tag{4.15}$$

In the following, we define infinitesimal equivariant eta forms. Now assume that $\dim M$ and $\dim Z$ are odd. Let Tr^{even} denote taking trace on the coefficients of even forms on B. Let T be the torsion tensor of $\nabla^{\oplus}$ and $c(T) = \sum_{1\le\alpha<\beta\le q} dy_\alpha dy_\beta c(T(\frac{\partial}{\partial y_\alpha},\frac{\partial}{\partial y_\beta}))$. Then the **infinitesimal equivariant eta form** is defined by

$$\widehat{\eta}_X = \int_0^\infty \frac{1}{\sqrt{\pi t}}\text{Tr}^{\text{even}}[\phi_t(D-\frac{1}{4}c(\frac{X}{t})+\frac{c(T)}{4})e^{-tH_{\frac{X}{t}}}]dt. \tag{4.16}$$

When n is even, we define the infinitesimal equivariant eta form by

$$\widehat{\eta}_X = \int_0^\infty \frac{1}{\sqrt{\pi t}}\text{Tr}^{\text{odd}}[\phi_t(D-\frac{1}{4}c(\frac{X}{t})+\frac{c(T)}{4})e^{-tH_{\frac{X}{t}}}]dt. \tag{4.17}$$

Let $e_1(x),\cdots e_n(x)$ denote the orthonormal frame of TZ. If $A(Y)$ is any 0 order operator depending linearly on $Y\in TZ$, we define the operator $(\nabla_{e_i}+A(e_i))^2$ as follows

$$(\nabla_{e_i}+A(e_i))^2 = \sum_1^n(\nabla_{e_i(x)}+A(e_i(x)))^2 - \nabla_{\sum_j \nabla_{e_j}e_j} - A(\sum_j \nabla_{e_j}e_j). \tag{4.18}$$

We introduce an auxiliary form dt. Let

$$\widetilde{H_X} = H_X - dt(D_X+\frac{c(T)}{4});\ h(x) = 1+\frac{1}{2}dt\sum_{j=1}^n x_i c(e_i), \tag{4.19}$$

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then we have

$$\widetilde{H_X} = -(\nabla_{e_i}^{G,X} + \frac{1}{2} < \omega(e_i)e_j, f_\alpha > e_j dy_\alpha + \frac{1}{4} < \omega(e_i) f_\alpha, f_\beta > dy_\alpha dy_\beta - \frac{1}{2} c(e_i)dt)^2 + \frac{rZ}{4}. \tag{4.20}$$

and

$$(h\rho)\widetilde{H_X}(h\rho)^{-1} = \rho H_X \rho^{-1} + dtu, \;\; O_G(u) \leq 0. \tag{4.21}$$

By the trick in Lemma 3.3, we get

Lemma 4.5 *When $t \to 0^+$, we have*

$$\mathrm{Tr}^{\mathrm{even}}[\phi_t(D - \frac{1}{4}c(\frac{X}{t}) + \frac{c(T)}{4})e^{-tH_{\frac{X}{t}}}] \sim O(t^{1/2}). \tag{4.22}$$

We introduce the following notations as in Lemma 3.4,

$$\mathcal{M}_{(j)} := \mathcal{M} \cap \oplus_{k+l \geq j} A^k(B, \mathrm{End}(\Gamma(S(TM_z))) \otimes \mathbb{C}[\mathfrak{g}^*]_{J,l};$$

$$\mathcal{N}_{(j)} := \mathcal{N} \cap \oplus_{k+l \geq j} A^k(B, \mathrm{End}(\Gamma(S(TM_z))) \otimes \mathbb{C}[\mathfrak{g}^*]_{J,l}.$$

$\mathcal{M}_{(j)}$ is the algebra generated by differential operators and smoothing operators acting on $\Gamma(S(TM_z))$ with coefficients in $\oplus_{k+l \geq j} A^k(B) \otimes \mathbb{C}[\mathfrak{g}^*]_{J,l}$ and $\mathcal{N}_{(j)}$ denotes the algebra generated by smoothing operators acting on $\Gamma(S(TM_z))$ with coefficients in $\oplus_{k+l \geq j} A^k(B) \otimes \mathbb{C}[\mathfrak{g}^*]_{J,l}$. Replace $\mathcal{M}_j$ and $\mathcal{N}_j$ in Lemma 2.34 in [Go] by $\mathcal{M}_{(j)}$ and $\mathcal{N}_{(j)}$, then we have

Lemma 4.6 *When $t \to +\infty$, we have*

$$\mathrm{Tr}^{\mathrm{even}}[\phi_t(D - \frac{1}{4}c(\frac{X}{t}) + \frac{c(T)}{4})e^{-tH_{\frac{X}{t}}}] \sim O(t^{-1}). \tag{4.23}$$

By Lemma 4.5 and Lemma 4.6, we get that infinitesimal equivariant eta forms are well-defined. We recall the definition of equivariant eta forms in [Wa2],

$$\widehat{\eta}(e^{-X}) = \int_0^\infty \frac{1}{\sqrt{\pi t}} \mathrm{Tr}^{\mathrm{even}}[\phi_t e^{-X}(D + \frac{c(T)}{4})e^{-t\mathcal{B}^2}]dt. \tag{4.24}$$

To conclude the paper, we announce a comparison formula between infinitesimal equivariant eta forms and equivariant eta forms. Its proof will appear elsewhere. Let $d_{rX}\theta_{rX} = d\theta_{rX} - ||rX||$ for $r > 0$. The vector field X_M is called geodesic if $\nabla_{X_M} X_M = 0$.

Theorem 4.7 *If the Killing field X_M is geodesic and has no zeros on M, then for $X \in \mathfrak{g}$ and small $r \neq 0$ and each $K > 0$, we have up to an exact form*

$$\widehat{\eta}_{rX} = \widehat{\eta}(e^{-rX}) + \int_{M/B} 2(2\pi i)^{-\frac{n+1}{2}} \frac{\theta_{rX}}{d_{rX}\theta_{rX}} \widehat{A}_{rX}(M/B) + O(r^K). \tag{4.25}$$

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