The three Steiner-Lehmus theorems

A. F. BEARDON

1. *One for all, all for one*

Steiner's proof of what is now called the *Steiner-Lehmus theorem* was published in 1844, the same year as the book *The three musketeers*, written by the French author Alexandre Dumas. The motto *One for all, all for one (Einer für alle, alle für einen; Un pour tous, tous pour un; Uno per tutti, tutti per uno)* of the three musketeers came into widespread use in Europe in the 19th century, and its essence is that the three musketeers are inseparable; each member pledges to support the group, and the group supports each member. Now there are three classical geometries of constant curvature, namely Euclidean, spherical and hyperbolic geometries, and one can argue that, like the three musketeers, these geometries should be considered as being inseparable; that is, an idea, theorem or proof in any one of them should automatically be considered in the other two. The issue here should be not only to decide whether a particular result is true, or false, in a given geometry, but *to understand which particular properties of the geometries make it so*.

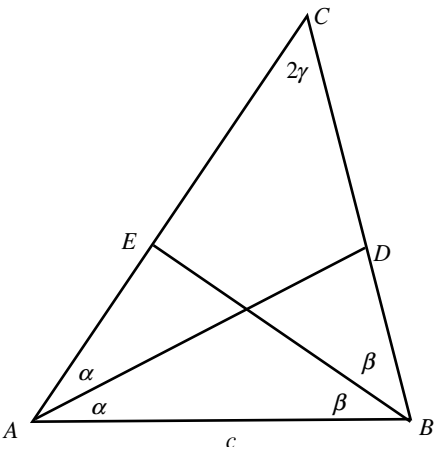


FIGURE 1

Here we shall consider just one result, namely the celebrated *Steiner-Lehmus theorem*. So what is the Steiner-Lehmus theorem? A Euclidean isosceles triangle has an axis of reflective symmetry, so the two angle bisectors of the equal angles are of equal length: in Figure 1, if $\alpha = \beta$ then $AD = BE$. The Steiner-Lehmus Theorem is the converse: *in any Euclidean triangle, if two angle bisectors are of equal length, then the two bisected angles are equal*: if $AD = BE$ then $\alpha = \beta$. Since the familiar property of isosceles triangles (two sides are equal if and only if their opposite angles are equal) is true in all three geometries (see Section 4), we should surely

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ask about the converse in each case. In each of the three proofs below we shall assume that $AD = BE$ and $\alpha > \beta$ and reach a contradiction. This will show that $\alpha \leqslant \beta$ and, by symmetry, $\beta \leqslant \alpha$ so that, in each of the three geometries, $\alpha = \beta$.

In 1840 C. L. Lehmus asked J. Steiner for a geometric proof of the converse, and Steiner quickly found (but did not immediately publish) a proof. *Nouvelles Annales de Mathématiques* of Paris published the problem in 1842, and two solutions, by Rougevain and by Grout de Saint-Paer, appeared in the same journal. Steiner published his proof in 1844 [1], when he also considered the question for external bisectors, and in spherical geometry. The question raised by Lehmus [2] has resulted in a long list of papers. For example, [3] contains a survey of sixty or so proofs given in the period 1840-1940; currently it is estimated that there are over eighty proofs in the literature, and this number still seems to be growing. One possible reason for this is that some of the earlier publications on this topic are not now easily obtainable; for example, from 1852 to 1874 about a dozen proofs appeared in the *Lady's and Gentleman's Diary*, an English publication devoted to poetry and mathematics.

Another reason for the proliferation of proofs is a continuing debate about direct versus indirect proofs: an indirect proof is one which uses a proof by contradiction, or a result which itself depends on a proof by contradiction. This debate began with Sylvester who, in 1852 [4], suggested that 'when a theorem depends on the necessary non-existence of real roots (within prescribed limits) of the analytical equation expressing the conditions of the question, no other form of proof than that of the *reductio ad absurdam* is possible in the nature of things.' In this context, Augustus de Morgan was already writing on direct and indirect proofs in 1852. More recently other writers have addressed the same issue but it is clear that the idea originates with Sylvester. Coxeter and Greitzer [5] state that 'papers on the Steiner-Lehmus theorem appeared in various journals in 1842, 1844 and 1848, and almost every year from 1854 until 1864, and with a good deal of regularity during the next hundred years .... A proof cannot properly claim to be direct if any one of the auxiliary theorems used has an indirect proof. Now, some of the simplest and most basic theorems have indirect proofs: consequently, if we insisted on complete directness, our store of theorems would be reduced to the merest trivialities. Is this observation any cause for sorrow?' Finally, G. H. Hardy [6] says that '*Reductio ad absurdam*, which Euclid loved so much, is one of the mathematician's finest weapons. It is a far finer gambit than any chess gambit; a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.' Readers can consider the importance of this issue for themselves, and we shall not discuss it any further. Here, all proofs are by contradiction, and our focus is on proofs that use as many common features of the three geometries as possible.

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2. *Miller's proof in the Euclidean case*

It is remarkable that Lehmus's question seems to have been perceived as a difficult problem, especially given the truly simple proof of the Euclidean case given by a schoolmaster Andrew Miller at Dundee High School in 1866 which is essentially as follows.

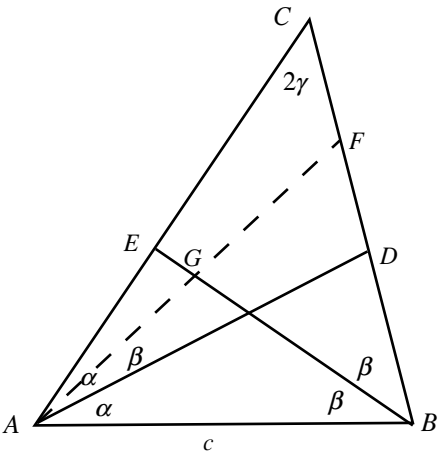


FIGURE 2

Proof: Consider Figure 2, where $AD = BE$ and $\alpha > \beta$. Draw the line AF, meeting BE at G, such that $\angle DAF = \beta$. As ΔADF is similar to ΔBGF (with the same angles), we have $BF/AF = BG/AD$. Thus, as $BG < BE = AD$ we have $BF < AF$. However, as $0 < 2\beta < \alpha + \beta < \pi - 2\beta$, and $\sin(\pi - 2\beta) = \sin 2\beta$, we have $\sin(\alpha + \beta) > \sin 2\beta > 0$ so, from the sine rule, $BF > AF$ which is a contradiction.

We have not examined every proof of the Steiner-Lehmus theorem, but surely this proof, which appeared in print in [7] as early as 1902, must be a candidate for the simplest proof of all. Where else in mathematics do we have a result with such a simple proof, published so long ago, which continues to have new proofs published with such consistent regularity?

3. *The three geometries*

The Steiner-Lehmus theorem was originally posed in Euclidean geometry. However, in his 1844 paper Steiner also considered the problem in spherical geometry, although one would hardly know this from the literature where, inexplicably, the spherical version is virtually ignored. For example, each of [3, 7, 8] gives a broad historical perspective of the problem but no more than a passing reference (if at all) to spherical geometry. Of course, spherical geometry had been well developed by the time Lehmus posed the problem (by Euler during the period 1753-1815, and long before him by Menelaus and others), but hyperbolic geometry was not as common

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a consideration in 1844 as it is now. For example, Klein announced his famous Erlangen programme in 1872, and Poincaré's decisive introduction of hyperbolic geometry into complex analysis came in 1881. Despite a reluctance by the author to publish yet more on this topic, this article was prompted by the note [9] in the *Mathematical Gazette* which contained the observation that the corresponding statement in hyperbolic geometry is claimed in [10] and [11] to be true, and claimed in [12] and [13] to be false. Here we use the same ideas as in [10] and [13] and we confirm that the Steiner-Lehmus theorem is indeed true in hyperbolic geometry. For the benefit of the reader, and in view of the apparent uncertainty, we give the details below. Indeed, up to a basic knowledge of the three classical geometries, we give a self-contained account of the Steiner-Lehmus theorem in all three geometries. Arguably, this is more instructive (and more interesting) than continuing the debate about direct versus indirect proofs.

4. *Triangles*

As beautiful as it is, Miller's proof uses similar triangles and so does not generalise to the other two geometries (where similarity is the same as congruence, and where the angle sum of a triangle is not constant). We shall assume familiarity with each of these geometries, and we start by describing the trigonometry in each of them. At this stage there is no need to say anything about Euclidean trigonometry. A triangle in *hyperbolic geometry* can have some or all of its vertices on the circle at infinity, but here we shall only consider those hyperbolic triangles whose vertices are strictly inside the hyperbolic plane; then the angles at the vertices are strictly positive, and each side of the triangle has a finite hyperbolic length. *Spherical geometry* is the geometry of the unit sphere in $\mathbb{R}^3$, and the distance between two points is the length of the shorter of the two arcs of the unique great circle that contains them. A spherical triangle is the intersection of three hemispheres (determined by three distinct great circles) and, as a consequence, all three sides, and all three angles, are less than π. For more details on hyperbolic geometry, and spherical geometry see (among many texts on these topics) [14, 15, 16, 17, 18, 19].

Throughout this discussion we shall consider a (Euclidean, hyperbolic or spherical) triangle ΔABC with vertices A, B and C, and sides AB, BC and CA of lengths c, a and b, respectively. Because of the nature of the Steiner-Lehmus theorem, it is convenient to let the angles at A, B and C be 2α, 2β and 2γ, respectively. Each geometry has its own particular trigonometry, and we shall now list the relevant formulae; of course, these apply to a general triangle but they are stated here for the triangle with angles 2α, 2β and 2γ.

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It is well known that the angle sum $2\alpha + 2\beta + 2\gamma$ of the triangle ABC satisfies

$$2\alpha + 2\beta + 2\gamma = \begin{cases} \pi & \text{in Euclidean geometry,} \\ \pi - \text{area}\,(\Delta ABC) & \text{in hyperbolic geometry,} \\ \pi + \text{area}\,(\Delta ABC) & \text{in spherical geometry.} \end{cases}$$

In Euclidean and hyperbolic geometries, $2\alpha + 2\beta + 2\gamma \leqslant \pi$; in spherical geometry, $2\alpha,\ 2\beta,\ 2\gamma \in (0, \pi)$, and it may be that $2\alpha + 2\beta + 2\gamma > \pi$. However, in all three geometries, $\alpha, \beta \in \left(0, \frac{1}{2}\pi\right)$ so that if $\alpha > \beta$ then $0 < \sin\beta < \sin\alpha$ and $0 < \cos\alpha < \cos\beta$.

Next, we have the cosine rules

$$c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos 2\gamma,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos 2\gamma,$$

and the sine rules

$$\frac{a}{\sin 2\alpha} = \frac{b}{\sin 2\beta} = \frac{c}{\sin 2\gamma},$$

$$\frac{\sinh a}{\sin 2\alpha} = \frac{\sinh b}{\sin 2\beta} = \frac{\sinh c}{\sin 2\gamma},$$

$$\frac{\sin a}{\sin 2\alpha} = \frac{\sin b}{\sin 2\beta} = \frac{\sin c}{\sin 2\gamma},$$

in Euclidean, hyperbolic and spherical geometries, respectively. Note that the sine rule in the spherical case differs from the other two sine rules in one important respect, namely that of the three functions x, $\sinh x$ and $\sin x$ in the numerators, x and $\sinh x$ are increasing functions but $\sin x$ is not. Because $0 < x,\ y < \pi$ does not imply that $\sin x < \sin y$, we need the next lemma.

Lemma 4.1: If $x > 0$, $y > 0$ and $x + y < \pi$, then $\sin x < \sin y$ if, and only if, $x < y$.

Proof: It is immediate from the graph of $x \to \sin x$ on $[0, \pi]$ that $\sin y > \sin x$ if, and only if, y lies between x and $\pi - x$. Given that $y < \pi - x$, this occurs if, and only if, $x < y$.

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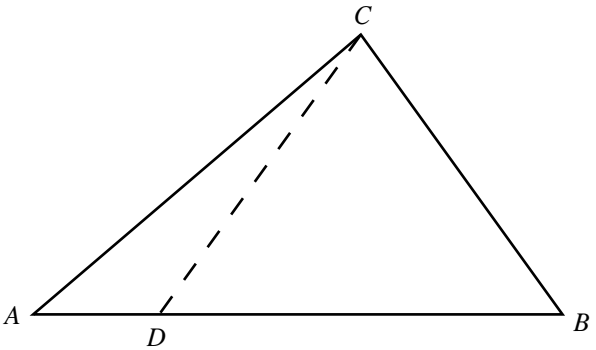


FIGURE 3

It seems that distinction between the sine rules necessitates a different approach in the spherical case. For example, we have already remarked that the familiar property of isosceles triangles (that two sides are equal if and only if their opposite angles are equal) is true in all three geometries. In Euclidean and hyperbolic geometries, this follows immediately from the sine rule. For spherical geometry a different proof is required; see [16, pp. 286-291]. More generally, in all three geometries it is true that the longer of two sides of a triangle is opposite the larger of the two angles opposite the sides. Again, the proofs in the Euclidean and hyperbolic cases, but not in the spherical case, follow from the sine rule. However, the following proof is valid in all three geometries. Suppose that, in Figure 3, in ΔABC the angle at C is greater than the angle at B. This implies that there is a point D between A and B such that in the triangle ΔBCD, the angles at B and C are equal. Thus $DC = DB$, so that $AC \leqslant AD + DC = AD + DB = AB$ as required.

5. *The basic triangle*

We shall now consider the basic triangle illustrated in Figure 4 which may lie in any of the three geometries. In each of the following sections, and in each of the three geometries, we shall assume that $BE = AD = \ell$, and $\alpha > \beta$, and (with an additional assumption in the spherical case) we shall reach a contradiction. As remarked earlier, this will imply that $\alpha = \beta$.

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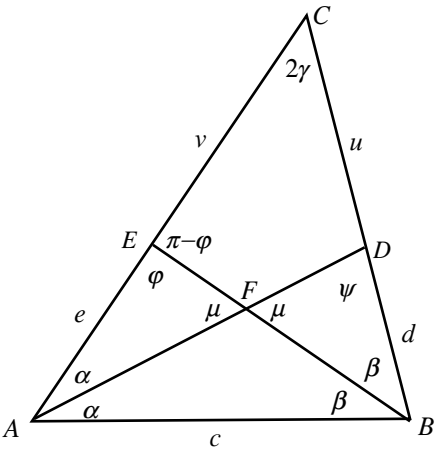


FIGURE 4

Lemma 5.1: Suppose that $BE = AD = \ell$ and $\alpha > \beta$, then, in all geometries, $a > b, d > e, BF > AF$ and $DF > EF$.

Proof: As the sides of lengths a and b are opposite the angles 2α and 2β, respectively, we see that $a > b$. Also, $a = d + u$ and $b = e + v$. If we use the cosine rule (in each geometry) to express d and e as functions of ℓ, c and α and β, respectively, we see that $d > e$ in all three geometries. Next, in the triangle ΔABF, the sides BF and AF are opposite the angles α and β, respectively, so that $BF > AF$. Finally, as $BF + EF = AF + DF$ and $BF > AF$, we see that $DF > EF$.

Lemma 5.2: Suppose that $BE = AD = \ell$ and $\alpha > \beta$. Then, in the Euclidean and hyperbolic geometries, $u > v$; in spherical geometry, $\sin u > \sin v$.

Proof: In Euclidean geometry, the sine rule applied to triangles ΔBCE and ΔACD gives

$$\frac{v}{\sin\beta} = \frac{\ell}{\sin 2\gamma}, \qquad \frac{\ell}{\sin 2\gamma} = \frac{u}{\sin\alpha}, \tag{1}$$

so that $u > v$. A similar argument in hyperbolic geometry gives $\sinh u > \sinh v$, so again $u > v$; in spherical geometry we obtain $\sin u > \sin v$.

As we shall see later, Steiner needed the following lemma (which he proved as follows).

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Lemma 5.3: In Figure 4 we have

$$\psi \begin{cases} < \varphi + (\alpha - \beta) & \text{in hyperbolic geometry;} \\ = \varphi + (\alpha - \beta) & \text{in Euclidean geometry;} \\ > \varphi + (\alpha - \beta) & \text{in spherical geometry.} \end{cases}$$

Proof: First, we reflect the triangle ΔAFE across a suitable line through F (and use the same labelling) to obtain Figure 5. The result follows in the Euclidean case because the angle sum in a triangle is π, and in the hyperbolic and spherical cases because area $(\Delta FAE) <$ area (ΔFBD).

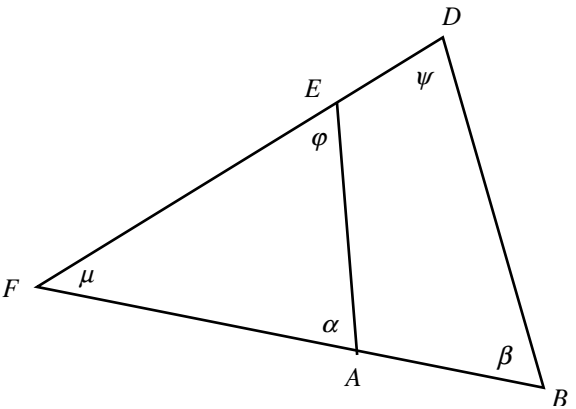


FIGURE 5

We shall see that the proofs of the Euclidean and hyperbolic versions of the Steiner-Lehmus theorem are essentially the same, and quite different from the proof of the spherical version.

6. *The Steiner-Lehmus theorem in Euclidean geometry*

This proof is essentially that in [20]. We assume that $\alpha > b$ and prove the contradictory inequalities $av < bu < av$. First, if we use (1) and the sine rule applied to ΔABC, we obtain

$$\frac{a}{b} = \frac{\sin 2\alpha}{\sin 2\beta} = \frac{u}{v}\left(\frac{\cos\alpha}{\cos\beta}\right) < \frac{u}{v}, \tag{2}$$

so that $av < bu$.

We shall now show that $av > bu$ or, equivalently (since $a = d + u$ and $b = e + v$), $dv > eu$. First, we apply the sine rule to the triangles ΔBEC and ΔBEA, and use the fact that $\sin\varphi = \sin(\pi - \varphi)$, to obtain $\frac{v}{a} = \frac{e}{c}$. Similarly, $\frac{u}{b} = \frac{d}{c}$; thus $dv = \frac{uc}{b}\frac{ae}{c} = \frac{a}{b}eu > eu$.

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7. *The Steiner-Lehmus theorem in hyperbolic geometry*

The proof below is a substantially shortened version of the proof in [10] which itself is a generalisation of the proof in [20]. We now regard the triangle in Figure 4 as a hyperbolic triangle and, following the ideas in the previous section, we obtain the desired contradiction by proving the contradictory inequalities $\sinh a \sinh v < \sinh b \sinh u < \sinh a \sinh v$.

First, if we use the hyperbolic sine rule, but otherwise argue exactly as in the first part of the Euclidean case, we obtain the first of these inequalities, namely, $\sinh a \sinh v < \sinh b \sinh u$. Next, if we follow the last part of the argument in the Euclidean case we obtain

$$\sinh d \sinh v > \sinh e \sinh u. \tag{3}$$

As $a = d + u$ and $b = e + v$ we have

$$\frac{\sinh a}{\sinh u} - \frac{\sinh b}{\sinh v} = (\cosh d - \cosh e) + \left(\cosh u \frac{\sinh d}{\sinh u} - \cosh v \frac{\sinh e}{\sinh v}\right), \tag{4}$$

so that it is enough to show that $\cosh d > \cosh e$ and $\cosh u > \cosh v$. However, these hold by Lemmas 5.1 and 5.2, so the proof is complete.

8. *Steiner's proof in Euclidean geometry*

Steiner's proof in spherical geometry is a generalisation of his proof in Euclidean geometry [1], so we shall describe this Euclidean proof first (see [8] for a 'free' English translation of Steiner's proof in the Euclidean case). In both cases he needs Lemma 5.3.

We now consider the Euclidean triangle in Figure 4, where $AD = BE$, $\alpha > \beta$, $d > e$ and $\psi > \varphi$. Steiner then shows that $e > d$, which contradicts $d > e$. The argument that $e > d$ is based on Figure 6. If we rotate the triangle ΔABE (in Figure 4) by an angle π about the midpoint of the segment AB we obtain the triangle labelled as ΔAKB in Figure 6 (with A, B, E moving to B, A, K, respectively). As all angles in the quadrilateral $AKBD$ are less than π, the quadrilateral is convex so we can join the vertices D and K to obtain the triangle on the right-hand side of Figure 6. Now as $AD = \ell = BE = AK$, we see that in the quadrilateral $AKBD$ we have $AD = AK$; thus AKD is an isosceles triangle, so that $\angle ADK = \angle AKD$. Since $\psi > \varphi$, we see that $\theta > \sigma$ in ΔDKB. Thus e (opposite the angle θ) is greater than d (opposite the angle σ); that is, $e > d$, and this gives the desired contradiction.

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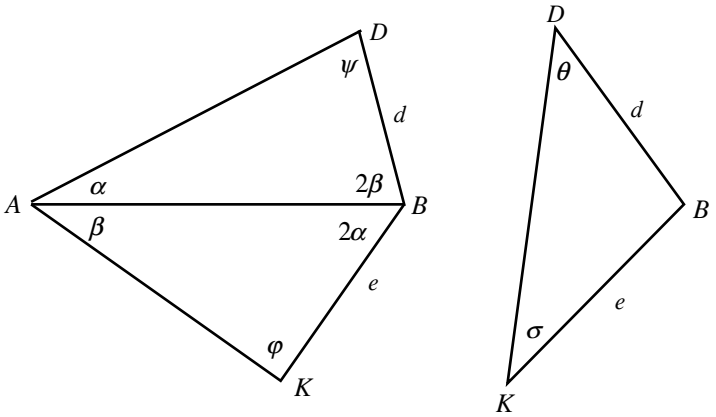


FIGURE 6

9. *The Steiner-Lehmus theorem in spherical geometry*

The following is essentially the same as (but slightly shorter than) Steiner's proof in the spherical case. We continue to use the same figures as before, but now these are interpreted in spherical geometry and, as before, we assume that $AD = BE = \ell$ and $\alpha > \beta$. Again, we have $d > e$ and $\psi > \varphi$. Steiner *now makes the assumption* (in the spherical case only) that $2\alpha + 2\beta < \pi$ (as he says, 'the angles at the base of the triangle are less than two right angles') and, with this, the quadrilateral *AKBD* is spherically convex. Steiner's proof now continues exactly as in the Euclidean case and again obtains a contradiction.

The question of whether there is an alternative proof in the spherical case which is closer to the common proof in the Euclidean and hyperbolic case remains open. It may be possible to replace Steiner's argument about areas by a more metric argument for there are very many formulae available in spherical trigonometry, and we end with some results of this type which may be of use.

Steiner's assumption that $\alpha + \beta < \frac{1}{2}\pi$ is equivalent to the inequality $a + b < \pi$ involving the lengths a and b of sides, but Steiner makes no reference to (or use of) this in [1]. To see this we refer to [18, §52], where it is shown that, in the context of the triangle in Figure 4,

$$\tan(\alpha + \beta) = \frac{\cos\frac{1}{2}(a - b)}{\cos\frac{1}{2}(a + b)} \cot\gamma.$$

Now it is shown in [18, §53] that $\cos\frac{1}{2}(a - b)$ and $\cot\gamma$ are positive, so that $\tan(\alpha + \beta)$ and $\cos\frac{1}{2}(a + b)$ have the same sign. It follows from this that $\alpha + \beta < \frac{1}{2}\pi$ if, and only if, $a + b < \pi$. Now we have seen that, in the Euclidean and hyperbolic cases, $u > v$, while in the spherical case $\sin u > \sin v$. Now, by Steiner's assumption, $u + v < a + b < \pi$, so that from Lemma 4.1, we also have $u > v$ in the spherical case too.

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Next, if we apply the sine rule to the triangles ΔABD and ΔABE we obtain (in all three geometries)

$$\frac{\sin\varphi}{\sin\psi} = \frac{\sin 2\alpha}{\sin 2\beta}.$$

Now $2\alpha + 2\beta < \pi$ in the Euclidean and hyperbolic cases, and also, if we accept Steiner's assumption, in the spherical case too. This, with Lemma 4.1, shows that in all cases (with Steiner's assumption) we have $\sin\varphi > \sin\psi$. This shows that in the spherical case we must have $\varphi + \psi > \pi$; indeed if not, then by Lemma 4.1 we would have $\varphi \geqslant \psi$ which is false in the spherical case. We end with the following question: should the proof in the spherical case really be so different from the other cases, and what can be said when Steiner's assumption that $\alpha + \beta < \frac{1}{2}\pi$ no longer holds?

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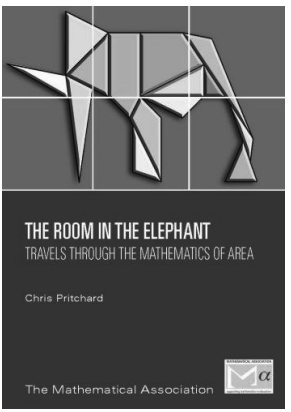
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