Macroeconomic Dynamics, **20**, 2016, 2173–2209. Printed in the United States of America.
doi:10.1017/S1365100515000061

OPTIMAL ENDOGENOUS GROWTH WITH NATURAL RESOURCES: THEORY AND EVIDENCE

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This paper considers an optimal endogenous growth model where the production function is assumed to exhibit increasing returns to scale and two types of resource (renewable and nonrenewable) are imperfect substitutes. Natural resources, labor, and physical capital are used in the final goods sector and in the accumulation of knowledge. Based on results in the calculus of variations, a direct proof of the existence of an optimal solution is provided. Analytical solutions for the planner case, balanced growth paths, and steady states are found for a specific CRRA utility and Cobb–Douglas production function. It is possible to have long-run growth where both energy resources are used simultaneously along the equilibrium path. As the law of motion of the technological change is not concave, reflecting the increasing returns to scale, so that the Arrow–Mangasarian sufficiency conditions do not apply, we provide a sufficient condition directly. Transitional dynamics to the steady state from the theoretical model are used to derive three convergence equations of output intensity growth rate, exhaustible resource growth rate, and renewable resource growth rate, which are tested based on OECD data on production and energy consumption.

Keywords: Endogenous Optimal Growth, Transitional Dynamics, Renewable Resource, Nonrenewable Resource

1. INTRODUCTION

An important question that has captivated the attention of environmental economists is whether growth is sustainable in the presence of natural resource scarcity. The issue concerns both academics and public decision makers, notably in the current context of increasing energy demands and the depletion of fossil fuels expected in the near future. The new growth theory gives the answer that with some

We are thankful to the anonymous referees, Jean-Pierre Amigues, Théophile Azomahou, Raouf Boucekkine, Aditya Goenka, André Grimaud, Cuong Le Van, Francesco Ricci, and François Salanié for helpful comments and suggestions. The usual disclaimer applies. Address correspondence to: Phu Nguyen-Van, BETA, CNRS and Université de Strasbourg, 61 avenue de la Forêt Noire, 67000 Strasbourg, France; e-mail: nguyen-van@unistra.fr.

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https://doi.org/10.1017/S1365100515000061 Published online by Cambridge University Press

technological properties, growth may be sustained in the long run even if resource stock is finite. This conclusion is summarized in Smulders (2005), "that a society willing to spend enough on R&D can realize a steady state of technological change sufficient to offset the diminishing returns from capital-resource substitution and sustain long-run growth," and in Bretschger (2005), "technological change has the potential to compensate for natural resource scarcity, diminishing returns to capital, poor input substitution, and material balance restrictions, but is limited by various restrictions like fading returns to innovative investments and rising research costs."

However, the literature does not pay enough attention to the existence of the proposed theoretical solutions and their empirical justifications. This is particularly due to the difficulty of building a testable version from a complex theoretical framework. Most existing theoretical studies focus separately on renewable and nonrenewable resources.[1] The roles of renewable and nonrenewable resources have been analyzed simultaneously in some recent studies, e.g. , André and Cerdá (2005), Gerlagh and van der Zwaan (2003), Grimaud and Rougé (2004, 2005, 2008), Growiec and Schumacher (2008), Tahvonen and Salo (2001), and Tsur and Zemel (2003). However, they are mainly related to the substitutability of natural resources (substitution between manmade and natural capital, substitution between renewable and nonrenewable resources) or conditioned on the usual concave production functions with diminishing or constant returns to scale in production inputs, which ensure sustainable economic growth, and are not explicitly concerned with empirical testing.

In this paper, we focus on the question of whether growth is sustainable in the presence of natural resource scarcity, substitution mechanisms of alternative energies, and technological progress. Under the general umbrella of growth models, we pay more attention especially to the issues arising in the allocation of resources over an infinite time horizon, where the production function is allowed to exhibit increasing returns to scale in knowledge capital. With optimal growth models, this issue is raised in the literature, where inputs in production are considered as "broad capitals" [Groth and Schou (2002, 2007)]. Our setting aims at allowance for substitutability between knowledge capital and various forms of inputs in production. Two types of natural resources (renewable and nonrenewable) are not perfect substitutes. This model explores the theory of nonconvex optimization, with the characterization of stability properties of the dynamical systems used to represent economic equilibrium process, to address the following questions in endogenous growth and natural resource economics: Is it possible to have long-run growth where both renewable and nonrenewable resources are simultaneously employed along the equilibrium path? What is the impact of environmental concerns on growth, and in particular, on the levels, the paths, or the growth rates of crucial variables such as capital, output, consumption, labor, and technological change? Is the steady-state equilibrium optimal along balanced growth paths?

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Most of the theoretical literature on energy transition is based on the assumption that two resources are perfect substitutes [e.g., Chakrovarty et al. (2006), Gerlagh (2011), Tahvonen and Salo (2001)]. In such models, resource use has two distinct phases: in the first phase, only the exhaustible resource is used, and in the second phase, the renewable substitute completely replaces the use of fossil fuels. In this construction there is no phase where both resources are consumed simultaneously. However, in reality, renewable resources such as solar energy cannot entirely replace petroleum in a number of uses at the present level of technological change. We then relax the assumption of perfect substitutability.[2] Our results show that there exists a steady-state growth equilibrium where both resources are simultaneously employed along the equilibrium path. We also analyze the transitional dynamics and the stability of equilibrium.

Concerning the technical problems in this paper, we present a rigorous proof of the existence of the optimal solution of a nonconvex model, which is often assumed in the many papers in the literature. As always, the arguments for existence of solutions rely on compactness of the feasible set and some form of continuity of the objective function. We first prove the uniform boundedness of the feasible set [assumptions in d'Albis et al. (2008)], which implies Lebesgue uniform integrability. The Dunford–Pettis theorem [Dunford and Schwartz (1967)], which characterizes Lebesgue uniform integrability and the relatively weak compactness of the feasible set, is needed in the proof. Then we prove that the set of feasible consumption paths is compact. Combined with compactness, upper semicontinuity of the objective function is all that is necessary for the existence of a maximum. Furthermore, when technical progress is interpreted broadly as a combination of knowledge capital, physical capital, human capital, and natural resources, the production function can have increasing returns to scale in technical progress. As Arrow sufficiency conditions cannot apply, because the Hamiltonian is not jointly concave in state variables, we prove a new type of transversality condition to obtain a direct sufficient condition following which any path that satisfies the necessary conditions is optimal.

Moreover, a recent strand of the literature concerns the search for empirical evidence for the proposed theoretical models. Pioneer work was done by Alvarez et al. (2005), Bretschger (2006), Brock and Taylor (2010), Miketa and Mulder (2005), and Mulder and De Groot (2007). Brock and Taylor (2010) proposed a Solow growth model with pollution together with empirical justification. They found evidence of an environmental Kuznets curve (an inverted U-shaped relationship between emissions and income) for OECD countries. They also showed that pollution emissions have a convergence feature, like income convergence in empirical growth studies. Alvarez et al. (2005) provided a Ramsey growth model with pollution, which is compatible with empirical findings about pollution convergence in a panel of European countries. Bretschger (2006) provided an empirical validation of the balanced growth path derived from an endogenous growth model with energy. The author also obtained the result that rising energy prices are not a

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threat to economic development. Miketa and Mulder (2005) and Mulder and De Groot (2007) found evidence for conditional convergence in energy productivity, i.e., convergence based on country-specific conditions, which seems to support the underlying Solow–Swan growth model. Under this approach, our paper performs an empirical test based on the analytical solutions from the theoretical model. For this purpose, the transitional equations derived from the model are estimated based on data on production and energy consumption of OECD countries.

The paper is organized as follows. Section 2 introduces the general optimal endogenous growth model. The existence of an optimal solution is shown in Section 3. Section 4 proves the existence of steady-state equilibrium, finds analytical optimal growth rates of all variables, and analyzes the transitional dynamics. In Section 5 we show that the steady state is indeed locally optimal by proving a direct sufficient condition. Section 6 presents the empirical test of an analytical model based on the results of transitional dynamics. Section 7 concludes.

2. THE MODEL

The model can be heuristically described as follows. The aggregate output is produced from labor, physical capital, and two types of natural resources: nonrenewable resources (e.g., fossil fuels) and renewable resources (solar, thermal, biomass, etc.). The final product is shared between consumption and investment in physical capital. The production function takes the form

$$Y = F(A, K, L_Y, Q, R) = A^{\theta} f(K, L_Y, Q, R),$$

where A, L_Y, K, Q, and R represent the technological level, labor input, physical capital input, nonrenewable resource (or fossil energy), and renewable resource (or nonfossil energy), respectively. As technical progress can have increasing returns to scale in the production function, the positive constant θ may take a value greater than 1. Various rationales for this assumption have been given, such as improvements in technical progress leading to further improvements or allowance for substitutability between technical progress and various forms of physical capital, labor, and natural resources. As it is not necessary, the time index is not included, to simplify our notation.

We assume the law of motion of technological change as

$$\dot{A} = \Psi(A, L_A),$$

where L_A is labor employed for research and Ψ is a knowledge production function. Normalizing the total flow of labor, we have

$$L_Y + L_A = 1.$$

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The final output can be allocated between consumption and investment (or capital accumulation),

$$\dot{K} = Y - C - \delta K = I - \delta K,$$

where $\delta \in (0, 1)$ is the depreciation rate of the stock of capital.

It is standard that the dynamics of the stock nonrenewable resource is

$$\dot{S}_Q = -Q,$$

where S_{Q_t} is the stock of the exhaustible resource at time t. It follows from this equation and the nonnegativity restriction on Q that

$$\int_0^\infty Q dt \leq S_{Q_0}.$$

The dynamics of stocks of the renewable resource is

$$\dot{S}_R = h(S_R) - R,$$

where h is a regeneration function.

The representative consumer derives her utility from consumption. Her utility function is given by

$$U = \int_0^\infty u(C)e^{-\rho t} dt.$$

3. EXISTENCE OF AN OPTIMAL SOLUTION

In this section, we prove the existence of a solution to the social planner's problem (P),

$$\max \int_0^\infty u(C)e^{-\rho t} dt,$$

subject to

$$\dot{S}_R = h(S_R) - R, \tag{1}$$

$$\dot{S}_Q = -Q, \tag{2}$$

$$\dot{A} = \Psi(A, L_A), \tag{3}$$

$$\dot{K} = A^\theta f(K, L_Y, Q, R) - C - \delta K, \tag{4}$$

$$L_A + L_Y = 1, \tag{5}$$

and $C \geq 0$, $K \geq 0$, $A \geq 0$, $0 \leq L_Y \leq 1$, $0 \leq L_A \leq 1$, given A_0, K_0, S_{Q_0}, S_{R_0}.

As always, the arguments for existence of solutions rely on compactness of the feasible set and some form of continuity of the objective function. We first prove the uniform boundedness of the feasible set [assumptions in d'Albis et al.

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(2008)], which implies Lebesgue uniform integrability. The Dunford–Pettis theorem [Dunford and Schwartz (1967)], which characterizes the Lebesgue uniform integrability and the relatively weak compactness of the feasible set, is needed in the proof. Then we prove that the set of feasible consumption paths is compact. Combined with compactness, upper semicontinuity of the objective function is all that is necessary for the existence of a maximum.

Let $L^1(e^{-\rho t})$ be the set of functions f verifying $\int_0^\infty |f(t)|\, e^{-\rho t} dt < \infty$. Recall that $f_i(t) \in L^1(e^{-\rho t})$ weakly converges to $f(t) \in L^1(e^{-\rho t})$ for the topology $\sigma(L^1(e^{-\rho t}), L^\infty)$ (written as $f_i \rightharpoonup f$) if and only if for every $q \in L^\infty$, $\int_0^\infty f_i q e^{-\rho t} dt$ converges to $\int_0^\infty f q e^{-\rho t} dt$ as $i \to \infty$ (written as $\int_0^\infty f_i q e^{-\rho t} dt \longrightarrow \int_0^\infty f q e^{-\rho t} dt$). When writing $f_i \longrightarrow f^*$, we mean that for every $t \in [0, \infty)$, $\lim_{i\to\infty} f_i(t) = f^*(t)$.

We make the following assumptions:

H1. *Function $u(C) : R_+ \to R$ is strictly concave, increasing, and continuous.*

H2. *Function $f(K, L_Y, Q, R) : R_+^4 \to R_+$ is continuously differentiable, increasing on all arguments, concave in L_Y, Q, R, and $\lim_{K\to+\infty} f_K \leq 0$. Moreover, it satisfies the Inada conditions $\lim_{K\to 0} f_K = \lim_{L_Y\to 0} f_{L_Y} = \lim_{Q\to 0} f_Q = \lim_{R\to 0} f_R = 0$.*

H3. *Function $\Psi(A, L_A) : R_+^2 \to R_+$ is continuously differentiable and increasing in both arguments. Moreover, there exists a constant b such that*

$$\Psi(A, L_A) \leq bA.$$

H4. *Function $h(S_R) : R_+ \to R_+$ is continuously differentiable and increasing and there exists a constant m such that $h(S_R) \leq mS_R$.*

H5. *There exist $\kappa, \varsigma, \pi \neq \infty$ such that $-\kappa \leq \dot{K}/K$ and $-\varsigma \leq \dot{S}_R/S_R, -\pi \leq \dot{S}_Q/S_Q$.*

H6. $\rho > \max\{b, m, b\theta\}$.

H1–H4 are standard. We do not need that f is concave in capital. Assuming $\Psi(A, L_A) \leq bA$ has been done in Chichilnisky (1981) and it is weaker than the standard assumption $\lim_{A\to\infty} \frac{\partial \Psi(A, L_A)}{\partial A} = 0$. This means that after certain levels of technical change the technology is constrained in its knowledge capital, represented by the depreciation parameter b. The assumption on function h (H4) is similar.

Assumption H5 is reasonable. It implies that it is not possible that the growth rate of physical capital or stock of renewable resource converges to $-\infty$ rapidly, and is weaker than those used in the literature, where κ is a physical depreciation rate [Chichilnisky (1981); d'Albis et al. (2008)]. Let us define the net investment : $I = \dot{K} - \delta K = A^\theta f(K, L_Y, Q, R) - C$. Then H5 implies that there exist $\kappa \geq 0, \kappa \neq \infty$ such that $I + (\kappa - \delta)K \geq 0$. Thus if the standard assumption of nonnegative investment holds (which means that capital goods cannot be converted back into consumption goods), then H5 holds with $\kappa = \delta$. Therefore the assumption of nonnegative investment is stronger than H5 (κ can take any value except for

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infinity). This assumption will be used to prove the boundedness of variables by some exponential functions that belong to $L^1(e^{-\rho t})$. H6 is similar to A4 in d'Albis et al. (2008), which ensures a finite value of the objective function and a maximal growth rate of the output that is less than the discount rate.

Now, the main idea for the proof of existence is to show that the control variables and derivatives of state variables weakly converge in a weak topology, whereas the state variables converge pointwise. The problem is that even if we have a weakly convergent sequence, the limit point may not be feasible. For pointwise convergent sequences, continuity is all that is necessary to prove feasibility. Therefore, concavity is not needed for state variables. We will show that the limit point is indeed optimal in the original problem. For a weakly convergent sequence, Mazur's Lemma is used to change to pointwise convergence. Jensen's inequality is used to eliminate the convex combination coefficients to prove feasibility. Thus, concavity with respect to control variables is crucial.

LEMMA 1. *Let us denote by* $\mathcal{K} = (L_A, L_Y, Q, R, S_Q, S_R, A, K, C)$ *the feasible path from* $A_0, K_0, S_{Q_0}, S_{R_0}$ *that satisfies (1)–(5) and* $C \geq 0, K \geq 0, A \geq 0, Q \geq 0, R \geq 0, 0 \leq L_Y \leq 1, 0 \leq L_A \leq 1$. *Then*

(i) $\mathcal{K}$ *is relatively weakly compact in* $L^1(e^{-\rho t})$.

(ii) State variables S_Q, S_R, A, K *are absolutely continuous.*

Proof. See the Appendix. ■

Because $\mathcal{K}$ is relatively compact in the weak topology $\sigma(L^1(e^{-\rho t}), L^\infty)$, a sequence X_i in $\mathcal{K}$ has convergent subsequences (denoted by X_i for simplicity) that weakly converge to limit points in $L^1(e^{-\rho t})$.

The following lemma shows that any weakly convergent sequence of control variables in $\mathcal{K}$ has a sequence of convex combinations of its members that converges pointwise to the same limit, whereas the limit of weak convergence coincides with the limit of pointwise convergence for state variables.

LEMMA 2. *(i) Let there exist state variables* $(K, A, S_R, S_Q)_i$ *in* $\mathcal{K}$ *and suppose that* $(K, A, S_R, S_Q)_i \rightharpoonup (K^*, A^*, S_R^*, S_Q^*)$. *Then* $(K, A, S_R, S_Q)_i \to (K^*, A^*, S_R^*, S_Q^*)$ *as* $i \to \infty$ *and* $(\dot{K}, \dot{A}, \dot{S}_R, \dot{S}_Q)_i \rightharpoonup (\dot{K}^*, \dot{A}^*, \dot{S}_R^*, \dot{S}_Q^*)$ *for the topology* $\sigma(L^1(e^{-\rho t}), L^\infty)$.

(ii) In addition, suppose that $\mathbf{Z}_i = (R, Q, \dot{K}, \dot{A}, \dot{S}_R, \dot{S}_Q)_i$ *in* $\mathcal{K}$ *and* $\mathbf{Z}_i \rightharpoonup \mathbf{Z}^* = (R^*, Q^*, \dot{K}^*, \dot{A}^*, \dot{S}_R^*, \dot{S}_Q^*)$ *in* $L^1(e^{-\rho t})$. *Then there exists a sequence of sets of real numbers* $\{\omega_{i(n)} \mid i = n, ..., \mathcal{N}(n)\}$ *such that* $\omega_{i(n)} \geq 0$ *and* $\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} = 1$ *such that the sequence* $(v_n)_{n \in N}$ *defined by the convex combination* $v_n = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \mathbf{Z}_i$ *converges pointwise to* $\mathbf{Z}^*$ *as* $n \to \infty$*; i.e., for every* $t \in [0, \infty)$, $\lim_{n \to \infty} v_n(t) = \mathbf{Z}^*(t)$. *It is clear that, because* $(L_A, L_Y,)_{i(n)} \in [0, 1]$, $(L_A, L_Y,)_{i(n)} \to (L_A^*, L_Y^*)$ *as* $n \to \infty$.

Proof. See the Appendix. ■

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We are now able to prove the existence of a solution to the social planner's problem. The difficulty is that although we have convergence of the feasible sequences, as only the closure of the feasible set is closed because $\mathcal{K}$ is only weakly compact, we do not know that the limit point is in fact feasible. This is shown in the following.

THEOREM 1. *Under Assumptions H1–H6, there exists a solution to the social planner's problem.*

Proof. Because u is concave, for any $\bar{c} > 0$, $u(C) - u(\bar{c}) \leq u'(\bar{c})(C - \bar{c})$. Thus, if $C \in L^1(e^{-\rho t})$, then $\int_0^\infty u(C)e^{-\rho t}dt$ is well defined because

$$\int_0^\infty u(C)e^{-\rho t}dt \leq \int_0^\infty [u(\bar{c}) - u'(\bar{c})\bar{c}]e^{-\rho t}dt + u'(\bar{c})\int_0^\infty Ce^{-\rho t}dt < +\infty.$$

Let us define $W = \sup_{C\in\mathcal{K}} \int_0^\infty u(C)e^{-\rho t}dt$. Assume that $W > -\infty$ (otherwise the proof is trivial). Let $C_i \in \mathcal{K}$ be the maximizing sequence of $\int_0^\infty u(C)e^{-\rho t}dt$ so that $\lim_{i\to\infty} \int_0^\infty u(C_i)e^{-\rho t}dt = W$. Because $\mathcal{K}$ is relatively weakly compact, suppose that $C_i \rightharpoonup C^*$ for some C^* in $L^1(e^{-\rho t})$. By Mazur's Lemma, there is a sequence of convex combinations

$$x_n = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}C_{i(n)} \to C^*, \omega_{i(n)} \geq 0, \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} = 1.$$

Because u is concave, we have

$$\limsup_{n\to\infty} u(x_n) = \limsup_{n\to\infty} u\left(\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}C_{i(n)}\right)$$

$$\leq \limsup_{n\to\infty}\left[u(C^*) + u'(C^*)\left(\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}C_{i(n)} - C^*\right)\right] = u(C^*).$$

Because this relation holds for almost all t, the integration w.r.t. $e^{-\rho t}dt$ gives

$$\int_0^\infty \limsup_{n\to\infty} u(x_n)e^{-\rho t}dt \leq \int_0^\infty u(C^*)e^{-\rho t}dt < +\infty.$$

Using a reverse Fatou's Lemma yields

$$\limsup_{n\to\infty} \int_0^\infty u(x_n)e^{-\rho t}dt \leq \int_0^\infty \limsup_{n\to\infty} u(x_n)e^{-\rho t}dt \leq \int_0^\infty u(C^*)e^{-\rho t}dt. \tag{6}$$

Moreover, by Jensen's inequality, we get

$$\limsup_{n\to\infty} \int_0^\infty u(x_n)e^{-\rho t}dt \geq \limsup_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \int_0^\infty u(C_{i(n)})e^{-\rho t}dt. \tag{7}$$

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But because $\int_0^\infty u(C_{i(n)})e^{-\rho t}dt \to W$, (6) and (7) imply that $\int_0^\infty u(C^*)e^{-\rho t}dt \geq W$.

So it remains to show that C^* is feasible. The task is now to show that there exist some $(K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ in $\mathcal{K}$ such that $(C^*, K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ satisfies (1)–(4).

Consider a feasible sequence $(K, L_A, L_Y, A, R, Q, S_R, S_Q)_{i(n)}$ in $\mathcal{K}$ associated with $C_{i(n)}$. According to Lemma 2 and Jensen's inequality, we have

$$
\begin{aligned}
C^* &= \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} C_{i(n)} \\
&\leq \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} [A_i^\theta f(K_i, L_{Y_i}, Q_i, R_i) - \delta K_i - \dot{K}_i] \\
&= \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} [\lim_{n\to\infty} A_i^\theta f(\lim_{n\to\infty} K_{i(n)}, \lim_{n\to\infty} L_{Y_i}, Q_i, R_i) - \delta \lim_{n\to\infty} K_{i(n)}] \\
&\quad - \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{K}_i \\
&\leq A^{*\theta} f(K^*, L_Y^*, \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} Q_i, \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} R_i) - \delta K^* \\
&\quad - \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{K}_i \\
&= A^{*\theta} f(K^*, L_Y^*, Q^*, R^*) - \delta K^* - \dot{K}^*.
\end{aligned}
$$

Therefore,

$$C^* \leq A^{*\theta} f(K^*, L_Y^*, Q^*, R^*) - \delta K^* - \dot{K}^*.$$

Applying a similar argument and using Jensen's inequality, we get

$$
\begin{aligned}
\dot{A}^* &= \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{A}_{i(n)} = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \Psi(\lim_{n\to\infty} A_i, \lim_{n\to\infty} L_{A_i}) = \Psi(A^*, L_A^*), \\
\dot{S}_R^* &= \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{S}_{R_i} = \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} [h(S_{R_i}) - R] = h(S_R^*) - R^*, \\
\dot{S}_Q^* &= \lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \dot{S}_{Q_i} = -\lim_{n\to\infty} \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} Q_i = -Q^*.
\end{aligned}
$$

Therefore, $(C^*, K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ satisfy (1)–(4).

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The proof is done. ■

D'Albis et al. (2008) also establish the existence of an optimal solution in an abstract model. Our proof is direct and more constructive with less stringent assumptions. Indeed, Lemma 1 proves the boundedness assumptions in their model. We have proven that the control variables and derivatives of state variables weakly converge in the weak topology $\sigma(L^1(e^{-\rho t}), L^\infty)$, whereas the state variables show pointwise convergence. For a pointwise convergent sequence, continuity is all that is necessary to prove feasibility. Therefore, concavity is not needed for the state variables. Related results on the existence of a solution can be found in Chichilnisky (1981), who used the theory of Sobolev weighted spaces and imposed a Caratheodory condition on the utility function.

4. EXISTENCE OF STEADY-STATE EQUILIBRIUM

Let $\sigma_C = -\frac{CU_{Cc}}{U_C}$ denote the elasticity of marginal utility and $F = A^\theta f(K, L_Y, Q, R)$. We have proven that (P) has a solution denoted by $(C^*, K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$. Then the solution is characterized by the Kuhn–Tucker conditions, Euler equations, and transversality conditions.

The current-value Hamiltonian is

$$H(C, K, Q, R, L_Y, A) = u(C) + \lambda(mS_R - R) - \mu Q + \nu(F - C - \delta K) + \omega b A^\phi (1 - L_Y),$$

where $\lambda, \mu, \nu, \omega$ are four costate variables.

The first-order conditions $\frac{\partial H}{\partial C} = 0$, $\frac{\partial H}{\partial Q} = 0$, $\frac{\partial H}{\partial R} = 0$, $\frac{\partial H}{\partial L_Y} = 0$ yield

$$\nu = U_C, \tag{8}$$

$$\mu = \nu F_Q, \tag{9}$$

$$\lambda = \nu F_R, \tag{10}$$

$$\omega = \frac{\nu F_{L_Y}}{bA^\phi}. \tag{11}$$

From the Euler equations $\frac{\partial H}{\partial K} = \rho\nu - \dot{\nu}$, $\frac{\partial H}{\partial S_R} = \rho\lambda - \dot{\lambda}$, $\frac{\partial H}{\partial S_Q} = \rho\mu - \dot{\mu}$, and $\frac{\partial H}{\partial A} = \rho\omega - \dot{\omega}$ we get

$$\frac{\dot{\nu}}{\nu} = \rho - F_K + \delta, \tag{12}$$

$$\frac{\dot{\mu}}{\mu} = \rho, \tag{13}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - m, \tag{14}$$

$$\dot{\omega} = [\rho - b\phi A^{\phi-1}(1 - L_Y)]\omega - \nu F_A.$$

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The transversality conditions are

$$\lim_{t\to+\infty} \lambda S_R^* e^{-\rho t} = \lim_{t\to+\infty} \mu S_Q^* e^{-\rho t} = \lim_{t\to+\infty} \nu K^* e^{-\rho t} = \lim_{t\to+\infty} \omega A^* e^{-\rho t} = 0. \tag{15}$$

Moreover, it is easy to see that, at the optimum, the Hotelling rule is satisfied:

$$\rho + \sigma_C \frac{\dot{C}^*}{C^*} = \frac{\dot{F}_Q}{F_Q},$$
$$\rho + \sigma_C \frac{\dot{C}^*}{C^*} = \frac{\dot{F}_R}{F_R} + h_{S_R}.$$

Before analyzing the full dynamic system, we look at the characterization of a balanced optimal growth path. Our model corresponds to a system with four state variables. As Kolstad and Krautkraemer (1993) remarked, "it is difficult or impossible to characterize the qualitative features of a dynamic model evolving three state variables without restrictive assumptions about the functional forms of important relationships"; we specify a set of restrictions imposed on preferences and production technology in order to have analytical results that can be tested with real data. We make the following assumptions:

H7.

$$u(C) = \begin{cases} \frac{C^{1-\varepsilon}-1}{1-\varepsilon}, & \text{if } \varepsilon \neq 1, \\ \ln C \text{ if } \varepsilon = 1. \end{cases}$$

H8. $f(L_Y, K, Q, R) = L_Y^{\gamma} K^{\xi} Q^{\alpha} R^{\beta}$, where $\gamma, \xi, \alpha, \beta \geq 0, \gamma + \xi + \alpha + \beta = 1$.
H9. $\Psi(A, L_A) = bA^{\phi} L_A$, where $b > 0, 0 < \phi \leq 1$.
H10. $h(S_R) = mS_R, m > 0$.

This specification satisfies H1–H6. Assumption H7 is a constant relative risk aversion (CRRA) utility function. Assumption H8 corresponds to a Cobb–Douglas production function. Assumption H9 is related to a function of knowledge accumulation that is widely used in the literature [see, e.g., Aghion and Howitt (1998), Jones (2006)]. Assumption H10 allows constant infinite growth for the renewable resource, which is not realistic for ecological restrictions. However, it may be reasonable for a type of of nonfossil energy such as solar energy, wind energy, or nuclear energy in which the stock of alternative energy can be considered infinite.

Let $g_\chi = \dot{\chi}/\chi$ denote the growth rate of any variable χ. We shall summarize the macroeconomic equilibrium in terms of five variables, $x = F/K, y = C/K, z = Q/S_Q, u = R/S_R, q = L_Y A^{\phi-1}, r = A^{\phi-1}$, from which other equilibrium rates

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g_F, g_K, g_C, g_{L_Y}, g_{L_A}, g_A, g_Q, g_{S_Q}, g_R, and g_{S_R} can be derived as in the following proposition.

PROPOSITION 1. *The optimal growth rates take the following values:*

$$
\begin{aligned}
g_A &= b(r-q),\\
g_K &= x-y-\delta,\\
g_C &= \frac{\xi x-\delta-\rho}{\varepsilon},\\
g_{S_Q} &= -z,\\
g_{S_R} &= m-u,\\
g_Q &= -y+\frac{b\theta r}{\xi}+\frac{m\beta+(1-\xi)\delta}{\xi},\\
g_R &= -y+\frac{b\theta r}{\xi}+\frac{m(\beta+\xi)+(1-\xi)\delta}{\xi},\\
g_{L_Y} &= -y+\frac{b\theta r}{\xi}+\frac{b\theta}{\gamma}q+\frac{m\beta+(1-\xi)\delta}{\xi},\\
g_F &= \xi x-y+\frac{b\theta r}{\xi}+\frac{m\beta+\delta(\alpha+\beta+\gamma)}{\xi}-\delta,\\
g_{L_A} &= \frac{q}{q-r}g_{L_Y}.
\end{aligned}
$$

Proof. See the Appendix. ■

A steady state satisfies that all rates of growth are constant. Let χ^* and g_χ^* denote respectively the value and the growth rate of any variable χ in the steady state.

Let us denote the matrix

$$
\mathbf{\Gamma}=\begin{bmatrix}
\xi-\varepsilon & \varepsilon & 0\\
(\xi-1)\xi & -\xi & b\theta\\
0 & -\xi\gamma & b\theta(\gamma+\xi)
\end{bmatrix}
$$

and its determinant as $\det\mathbf{\Gamma}$.

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PROPOSITION 2. *If* $\det \mathbf{\Gamma} \neq 0$ *then there exist steady state equilibrium growth rates that take the following values:*

$$g_Q^* = g_{S_Q}^* = -y^* + \frac{b\theta r^*}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi},$$

$$g_R^* = g_{S_R}^* = -y^* + \frac{b\theta r^*}{\xi} + \frac{m(\beta+\xi) + (1-\xi)\delta}{\xi},$$

$$g_F^* = g_K^* = g_C^* = \frac{\xi x^* - \delta - \rho}{\varepsilon},$$

$$g_{L_Y}^* = g_{L_A}^* = 0,$$

$$g_A^* = b(r^* - q^*),$$

where if $\phi = 1$ *then*

$$x^* = \frac{b\theta + m\beta + \delta(1-\xi)}{\xi(1-\xi)},$$

$$y^* = \frac{(\varepsilon - \xi)[b\theta + m\beta + \delta(1-\xi)] + \frac{(1-\varepsilon)\delta + \rho}{\varepsilon}}{\varepsilon\xi(1-\xi)},$$

$$q^* = [y^* - \frac{m\beta + (1-\xi)\delta + b\theta}{\xi}]\frac{\gamma}{\theta b},$$

$$r^* = 1,$$

and if $\phi < 1$ *then* $r^* = q^*$ *and* x^*, y^*, q^* *are solutions of* $\mathbf{\Gamma X} = \mathbf{\Sigma}$, *where*

$$\mathbf{X} = \begin{bmatrix} x^* \\ y^* \\ q^* \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} \delta + \rho - \varepsilon\delta \\ -m\beta - \delta(1-2\xi) - \xi\delta \\ -m\beta\gamma - (1-\xi)\gamma\delta \end{bmatrix}.$$

Proof. See the Appendix. ■

By substituting the value of x^* into the values of $g_F^* = g_C^*$, we have the following result.

COROLLARY 1. *There exists a steady state equilibrium with* $g_F^* = g_C^* > 0$ *if* $\det \mathbf{\Gamma} \neq 0$ *and* $\frac{b\theta + m\beta + \delta(1-\xi)}{(1-\xi)} > \delta + \rho$. *This condition is true for high degrees of substitution of technical progress* (θ) *or the maximal capacity of natural regeneration of renewable resources* (m).

We now study the dynamic behavior of the nonlinear system characterized by the behavior of the linearized system around the steady state. We shall summarize the macroeconomic equilibrium in terms of four stationary variables, $x = F/K$, $y = C/K$, $z = Q/S_Q$, $u = R/S_R$, $q = A^{\phi-1}L_Y$, $r = A^{\phi-1}$, from which other equilibrium rates can be derived as in Proposition 1. Let us define

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$\mathbf{h} = (x, y, z, u, q, r)$. In the neighborhood of the steady state, we can write $\dot{\mathbf{h}} = \mathbf{J}(\mathbf{h} - \mathbf{h}^*)$, where $\mathbf{h}^* = (x^*, y^*, z^*, u^*, q^*, r^*)$ and $\mathbf{J}$ is the Jacobian matrix evaluated at the steady state:

$$\mathbf{J} = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y & \partial\dot{x}/\partial z & \partial\dot{x}/\partial u & \partial\dot{x}/\partial q & \partial\dot{x}/\partial r \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y & \partial\dot{y}/\partial z & \partial\dot{y}/\partial u & \partial\dot{y}/\partial q & \partial\dot{y}/\partial r \\ \partial\dot{z}/\partial x & \partial\dot{z}/\partial y & \partial\dot{z}/\partial z & \partial\dot{z}/\partial u & \partial\dot{z}/\partial q & \partial\dot{z}/\partial r \\ \partial\dot{u}/\partial x & \partial\dot{u}/\partial y & \partial\dot{u}/\partial z & \partial\dot{u}/\partial u & \partial\dot{u}/\partial q & \partial\dot{u}/\partial r \\ \partial\dot{q}/\partial x & \partial\dot{q}/\partial y & \partial\dot{q}/\partial z & \partial\dot{q}/\partial u & \partial\dot{q}/\partial q & \partial\dot{q}/\partial r \\ \partial\dot{r}/\partial x & \partial\dot{r}/\partial y & \partial\dot{r}/\partial z & \partial\dot{r}/\partial u & \partial\dot{r}/\partial q & \partial\dot{r}/\partial r \end{pmatrix}.$$

By investigating the number of negative eigenvalues of the Jacobian, we obtain the following proposition:

PROPOSITION 3. *There is a saddlepoint of stability.*

Proof. See the Appendix. ■

Moreover, we can show that both types of resources, renewable and nonrenewable, are simultaneously employed along the equilibrium path.

PROPOSITION 4. *There can be a long-run growth where both resources are simultaneously employed along the equilibrium path.*

Proof. As the Inada conditions are satisfied, two resources used are strictly positive optimal solutions. Therefore two resources are used simultaneously in finite time. It follows from the transversality conditions that $\lim_{t\to\infty} \mu S_Q^* e^{-\rho t} = 0$, where $\mu = \mu(0)e^{\rho t}$ and $S_Q^*(t) = S_Q^*(0)e^{g_Q^* t}$. Thus $\lim_{t\to\infty} \mu(0) S_Q^*(0) e^{g_Q^* t} = 0$, which implies that $g_Q^* < 0$.

Similarly, because $\lim_{t\to\infty} \lambda S_R^* e^{-\rho t} = 0$, where $\lambda = \lambda(0)e^{(\rho - m)t}$, we get $\lim_{t\to\infty} \lambda(0) S_R^*(0) e^{(g_R^* - m)t} = 0$ or $g_R^* < m$. From Proposition 2, we have shown that there exists a steady state $(R^s(t), Q^s(t))$ of the renewable energy resource and the nonrenewable energy resource, where the rates of growth $g_R^* = \frac{\dot{R}^s(t)}{R^s(t)}$ and $g_Q^* = \frac{\dot{Q}^s(t)}{Q^s(t)}$ are constant. Indeed, $R^s(t) = R_0 e^{g_R^* t} > 0$ and $Q^s(t) = Q_0 e^{g_Q^* t} > 0$. Along such a path, because $g_Q^* < 0$ and $g_R^* < m$, long-run growth of the nonrenewable resource must be nonnegative but approach zero as $t \to \infty$ and the renewable resource grows with an exponential bound, $R^s(t) < R_0 e^{mt}$. Moreover, when $\phi < 1$, the steady state of technological change is constant because $A^s(t) = A_0 e^{b(r^* - q^*)t} = 0$. As the production function exhibits increasing returns to scale in technological change and the two resources are imperfect substitutes, there can be long-run growth where both resources are used simultaneously. ■

5. SUFFICIENT CONDITIONS

We have shown that a solution to the social planner's problem and a steady state growth equilibrium exist. And the steady state can be a saddlepoint of stability.

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Note that θ may be greater than 1, and the maximal Hamiltonian is not concave in every state variable, so the Arrow or Mangasarian sufficiency theorem does not apply in our model. In such a dynamic model with nonconcave maximal Hamitonian, Kuhn–Tucker first-order conditions, together with transversality conditions, are necessary, whereas sufficient conditions for an optimal solution are still a conjecture. We show directly that the model satisfies the Leitmann–Stalford (1971) sufficiency conditions for a nonconcave Hamiltonian, and hence the steady state is optimal.

THEOREM 2 *(Leitmann–Stalford Theorem). Consider the problem*

$$\max \int_0^\infty e^{-\rho t} g_0(x(t), z(t))dt,$$

subject to

$$\dot{x}(t) = g(x(t), z(t)), x(0) = x_0, z(t) \in Z.$$

Define the current-value Hamiltonian

$$H(x, z, \Lambda) = g_0(x(t), z(t)) + \langle \Lambda, g(x(t), z(t)) \rangle,$$

where $\langle \Lambda, g(x(t), z(t)) \rangle$ *is the inner product in* R^n. *Let* $z^* \in Z$, *let* $x^* = x(z^*)$ *be the corresponding trajectory, and let* $\Lambda : [0, \infty) \to R^n$ *be absolutely continuous. Let the following conditions be fulfilled for every* $u \in Z$ *and* $x = x(z)$*:*

(i)

$$\int_0^\infty e^{-\rho t}[H(x^*(t), z^*(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) + \langle \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t) \rangle]dt \geq 0.$$

(ii) $\lim_{t\to\infty} e^{-\rho t} \langle \Lambda, x^*(t) - x(t) \rangle \leq 0.$

Then (x^*, z^*) *is an optimal solution.*

If the co-state variable is nonzero, then the $\Lambda(t)$ can be set equal to the co-state variable that satisfies the first-order necessary conditions [see Leitmann and Stalford (1971, Remark 3.1)].

PROPOSITION 5. *Define the state variable* $x^*(t) = (S_R^*, S_Q^*, K^*, A^*)$, *where* $x_0^* = (S_{R_0}, S_{Q_0}, K_0, A_0)$ *is given, and the control variable* $z^*(t) = (C^*, Q^*, R^*, L_Y^*, L_A^*)$. *Assume that* $U_C, F_R, F_{L_Y}, F_K, F_A$ *belong to* $L^1(e^{-\rho t})$ *when* $x^*(t), z^*(t)$ *lie in* $L^1(e^{-\rho t})$. *If there exists* $\Lambda(t) = (\lambda(t), \mu(t), \nu(t), \omega(t))$ *that satisfies conditions (8)–(15) then* (x^*, z^*) *is an optimal solution.*

Proof. Because $S_Q \to 0$ as $t \to \infty, \lim_{t\to\infty} e^{-\rho t}\mu(S_Q^* - S_Q) = \lim_{t\to\infty} \mu_0(S_Q^* - S_Q) = 0$. It follows from Lemma 1 that the co-state variables ν, λ, ω are absolutely continuous and belong to $L^1(e^{-\theta t})$. It follows from Assumption H2 that there exists $\hat{k}$ such that $K \leq \hat{k}$. As $\nu(t) \in L^1(e^{-\theta t})$, we have

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$\lim_{t\to\infty} e^{-\rho t} v(K^* - K) = 0$. Similarly, we can prove that $\lim_{t\to\infty} e^{-\rho t} \Lambda(x^* - x) = 0$, where $x \in (S_R^*, A^*)$.

Recall the current Hamiltonian,

$$H(\Lambda, x, z) = u(C) + \lambda(mS_R - R) - \mu Q + \nu(F - C - \delta K) + \omega b A^{\phi}(1 - L_Y).$$

For any $x(t), z(t)$ feasible with $x(0) = x_0^*$, we have

$$\begin{aligned}
&\int_0^{\infty} e^{-\rho t}[H(x^*(t), z^*(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) \\
&\quad + \langle \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle]dt \\
&= \int_0^{\infty} e^{-\rho t}[H(x^*(t), z^*(t), \Lambda(t)) - H(x^*(t), z(t), \Lambda(t))]dt \\
&\quad + \int_0^{\infty} e^{-\rho t}[H(x^*(t), z(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) \\
&\quad + \langle \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle]dt.
\end{aligned}$$

Because u and F are concave w.r.t. control variables, we have

$$u(C^*) - u(C) \geq u_C(C^* - C), \tag{16}$$

$$F(z^*) - F(z) \geq F_R(R^* - R) + F_Q(Q^* - Q) + F_{L_Y}(L_Y^* - L_Y). \tag{17}$$

Looking at the first term, using (8)–(14) and (16), (17), we easily obtain

$$H(x^*(t), z^*(t), \Lambda(t)) - H(x^*(t), z(t), \Lambda(t)) \geq 0.$$

Thus,

$$\int_0^{\infty} e^{-\rho t}[H(x^*(t), z^*(t), \Lambda(t)) - H(x^*(t), z(t), \Lambda(t))]dt \geq 0. \tag{18}$$

Because

$$\begin{aligned}
&H(x^*(t), z(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) \\
&\quad = \langle \Lambda, g(x^*(t), z(t)) - g(x(t), z(t))\rangle = \langle \Lambda, \dot{x}^*(t) - \dot{x}(t))\rangle,
\end{aligned}$$

the next term can be read as

$$\begin{aligned}
&H(\Lambda, x, z) = u(C) + \lambda(mS_R - R) - \mu Q \\
&\quad + \nu(F - C - \delta K) + \omega b A^{\phi}(1 - L_Y).
\end{aligned}$$

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Hence,

$$\begin{aligned}
&\int_0^\infty e^{-\rho t}[H(x^*(t), z(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t))]dt \\
&\quad + \int_0^\infty e^{-\rho t}\langle \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle dt \\
&= \int_0^\infty e^{-\rho t}\langle \Lambda, \dot{x}^*(t) - \dot{x}(t)\rangle dt + \int_0^\infty e^{-\rho t}\langle \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle dt \\
&= \int_0^\infty e^{-\rho t}[\lambda(\dot{S}_R^* - \dot{S}_R) + \mu(\dot{S}_Q^* - \dot{S}_Q) + \upsilon(\dot{K}^* - \dot{K}) + \omega(\dot{A}^* - \dot{A})]dt \\
&\quad + \int_0^\infty e^{-\rho t}[(\dot{\lambda} - \rho\lambda)(S_R^* - S_R) \\
&\quad + (\dot{\mu} - \rho\mu)(S_Q^* - S_Q) + (\dot{\upsilon} - \rho\upsilon)(K^* - K) + (\dot{\omega} - \rho\omega)(A^* - A)]dt.
\end{aligned} \tag{19}$$

Because $e^{-\rho t}(\dot{\lambda} - \rho\lambda) = \frac{d(e^{-\rho t}\lambda)}{dt}$, we have

$$\begin{aligned}
&\int_0^\infty e^{-\rho t}[\lambda(\dot{S}_R^* - \dot{S}_R) + (\dot{\lambda} - \rho\lambda)(S_R^* - S_R)]dt \\
&= \int_0^\infty [\frac{d(e^{-\rho t}\lambda)(S_R^* - S_R)}{dt}]dt = \lim_{t\to\infty} e^{-\rho t}\lambda(S_R^* - S_R) - \lambda(0)(S_{R_0}^* - S_{R_0}) \\
&= \lim_{t\to\infty} e^{-\rho t}\lambda(S_R^* - S_R) = 0.
\end{aligned}$$

With the same reasoning for the second and third terms in (19), we have

$$\begin{aligned}
&\int_0^\infty e^{-\rho t}[H(x^*(t), z(t), \Lambda(t)) - H(x(t), z(t), \Lambda(t)) \\
&\quad + \langle \dot{\Lambda} - \rho\Lambda, x^*(t) - x(t)\rangle]dt \\
&= \lim_{t\to\infty} e^{-\rho t}\langle \Lambda, x^*(t) - x(t)\rangle \le 0.
\end{aligned} \tag{20}$$

It follows from (18) and (20) that condition (i) in Theorem 2 is satisfied. ■

As the steady state satisfies the necessary conditions, we have shown that it is indeed optimal. Thus, we have the following result.

THEOREM 3. *All the steady states are locally optimal.*

The Leitmann–Stalford theorem is a powerful result, but has not been used much in the economics literature, as the conditions are difficult to verify. To use the Leitmann–Stalford conditions, we show from Lemma 1 that control and state variables are bounded and belong to $L^1(e^{-\rho t})$. This implies that co-state variables $\Lambda \in L^1(e^{-\rho t})$. This enables us to show the condition

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$\lim_{t\to\infty} e^{-\rho t}\langle \Lambda, x^*(t) - x(t)\rangle = 0$. This is crucial, as when we check the maximality of the Hamiltonian, we can decompose it into two parts: the first term just relies on the control variables and we have concavity in the objective function in control variables, and thus, using standard results, the difference between the candidate solution and any other solution is nonnegative; and the second term depends on the co-state and the state variables as given previously. Recall that the nonconcavity in the problem arises from the law of evolution of state variables only. As this term converges to zero, we are able to obtain sufficiency of the first-order conditions. Thus, three things turn out to be important in our problem: the boundedness of the state and control and hence of the co-state variables; concavity of the objective in the control variables; and the ability to separate the control and state/co-state variables in the Hamiltonian.

6. ECONOMETRIC ESTIMATION

6.1. Estimated Equations

Consider the Jacobian matrix of the model. Let $\mathbf{v}_i$, $i = 1, ..., 6$, denote the eigenvectors corresponding to six eigenvalues λ_i. We may write the general solution of $\dot{\mathbf{h}} = \mathbf{J}(\mathbf{h} - \mathbf{h}^*)$, where $\mathbf{h}^* = (x^*, y^*, z^*, u^*, q^*, r^*)$, as follows:

$$\mathbf{h}(t) - \mathbf{h}^* = \sum_{i=1}^{6} a_i \mathbf{v}_i e^{\lambda_i t},$$

where parameters a_i are determined by the initial conditions $\mathbf{h}(0) - \mathbf{h}^* = \sum_{i=1}^{6} a_i \mathbf{v}_i$. The optimal path in the neighborhood of the steady state is located in the stable subspace which corresponds to the negative eigenvalues. Thus, if $\phi = 1$, we can write $\mathbf{h}(t) - \mathbf{h}^* = c_1 \mathbf{v}_1 e^{\lambda_1 t}$, where $\lambda_1 < 0$.

We are now in a position to discuss the empirical implications of the theoretical model. We focus our analysis on the growth rates $g_{F/K}$, g_Q, and g_R on the transition path. We consider only the case $\phi = 1$, where all of these growth rates depend only on $x \equiv F/K$ and $y \equiv C/K$.[3] From the previous analysis for $\phi = 1$, as the components of $\mathbf{h}$ are independent of each other, we can write the approximations for x and y as follows:

$$x_t - x^* = c_1 v_{1x} e^{\lambda_1 t}, \tag{21}$$

$$y_t - y^* = c_1 v_{1y} e^{\lambda_1 t}, \tag{22}$$

where v_{1x} and v_{1y} are two components of $\mathbf{v}_1$. By using the initial conditions, we get

$$c_1 v_{1x} = x_0 - x^*,$$

$$c_1 v_{1y} = y_0 - y^*.$$

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Therefore, we obtain the following solution:

$$x_t = (1 - e^{\lambda_1 t})x^* + e^{\lambda_1 t} x_0, \tag{23}$$

$$y_t = (1 - e^{\lambda_1 t})y^* + e^{\lambda_1 t} y_0. \tag{24}$$

Recall that we have

$$g_x = (\xi - 1)x + \frac{b\theta + m\beta + \delta(1-\xi)}{\xi},$$

$$g_Q = -y + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi},$$

$$g_R = -y + \frac{b\theta}{\xi} + \frac{m(\beta + \xi) + (1-\xi)\delta}{\xi}.$$

We substitute (23) and (24) in these expressions and use the definitions $x_0 = F_0/K_0$, $y_0 = C_0/K_0$, and $g_x = g_{F/K} = \frac{1}{T}\ln(F_T/F_0) - \frac{1}{T}\ln(K_T/K_0)$, $g_Q = \frac{1}{T}\ln(Q_T/Q_0)$, and $g_R = \frac{1}{T}\ln(R_T/R_0)$, which are respectively the average growth rates of F/K, Q, and R, between 0 and T. This results in the transitional dynamics of g_χ, $\chi = F/K$, Q, R toward the steady state of the economy.[4]

First, $g_{F/K}$ is given by[5]

$$\begin{aligned} g_{F/K} &= (\xi - 1)x_T + \frac{b\theta + m\beta + \delta(1-\xi)}{\xi} \\ &= \frac{b\theta + m\beta + \delta(1-\xi)}{\xi} + (\xi - 1)(1 - e^{\lambda_1 T})x^* + (\xi - 1)e^{\lambda_1 T} x_0 \\ &= \alpha_0 + \alpha_1 F_0/K_0 + \varepsilon_{F/K}, \end{aligned} \tag{25}$$

with

$$\alpha_0 = \frac{b\theta + m\beta + \delta(1-\xi)}{\xi} + (\xi - 1)(1 - e^{\lambda_1 T})x^*,$$

$$\alpha_1 = (\xi - 1)e^{\lambda_1 T}.$$

Concerning g_Q, we have

$$\begin{aligned} g_Q &= -(1 - e^{\lambda_1 T})y^* - e^{\lambda_1 T} y_0 + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} \\ &= \beta_0 + \beta_1 C_0/K_0 + \varepsilon_Q, \end{aligned} \tag{26}$$

where

$$\beta_0 = -(1 - e^{\lambda_1 T})y^* + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi},$$

$$\beta_1 = -e^{\lambda_1 T}.$$

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Similarly, g_R is given by

$$\begin{aligned} g_R &= -y + \frac{b\theta}{\xi} + \frac{m(\beta+\xi)+(1-\xi)\delta}{\xi} \\ &= -(1-e^{\lambda_1 T})y^* - e^{\lambda_1 T} y_0 + \frac{b\theta}{\xi} + \frac{m(\beta+\xi)+(1-\xi)\delta}{\xi} \\ &= \gamma_0 + \gamma_1 C_0/K_0 + \varepsilon_R, \end{aligned} \tag{27}$$

where

$$\gamma_0 = \frac{b\theta}{\xi} + \frac{m(\beta+\xi)+(1-\xi)\delta}{\xi} - (1-e^{\lambda_1 T})y^*,$$
$$\gamma_1 = -e^{\lambda_1 T}.$$

Equations (25), (26), and (27) represent three cross-sectional regressions where $\varepsilon_{F/K}$ and ε_Q and ε_R are the corresponding error terms.[6] These equations may be estimated using ordinary least squares as in Mankiw et al. (1992). However, as emphasized by Islam (1995) and subsequent studies on income convergence, we will lose information contained by the data, as we need only observations of the initial and final dates, i.e., dates 0 and T, and the sample size is then reduced to to the number of countries.

An alternative approach is to transform the model into a panel structure. In particular, we can rewrite equations (25), (26), and (27) as follows:

$$g_{(F/K)_{it}} = \alpha_0 + \alpha_1 F_{i,t-1}/K_{i,t-1} + \varepsilon_{(F/K)_{it}}, \tag{28}$$

$$g_{Q_{it}} = \beta_0 + \beta_1 C_{i,t-1}/K_{i,t-1} + \varepsilon_{Q_{it}}, \tag{29}$$

$$g_{R_{it}} = \gamma_0 + \gamma_1 C_{i,t-1}/K_{i,t-1} + \varepsilon_{R_{it}}, \tag{30}$$

where $i = 1, ..., N$ and $t = 1, ..., T$. In our panel data framework, the error terms $\varepsilon_{\chi_{it}}$, $\chi = F/K, Q, R$, include country and time effects, i.e., $\varepsilon_{\chi_{it}} = \mu_i + \lambda_t + u_{\chi_{it}}$, where μ_i and λ_t denote country heterogeneity and time heterogeneity, respectively, and $u_{\chi_{it}}$ is the idiosyncratic error. Moreover, the model predicts that the coefficients α_1, β_1, and γ_1 are negative.

Estimates of these equations can be obtained with standard panel methods (within estimation for fixed effects, generalized least squares for random effects), which assume the *strict exogeneity* of regressors, i.e.,

$$E[(F_s/K_s)\varepsilon_{(F/K)_t}] = E[(C_s/K_s)\varepsilon_{Q_t}] = E[(C_s/K_s)\varepsilon_{R_t}] = 0, \ \forall s, t.$$

However, this assumption may be faulty, as regressors can be correlated with some unobserved factors or with future values of the dependent variable. This is probably the case when we study macroeconomic data. For example, we may think that current consumption and capital stock may have some impact not only

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on current income and energy consumption but also on their future values. This situation arises when regressors are *predetermined*, i.e.,

$$E[(F_s/K_s)\varepsilon_{(F/K)_t}] = E[(C_s/K_s)\varepsilon_{Q_t}] = E[(C_s/K_s)\varepsilon_{R_t}] = 0,\ \forall s < t-1.$$

In this case, the model can be estimated using generalized methods of moments [see, e.g., Arellano and Bond (1991), Baltagi (2005), and Lee (2002)]. This is also the approach adopted in our estimation strategy.

6.2. Data

The data cover twenty-seven OECD countries for the period 1977-1997.[7] Data on nonrenewable energy consumption Q and renewable energy consumption R are collected from the International Energy Agency (IEA). nonrenewable energy consumption, Q, is measured as the sum of consumption of gas, and liquid fuels (in metric tons oil equivalent, TOE). We assume that renewable energy consumption, R, corresponds to the sum of nuclear energy, hydroelectricity, geothermal energy, renewable fuels and waste, solar energy, wind energy, and energy from tide, waves, and ocean (also in TOE).

Data on production F, consumption C, physical capital stock K, and population are collected from the Penn World Table 6.1 [Heston et al. (2002)]. We note that all figures, except population, are expressed in PPP and 1996 prices for international comparison purposes. We use the population series and the series on real GDP per capita in 1996 prices (series RGDPL) to produce the volume of real GDP in 1996 prices. We compute the product of real GDP and investment share of real GDP, on one hand, and the product of real GDP and consumption share of real GDP, on the other hand, which correspond respectively to investment and consumption in real terms. Calculation of physical capital stock is based on the investment series (i.e., investment share of real GDP) following the perpetual inventory method.[8] Data are given in per capita terms to neutralize possible scale effects due to the difference in population size observed between countries.

Descriptive statistics of the data are summarized in Table 1. Evolution of the averages of ratios F/K and C/K is displayed in Figure 1. These ratios have similar patterns, with two dips around 1982 and 1993, except that F/K has stronger variation than C/K (standard deviation of F/K, 0.121, is higher than that of C/K, 0.098). Series on consumption of nonrenewable and renewable resources are presented in Figure 2. The average of consumption of renewable energies R is much lower than that of nonrenewable energies Q, which increased over the whole period of the study, whereas the latter considerably decreased in the late 1970s until the dip in 1983 and increased thereafter.

For the estimates, as in most empirical studies on economic convergence [see Durlauf et al. (2005) for a survey], we use data corresponding to the five-year

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TABLE 1. Descriptive statistics, 1977–1997

Variable	Mean	Std. dev.	Min	Max
Consumption per capita	10.558	3.473	2.757	21.505
Capital stock per capita	46.629	19.066	5.247	89.764
GDP per capita	16.592	5.591	4.325	37.917
Consumption–capital ratio C/K	0.254	0.098	0.129	0.730
Output–capital ratio F/K	0.391	0.121	0.190	0.911
Nonrenewable energy consumption Q	3.145	1.706	0.496	10.637
Renewable energy consumption R	0.864	1.097	0.004	5.467
Number of countries		27		
Number of years		21		

Notes: Data on energy consumption are collected from the IEA for the period 1977–1997. Consumption of renewable and nonrenewable energies is expressed in metric tons oil equivalent (TOE). Economic data are drawn from the Penn World Table 6.1 [see Heston et al. (2002)]. GDP, consumption, and capital stock are measured in thousands of U.S. dollars and 1996 prices. All figures are in per capita terms.

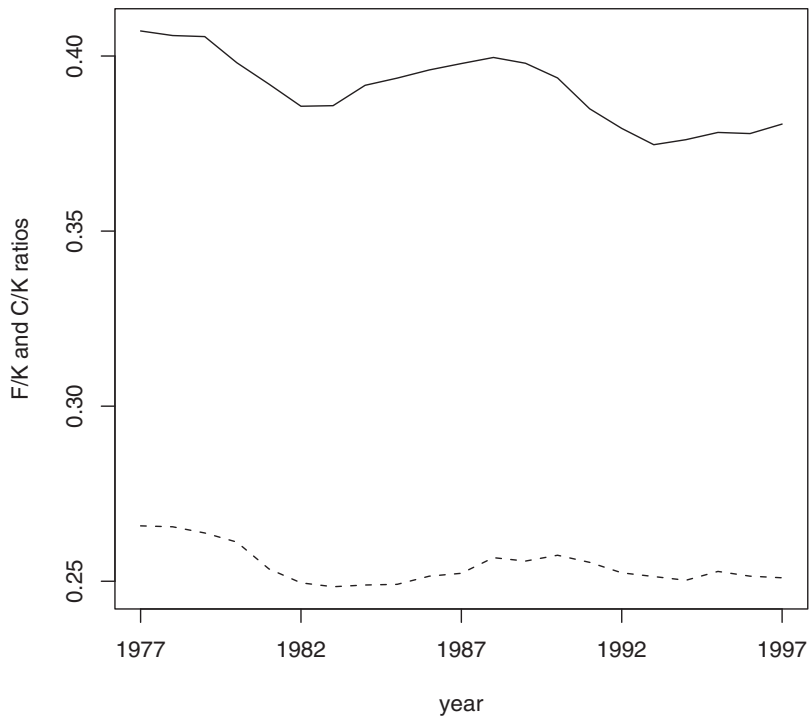


FIGURE 1. Averages of ratios F/K (solid line) and C/K (dashed line), period 1977–1997.

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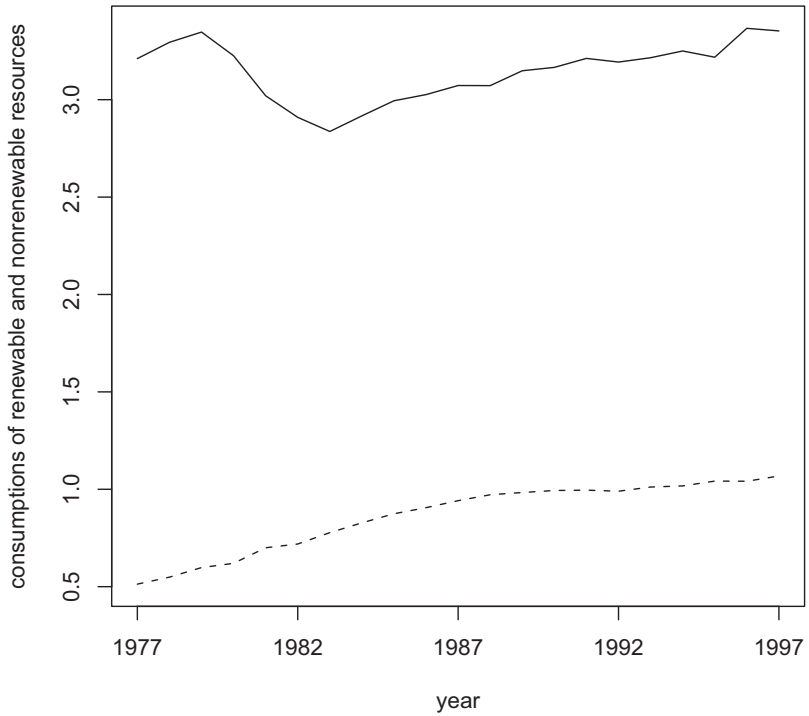


FIGURE 2. Average consumption per capita of nonrenewable energies Q (solid line) and renewable energy R (dashed line), period 1977–1997.

interval (i.e., data from 1977, 1982, 1987, 1992, and 1997) in order to reduce business cycle effects. Hence the length between t and $t-1$ is equal to 5 and $g_{\chi_{it}} = \frac{1}{5}\ln(\chi_t/\chi_{t-1})$. The sample size is 108 observations ($N = 27$, $T = 4$). Distributions of average annual growth rates (computed from these time intervals) of the output to capital ratio F/K, $g_{F/K}$, renewable energy consumption per capita, g_R, and nonrenewable energy consumption per capita, g_Q, are reported in Figures 3–5, respectively. The distribution of these growth rates' sensitivity changes over time. We also observe that the dispersion of g_Q and g_R diminishes throughout the period of study.

6.3. Estimation Results

Estimation results by GMM are reported in Table 2. As GMM use the first-difference transformation, the intercept and country effects (which are not separately identified) are deleted from the regressions and therefore not estimated. Furthermore, the assumption of predetermined regressors,

$$E[(F_s/K_s)\varepsilon_{(F/K)_t}] = E[(C_s/K_s)\varepsilon_{Q_t}] = E[(C_s/K_s)\varepsilon_{R_t}] = 0,\ \forall s < t-1,$$

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TABLE 2. Estimation results

Equation	Variable	Coefficient	Sargan	AR(1)	AR(2)	Wald
$g_{(F/K)_t}$	$(F/K)_{t-1}$	−0.174** (0.060)	0.229	−3.08**	−1.15	17.4**
g_{Q_t}	$(C/K)_{t-1}$	−0.015 (0.067)	0.883	−2.86**	1.41*	17.8**
g_{R_t}	$(C/K)_{t-1}$	−0.275* (0.166)	5.7	−2.37**	−0.553	6.52*

Notes: Regressions include country effects and year effects. Overidentifying restrictions is tested by the Sargan test. AR(1) and AR(2) tests are the Arellano and Bond (1991) tests for serial correlation of order 1 and 2, respectively. The Wald test is for significance of year dummies. Estimation results are obtained by GMM with robust standard errors (in parentheses).
* and ** Significance levels of 10% and 5%, respectively.

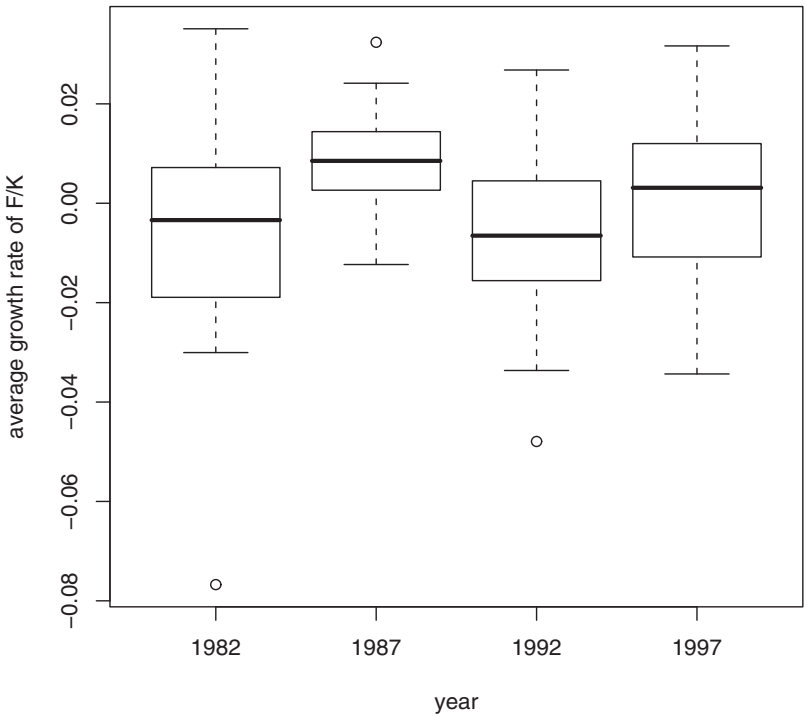


FIGURE 3. Average growth rate of F/K, period 1977–1997. The box plots relative to 1982, 1987, 1992, and 1997 represent the distributions of the average annual growth rate observed for the periods 1977–1982, 1982–1987, 1987–1992, and 1992–1997, respectively.

allows us to use all values of F_s/K_s and C_s/K_s, $s = 1, ..., t-2$, as possible instruments for F_t/K_t and C_t/K_t, respectively. As a consequence, there are finally 81 observations used in the estimations.

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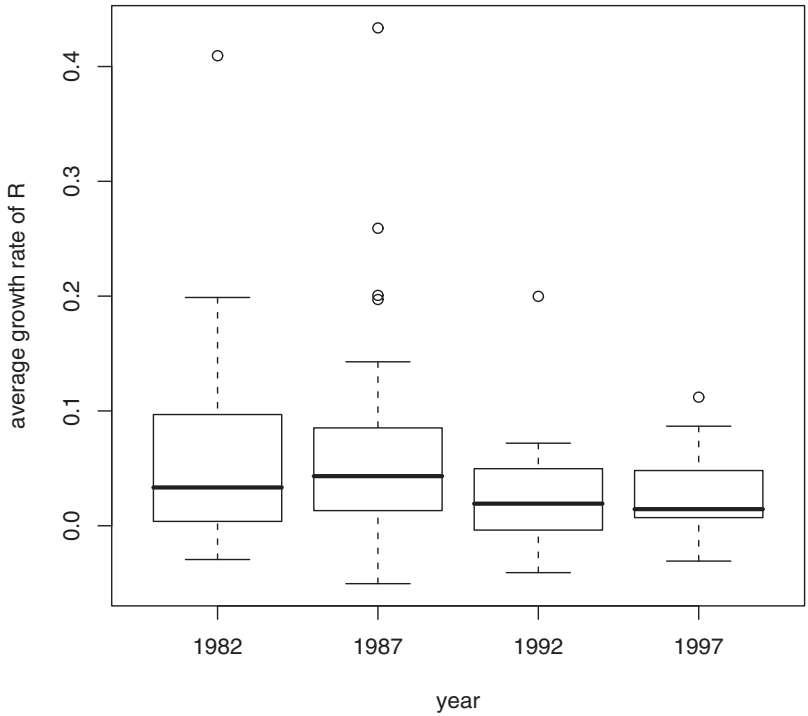


FIGURE 4. Average growth rate of renewable resource consumption per capita g_R, period 1977–1997. The box plots relative to 1982, 1987, 1992, and 1997 represent the distributions of the average annual growth rate observed for the periods 1977–1982, 1982–1987, 1987–1992, and 1992–1997, respectively.

Empirical results based on OECD data confirm the prediction of the theoretical model. Indeed, estimation results show that coefficients α_1, β_1, and γ_1 are negative, as expected. Coefficients α_1 and γ_1 are statistically significant at the 5% and the 10% levels, respectively, whereas β_1 is insignificant. The Wald test confirms the existence of time effects in all regressions. The Sargan specification test for overidentifying restrictions (relative to the use of instrumental variables) is always satisfied in either set of regressions, as these overidentifying restrictions are not rejected.

It should be noted that the GMM estimator is consistent with an AR(1) process for the regression residuals but not consistent with an AR(2). We use the Arellano–Bond (1991) tests to examine this issue. Test results reject the absence of autocorrelation of order 1 but do not reject that of order 2 for regressions with $g_{F/K}$ and g_R, suggesting that estimates are consistent in these cases. Concerning g_Q, the specification does not seem robust, as Arellano–Bond test does

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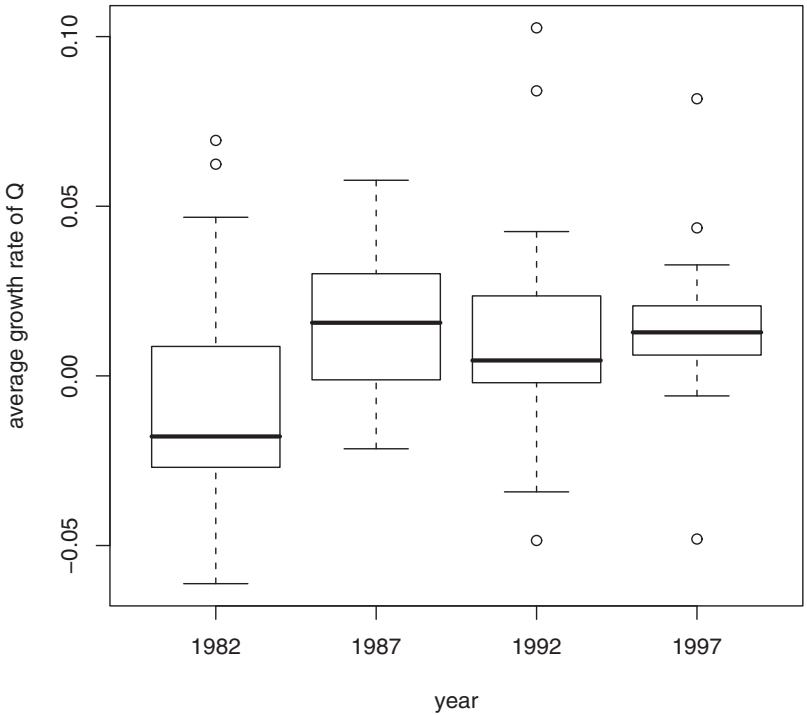


FIGURE 5. Average growth rate of nonrenewable resource consumption per capita g_Q, period 1977–1997. The box plots relative to 1982, 1987, 1992, and 1997 represent the distributions of the average annual growth rate observed for the periods 1977–1982, 1982–1987, 1987–1992, and 1992–1997, respectively.

not confirm (but only at the 10% level) the absence of an AR(2) process in the residuals.

7. CONCLUSION

The paper is an attempt to explore theoretically the interaction between growth, endogenous technical change, and consumption of renewable and nonrenewable resources. When the production function exhibits increasing returns to scale in technological change and two natural resources are imperfect substitutes, there can be long-run growth where both resources are used simultaneously along the equilibrium path. A proof of existence of optimal solution and sufficient conditions are provided when the Arrow–Mangasarian conditions do not apply. We directly show sufficient conditions and deduce that the steady states that satisfy the necessary conditions are indeed locally optimal. We also characterize the balanced growth paths together with the transitional dynamics for an analytical model, which are shown to helpful for empirical analysis by using panel data on OECD

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countries. Our estimation strategy is appealing beause it accounts for country and time heterogeneities and it allows a more flexible assumption about regressors than the usual assumption of strictly exogenous regressors in the standard empirical framework.

In a further work, it would be of particular interest to study the identification issue for the empirical specification that can be derived from the theoretical model when $\phi < 1$ or from a more general case. Moreover, it would be promising to consider externalities from resource use in utility or production in order to improve the realism of the modeling. Competitive equilibrium and public policy would also require a particular attention.

NOTES

1. The literature to be cited here is very abundant. See Barbier (1999), Bretschger (2005), Brock and Taylor (2006), Kolstad and Krautkraemer (1993), Scholz and Ziemes (1999), and Smulders (2005), among others, for some literature overviews.

2. Long (2014) also adopted imperfect substitutability between clean and dirty fuels to analyze the green paradox. The author found that a green paradox outcome can likely occur with a moderate or high degree of substitutability, whereas such an outcome cannot exist if the degree of substitutability is near zero.

3. It should be noticed that when $\phi < 1$, the model is not identified and then estimation becomes impossible unless some restrictions are imposed.

4. The reason for using $g_{F/K}$ instead of g_F is that the approximation of the latter contains on the right-hand side both F/K and C/K, which are highly correlated. Hence, regression of g_F on F/K and C/K will face a problem of multicollinearity.

5. Index i is dropped for simplicity's sake.

6. These error terms may correspond to omitted variables or unobserved factors that can affect the growth rates $g_{F/K}$, g_Q, and g_R. They can also represent measurement errors in these growth rates.

7. The data include Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Italy, Japan, Korea, Luxembourg, Mexico, the Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, Switzerland, Turkey, the United Kingdom, and the United States.

8. The perpetual inventory equation is $K_t = I_t + (1-\delta)K_{t-1}$, where I_t is the investment flow. The initial capital stock is given by $K_0 = I_0/(g_I + \delta)$, where g_I is the average geometric growth rate of investment from the initial date. The depreciation rate, δ, is often set to 4 to 6%. In our paper, changing δ from 4% to 6% does not modify the qualitative conclusion.

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APPENDIX

A.1. PROOF OF LEMMA 1

(i) By (1) and assumption H4 we have $\dot{S}_R \leq h(S_R) \leq mS_R$ and we get $\dot{S}_R/S_R \leq m$. Thus, there exists $\overline{S}$ such that

$$\begin{aligned} 0 \leq S_R &\leq \overline{S}e^{mt}, \\ \dot{S}_R &\leq m\overline{S}e^{mt}. \end{aligned} \tag{A.1}$$

Thus, S_R belongs to the space $L^1(e^{-\rho t})$ because

$$0 \leq \int_0^\infty S_R e^{-\rho t}dt \leq \overline{S}\int_0^\infty e^{(m-\rho)t}dt < +\infty.$$

According to H4, $-\dot{S}_R \leq \varsigma S_R \leq \overline{\varsigma S}e^{mt}$. It follows from (A.1) that $\left|\dot{S}_R\right| \leq \max\{m\overline{S}, \varsigma\overline{S}\}e^{mt}$ and

$$\int_0^\infty \left|\dot{S}_R\right| e^{-\rho t}dt \leq \max\{m\overline{S}, \varsigma\overline{S}\}\int_0^\infty e^{(m-\rho)t}dt < +\infty,$$

because $0 \leq R = h(S_R) - \dot{S}_R \leq (m+\varsigma)\overline{S}e^{mt}$. Therefore we have

$$0 \leq \int_0^\infty Re^{-\rho t}dt \leq (m+\varsigma)\overline{S}\int_0^\infty e^{(m-\rho)t}dt < +\infty.$$

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It follows from (2) that $0 \le \int_0^t Q_s ds = -\int_0^t \dot{S}_{Q_s} ds = S_{Q_0} - S_Q \le S_{Q_0}$. Thus $\left|\dot{S}_Q\right| = Q$, $S_Q \le S_{Q_0}$ and $Q = -\dot{S}_Q \le \pi S_Q$. We then have

$$\int_0^\infty S_Q e^{-\rho t} dt \le S_{Q_0} \int_0^\infty e^{-\rho t} dt < +\infty,$$
$$\int_0^\infty Q e^{-\rho t} dt = \int_0^\infty \left|\dot{S}_Q\right| e^{-\rho t} dt \le \pi S_{Q_0} \int_0^\infty e^{-\rho t} dt < +\infty.$$

Because $\lim_{K\to+\infty} f_K(K, 1, S_{Q_0}, S_{R_0}) \le 0$, for any $\zeta \in (0, \rho - b\theta)$ there exists a constant B_0 such that

$$f(K, L_Y, Q, R) \le B_0 + \zeta K.$$

It follows that

$$\dot{K} \le B_0 + \zeta K.$$

Multiplying by $e^{-\zeta s}$ we get $e^{-\zeta s}\dot{K} - \zeta K e^{-\zeta s} \le B_0 e^{-\zeta s}$. Then we get

$$e^{-\zeta t} K = \int_0^t \frac{\partial(e^{-\zeta s} K)}{\partial s} ds + K_0 \le \int_0^t B_0 e^{-\zeta s} ds + K_0 = \frac{-B_0 e^{-\zeta t}}{\zeta} + \frac{B_0 + \zeta K_0}{\zeta}.$$

This implies that there exists a constant B_1 such that $K \le B_1 e^{\zeta t}$. Hence $\int_0^\infty K e^{-\rho t} dt \le \int_0^\infty B_1 e^{(\zeta-\rho)t} dt < +\infty$.

Furthermore, because $-\dot{K} \le \kappa K$ and $\dot{K} \le B_0 + \zeta K \le B_0 + \zeta B_1 e^{\zeta t}$, there exists a constant B_2 such that $\left|\dot{K}\right| \le B_2 e^{\zeta t}$. Thus,

$$\int_0^\infty \left|\dot{K}\right| e^{-\rho t} dt \le \int_0^\infty B_2 e^{(\zeta-\rho)t} dt < +\infty.$$

Because $\Psi(A, L_A) \le bA$, we have $\dot{A}/A \le b$. There exists a constant D_1 such that $A \le D_1 e^{bt}$. Moreover, we have $0 \le \dot{A} \le bA \le D_1 e^{bt}$. Therefore, $A, \left|\dot{A}\right|$ belong to $L^1(e^{-\rho t})$ because

$$0 \le \int_0^\infty A e^{-\rho t} dt \le D_1 \int_0^\infty e^{(b-\rho)t} dt < +\infty$$
$$0 \le \int_0^\infty \left|\dot{A}\right| e^{-\rho t} dt \le D_1 \int_0^\infty e^{(b-\rho)t} dt < +\infty.$$

As $-\dot{K} \le \kappa K$, we have

$$\begin{aligned} C &\le A^\theta f(K, L_Y, Q, R) + (\kappa - \delta)K \\ &\le {D_1}^\theta e^{b\theta t}(B + \zeta K) + (\kappa - \delta) B_1 e^{\zeta t} \\ &\le {D_1}^\theta e^{b\theta t}(B + \zeta B_1 e^{\zeta t}) + (\kappa - \delta) B_1 e^{\zeta t} \\ &= {D_1}^\theta \zeta B_1 e^{(b\theta+\zeta)t} + {D_1}^\theta B e^{b\theta t} + (\kappa - \delta) B_1 e^{\zeta t}. \end{aligned}$$

Thus, we can choose a positive constant $D_2 \ge {D_1}^\theta B + {D_1}^\theta \zeta B_1 + (\kappa - \delta) B_1$. Thus

$$C \le D_2 e^{(b\theta+\zeta)t} < D_2 e^{\rho t},$$

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which implies that

$$0 \leq \int_0^{\infty} Ce^{-\rho t}dt < +\infty.$$

We have proven that $\mathcal{K}$ is uniformly bounded on $L^1(e^{-\rho t})$.

Moreover, $\lim_{a\to\infty}\int_a^{\infty} Ke^{-\rho t}dt \leq \lim_{a\to\infty}\int_a^{\infty} B_1 e^{(\zeta-\rho)t}dt = 0$. This property is true for other variables in $\mathcal{K}$. Therefore $\mathcal{K}$ satisfies the Dunford–Pettis theorem and it is relatively compact in the weak topology $\sigma(L^1(e^{-\rho t}), L^{\infty})$.

(ii) We have shown that every state variables and its derivative lie in $L^1(e^{-\rho t})$. Thus they belong to Sobolev space, which is integrable and absolutely continuous [Maz'ja (1985)].

A.2. PROOF OF LEMMA 2

For any state variable $X_i \in \{(K, A, S_R, S_Q)_i\}$ in $\mathcal{K}$, by hypothesis $X_i \rightharpoonup X^*$. We first claim that, for $t \in [0, \infty)$, $\int_0^t X_i dt \to \int_0^t X^* dt$. Note that $X_i \rightharpoonup X^*$ for the topology $\sigma(L^1(e^{-\rho t}), L^{\infty})$ if and only if, for every $Y \in L^{\infty}$, $\int_0^{\infty} X_i Y e^{-\rho t}dt \to \int_0^{\infty} X^* Y e^{-\rho t}dt$.

Pick any t in $[0, \infty)$ and let

$$Y(s) = \begin{cases} \frac{1}{e^{-\rho t}} & \text{if } s \in [0, t] \\ 0 & \text{if } s > t. \end{cases}$$

Therefore $Y(s) \in L^{\infty}$ and we get $\int_0^t X_i(s)ds = \int_0^{\infty} X_i(s)Y(s)e^{-\rho s}ds \to \int_0^{\infty} X^*(s)Y(s)e^{-\rho s}ds = \int_0^t X^*(s)ds$.

Given that $K_i \rightharpoonup K^*$ and $\dot{K}_i \rightharpoonup y$ weakly for some y in $L^1(e^{-\rho t})$, by the preceding claim, for all $t \in [0, \infty)$ we have $\int_0^t K_i ds \to \int_0^t y ds$. This implies that, for a fixed t, $K_i \to \int_0^t y ds + K_0$. Thus $\int_0^t y ds + k_0 = K^*$. Therefore $\dot{K}^* = y$ or $\dot{K}_i \rightharpoonup \dot{K}^*$. The same reasoning applies for $(A, S_R, S_Q)_i$ in $\mathcal{K}$.

(ii) A direct application of Mazur's Lemma.

A.3. PROOF OF PROPOSITION 1

By (8) and $\dot{A}/A = bA^{\phi-1}(1 - L_Y)$ we get

$$\dot{\omega} = (\rho - \phi g_A)\omega - U_C F_A. \tag{A.2}$$

From the identities $\dot{S}_R = mS_R - R$, $\dot{S}_Q = -Q$, and $\dot{K} = F - C - \delta K$, we obtain

$$\begin{aligned} g_{S_Q} &= -z, \\ g_{S_R} &= m - u, \\ g_K &= x - y - \delta, \\ g_A &= b(r - q). \end{aligned}$$

Because $F = A^{\theta} L_Y^{\gamma} K^{\xi} Q^{\alpha} R^{\beta}$, we have

$$F_K = \xi F/K = \xi x,$$

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$$\frac{\dot{F}_Q}{F_Q} = \theta g_A + \gamma g_{L_Y} + \xi g_K + (\alpha - 1) g_Q + \beta g_R, \tag{A.3}$$

$$\frac{\dot{F}_R}{F_R} = \theta g_A + \gamma g_{L_Y} + \xi g_K + \alpha g_Q + (\beta - 1) g_R, \tag{A.4}$$

$$\frac{\dot{F}_{L_Y}}{F_{L_Y}} = \theta g_A + (\gamma - 1) g_{L_Y} + \xi g_K + \alpha g_Q + \beta g_R. \tag{A.5}$$

Equation (8), together with (12), yields

$$\rho - \frac{\dot{U}_C}{U_C} = F_K - \delta = \xi x - \delta. \tag{A.6}$$

It is easy to check that

$$\frac{\dot{U}_C}{U_C} = -\varepsilon \frac{\dot{C}}{C} = -\varepsilon g_C. \tag{A.7}$$

Thus,

$$g_C = \frac{\xi x - \delta - \rho}{\varepsilon}.$$

By logarithmic differentiation (9) and together with (13) we have

$$\frac{\dot{F}_Q}{F_Q} = \rho - \frac{\dot{U}_C}{U_C} = \xi x - \delta.$$

It follows from (10) and (14) that

$$\frac{\dot{F}_R}{F_R} = \rho - \frac{\dot{U}_C}{U_C} - m = \xi x - \delta - m.$$

From (11) and (A.2) we get

$$\begin{aligned}
&\frac{\dot{U}_C}{U_C} + \frac{\dot{F}_{L_Y}}{F_{L_Y}} - \phi g_A = \rho - \phi g_A - \frac{U_C F_A}{\omega} \\
&= \rho - \phi g_A - \frac{U_C F_A}{U_C F_{L_Y}} A^{\phi} = \rho - \phi g_A - \frac{\theta F/A}{\gamma F/L_Y} A^{\phi} \\
&= \rho - \phi g_A - \frac{\theta}{\gamma} L_Y A^{\phi - 1} = \rho - \phi g_A - \frac{b\theta}{\gamma} q.
\end{aligned}$$

Thus,

$$\frac{\dot{F}_{L_Y}}{F_{L_Y}} = \rho - \frac{\dot{U}_C}{U_C} - \frac{b\theta}{\gamma} q = \xi x - \delta - \frac{b\theta}{\gamma} q.$$

Now, it follows from (A.3)–(A.5) that we have a system of equations with three variables g_Q, g_R, and g_{L_Y}:

$$\gamma g_{L_Y} + (\alpha - 1) g_Q + \beta g_R = \xi y - b\theta (r - q) + (\xi - 1)\delta = T_1, \tag{A.8}$$

$$\gamma g_{L_Y} + \alpha g_Q + (\beta - 1) g_R = \xi y - m - b\theta (r - q) + (\xi - 1)\delta, \tag{A.9}$$

$$(\gamma - 1) g_{L_Y} + \alpha g_Q + \beta g_R = \xi y - \frac{b\theta}{\gamma} q - b\theta (r - q) + (\xi - 1)\delta. \tag{A.10}$$

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From (A.8) and (A.9) we have $-g_Q + g_R = m$. From (A.8)—(A.10) we get $\gamma g_{L_Y} = b\theta q + \gamma g_Q$. Replacing this equation in (A.8), we have

$$(\gamma + \alpha - 1)g_Q + \beta g_R = T_1 - b\theta q.$$

We then have two equations to find g_Q and g_R:

$$g_Q = \frac{m\beta + b\theta q - T_1}{\xi} = \frac{b\theta r - \xi y + m\beta + (1-\xi)\delta}{\xi},$$
$$g_R = g_Q + m = \frac{b\theta r - \xi y + m(\beta + \xi) + (1-\xi)\delta}{\xi},$$

and we have

$$g_{L_Y} = \frac{b\theta}{\gamma} q + g_Q = \frac{b\theta r - \xi y + m\beta + (1-\xi)\delta}{\xi} + \frac{b\theta}{\gamma} q.$$

By log-differentiating $F = A^\theta L_Y^\gamma K^\xi Q^\alpha R^\beta$ we get

$$\begin{aligned} g_F &= \theta g_A + \gamma g_{L_Y} + \xi g_K + \alpha g_Q + \beta g_R \\ &= \gamma g_{L_Y} + (\alpha - 1)g_Q + \beta g_R + \theta g_A + \xi g_K + g_Q \\ &= T_1 + \theta g_A + \xi g_K + g_Q \\ &= \xi y + (\xi - 1)\delta + \xi(x - y - \delta) + \frac{b\theta r - \xi y + m\beta + (1-\xi)\delta}{\xi} \\ &= \xi x - y + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(1 - 2\xi)}{\xi} \\ &= \xi x - y + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(\alpha + \beta + \gamma)}{\xi} - \delta. \end{aligned}$$

Finally, because $L_Y = \frac{q}{r}$,

$$g_{L_A} = \frac{\dot{L}_A}{L_A} = -\frac{\dot{L}_Y}{1 - L_Y} = \frac{L_Y}{L_Y - 1} g_{L_Y} = \frac{q}{q - r} g_{L_Y}.$$

A.4. PROOF OF PROPOSITION 2

In the steady state, $g_A^* = b(r^* - q^*)$ is constant. Therefore, because g_C^* and g_K^* are constant, it follows that x^* and y^* are constant. It also follows from constant g_Q^* that r^* is constant. Thus q^*, z^*, and u^* are also constant. As $\dot{S}_Q/S_Q = -Q/S_Q$ and $g_{S_Q}^*$ is constant, we have $g_Q^* = g_{S_Q}^*$. Similarly, we have $g_R^* = g_{S_R}^*$. Because $x = F/K$ and $y = C/K$, we have $g_C^* = g_F^* = g_K^*$. Moreover, $L_Y^* = q^*/r^*$ is constant, which implies that L_A^* is constant. Thus, we get

$$g_{L_Y}^* = g_{L_A}^* = 0.$$

Because $r^* = A^{*\phi - 1}$ is constant, we have $(\phi - 1)g_A^* = 0$ or

$$(\phi - 1)b(r^* - q^*) = 0.$$

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This equation, together with $g_C^* = g_K^*$, $g_F^* = g_K^*$, and $g_{L_Y}^* = 0$, yields

$$\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta, \\
\xi x^* - y^* + \frac{b\theta r^* + m\beta + \delta(1-2\xi)}{\xi} &= x^* - y^* - \delta, \\
-y^* + \frac{m\beta + (1-\xi)\delta + b\theta r^*}{\xi} + \frac{b\theta}{\gamma} q^* &= 0, \\
(\phi - 1)(r^* - q^*) &= 0.
\end{aligned}$$

We consider two cases:

(a) If $\phi = 1$, $r^* = A^{*\phi-1} = 1$, we have

$$\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta \\
\xi x^* - y^* + \frac{b\theta + m\beta + \delta(1-2\xi)}{\xi} &= x^* - y^* - \delta \\
-y^* + \frac{m\beta + (1-\xi)\delta + b\theta}{\xi} + \frac{b\theta}{\gamma} q^* &= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
x^* &= \frac{b\theta + m\beta + \delta(1-\xi)}{\xi(1-\xi)}, \\
y^* &= \frac{(\varepsilon - \xi)[b\theta + m\beta + \delta(1-\xi)] + \frac{(1-\varepsilon)\delta + \rho}{\varepsilon}}{\varepsilon\xi(1-\xi)}, \\
q^* &= \left[y^* - \frac{m\beta + (1-\xi)\delta + b\theta}{\xi} \right] \frac{\gamma}{\theta b}.
\end{aligned}$$

(b) If $\phi \neq 1$, $A^* = (r^*)^{1/\phi-1}$, and thus $r^* = q^*$. Hence, we have three equations that determine the optimal growth rates in the steady state:

$$\begin{aligned}
(\xi - \varepsilon)x^* + \varepsilon y^* &= \delta + \rho - \varepsilon\delta \\
(\xi - 1)\xi x^* - \xi y^* + b\theta q^* &= -m\beta - \delta(1-2\xi) - \xi\delta \\
-\xi\gamma y^* + (b\theta\gamma + \xi b\theta) q^* &= -m\beta\gamma - (1-\xi)\gamma\delta \\
r^* &= q^*,
\end{aligned}$$

which correspond to $\mathbf{\Gamma X} = \mathbf{\Sigma}$.

A.5. PROOF OF PROPOSITION 3

Case 1. If $\phi = 1$. In this case $r = 1$, and we just need to analyze the dynamic system x, y, z, u, q. By logarithmic differentiation and using Proposition 1

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we get

$$\dot{x} = (g_F - g_K)x = \left[(\xi - 1)x + \frac{b\theta}{\xi} + \frac{m\beta + \delta(1-\xi)}{\xi}\right]x,$$

$$\dot{y} = (g_C - g_K)y = \left[\frac{(\xi - \epsilon)x + (\varepsilon - 1)\delta - \rho}{\varepsilon} + y\right]y,$$

$$\dot{z} = (g_Q + z)z = \left[-y + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + z\right]z,$$

$$\dot{u} = (g_R + u - m)u = \left[-y + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + u\right]u,$$

$$\dot{q} = (g_{L_Y})q = \left[-y + \frac{b\theta}{\xi} + \frac{b\theta}{\gamma}q + \frac{m\beta + (1-\xi)\delta}{\xi}\right]q.$$

The dynamics of $\mathbf{h} = (x, y, z, u, q, r)$ is described by the preceding system.

From the theory of linear approximation we know that in the neighborhood of the steady state, the dynamic behavior of the nonlinear system is characterized by the behavior of the linearized system around the steady state $\dot{\mathbf{h}} = \mathbf{J}(\mathbf{h} - \mathbf{h}^*)$, where $\mathbf{h}^* = (x^*, y^*, z^*, u^*, q^*, r^*)$ and $\mathbf{J}$ is the Jacobian matrix evaluated in the steady state, i.e.,

$$\mathbf{J} = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y & \partial\dot{x}/\partial z & \partial\dot{x}/\partial u & \partial\dot{x}/\partial q \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y & \partial\dot{y}/\partial z & \partial\dot{y}/\partial u & \partial\dot{y}/\partial q \\ \partial\dot{z}/\partial x & \partial\dot{z}/\partial y & \partial\dot{z}/\partial z & \partial\dot{z}/\partial u & \partial\dot{z}/\partial q \\ \partial\dot{u}/\partial x & \partial\dot{u}/\partial y & \partial\dot{u}/\partial z & \partial\dot{u}/\partial u & \partial\dot{u}/\partial q \\ \partial\dot{q}/\partial x & \partial\dot{q}/\partial y & \partial\dot{q}/\partial z & \partial\dot{q}/\partial u & \partial\dot{q}/\partial q \end{pmatrix}.$$

Note that x^*, y^*, z^*, u^*, q^* are stationary variables; i.e., if $\dot{x} = f(\mathbf{h})x$, then $f(\mathbf{h}^*) = 0$ and

$$\frac{\partial\dot{x}}{\partial x}(\mathbf{h}^*) = \frac{\partial f(\mathbf{h}^*)}{\partial x}x^*.$$

Thus, we get the Jacobian matrix

$$\mathbf{J} = \begin{pmatrix} (\xi - 1)x^* & 0 & 0 & 0 & 0 \\ (\frac{\xi - \varepsilon}{\varepsilon})y^* & y^* & 0 & 0 & 0 \\ 0 & -z^* & z^* & 0 & 0 \\ 0 & -u^* & 0 & u^* & 0 \\ 0 & -q^* & 0 & 0 & \frac{b\theta}{\gamma}q^* \end{pmatrix}.$$

The characteristic roots λ_k, $k = 1, .., 5$, are the solutions of the characteristic equation $|\mathbf{J} - \lambda\mathbf{U}| = 0$, where $\mathbf{U}$ is the 5×5 unit matrix. We can write at $\mathbf{h}^*$ that

$$\left(\frac{b\theta}{\gamma}q^* - \lambda_5\right)(u^* - \lambda_4)(z^* - \lambda_3)(y^* - \lambda_2)[(\xi - 1)x^* - \lambda_1] = 0.$$

It is easy to see that only $\lambda_1 = (\xi - 1)x^* < 0$, whereas the others are positive.

Case 2. If $\phi \neq 1$, we must analyze the dynamic system of x, y, z, u, q, and $r = A^{\phi-1}$. Because $\dot{r}/r = (\phi - 1)g_A$, we know that $g_A^* = 0$, which implies that r is a stationary

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variable. Moreover, $r^* = q^*$. We have

$$\dot{x} = (g_F - g_K)x = \left[(\xi - 1)x + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(1-\xi)}{\xi}\right]x,$$

$$\dot{y} = (g_C - g_K)y = \left[\frac{(\xi - \epsilon)x + (\varepsilon - 1)\delta - \rho}{\varepsilon} + y\right]y,$$

$$\dot{z} = (g_Q + z)z = \left[-y + \frac{b\theta r}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + z\right]z,$$

$$\dot{u} = (g_R + u - m)u = \left[-y + \frac{b\theta r}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + u\right]u,$$

$$\dot{q} = ((\phi - 1)g_A + g_{L_Y})q = \left[(\phi - 1)b(r - q) - y + \frac{b\theta r}{\xi} + \frac{b\theta}{\gamma}q + \frac{m\beta + (1-\xi)\delta}{\xi}\right]q,$$

$$\dot{r} = [(\phi - 1)g_A]r = [b(\phi - 1)(r - q)]r.$$

It is easy to get

$$\boldsymbol{J} = \begin{pmatrix} (\xi-1)x^* & 0 & 0 & 0 & 0 & \frac{b\theta}{\xi}x^* \\ \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^* & 0 & 0 & 0 & 0 \\ 0 & -z^* & z^* & 0 & 0 & \frac{b\theta}{\xi}z^* \\ 0 & -u^* & 0 & u^* & 0 & \frac{b\theta}{\xi}u^* \\ 0 & -q^* & 0 & 0 & [(1-\phi)b + \frac{b\theta}{\gamma}]q^* & [(\phi-1)b + \frac{b\theta}{\xi}]q^* \\ 0 & 0 & 0 & 0 & b(1-\phi)r^* & b(\phi-1)r^* \end{pmatrix}.$$

The characteristic roots λ_k, $k = 1, .., 6$ are the solutions of the characteristic equation

$$|\mathbf{J} - \lambda\mathbf{U}| = 0, \tag{A.11}$$

where $\mathbf{U}$ is the 6×6 unit matrix. Equation (A.11) is equivalent to

$$(z^* - \lambda)(u^* - \lambda)\det \boldsymbol{M} = 0,$$

where

$$\boldsymbol{M} = \begin{pmatrix} (\xi-1)x^* - \lambda & 0 & 0 & \frac{b\theta}{\xi}x^* \\ \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^* - \lambda & 0 & 0 \\ 0 & -q^* & \left[(1-\phi)b + \frac{b\theta}{\gamma}\right]q^* - \lambda & \left[(\phi-1)b + \frac{b\theta}{\xi}\right]q^* \\ 0 & 0 & b(1-\phi)r^* & b(\phi-1)r^* - \lambda \end{pmatrix}.$$

We then get two positive solutions, $\lambda = z^*$ and $\lambda = u^*$, immediately. We have

$$\det \boldsymbol{M}(\lambda) = [(\xi-1)x^* - \lambda](y^* - \lambda)\begin{vmatrix} [(1-\phi)b + \frac{b\theta}{\gamma}]q^* - \lambda & [(\phi-1)b + \frac{b\theta}{\xi}]q^* \\ b(1-\phi)r^* & b(\phi-1)r^* - \lambda \end{vmatrix}$$

$$- \frac{b\theta}{\xi}x^*\begin{vmatrix} \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^* - \lambda & 0 \\ 0 & -q^* & [(1-\phi)b + \frac{b\theta}{\gamma}]q^* - \lambda \\ 0 & 0 & b(1-\phi)r^* \end{vmatrix}$$

$$= [(\xi-1)x^* - \lambda](y^* - \lambda)\det \boldsymbol{N}(\lambda) + \frac{b\theta}{\xi}\frac{(\xi - \varepsilon)}{\varepsilon}b(1-\phi)x^*y^*q^*r^*,$$

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where

$$\det \boldsymbol{N}(\lambda) = \begin{vmatrix} [(1-\phi)b + \frac{b\theta}{\gamma}]q^* - \lambda & [(\phi-1)b + \frac{b\theta}{\xi}]q^* \\ b(1-\phi)r^* & b(\phi-1)r^* - \lambda \end{vmatrix}.$$

Hence, $\det \boldsymbol{M}(\lambda)$ is a polynomial of degree 4,

$$\det \boldsymbol{M}(\lambda) = H(\lambda) = \lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda^2 + c_4\lambda + c_5.$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be solutions of $H(\lambda) = 0$. Then by Viete's theorem, we get

$$\begin{aligned} H(0) &= \lambda_1\lambda_2\lambda_3\lambda_4 \\ &= (\xi-1)x^*y^*\left[\left(\frac{b\theta}{\gamma} + \frac{b\theta}{\xi}\right)b(\phi-1)r^*q^*\right] - \frac{b\theta}{\xi}x^*\frac{(\xi-\varepsilon)}{\varepsilon}y^*b(\phi-1)q^*r^* \\ &= [b(\phi-1)r^*q^*b\theta x^*y^*]\left[\frac{(\xi-1)(\xi+\gamma)}{\gamma} - \frac{(\xi-\varepsilon)}{\varepsilon}\right] \\ &= [b(\phi-1)r^*q^*b\theta x^*y^*]\left[\frac{(\xi-1)\xi\varepsilon + (\xi-1)\gamma\varepsilon - \gamma(\xi-\varepsilon)}{\gamma\varepsilon}\right] \\ &= [b(\phi-1)r^*q^*b\theta x^*y^*]\left[\frac{(\xi-1)\xi\varepsilon + \xi\gamma(\varepsilon-1)}{\gamma\varepsilon}\right] \\ &= [b(1-\phi)\theta r^*q^*bx^*y^*]\left[\frac{\xi(\alpha+\beta)\varepsilon+\gamma)}{\gamma\varepsilon}\right] > 0. \end{aligned}$$

It follows from Proposition 2 that

$$(\xi-1)x^* + y^* = \frac{b\theta q^*}{\gamma}.$$

Therefore, $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = (\xi-1)x^*+y^*+[(1-\phi)b+\frac{b\theta}{\gamma}]q^*+b(\phi-1)r^* = \frac{2b\theta q^*}{\gamma} > 0$ and equation $H(\lambda) = 0$ has either two negative solutions or no negative solution.

If $\xi < \varepsilon$, it follows that $H((\xi-1)x^*) = \frac{b\theta}{\xi}\frac{(\xi-\varepsilon)}{\varepsilon}b(1-\phi)x^*y^*q^*r^* < 0$.

It results that $H((\xi-1)x^*)H(0) < 0$ and there is a negative $\lambda \in ((\xi-1)x^*, 0)$. Because $\lim_{\lambda\to-\infty} H(\lambda) > 0$, there is another negative $\lambda \in (-\infty, (\xi-1)x^*)$.

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