3 Rockets

Space exploration to date relies heavily on rockets. Although most rockets have been driven by chemical propellants, there are many other types of rocket, such as ion engines and nuclear rockets. In this chapter we look at the general principles of a rocket, regardless of the mechanism used to generate thrust. We derive the important rocket equation and examine its consequences for space travel.

3.1 The rocket equation

The principle of a rocket is to eject something in one direction in order to move the spacecraft in the opposite direction. This occurs on the basis of the conservation of momentum, and a mathematical description of this leads to a fundamental equation in astronautics.

Consider a rocket of mass M moving with velocity V relative to some inertial reference frame (figure 3.1). The rocket ejects a packet of propellant of mass dm at velocity v_e relative to the rocket, with the result that the rocket's speed increases to $V + dV$ relative to the reference frame. Assuming there are no external forces, in particular no gravity, then from the law of conservation of momentum, the initial momentum equals the final momentum. From figure 3.1 this is

$$MV = dm(V - v_e) + (M - dm)(V + dV) \tag{3.1a}$$

$$0 = MdV - v_e dm \tag{3.1b}$$

where $v_e > 0$ is directed to the left, and we've neglected the $dmdV$ term in the second line on the grounds that we are dealing with differentials. Assuming mass is conserved, then the change in the mass of the rocket is

$$dM = -dm. \tag{3.2}$$

Putting together these two equations to eliminate dm we get

$$\frac{dV}{v_e} = -\frac{dM}{M}. \tag{3.3}$$

This tells us how the instantaneous velocity of the rocket changes as a function of the propellant velocity and change in rocket mass. We integrate this from an initial condition when the rocket is moving at velocity V_i with total mass M_i (spacecraft plus propellant) to a final condition when it is moving at velocity V_f with mass M_f, after having ejected propellant of mass $M_p = M_i - M_f$. This gives

$$\Delta v = v_e \ln\left(\frac{M_i}{M_f}\right), \tag{3.4}$$

where $\Delta v = V_f - V_i$ is the change in velocity of the rocket in the reference frame.

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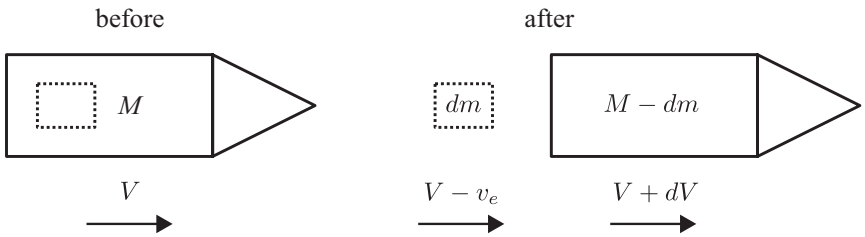


Fig. 3.1 Derivation of the rocket equation. The rocket and propellant together initially have mass M and move at velocity V to the right ('before'). The rocket then ejects propellant of mass dm at velocity v_e relative to the rocket ('after'). This accelerates the rocket to velocity $V + dV$, leaving the propellant moving at velocity $V - v_e$. All quantities are positive and all velocities are relative to the observer's inertial reference frame.

Equation 3.4 is the rocket equation. It shows that the change in velocity of the rocket depends only on its initial-to-final mass ratio and on the exhaust velocity of the propellant. The equation is plotted in figure 3.2. We see that Δv increases only very slowly with mass ratio for a given v_e. To achieve a Δv equal to v_e we need a mass ratio of 2.7, i.e. a propellant mass 1.7 times the rest of the spacecraft. To change the velocity of the rocket by $4v_e$, a propellant mass around 55 times the mass of the empty rocket is required.

We shall see in chapter 5 that the maximum value of v_e attainable with chemical propellants is about $5\,\mathrm{km\,s^{-1}}$, so even with a mass ratio of 55 a rocket would only achieve a Δv of $20\,\mathrm{km\,s^{-1}}$. Assuming our rocket started in the Earth's orbit around the Sun, with a velocity of $30\,\mathrm{km\,s^{-1}}$, the rocket's final velocity (after a rapid engine burn) would be $50\,\mathrm{km\,s^{-1}}$. This is only just enough to escape from the solar system (see section 4.1.4), leaving the rocket to travel to the next star at just a few $\mathrm{km\,s^{-1}}$. Even a mass ratio of 1000 would yield a Δv of only $35\,\mathrm{km\,s^{-1}}$, and gravity assists, discussed in section 4.3, do not substantially improve this. To get to the nearest star in a human lifetime, a spacecraft needs to achieve a velocity of about $\Delta v = 4.25\,\mathrm{ly}/50\,\mathrm{yr} = 25\,000\,\mathrm{km\,s^{-1}}$. From equation 3.4, we see that the mass ratio required to achieve this with $v_e = 5\,\mathrm{km\,s^{-1}}$ is $M_i/M_f = \exp(5000)$, or 10^{2170}. Even with a payload mass of only 1 g, this is far more mass than in the known universe (a mere 10^{53} kg).

The rocket equation tells us that we need exponentially more propellant to achieve linear increments in velocity. The physical reason for this is that the rocket is using some of its propellant to accelerate the not-yet-ejected propellant. The pursuit of high velocity rockets must therefore focus on achieving a large v_e. This is not attainable with chemical rockets (see section 5.4).

It is important to appreciate the assumptions made when deriving the rocket equation. First, we assumed no external forces are acting, so this rocket equation does not apply for launch from the surface of a planet. We shall relax this assumption in section 3.10. Second, we assumed that the propellant is ejected continuously, at both a constant rate and at a constant velocity relative to the rocket. Third, we derived the rocket equation using classical mechanics, not special relativity, and so are implicitly assuming that Δv is much less than the speed of light. This will be addressed in section 7.5.

We have not, however, made any assumptions about what the propellant is or what is causing it to be ejected. The rocket equation therefore applies equally well to a nuclear or ion drive as it does to a chemical thermal rocket, and even applies to just spraying an inert gas through a hole. We will, however, have to pay attention to the definition of v_e when there is

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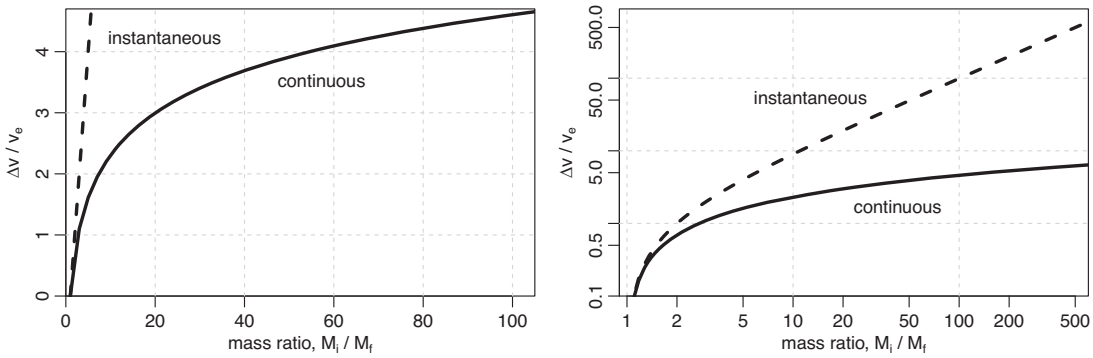


Fig. 3.2 The solid line shows the classical rocket equation, $\Delta v = v_e \ln(M_i/M_f)$, where Δv is the change in velocity of the rocket when ejecting propellant continuously at velocity v_e such that the rocket's mass decreases from M_i to M_f. The dashed line shows the change in velocity when the propellant is ejected in a single instantaneous event (equation 3.5). The left panel uses linear–linear axes whereas the right one uses log–log axes over a wider range.

a significant conversion of mass into energy, or when there are mass/energy losses. We will look at these in section 7.8.

To relax the second assumption mentioned above we could re-derive the rocket equation with time-dependent values of v_e and $\dot{m}$. It turns out that if v_e can be increased as the rocket accelerates, then this minimizes the kinetic energy given to the exhaust. This increases the kinetic energy of the rocket and thus the energy efficiency (defined in section 3.6). This so-called 'perfect' rocket is not achievable in practice with a chemical thermal rocket. It could be more straightforward with other types of rocket that we will discuss in later chapters.

We can also ask what would change if we ejected the propellant not continuously, but in a single discrete chunk. Because we then do not have to use propellant to accelerate any remaining propellant, it is clear that this 'instantaneous' rocket will achieve a larger Δv for a given v_e and mass of propellant. This follows from momentum conservation (equation 3.1a) with $M = M_i$, $dV = \Delta v$, and $dm = M_i - M_f$, which gives

$$\Delta v = v_e \left(\frac{M_i}{M_f} - 1 \right) \quad \text{(instantaneous rocket)}. \tag{3.5}$$

The velocity is now linear in the mass ratio and so increases much faster than the logarithm in the usual rocket equation. This is shown in figure 3.2 by the dashed line. With $v_e = 5\,\mathrm{km\,s^{-1}}$, a mass ratio of 55 would now yield a Δv of $270\,\mathrm{km\,s^{-1}}$. Ejecting all the mass in one go produces considerably larger velocity increments and – as we will see in section 3.9 – a higher efficiency. Unfortunately, such an approach is impractical and could not be achieved with a conventional engine.

3.2 Braking at the target star

The rocket equation tells us how much propellant is needed to accelerate by Δv. But how much propellant is needed to also decelerate back to the original velocity, in order to rendezvous at a target star, for example? For this we need two separate boosts. First we accelerate

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by Δv, then we turn the rocket around and decelerate by Δv. Let M_m be the mass of the rocket after the first boost. From the rocket equation we have for the first boost

$$\Delta v = v_e \ln\left(\frac{M_i}{M_m}\right) . \tag{3.6}$$

For the second boost we have

$$\Delta v = v_e \ln\left(\frac{M_m}{M_f}\right) . \tag{3.7}$$

Combining these we get

$$\frac{M_i}{M_f} = \left(\frac{M_m}{M_f}\right)^2 . \tag{3.8}$$

The mass ratio required to decelerate back to the initial velocity is the *square* of that required to accelerate in the first place. Decelerating at the target system is therefore very costly in terms of mass budget. Note that the mass after the first boost is the geometric mean of the initial and final masses, $M_m = \sqrt{M_i M_f}$.

In terms of propellant required, it of course does not matter in which direction the engine is pointing. The propellant required for the journey is the same as that required to accelerate to $2\Delta v$, which of course requires the same mass ratio,

$$2\Delta v = 2v_e \ln\left(\frac{M_i}{M_m}\right) = v_e \ln\left(\frac{M_i}{M_m}\right)^2 . \tag{3.9}$$

In general, if we require a mass ratio R to change the velocity by Δv, then to change the velocity by $n\Delta v$, we need a mass ratio of R^n.

3.3 The thrust equation

Newton's second law tells us that the force exerted by the rocket on the propellant – and therefore also by the propellant on the rocket (Newton's third law) – is equal to the rate of change of momentum. When propellant is ejected at mass flow rate $\dot{m}$ at velocity v_e relative to the rocket, the magnitude of this force, or thrust, is

$$F = \dot{m} v_e, \tag{3.10}$$

where in general both m and v_e are time dependent. Thrust is important for two reasons. First, when manoeuvring in a gravitational field, a minimum thrust is required to overcome the force of gravity. This is most apparent when launching from the surface of a planet, where the thrust must exceed the weight of the rocket for it to get off the launch pad (see section 3.10). Second, even in the absence of gravity, a larger thrust means a larger instantaneous acceleration (for a given mass), and so the faster the rocket will be after a certain period of time. How can we reconcile this with the rocket equation, which tells us how fast the rocket will be moving after a certain amount of propellant has been expended? At any instant, the acceleration of a rocket of mass M is

$$\frac{dV}{dt} = \frac{F}{M} = \frac{\dot{m} v_e}{M} \tag{3.11a}$$

$$= -\frac{dM}{dt}\frac{v_e}{M}, \tag{3.11b}$$

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which reduces to equation 3.3, the differential form of the rocket equation. Hence the rocket equation is the integral of the thrust equation when assuming constant v_e (but not necessarily constant $\dot{m}$). This should not be surprising given that both arise from Newton's second law. Nonetheless, the two equations are still separately useful because they describe different aspects of the rocket. The rocket equation tells us what Δv is ultimately achievable, whereas the thrust equation tells us how fast it can be attained. Both aspects are relevant to getting to a destination quickly, as we will now see.

3.4 Travel time and distance for a constant mass rate

The rocket equation gives the velocity of the rocket, $\Delta v = V(t)$, as a function of the current mass of the rocket, $M_f = M(t)$, which comes from integrating the variable acceleration (equation 3.11b) over time. Integrating again over time gives the distance travelled by the rocket,

$$d(t) = \int_0^t V(t')dt'. \tag{3.12}$$

Adopting a constant rate of mass ejection, the mass decreases linearly with time. Specifically, if $M_i - M_f$ is the total mass of propellant ejected over some time interval, then the mass of the rocket at any time t within this interval is

$$M(t) = \begin{cases} M_i - \dot{m}t & \text{if} \quad \dot{m}t < M_i - M_f \\ M_f & \text{otherwise.} \end{cases} \tag{3.13}$$

Putting this into the rocket equation gives

$$V(t) = \begin{cases} v_e \ln\left(1 - \dfrac{\dot{m}t}{M_i}\right)^{-1} & \text{if} \quad t < (M_i - M_f)/\dot{m} \\ v_e \ln\left(\dfrac{M_i}{M_f}\right) & \text{otherwise.} \end{cases} \tag{3.14}$$

This is plotted in figure 3.3. The horizontal axis is the fraction of the initial mass ejected as propellant, mt/M_i. This is shown for two cases: one in which the propellant is 99% of the total mass ($M_i/M_f = (99+1)/1 = 100$), and the other in which the propellant is 90% of the total mass ($M_i/M_f = (9+1)/1 = 10$). We see that most of the change in velocity comes later in the mass ejection, when the rocket is lightest.

Inserting equation 3.14 into equation 3.12 and integrating, we get the following expression for how far the rocket travels during its acceleration, once its mass has reduced to M_f (see exercises):

$$d_{\text{acc}} = v_e \frac{M_i}{\dot{m}}\left[1 - \frac{M_f}{M_i}\left(1 + \ln\frac{M_i}{M_f}\right)\right] \quad \text{or} \tag{3.15a}$$

$$= v_e \frac{M_f}{m}\left[\frac{M_i}{M_f} - 1 - \ln\left(\frac{M_i}{M_f}\right)\right]. \tag{3.15b}$$

Although equation 3.15b is a function of four variables, we can consider just three: the exhaust velocity v_e, the mass ratio $R = M_i/M_f$, and $M_i/\dot{m}$, which is the time it would take to eject a mass equal to the initial mass of the rocket.

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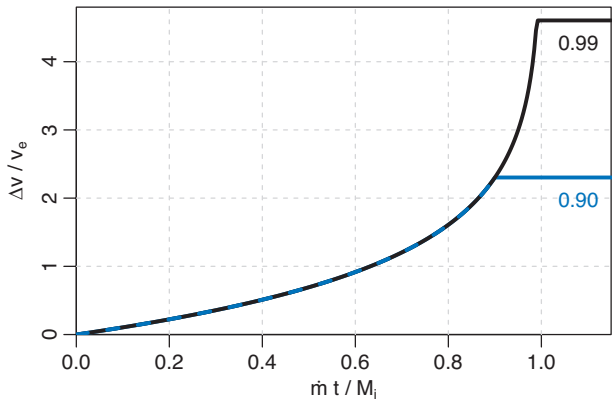


Fig. 3.3 The variation in velocity of a rocket (in units of v_e) of initial mass M_i with time t as it ejects mass at constant rate $\dot{m}$, shown as a function of the fraction of mass ejected so far (equation 3.14). The upper line is for a propellant mass fraction of 0.99, so the rocket accelerates up until time t given by $\dot{m}t/M_i = 0.99$. The lower line is for 0.90.

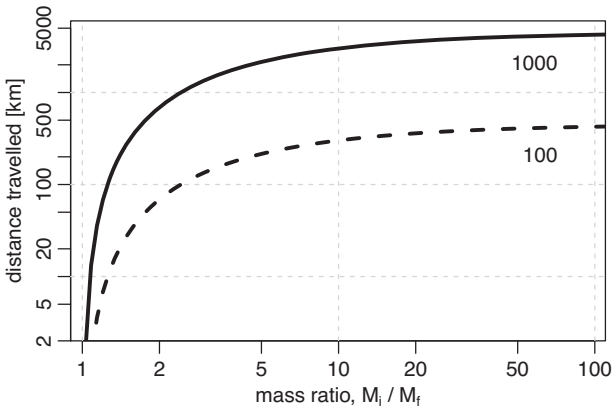


Fig. 3.4 The distance travelled by a rocket during its acceleration when ejecting propellant at a constant velocity $v_e = 4.5$ km s^{-1} and constant rate $\dot{m}$ (equation 3.15b). The solid line is for $M_i/\dot{m} = 1000$ s and the dashed line is for $M_i/\dot{m} = 100$ s, where M_i is the initial mass of the rocket.

The distance travelled is plotted in figure 3.4 as a function of the mass ratio for fixed v_e and two values of M_i/m. Consider the behaviour along one of the curves. The larger the mass ratio R, the more propellant is ejected, so the larger the change in velocity. The time taken to accelerate is

$$t_{\text{acc}} = \frac{M_i - M_f}{\dot{m}} = \frac{M_i}{\dot{m}}\left(1 - \frac{1}{R}\right) = \frac{M_f}{\dot{m}}(R-1). \tag{3.16}$$

Increasing the mass ratio at constant $M_i/\dot{m}$ extends the acceleration time, so a larger distance is covered.

We see from equation 3.15b that the distance travelled during the acceleration is *inversely* proportional to the mass ejection rate, $\dot{m}$. This might seem counter-intuitive, because for given v_e, a smaller m means a smaller thrust and so a smaller acceleration. But if M_i and M_f are fixed, then the final Δv is the same regardless of $\dot{m}$, from the rocket equation. If the rocket uses up the propellant more slowly, it spends more time accelerating and so travels

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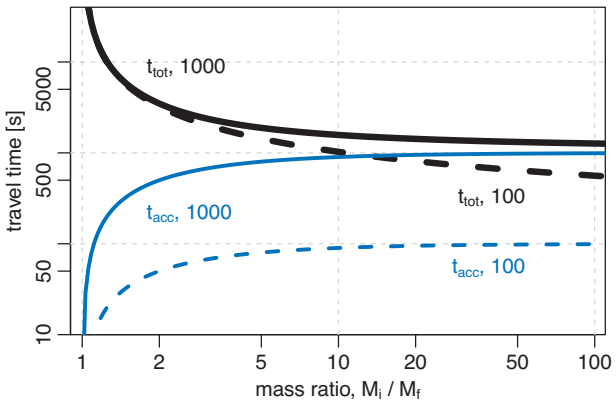


Fig. 3.5 The time taken by a rocket to travel a distance of 10 000 km (equation 3.17a). The rocket ejects propellant at constant velocity $v_e = 4.5\ \mathrm{km\ s^{-1}}$ and constant rate $\dot{m}$. It accelerates until it has exhausted its propellant, then continues at constant velocity. The thick upper line shows the total time (t_{tot}) taken to reach the distance. The thin lower line shows how much of this time the rocket spends accelerating (t_{acc}) to achieve its cruise velocity of $v_e \ln(M_i/M_f)$. The solid line is for $M_i/\dot{m} = 1000$ s and the dashed line is for $M_i/\dot{m} = 100$ s, where M_i is the initial mass of the rocket.

further during this acceleration. In the limit of an infinitesimally small mass ejection rate, the rocket will spend an arbitrarily long time at a non-zero velocity and so travel arbitrarily far. Of course, it would take such a rocket much longer to travel that distance, and for travelling to a given star we are particularly interested in minimizing the time.

When we want to travel further than the acceleration distance, the remainder of the journey will be spent cruising at a constant velocity Δv relative to the initial rest frame. If the total distance is d_{tot}, then the time spent cruising is $(d_{\mathrm{tot}} - d_{\mathrm{acc}})/\Delta v$. The total travel time is therefore

$$t_{\mathrm{tot}} = \frac{M_i}{\dot{m}}\left(1 - \frac{1}{R}\right) + \frac{d_{\mathrm{tot}} - d_{\mathrm{acc}}}{v_e \ln R} \quad \text{or} \tag{3.17a}$$

$$= \frac{M_f}{m}(R-1) + \frac{d_{\mathrm{tot}} - d_{\mathrm{acc}}}{v_e \ln R} \tag{3.17b}$$

for $d_{\mathrm{tot}} \geq d_{\mathrm{acc}}$, where again $R = M_i/M_f$. This is plotted as the thick line in figure 3.5 as a function of R for two values of $M_i/\dot{m}$. The larger the mass ratio, the quicker the rocket reaches the destination, because the cruise velocity is larger. The smaller $M_i/\dot{m}$, the faster the propellant is ejected, and so the quicker the rocket reaches Δv. The rocket spends more of the journey at a larger velocity, compared to when it has a lower mass ejection rate. The thin lines in figure 3.5 show the amount of time spent accelerating (equation 3.16). For a larger mass ratio (for given $M_i/\dot{m}$) the rocket spends not only more time accelerating, but also a larger fraction of the total time. Finally, even though v_e and $\dot{m}$ are constant, and so the thrust is constant (equation 3.10), the rocket's acceleration is not because its mass is varying. The time dependence of the acceleration is given by equations 3.11a and 3.13.

Figure 3.6 shows the total time taken for a journey to Proxima Cen ($d_{\mathrm{tot}} = 4.25$ ly) as a function of v_e for different configurations. The thick line is for a rocket with a mass ratio $R = 10$ and time to exhaust all the propellant at constant $\dot{m}$ set to $t_{\mathrm{acc}} = 10$ years. With $v_e = 0.043\,c$, for example, the journey takes 50 years, and the cruise velocity is $0.043 \ln(10) = 0.099\,c$. If the time spent accelerating is increased by a factor of four to 40 years (the dotted

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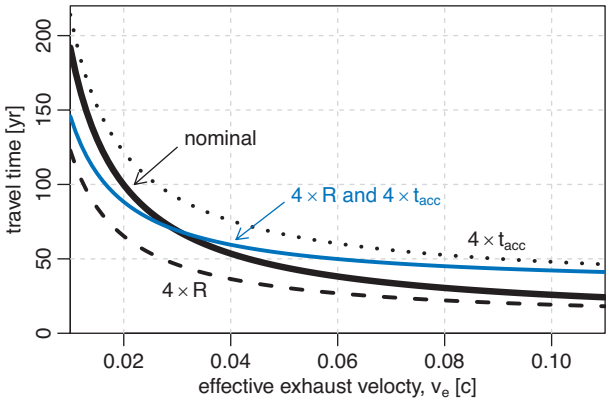


Fig. 3.6 The time taken by a rocket to travel to Proxima Cen ($d_{\text{tot}} = 4.25$ ly) as a function of v_e. The thick solid line labelled 'nominal' is for a mass ratio $R = 10$ and acceleration time $t_{\text{acc}} = 10$ yr. The dashed and dotted lines increase each of these by a factor of four, respectively, holding the other constant. The thin solid line increases both of these by a factor of four over the nominal.

line), the total travel time is increased, although not by much at low v_e (at $v_e = 0.043\,c$ the time is now 72 years). If instead the mass ratio is increased by a factor of four (the dashed line), the travel time is shorter, but not by much at high v_e (at $v_e = 0.43\,c$ the time is now 34 years). If we increase both acceleration time and mass ratio by a factor of four, then the travel time is given by the thin solid line in figure 3.6. The travel time is shorter than the nominal case at low v_e, but longer at high v_e.

3.5 Specific impulse and effective exhaust velocity

The term *specific impulse* is often encountered in astronautics, and although I will rarely use it, it is worth knowing that it has two mathematical definitions. The first is the change of momentum of the rocket per unit mass of propellant expelled. The change of momentum is the force integrated over time, so when keeping all terms in the thrust equation $F = \dot{m} v_e$ constant, this is

$$F \delta t = \delta m\, v_e. \tag{3.18}$$

The specific impulse is therefore just v_e, and has units of velocity. For a rocket that derives its impulse entirely from ejected mass, the specific impulse is just the velocity of its propellant. However, in some cases, such as when the propellant contains a mixture of particles moving at different speeds, the exhaust velocity must be replaced with a representative value known as the *effective exhaust velocity*. I will often use this term, rather than simply 'exhaust velocity', to remind us that they are not always the same.

The second definition of specific impulse is the change of momentum per unit *weight* of propellant. In this case, the specific impulse is v_e/g, where g is the acceleration due to gravity, and this version has units of time. Normally we take g as the value at the Earth's surface. As this is irrelevant to interstellar travel, I will not use this definition of specific impulse. It tends

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to be used in the context of jet engines operating in the Earth's atmosphere, where air as well as fuel make up the propellant. In that case the effective exhaust velocity is very different from the actual exhaust velocity.

3.6 Energy and energy efficiency

How much energy do we need to accelerate a rocket to a certain speed? Consider first the view from the instantaneous rest frame of the rocket during its acceleration. At every moment the rocket ejects a packet of propellant of mass δm with velocity v_e relative to the rocket. Each packet has an energy of $(1/2)\delta m v_e^2$. The total mass ejected is $M_i - M_f$, so the total energy that the rocket must supply to eject this (assuming no losses) is

$$E_{\text{required}} = \frac{1}{2}(M_i - M_f)v_e^2 = \frac{1}{2}M_f v_e^2(R-1), \tag{3.19}$$

where $R = M_i/M_f$. As the rocket is the only source of energy, E_{required} is the energy stored in the rocket fuel at the beginning, when the rocket is at rest in the initial rest frame. From the point of view of the continuously changing instantaneous rest frame of the rocket, all of this fuel energy goes into the propellant (the kinetic energy of the rocket in this frame is zero). But from the point of view of the initial rest frame, some of the fuel energy goes into the propellant and some into the rocket. Due to conservation of momentum in the initial rest frame, not all of the energy can go into the rocket's motion. Once the rocket has exhausted its propellant, it has reached a velocity Δv in the initial rest frame, so has a kinetic energy of

$$E_{\text{rocket}} = \frac{1}{2}M_f(\Delta v)^2 = \frac{1}{2}M_f v_e^2 \ln^2 R, \tag{3.20}$$

having used the rocket equation (equation 3.4) in the second step. Seen from the initial rest frame, propellant is moving with a range of (signed) velocities from $-v_e$ to $\Delta v - v_e$ (and not uniformly distributed over this range because the rocket did not have constant acceleration). From conservation of energy, the total kinetic energy of the propellant in the initial rest frame is $E_{\text{required}} - E_{\text{rocket}}$.

We need to be careful when considering conservation of energy in different reference frames because (kinetic) energy is a relative quantity. What we have done above is equate the 'before' and 'after' energy in the initial rest frame, having determined the energy required in the rocket's rest frame.

A useful measure of the energy efficiency is the ratio of the rocket's kinetic energy to the energy it supplied. This is

$$\frac{E_{\text{rocket}}}{E_{\text{required}}} = \frac{\ln^2 R}{R-1} \tag{3.21}$$

and is plotted as the solid line in figure 3.7 (the dashed line will be derived in section 3.9). For low mass ratios, only a small fraction of the total energy ends up as kinetic energy of the rocket. At a mass ratio of $R = 4.92$, or equivalently $\Delta v/v_e = 1.59$, the efficiency reaches its maximum value of 0.65. It is not possible to put all of the energy into the rocket. In order to conserve both energy and momentum, at least a third of the energy supplied must end up in the propellant. At higher mass ratios than this optimum, the rocket will attain a larger velocity, but the efficiency is lower. Note that this is an ideal theoretical efficiency which does

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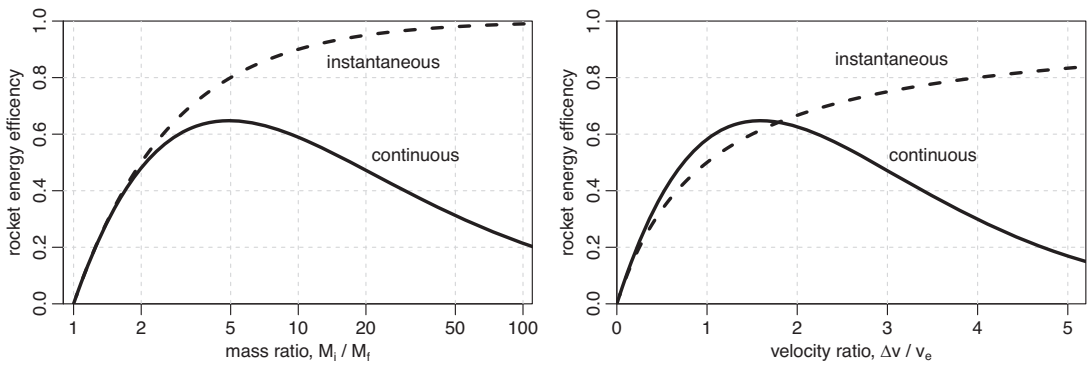


Fig. 3.7 The ratio of the kinetic energy of the rocket (in the initial rest frame) to the energy expended by the rocket (to accelerate it and the propellant) plotted against the mass ratio (left) and the velocity ratio (right). The solid line is for the standard rocket equation, with efficiency given by equation 3.21. The dashed line is for the instantaneous rocket as given by equation 3.32. Note that the two types of rocket do not have the same relationship between mass ratio and velocity ratio.

not take into account any practical losses that a real engine would have, for example due to a finite thermodynamic efficiency in a heat engine, or neutron losses in a fission or fusion engine.

3.7 Power

The power of a rocket is the rate at which energy flows out via the propellant. From the expression for kinetic energy for a constant mass flow rate $\dot{m}$, it follows that the power is

$$P = \frac{1}{2}\dot{m}v_e^2 \,. \tag{3.22}$$

Like energy and velocity, power is not absolute but is defined relative to a reference frame, here the instantaneous rest frame of the rocket. Using the thrust equation 3.10 we can relate the power to the thrust as

$$F = \frac{2P}{v_e} \,. \tag{3.23}$$

This tells us that lowering the exhaust velocity for a given power increases the thrust. This might seem counter-intuitive until we realize that by pushing the propellant out more slowly at constant power we must increase the mass flow rate, from equation 3.22. When working with these various equations – for velocity, thrust, and power – we must always be aware which quantities are being held constant. Equation 3.23 tells us that in order to maxmize the thrust for a given power (for example at launch), we should expel the propellant relatively slowly (small v_e). Equation 3.22 then tells us that this requires a high mass flow rate, which for a low exhaust velocity implies a large engine or a heavy propellant.

Note that P above is the power of the propellant in the instantaneous rocket rest frame, and is therefore the power that the rocket must provide. The power of the engines themselves must inevitably be higher due to efficiency limitations.

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It can also be informative to consider the power per unit mass flow rate, $P/\dot{m}$. From equation 3.22 this is

$$\frac{P}{m} = \frac{1}{2}v_e^2, \tag{3.24}$$

which is the kinetic energy per unit mass (specific energy) of the propellant. For a chemical rocket the specific energy is determined by the fuel and the engine. A liquid hydrogen-oxygen propellant mix that provides $v_e = 4.5\,\mathrm{km\,s^{-1}}$ has a specific energy of $10.1\,\mathrm{MJ\,kg^{-1}}$.

3.8 Staging

The tanks that hold the rocket propellant have a mass that can be a significant fraction of the complete fuelled rocket. If the propellant is stored in multiple tanks, each tank can be jettisoned as soon as it is empty to avoid wasting propellant on accelerating it further. Dropping surplus mass is the concept of staging.

For reasons that will be discussed in chapter 5, each stage of a thermal rocket often has its own engines too (figure 3.8). Once the propellant in that stage is exhausted, the empty tanks together with the engines are jettisoned, and then the engines in the next stage fire. We are most familiar with this in launches from the Earth's surface.

Figure 3.9 demonstrates the principle with two stages. From the rocket equation we can compare two alternative scenarios: (a) using the entire rocket as a single stage (i.e. without

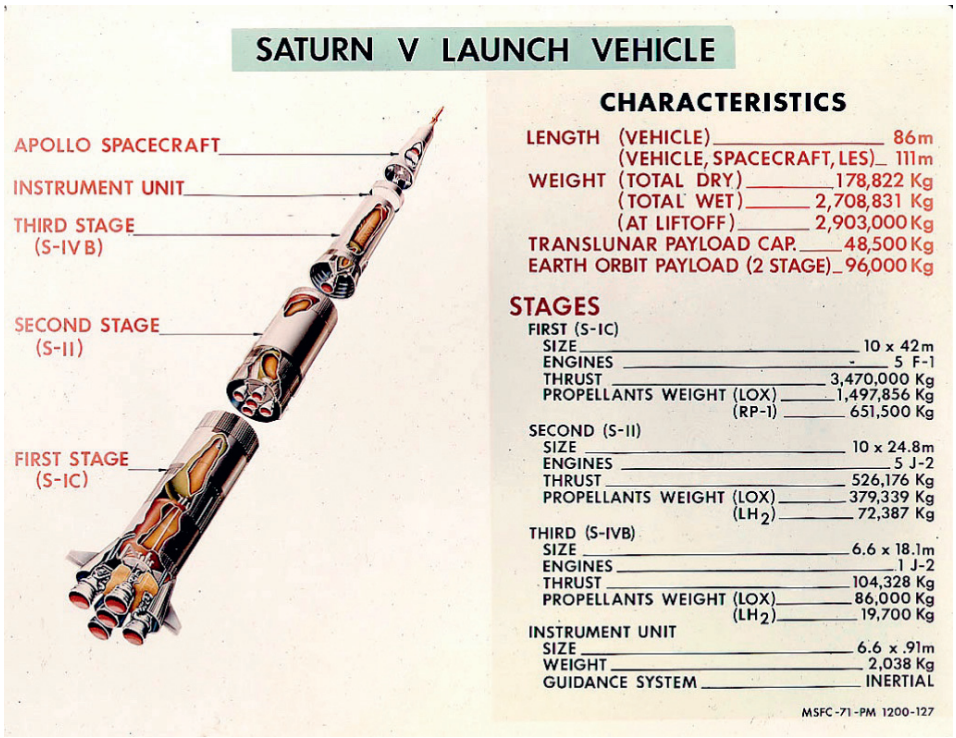
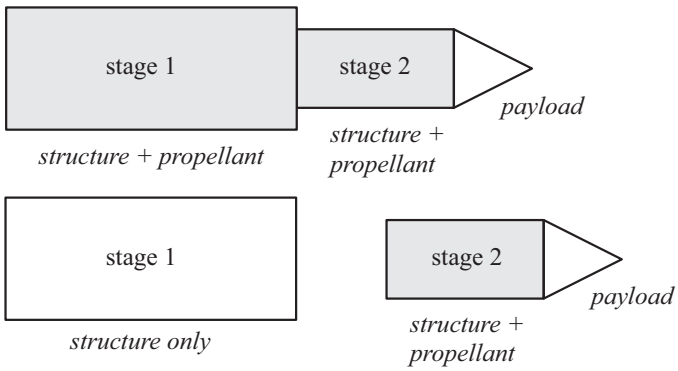


Fig. 3.8 The different stages of the Saturn V rocket that carried the Apollo spacecraft to the Moon. Credit: NASA.

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Principle of rocket stages. Top: The rocket is initially full of propellant. Bottom: Once a stage has exhausted all of its propellant, it is ejected to remove the mass of the empty tanks. Fig. 3.9

jettisoning stage 1), and (b) using up the propellant first in stage 1, jettisoning it, then using the propellant in stage 2. Let us assume the engines in each stage have the same exhaust velocity v_e. We then only need to consider the initial-to-final mass ratio in order to compare the Δv in these two scenarios. We define f as the ratio of the mass of the structure (empty tanks and engines) to the propellant in each stage; for simplicity we'll make f the same in both stages too. The mass of the propellant in the two stages is m_1 and m_2, respectively, and m_{pl} is the mass of the payload.

(a) The initial-to-final mass ratio of the rocket when using it as a single stage is

$$R_s = \frac{m_{pl} + m_2(f+1) + m_1(f+1)}{m_{pl} + m_2 f + m_1 f} . \tag{3.25}$$

From the rocket equation the velocity change Δv_s is

$$\frac{\Delta v_s}{v_e} = \ln R_s . \tag{3.26}$$

(b) When using the stages separately, the mass ratio of stage 1 is

$$R_1 = \frac{m_{pl} + m_2(f+1) + m_1(f+1)}{m_{pl} + m_2(f+1) + m_1 f} \tag{3.27}$$

and the mass ratio of stage 2 is

$$R_2 = \frac{m_{pl} + m_2(f+1)}{m_{pl} + m_2 f} . \tag{3.28}$$

From the rocket equation we compute the Δv given by each of these stages separately and then sum them to get the total Δv, which is

$$\frac{\Delta v_1 + \Delta v_2}{v_e} = \ln R_1 + \ln R_2. \tag{3.29}$$

To compare scenarios (a) and (b), let us adopt specific values of $m_1 = 30 m_{pl}$ and $m_2 = 5 m_{pl}$ and examine how the velocities vary as a function of f. This is plotted in figure 3.10. As expected, the two-stage scenario (thick solid line) gives a larger Δv than the single-stage scenario (thin solid line) for all $f > 0$, even though both use the same amount of propellant. This difference of course drops to zero in the limit of a zero mass structure, as then nothing is gained by jettisoning it. The advantage of staging can be quite significant: for $f = 0.1$,

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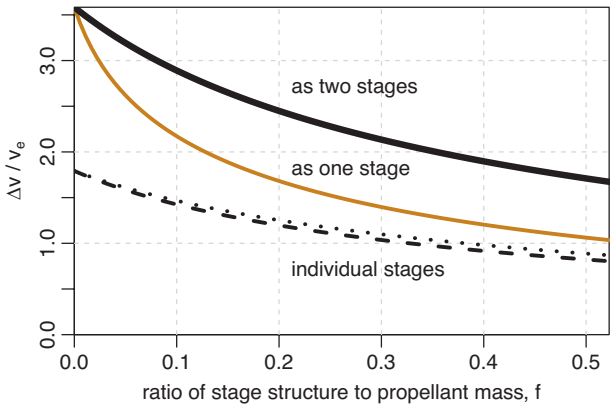


Fig. 3.10 Two-stage rocket performance for the case $m_1 = 30m_{pl}$ and $m_2 = 5m_{pl}$. The thick solid line shows the Δv delivered using two stages according to equation 3.29 as a function of the ratio of the mass of the structure (empty tanks and engines) to the propellant in the stages. The dashed and dotted lines show the Δv delivered by stages 1 and 2, respectively. The thin solid line shows, for comparison, the Δv delivered when using the same rocket as a single stage (i.e. without jettisoning the first stage when empty), as given by equation 3.26.

$\Delta v/v_e$ of the two stages is 2.89 compared to 2.17 when using them as one stage, a gain of a factor of 1.33. As the structure gets more massive (increasing f), Δv of course decreases for both scenarios, but the advantage of two stages over a single stage is more pronounced: at $f = 0.5$, $\Delta v/v_e$ is 1.71 vs 1.06, a gain of a factor of 1.61.

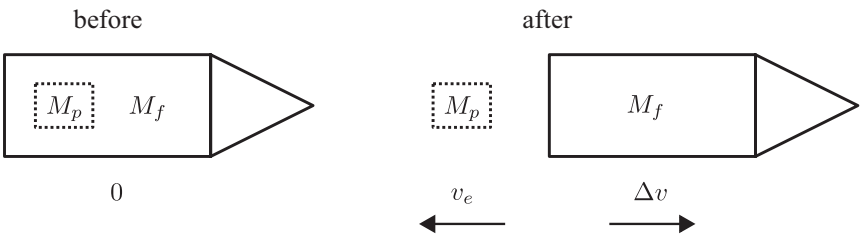
This illustrates the advantage of staging. When designing a real mission, we would be given m_{pl} and know the $\Delta v_1 + \Delta v_2$ required, and would then find the value of m_1/m_2 that minimizes $m_1 + m_2$. In practice f, and possibly v_e, would not be the same for each stage.

We have examined only two stages in order to illustrate the concept. A larger Δv for a given mass is achieved using more stages, and in principle the more stages, the better. There are practical and economic issues with using an ever larger number of stages, however, not least because each requires its own engines, plus the overall risk of failure increases. For launches from the Earth's surface that escape Earth's gravity, three stages has proven to be optimum, also for reasons of thrust and nozzle optimization in chemical rockets, as will be discussed in chapter 5.

3.9 Instantaneous rockets

At the end of section 3.1, we saw that by ejecting all of the propellant as a single chunk we achieve a higher Δv for a given mass ratio and effective exhaust velocity (figure 3.2 and equation 3.5). This physical situation is shown in figure 3.11. Suppose the rocket provides an energy E in order to simultaneously accelerate the rocket of mass M_f to a velocity Δv and the propellant of mass M_p to a velocity v_e, both velocities being relative to the initial rest frame of the rocket (and both positive). The equations of conservation of momentum and energy are

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The instantaneous rocket. The rocket of mass M_f and its propellant of mass M_p are initially at rest. The rocket then ejects the propellant at velocity v_e relative to the rocket, which accelerates the rocket to velocity Δv. All quantities are positive. **Fig. 3.11**

$$M_p v_e = M_f \Delta v, \tag{3.30a}$$

$$E = \frac{1}{2} M_p v_e^2 + \frac{1}{2} M_f (\Delta v)^2 . \tag{3.30b}$$

Combining these to eliminate M_p gives the rocket's velocity for a given energy:

$$(\Delta v)^2 = \frac{2E}{M_f} \left(\frac{1}{1 + M_f/M_p} \right) = \frac{2E}{M_f} \left(1 - \frac{1}{R} \right) . \tag{3.31}$$

The energy of the rocket (in the initial rest frame) is $(1/2)M_f(\Delta v)^2$ and the energy required is E. Thus the efficiency of the instantaneous rocket is

$$\frac{E_{\text{rocket}}}{E_{\text{required}}} = 1 - \frac{1}{R} . \tag{3.32}$$

The variation of this efficiency with the mass ratio R is shown in the left panel of figure 3.7 as the dashed line. Whereas the efficiency of the continuous rocket (solid line) has a maximum and then decreases at large mass ratios and velocity ratios ($\Delta v/v_e$), the efficiency of the instantaneous rocket continues to increase, asymptoting to unity. Note that in the left panel, the efficiency of the instantaneous rocket is always higher than that of the continuous rocket, which is not the case in the right panel. This is because a given mass ratio does not correspond to the same velocity ratio for both types of rockets. They have different rocket equations, so a given mass ratio in the left panel corresponds to two different velocity ratios in the right panel.

3.10 Launching from the surface of a planet

So far we have considered a rocket in space, free from external forces such as gravity. The derivation of the rocket equation in section 3.1 needs to be modified to determine the motion of a rocket launched from a planet's surface. If g is the local acceleration due to gravity, then for a rocket of mass M moving vertically (parallel to the gravitational force), Newton's second law gives

$$\frac{dV}{dt} = \frac{F}{M} - g, \tag{3.33}$$

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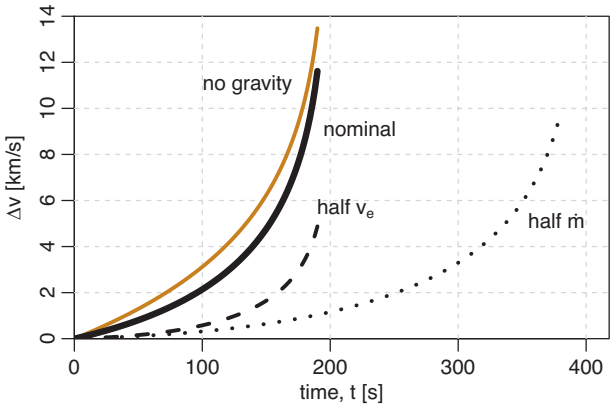


Fig. 3.12 The variation of velocity with time for a rocket with $M_f = 10$ t, $M_i = 200$ t launched vertically in a constant gravitational field of $g = 9.8\,\mathrm{m\,s^{-2}}$ up until it has exhausted all of its propellant. The thick solid line labelled 'nominal' is for $v_e = 4.5\,\mathrm{km\,s^{-1}}$, $\dot{m} = 1000\,\mathrm{kg\,s^{-1}}$. The dashed line shows the case for half v_e, and the dotted line shows the case for half $\dot{m}$. The thin solid line shows the gravity-free case ($g = 0$) for comparison.

where F is the rocket thrust. From the thrust equation (equation 3.10) and recalling that $\dot{m} = -dM/dt$, this becomes

$$dV = -\frac{dM}{M} v_e - g dt. \tag{3.34}$$

Let us assume g remains constant as the rocket rises – which is not such a bad approximation for a launch into a very low orbit (but see below) – and take v_e as constant. The integral of the above equation from the initial mass M_i at time $t = 0$ to the mass $M(t)$ at time t is

$$V = v_e \ln\left(\frac{M_i}{M(t)}\right) - gt. \tag{3.35}$$

If we further assume $\dot{m}$ is constant, then using equation 3.13 we get the following expression for the variation of velocity with time

$$V = v_e \ln\left(\frac{M_i}{M_i - \dot{m}t}\right) - gt. \tag{3.36}$$

The thick solid line in figure 3.12 shows how the velocity varies during launch when $g = 9.8\,\mathrm{m\,s^{-2}}$, $v_e = 4.5\,\mathrm{km\,s^{-1}}$, $\dot{m} = 1000\,\mathrm{kg\,s^{-1}}$, $M_f = 10\,\mathrm{t}$, and the mass ratio is $R = 20$. The rocket starts at rest at $t = 0$. The end point of the line is at the time when all the propellant has been exhausted, which is $t = (M_i - M_f)/\dot{m} = 190\,\mathrm{s}$. The final velocity is $11.62\,\mathrm{km\,s^{-1}}$. Comparing this to the no gravity case (thin solid line), we see the effect of gravity reducing the velocity throughout the launch. The dashed line in the figure shows the launch profile when halving the effective exhaust velocity: the final velocity is only $4.88\,\mathrm{km\,s^{-1}}$, less than half the nominal case. The dotted line shows the launch profile for half the mass flow rate. As the propellant mass is unchanged, the travel time is now twice as long. However, the final velocity is now $9.76\,\mathrm{km\,s^{-1}}$, less than in the nominal case. This is because gravity has acted for longer and so causes a larger overall deceleration.

To get into a low Earth orbit a spacecraft needs a Δv of around $10\,\mathrm{km\,s^{-1}}$. The above example shows that rockets with too low an effective exhaust velocity, for a given mass ratio, will not reach orbit at all. Indeed, for a rocket to even get off the ground, the thrust must exceed its weight; that is, it must have $\dot{m}v_e > M_i g$. Some values of the parameters in

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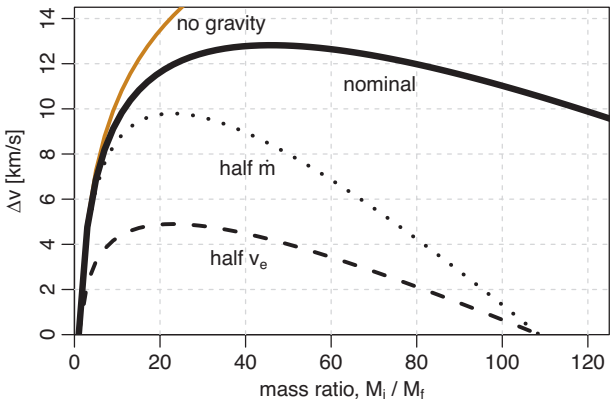


Fig. 3.13 The rocket equation (equation 3.37b) for a rocket launched vertically in a constant gravitational field with constant v_e, $\dot{m}$, and M_f. The thick solid line labelled 'nominal' is for $g = 9.8\,\mathrm{m\,s^{-2}}$, $v_e = 4.5\,\mathrm{km\,s^{-1}}$, $\dot{m} = 1000\,\mathrm{kg\,s^{-1}}$, and $M_f = 10\,\mathrm{t}$. The dashed line shows the case for half v_e, and the dotted line shows the case for half $\dot{m}$ (or equivalently for twice g). The thin solid line shows the force-free rocket equation for comparison.

equation 3.36 yield negative velocities. These correspond to less thrust than weight, and so the rocket would fall if the ground were not there to hold it.

We can rewrite equation 3.36 in terms of the final mass to get an expression for the Δv attained once all the propellant has been exhausted:

$$V = v_e \ln\left(\frac{M_i}{M_f}\right) - g\frac{M_i}{\dot{m}}\left(1 - \frac{M_f}{M_i}\right) \quad \text{or} \tag{3.37a}$$

$$V = v_e \ln\left(\frac{M_i}{M_f}\right) - g\frac{M_f}{\dot{m}}\left(\frac{M_i}{M_f} - 1\right) . \tag{3.37b}$$

This is the constant gravity rocket equation. The first term on the right is the familiar one from the force-free rocket equation. The second term is the deceleration caused by gravity.

The variation of V with the mass ratio $R = M_i/M_f$ is shown in figure 3.13. The thick solid line is for $g = 9.8\,\mathrm{m\,s^{-2}}$, $v_e = 4.5\,\mathrm{km\,s^{-1}}$, $\dot{m} = 1000\,\mathrm{kg\,s^{-1}}$, and $M_f = 10\,\mathrm{t}$, which gives $gM_f/\dot{m} = 98\,\mathrm{m\,s^{-1}}$. As R is varying and M_f is fixed, M_i is varying. We see there is a maximum Δv that can be achieved. By differentiation of equation 3.37b, we find that this occurs at a mass ratio of $R = v_e\dot{m}/gM_f$, which in this case is 45.9. This variation is different from the gravity-free rocket equation, in which the velocity increases monotonically with the mass ratio (the thin solid line). A consequence of this maximum is that there are now two possible values of the mass ratio that will deliver a specified Δv (as long as that is not too large), as we can see from the figure. We can find these by solving equation 3.37b for M_i/M_f. This has to be done numerically. For example, to achieve $\Delta v = 10\,\mathrm{km\,s^{-1}}$ for the case shown, we require either $M_i/M_f = 11.6$ or 118. These correspond to initial masses of $M_i = 116\,\mathrm{t}$ and 1180 t, respectively, and to travel times of 106 s and 1170 s, respectively. The lower mass solution is obviously more desirable. The higher mass one is actually too heavy to get off the ground initially. Mathematically the velocity from equation 3.35 is initially negative, before reaching a minimum and then increasing to positive values. In practice the ground prevents the rocket from falling, so this corresponds to the rocket sitting on the launch pad burning off fuel for a while before taking off.

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The thin solid line in figure 3.13 shows the gravity-free rocket equation for comparison, which is independent of $gM_f/\dot{m}$. As expected, the mass ratios required when considering gravity are higher than when neglecting it. Whereas the gravity rocket needs a mass ratio of 11.6 to reach $\Delta v = 10\,\mathrm{km\,s^{-1}}$, the gravity-free rocket equation requires only $\exp(10/4.5) = 9.2$.

It must be stressed that these values are not applicable to real launches, because the derivation of the above gravity rocket equation has ignored several practical and important aspects of launching. The first is the assumption of constant gravity. In the nominal example in figure 3.12, the rocket will move of the order of 1000 km in its 190 s journey, a distance over which the Earth's gravity drops from $9.8\,\mathrm{m\,s^{-2}}$ at the surface to $7.3\,\mathrm{m\,s^{-2}}$. Second, air resistance is quite significant. Third, rockets are not launched vertically, but instead are given a significant horizontal component in order to get them into a Keplerian orbit. Such factors mean that the required mass ratios are larger (and flight times longer) than estimated here.

3.11 Summary

The rocket equation gives the change in velocity Δv of a rocket when it expels propellant at constant velocity relative to the rocket (equation 3.4). The Δv is independent of both how quickly the propellant is ejected and the means used to accelerate it.

The logarithmic dependence of Δv on the ratio of the initial mass (spacecraft including propellant) to final mass (spacecraft without propellant) means that in practice Δv is limited to a few times the effective exhaust velocity v_e of the propellant. To reach $25\,000\,\mathrm{km\,s^{-1}}$ ($0.08\,c$) with a mass ratio of 100, for example, we need $v_e = 5400\,\mathrm{km\,s^{-1}}$. This logarithmic dependence also means that the mass ratio required to accelerate to velocity Δv and then decelerate back to zero is the square of that required to accelerate to Δv. Rockets intended for a rendezvous mission are therefore much more massive than those intended for a flyby.

Staging is a common way to increase Δv, by jettisoning expended mass such as empty fuel tanks.

To get to the stars quickly we not only need large v_e, but we need to achieve the Δv quickly. This means expelling the propellant rapidly (large $\dot{m}$), which corresponds to a large thrust (equation 3.10). In later chapters we will see what technologies can be used to achieve a large v_e and/or large $\dot{m}$.

3.12 Exercises

1. Show by integration that equation 3.11b leads to equation 3.4 when assuming that mass is exhausted at a constant rate.
2. Derive equation 3.15b, the expression for the distance a rocket travels while exhausting propellant.
3. Compute the time taken to travel from Earth to Proxima Cen for $v_e = 0.05\,c$ and mass ratio $R = 5$, with the constant mass flow rate $\dot{m}$ adjusted so that the spacecraft accelerates from rest to its maximum velocity (having exhausted all its fuel) at the destination.

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4. In section 3.2, a rocket with constant v_e and mass ratio $R = M_i/M_f$ first accelerates to Δv then decelerates, arriving at its destination at rest. Show that if the rocket turnaround point is placed midway between origin and destination, this requires that $\dot{m}$ (the mass ejection rate) in the acceleration phase be R times larger than $\dot{m}$ in the deceleration phase.

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