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Finite entropy for multidimensional cellular automata

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Abstract. Let $X = S^{\mathbb{G}}$ where $\mathbb{G}$ is a countable group and S is a finite set. A cellular automaton (CA) is an endomorphism $T : X \to X$ (continuous, commuting with the action of $\mathbb{G}$). Shereshevsky [Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms. *Indag. Math. (N.S.)* **4**(2) (1993), 203–210] proved that for $\mathbb{G} = \mathbb{Z}^d$ with $d > 1$ no CA can be forward expansive, raising the following conjecture: for $G = \mathbb{Z}^d$, $d > 1$, the topological entropy of any CA is either zero or infinite. Morris and Ward [Entropy bounds for endomorphisms commuting with K actions. *Israel J. Math.* **106** (1998), 1–11] proved this for linear CAs, leaving the original conjecture open. We show that this conjecture is false, proving that for any d there exists a d-dimensional CA with finite, non-zero topological entropy. We also discuss a measure-theoretic counterpart of this question for measure-preserving CAs.

1. *Introduction*

Cellular automata form a class of dynamical systems that have been extensively studied since the 1940s, going back to some work of von Neumann [**15**] and others. For some survey papers on different aspects of cellular automata see [**2, 3, 9, 16**]. The following is an intuitive description of a cellular automaton. Consider an infinite mesh of finite state machines, called *cells*, interconnected locally with each other. These cells change their states synchronously depending on the states of some nearby cells, as determined by a local update rule. All the cells use the same update rule so that the system is homogeneous.

The *entropy* of a cellular automaton is, informally, the rate of information flow required to describe the long-term evolution of any finite number of cells. If the mesh of cells is a one-dimensional array (bi-infinite or one-sided), a simple argument shows that the entropy must be finite. Intuitively, when the cells are arranged in a lattice of dimension 2 or more, it can be expected that the entropy is either zero (the evolution of a finite number of cells is eventually determined by the history of these cells) or infinite (the rate of information to describe the evolution of a set F of cells grows to infinity with the cardinality of F). This intuition, backed up by some supporting results, gave rise to the following conjecture [**4, 11**], attributed to Shereshevsky.

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CONJECTURE 1.1. *A d-dimensional cellular automaton, with $d > 1$, cannot have finite, positive entropy.*

The main result of this paper is a counter-example to this conjecture. For any $d \geq 2$ we describe a $\mathbb{Z}^d$ cellular automaton and give non-trivial bounds in its topological entropy. An important tool implied in this construction is a certain aperiodic set of tiles, associated to a substitution system. The cellular automaton we describe was introduced by Kari [**8**] for $d = 2$, to prove certain undecidability results on cellular automata.

The paper is organized as follows. In §2 we introduce notation and give brief definitions of cellular automata, subshifts and entropy. Section 2 concludes with previous results and background on the entropy problem for cellular automata. Section 3 consists of a construction of an automaton with non-trivial topological entropy, along with entropy estimations. Section 4 contains an explanation of a key ingredient for this construction—the so-called 'Hilbert tiles'. Some tools developed in §§3 and 4 are applied in §5 to study measure-theoretic entropy of surjective cellular automata. The last section of this paper contains some further comments and open questions.

2. *Preliminaries and background*

2.1. *Subshifts and cellular automata.* Let S be a finite set, which we call the possible *states* of each cell. A *configuration* is an element $x \in S^{\mathbb{G}}$, where $\mathbb{G}$ is a countable set. For $n \in \mathbb{G}$ we denote by x_n the state of the cell at coordinate n in the configuration x. If $F \subset \mathbb{G}$, we denote by x_F the restriction of x to F—the states of those cells located in F. We put the product topology on $S^{\mathbb{G}}$, which is compact and metrizable. If $\mathbb{G}$ is a semigroup, we can define an action σ of the group $\mathbb{G}$ on X by translations: for $h, g \in G$ and $x \in S^{\mathbb{G}}$, $(\sigma_g x)_h = x_{hg}$. Each map σ_g is a continuous transformation of X, and if $\mathbb{G}$ is a group, $\{\sigma_g : g \in \mathbb{G}\}$ are homeomorphisms. The classic cases are $\mathbb{G} = \mathbb{Z}$ and $\mathbb{G} = \mathbb{N}$, in which case the action σ is generated by one transformation, called the *shift map*. In this case, $S^{\mathbb{Z}}$ (or $S^{\mathbb{N}}$) is called a *full shift* space (one-sided or two-sided, respectively). If $\mathbb{G} = \mathbb{Z}^d$ where $d \geq 1$ then $S^{\mathbb{Z}^d}$ is a *d-dimensional full shift*. A subset $X \subset S^{\mathbb{Z}^d}$ which is closed and invariant to the action of σ is called a $\mathbb{Z}^d$-subshift. Any subshift can be described by a set of *forbidden blocks* $FB = \{b_1, \ldots b_n, \ldots\}$ where $b_i \in S^{F_i}$ and $F_i \subset \mathbb{Z}^d$ are finite sets as follows:

$$X = \{x \in S^{\mathbb{Z}^d} : x_{F_i+m} \neq b_i \ \forall\, m \in \mathbb{Z}^d, i \in \mathbb{N}\}.$$

A subshift X is called *a subshift of finite type* (SFT) if there exists a finite set FB of forbidden blocks for X. We say that a subshift $X \subset S^{\mathbb{Z}^d}$ is a *factor* of a subshift $Y \subset \tilde{S}^{\mathbb{Z}^d}$ (or Y *extends* X) if there exists a surjective map $\pi : Y \to X$ which is continuous and commutes with the shift action σ.

For a countable group $\mathbb{G}$, a *$\mathbb{G}$-cellular automaton* is a pair $(S^{\mathbb{G}}, T)$ where $T : S^{\mathbb{G}} \to S^{\mathbb{G}}$ is a continuous transformation which commutes with the $\mathbb{G}$-action σ of translations. We abbreviate CA for cellular automaton. The transformation T is always of the form $(Tx)_n = t(x_{F+n})$ where $F \subset \mathbb{G}$ is a finite set, and $t : S^F \to S$ is called the *local rule* of the CA. Mostly, we restrict attention to the case $\mathbb{G} = \mathbb{Z}^d$.

Often a wider definition of a $\mathbb{Z}^d$-CA is used, and any continuous, shift commuting transformation of an SFT is called a $\mathbb{Z}^d$-CA. In this paper we deal specifically with CA on a full shift.

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2.2. *Topological and measure-theoretic entropy.* Let (X, T) be a CA with $X = S^{\mathbb{G}}$, and F a finite subset of $\mathbb{G}$. We denote

$$W(F, n, T) := \{((x)_F, (Tx)_F, \ldots, (T^{n-1}x)_F) : x \in X\},$$

i.e. $W(F, n, T)$ is the set of possible configurations for cells inside F along n iterations of T.

The *topological entropy* of (X, T) is

$$h_{\text{top}}(X, T) = \sup_{F \subset \mathbb{G}} \lim_{n \to \infty} \frac{1}{n} \log |W(F, n, T)|,$$

where the supremum above is over all finite subsets F of $\mathbb{G}$, and the limit above exists by subadditivity of the sequence $\log |W(F, n, T)|$.

For $w = (w_0, \ldots, w_{n-1}) \in W(F, n, T)$, let $[w] = \{x \in X : (T^i x)_F = w_i, 0 \leq i < n\}$. Let $\mathcal{P}(X, T)$ be the set of probability measures on the Borel σ-algebra $\mathcal{B}$ of X which are invariant to T ($\mu(T^{-1}B) = \mu(B)$ any Borel $B \subset X$, for $\mu \in \mathcal{P}(X, T)$). The measure-theoretic entropy of (X, μ, T) is

$$h_\mu(X, \mathcal{B}, T) = \sup_{F \subset \mathbb{G}} \lim_{n \to \infty} \frac{1}{n} H_n,$$

with

$$H_n = - \sum_{w \in W(F,n,T)} \mu([w]) \log(\mu([w])),$$

where the supremum is again over all finite subsets F of $\mathbb{G}$, and the limit above exists by subadditivity of the sequence H_n. Both the quantities $h_{\text{top}}(X, T)$ and $h_\mu(X, \mathcal{B}, T)$ indicate the 'information flow' of the long-term evolution of any finite number of cells by T. The conceptual difference is that the topological entropy gives the rate of information required to describe any evolution of finitely many cells, where as $h_\mu(X, \mathcal{B}, T)$ gives the rate of information required to describe a 'typical' evolution, where 'typical' is with respect to the probability μ.

An important result connecting the topological entropy and the measure-theoretic entropy, known as the *variational principle*, states that

$$h_{\text{top}}(X, T) = \sup_{\mu \in \mathcal{P}(X,T)} h_\mu(X, \mathcal{B}, T).$$

Entropy of a transformation (topological or measure-theoretic) is defined in a broader context. See [**5**] for definitions, proofs and a detailed discussion in the context of compact spaces.

2.3. *Substitution systems and tilings.* A $\mathbb{Z}^d$*-tiling system* consists of a finite set S of 'square tiles' and some adjacency rules $R \subset S^F$ with $F = \{0, \pm e_1, \ldots, \pm e_d\}$, which determine when a tile s_1 is allowed to be placed next to a tile s_2 (and in which directions). A configuration $x \in S^F$ for some $F \subset \mathbb{Z}^d$ is *valid* at $n \in F$ if the neighbors of the cell at n obey the adjacency rules: $x_{F+n} \in R$. Evidently, the set of infinite valid configurations in $S^{\mathbb{Z}^d}$ are an SFT. We call this SFT the associated subshift of (S, R) and denote it by X_R.

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A $\mathbb{Z}^d$-*substitution system* consists of a finite set S and a substitution rule ρ which is a function $\rho : S \to S^{F_k}$ where $F_k = \{1, \dots, k\}^d$. The function ρ naturally extends to a map $\rho^n : S \to S^{F_{k^n}}$ by applying n iterations of ρ, and to a map $\rho : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$. For $F \subset \mathbb{Z}^d$ and a configuration $x_F \in s^F$, we say that x_F is *admissible* for the substitution system (S, ρ), if it appears as a subconfiguration of $\rho^n(s)$ for some $s \in S$ and $n \in \mathbb{N}$. To a substitution system (S, ρ) there is an associated subshift X_ρ, whose forbidden blocks are the non-admissible blocks for (S, ρ).

We say that a substitution system has the *unique derivation property* if for any $x \in X_\rho$ there exist a unique $n \in F_k$ and $y \in X_\rho$ such that $x = \sigma_n \rho(y)$. Equivalently, this means that, given a part of a finite configuration $x \in S^{F_{k^n}}$ such that $x = \rho^n(s)$ for some $s \in S$ which has been shifted, the shift can be recovered uniquely modulo $k\mathbb{Z}^d$. Mozes [**12**], based on some earlier work of Robinson [**13**], devised an algorithm for implementing a subshift associated with a substitution system via local constraints—a tiling system. Namely, Mozes proved the following theorem.

THEOREM 2.1. [**12**, Theorem 4.5] *Let $X_\rho \subset S^{\mathbb{Z}^d}$ be the $\mathbb{Z}^d$-subshift associated to a substitution rule $s : S \to S^{F_k}$ with unique derivation. If $d > 1$, then there exists a tiling system $(\tilde{S}, R)$ such that the associated subshift X_R extends X_ρ.*

To be precise, Mozes proved this theorem for $d = 2$, but the proof extends to any $d > 1$. In fact, Goodman-Strauss [**6**] proved a more general result, relaying on Mozes' techniques, which implies Theorem 2.1 for any $d > 1$. For $d = 1$, Theorem 2.1 does not hold: there are classic examples of one-dimensional substitution systems which are not factors of an SFT.

2.4. *The CA entropy problem.* We turn to explain some motivation for Conjecture 1.1. A $\mathbb{G}$-CA $(S^{\mathbb{G}}, T)$ is *expansive* if there exists some finite $F \subset \mathbb{G}$ such that for any $x \neq y \in S^{\mathbb{G}}$ there exists $n \geq 0$ such that $(T^n x)_F \neq (T^n y)_F$. Expansiveness of a CA means that, by observing the states of some finite number of cells under iterations of F, we can eventually distinguish between any two configurations. It can be shown that an expansive CA always has finite, non-zero topological entropy. Shereshevsky [**14**] proved that a $\mathbb{Z}^d$-CA with $d > 1$ cannot be expansive, which gave raise to Conjecture 1.1, stated in the introduction. Motivated by this, Morris and Ward [**11**] proved a result on entropy of group automorphisms, which implies this conjecture for some subclass of cellular automata called *linear cellular automata*. This result has been refined [**4**], to show that linear CA are either 'sensitive to initial conditions', in which case the entropy is infinity, or equicontinuous, in which case the entropy is zero. Lakshtanov and Langvagen [**10**] prove that any multidimensional cellular automaton which admits a *spaceship* has infinite entropy. A *spaceship* for a CA (X, T) is a configuration $x \in X$ which differs from a σ-invariant configuration in only finitely many cells, and such that $T^k x = \sigma_n x$ for some $n \in \mathbb{Z}^d \setminus \{0\}$ and some integer $k > 0$, but x is not itself σ-invariant.

We point out that for any $d \geq 1$ there are trivial examples of continuous, shift commuting transformations $T : X \to X$ with finite, positive entropy where $X \subset S^{\mathbb{Z}^d}$ is an SFT.

Let

$$X = \{x \in S^{\mathbb{Z}^d} : x_n = x_{n+e_i} \ \forall\, 1 < i \leq d\},$$

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where $\{e_1, \ldots, e_d\}$ are the standard generators of $\mathbb{Z}^d$ and $T(x) = \sigma_{e_1} x$ is the shift in the direction of e_1. In this case, the example is essentially one-dimensional, and the topological entropy of T is $\log |S|$.

3. *Surjective $\mathbb{Z}^d$ CA with finite non-zero entropy*

In this section we present a construction of a surjective $\mathbb{Z}^d$-CA, associated with a set of directed tiles S and a finite group Γ. This is a simple generalization of Kari's CA [**8**]. We prove that, if S has some special properties, the associated CA has non-trivial entropy. The existence of a set of directed tiles with the required properties is proved in §4.

A set of *directed tiles* is a tiling system with a forward direction $d(s) \in \{\pm e_1, \ldots, \pm e_d\}$ associated to each tile $s \in S$. Given a configuration $x \in S^{\mathbb{Z}^2}$, a path defined by x is a sequence $p_1, p_2, \ldots$ with $p_n \in \mathbb{Z}^d$ obtained by traversing the forward directions of x: $p_{n+1} = p_n + d(x_{p_n})$. Given $x \in S^{\mathbb{Z}^2}$, a path $\{p_n\}$ is valid if x is valid at every p_n. A set of directed tiles S such that no valid path in $x \in S^{\mathbb{Z}^d}$ forms a loop is called an *acyclic* set of tiles.

Fix a finite group Γ. The group operation of Γ is written here in additive notation. Let S be a set of directed tiles. Recall that, for $s \in S$, $d(s)$ denotes the forward direction of the tile s.

Define $T_{S,\Gamma} : X \to X$ with $X = (S \times \Gamma)^{\mathbb{Z}^d}$ as follows:

$$T_{S,\Gamma}(x, y)_n := \begin{cases} (x_n, y_n + y_{n+d(x_n)}) & \text{if } x \text{ is valid at } n, \\ (x_n, y_n) & \text{otherwise.} \end{cases} \tag{3.1}$$

In this cellular automaton cells 'transmit information' along valid paths. We prove that when the number of forward-infinite valid paths is bounded, the topological entropy is finite.

For $x \in X$, denote by $\gamma(x)_n$ the Γ-part of the state of the cell at n in x, and by $s(x)_n$ the S-part of the state.

LEMMA 3.1. *With S an acyclic set of tiles as above, and $T_{S,\Gamma}$ defined according to equation (3.1), $T_{S,\Gamma}$ is surjective.*

Proof. It is sufficient to prove that the image of $T_{S,\Gamma}$ is dense in X, that is, to any finite $F \subset \mathbb{Z}^d$ and any $y \in X$ there is some $x \in X$ such that $(T_{S,\Gamma} x)_F = y_F$. Let $y \in X$ and $F \subset \mathbb{Z}^d$ be a finite set. We describe $x \in X$ as follows. Set $s(x)_n = s(y)_n$ for any $n \in \mathbb{Z}^d$. For $n \in \mathbb{Z}^d \setminus F$, set $\gamma(x)_n = 0$ (the identity of the group Γ). It remains to define the $\gamma(x)_n$ for $n \in F$. We define $\gamma(x)_n$ iteratively, according to the length of maximal valid path in x beginning at n, which remains in F. Since S is acyclic and F is finite, any such path must be finite. If x is not valid at n, or the successor of n in x is not in F, define $\gamma(x)_n = \gamma(y)_n$. Suppose now that the length of the maximal valid path in x beginning at n is $k > 1$, and we have defined $\gamma(x)_{n+d(x_n)}$, So we define $\gamma(x)_n = \gamma(y)_n - \gamma(x)_{n+d(x_n)}$. It can now be verified that indeed $(T_{S,\Gamma} x)_F = y_F$. □

Suppose S is an acyclic set of tiles, and $w \in S^{\mathbb{Z}^d}$. Let $X_w = \{(x, y) \in S^{\mathbb{Z}^d} \times \Gamma^{\mathbb{Z}^d} : x = w\}$ and identify it with naturally as a subset of X. The set X_w is a closed subset of

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X, and is invariant under $T_{S,\Gamma}$. Define a directed graph $G_w = (V_w, E_w)$ with vertex set $V_w = \mathbb{Z}^d$ and the edges

$$E_w = \{(n, n + d(w_n)) : n \in \mathbb{Z}^d, w \text{ is valid at } n\}.$$

We say that $K \subset V_w$ is *connected* in G_w if for any two cells $a, b \in K$ there exists a common $c \in V_w$ with directed paths in G_w from a to c and from b to c. In other words, the maximal forward-paths of a and b eventually coincide for every $a, b \in K$. A *connected component* of G_w is a connected set in G_w which is maximal with respect to inclusion. As G_w is an acyclic directed graph with out-degree at most 1, it is a disjoint union of its connected components. A connected component K in G_w is *forward-infinite* if, starting at some (hence any) $a \in K$, there exists a forward-infinite directed path in G_w. Note that an infinite connected component is not necessarily forward-infinite.

It follows from the definition in (3.1) and the above discussion that for any connected component K of G_w, and any $x \in X_w$, the CA $T_{S,\Gamma}$ acts on x_K independently of the states outside K.

Fix $w \in S^{\mathbb{Z}^d}$. Suppose G_w has c forward-infinite connected components, denoted by $K_1, \ldots, K_c$. Let $K_0 = \mathbb{Z}^d \setminus \bigcup_{i=1}^{c} K_c$ denote the union of cells which are not part of a forward-infinite connected component of G_w. Let (X_i, T_i) denote the system corresponding to those cells in K_i with the action of $T_{S,\Gamma}$. The systems (X_i, T_i) are factors of $(X_w, T_{S,\Gamma})$.

For $w \in S^{\mathbb{Z}^d}$, denote by $I(w)$ the number of forward-infinite connected components of G_w, which is equal to the maximal number of pairwise disjoint, forward-infinite valid paths in w. For a directed set of tiles S, let

$$I(S) = \sup_{w \in S^{\mathbb{Z}^d}} I(w).$$

LEMMA 3.2. *Let (X_0, T_0) be the system corresponding to the cells which are not part of a forward-infinite connected component; then $h_{\text{top}}(X_0, T_0) = 0$.*

Proof. Since any valid path in X_0 is forward-finite, a simple induction on the length of the path shows that the state of the cells in a valid path of length at most k is $|\Gamma|^k$-periodic. The system (X_0, T_0) is isomorphic to an inverse limit of periodic systems: the kth system in this sequence consists of the cells in K_0 with forward-path in G_w of length at most k. Thus, (X_0, T_0) has 0 topological entropy. □

LEMMA 3.3. *Fix $w \in S^{\mathbb{Z}^d}$ and let X_w, G_w and $T_{S,\Gamma}$ be as above. If G_w contains exactly c forward-infinite connected components, then $h_{\text{top}}(X_w, T_{S,\Gamma}) = c \log |\Gamma|$.*

Proof. With the above notation, since T acts independently on cells in different connected components, $(X_w, T_{S,\Gamma})$ is isomorphic to $\prod_{i=0}^{c}(X_i, T_i)$, and so

$$h_{\text{top}}(X_w, T_{S,\Gamma}) = \sum_{i=0}^{c} h_{\text{top}}(X_i, T_i).$$

By Lemma 3.2, the topological entropy of (X_0, T_0) is 0.

We now show that each of the other systems (X_i, T_i) has topological entropy $\log |\Gamma|$. Let $K = K_i$ be a forward-infinite component in G_w. Let $J_0 = \{p_1, p_2, \ldots\} \subset \mathbb{Z}^d$ be some

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forward-infinite path in G_w. Inductively define a sequence $J_n \subset K$ as follows. If $J_n = K$, let $J_{n+1} = J_n$; otherwise choose a cell a_n in $K \setminus J_n$ whose successor in G_w is in J_n and let $J_{n+1} = J_n \cup \{a_n\}$. We see that $J_0 \supseteq J_1 \supseteq \cdots J_n \supseteq \cdots$ and $K = \bigcap_{n\geq 0} J_0$. For each $n \geq 0$, T acts on the cells in J_n independently of the other cells. Now, the restriction of T to J_0 is isomorphic to the $\mathbb{N}$-CA on $\Gamma^{\mathbb{N}}$ defined by: $(Tx)_n = x_n + x_{n+1}$. It clearly has topological entropy $\log|\Gamma|$. The restriction of T to the cells of J_n is a finite-to-one extension of the previous system, thus has equal entropy. The restriction of T to the cells of K is an inverse limit of these systems, so it also has topological entropy $\log|\Gamma|$. □

In the next section we describe Kari's directed set of tiles from [**8**], called S_H. The tile set S_H is acyclic and has $0 < I(S_H) < \infty$. In fact, we show that this construction can be carried out in any dimension $d > 1$. For this set of tiles S_H and some fixed group Γ, denote $T_H = T_{S_H,\Gamma}$. Assuming this, we have our main result.

THEOREM 3.4. *For any $d \geq 1$ there exists a surjective $\mathbb{Z}^d$-CA with positive, finite topological entropy.*

Proof. Let S_H be an acyclic set of directed tiles with $0 < I(S_H) < \infty$. By Lemma 3.3, this means that $\sup_{w\in S_H^{\mathbb{Z}^2}} h_{\text{top}}(X_w, T_H) = I(S_H)\log|\Gamma| < \infty$. Since $X = \biguplus_{w\in S_H^{\mathbb{Z}^2}} X_w$, we have that $h_{\text{top}}(X, T_H) = \sup_{w\in S_H^{\mathbb{Z}^2}} h_{\text{top}}(X_w, T_H)$. In §4 we will prove the existence of S_H as above, which will complete the proof. □

Before describing the set of tiles required to complete the proof of Theorem 3.4, we note that a slight modification of the above construction yields the following result about the possible values the topological entropy of a multidimensional CA can attain.

PROPOSITION 3.5. *For any $d \geq 1$, the set of entropies of surjective d-dimensional cellular automata is dense in* $[0, \infty)$.

Proof. Since a product of surjective d-dimensional cellular automata is also a surjective d-dimensional CA with topological entropy equal to the sum of the entropies, it is sufficient to prove that there exists a surjective d-dimensional CA with arbitrarily small positive topological entropy.

Retaining the notation from the beginning of this section, let Γ be some finite group, S an acyclic set of tiles with $0 < I(S) < \infty$ and $\mathbb{Z}_m$ the cyclic group of order m. We define a CA

$$T_{S,\Gamma,m} : (S \times \Gamma \times \mathbb{Z}_m) \to (S \times \Gamma \times \mathbb{Z}_m),$$

by

$$T_{S,\Gamma,m}(x, y, t)_n := \begin{cases} (x_n, y_n + y_{n+d(x_n)}, 1) & \text{if } x \text{ is valid at } n \text{ and } t = 0, \\ (x_n, y_n, t+1) & \text{otherwise.} \end{cases} \tag{3.2}$$

Evidently, $T_{S,\Gamma,m}$ is surjective (the proof is similar to Lemma 3.1). Also note that $T^m_{S,\Gamma,m}$ is conjugate to $T_{S,\Gamma} \times \text{id}$, and so

$$h_{\text{top}}(T_{S,\Gamma,m}) = \frac{1}{m} h_{\text{top}}(T_{S,\Gamma}).$$

Since $0 < h_{\text{top}}(T_{S,\Gamma}) < \infty$ and m was arbitrary, the proof of this proposition is complete. □

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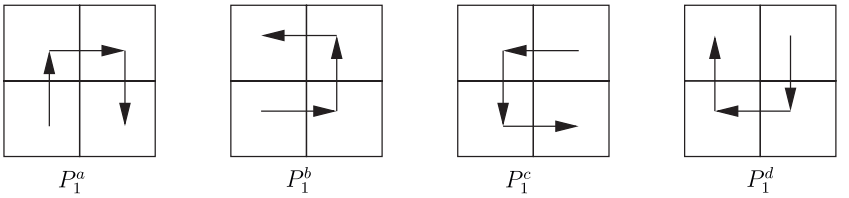


FIGURE 4.1. The four basic paths through the squares of size 2×2 tiles.

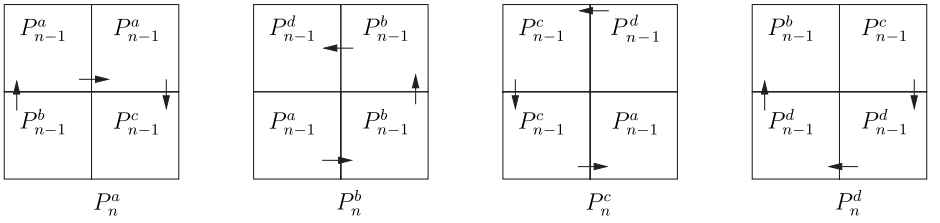


FIGURE 4.2. The inductive step n in the definition of a Hilbert path.

4. *Kari's tiles and Hilbert space filling paths*

In this section we describe Kari's set of directed tiles S_H, which has the properties announced above. In order to prove our main result, we only need two properties from this set of tiles S_H. The first property is $0 < I(S_H)$, that is, there exists $x \in S_H^{\mathbb{Z}^2}$ with a forward-infinite valid path. The second property, which is harder to prove, is $I(S_H) < \infty$. This property of S_H follows from a lemma proved by Kari (quoted here as Lemma 4.2). In the first part of this section (§4.1) we describe a substitution system which is associated with Kari's set of tiles. Section 4.2 contains a technical description of Kari's tiles, and it may be skipped by readers who are familiar with [**8**]. The last part of this section contains a proof that Kari's tiles have a bounded number of forward-infinite valid paths.

4.1. *Hilbert space filling paths.*

Before describing Kari's tiles, let us describe a certain substitution system, which is closely related with these tiles. Consider the discrete version of Hilbert's plane filling curve. This is a path in $\mathbb{Z}^2$ which, starting at $(0, 0)$, visits each point of the non-negative quarter of $\mathbb{Z}^2$ once. The following is an inductive definition of this path. Define four basic paths which visit each point in a $2^n \times 2^n$ square. We denote these paths by P_n^a, P_n^b, P_n^c and P_n^d. Path P_1^a is defined as $(0, 0) \to (0, 1) \to (1, 1) \to (1, 0)$. The other basic paths are obtained by rotations and reflections. These four basic paths are described in Figure 4.1. Path P_{n+1}^a is obtained by walking according to P_n^b, P_n^a, P_n^a and P_n^c. Similarly, paths P_{n+1}^b, P_{n+1}^c and P_{n+1}^d are defined inductively as shown in Figure 4.2. For example, paths P_1^d and P_2^a are illustrated in Figure 4.3.

A $\mathbb{Z}^d$-Hilbert path can also be defined in an analogous manner. The basic paths are obtained by traversing the vertices of a d-dimensional hypercube using Gray code. In dimension d there are $2d$ basic paths, corresponding to the d generators and their inverses (see Alber and Niedermeier [**1**] for more on $\mathbb{Z}^d$-Hilbert type paths). For simplicity, we first consider only the case $d = 2$.

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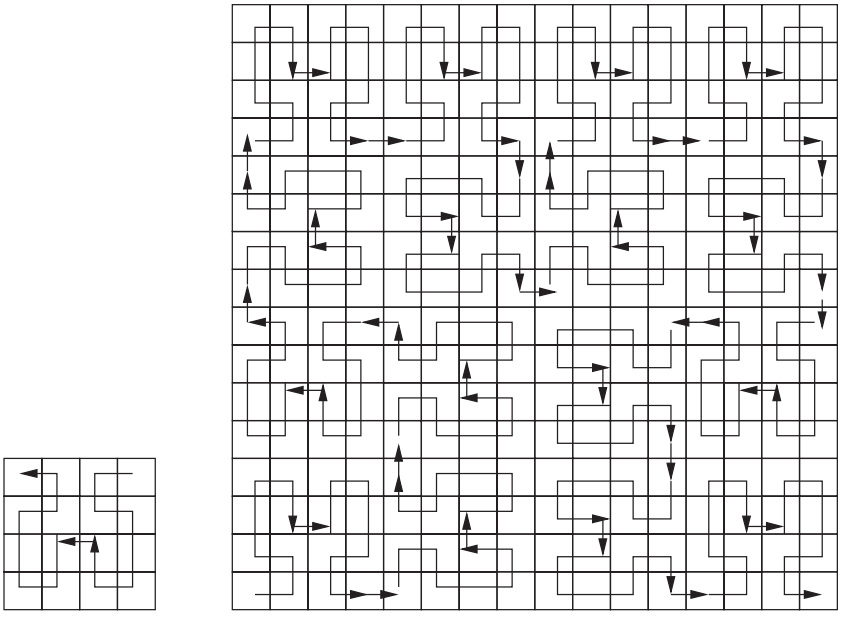


FIGURE 4.3. Hilbert curves for squares of sizes $2^2 \times 2^2$ and $2^4 \times 2^4$.

We can associate a substitution system to the Hilbert path. There are 12 tiles in this system, each corresponding to a basic path with the inward and outward directions indicated. The substitution rules reflect the inductive step of the definition of this path. The tiles and one of the rules of this substitution system are described in Figure 4.4. The other substitution rules follow in a symmetric manner, according to the definition of the Hilbert path.

LEMMA 4.1. *The substitution system associated with the Hilbert tiles (described in Figure 4.4) has unique derivation.*

Proof. It can be directly verified that any tile in the interior of a valid configuration is a part of a unique path of length 4 which fills a 2×2 square. Now consider a $\mathbb{Z}^2$ configuration in the associated subshift. The lattice $\mathbb{Z}^2$ is partitioned into 2×2 squares each of which is tiled by a path of length 4. This partition uniquely determines the substitution rule applied to derive the $\mathbb{Z}^2$ configuration. □

By the result of Mozes (Theorem 2.1), there exist a set of tiles which implements the Hilbert substitution system. We do not know, however, if any tiling system S which implements the Hilbert substitution system has the property that $0 < I(S) < \infty$. Kari's tiles, which are a specific set of tiles which implement this substitution system, do have this property. We suspect that by applying the algorithm described by Mozes in [**12**] on the Hilbert substitution system, one also ends up with a set of tiles with the required properties.

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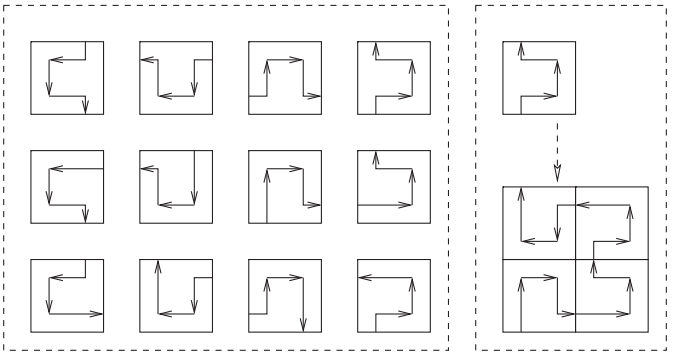


FIGURE 4.4. The tiles for the Hilbert substitution system, and a substitution rule.

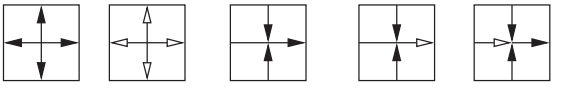


FIGURE 4.5. The five basic labels of tiles: (a) bold cross, (b) blank cross, (c) bold arm, (d) blank arm and (e) mixed arm.

4.2. *Description of Kari's tiles.* We now describe Kari's tiles, which we denote by S_H. This directed set of tiles has 'direction' labels, which are defined in such a manner that any valid path follows part of the Hilbert path. The construction is similar to an aperiodic set of tiles constructed by Robinson [**13**], which is also at the heart of Mozes' result [**12**].

Each tile has a *basic label* which is either a *blank cross*, a *bold cross*, a *blank arm*, a *bold arm*, or a *mixed arm*. The five types of basic labels are represented in Figure 4.5. A tile with a basic label which is an arm can face in one of the four main directions (north, south, east or west). Each cross also has an *orientation label* in NE, NW, SE, SW referring to the four cornerwise directions. The tiles labeled with an arm contain one arrow which faces outwards, and two arrows facing inwards. We call the arrow facing outwards a *principal arrow* and the arrows facing inwards *side arrows*. Each arrow in an arm also contains an orientation label in NE, NW, SE, SW. The principal arrow of an arm can be labeled with any of the four orientations.

There are restrictions on the orientation labels of the side arrows, determined by the direction of the arm, and the type of the arm (see Figure 4.6). For a bold arm in which the principal arrow is horizontal, there are two possibilities: either the upper side arrow is labeled with SE and the lower one with NE, or the upper arrow is labeled with SW and the lower one with NW. If the arm is vertical, then either the left arrow is NE and the right arrow is NW or the left arrow is SE and the right arrow is SW. The possible orientation labels for size arrows in bold arms are shown in Figure 4.6. The orientation labels for size arrows in a blank arm are similar to a bold arm (Figure 4.6). The possible orientation labels for arrows in mixed arms are shown in Figure 4.7.

Each arrow head also has a *Hilbert label* in $\{a, b, c, d\}$. The only restriction imposed on the Hilbert labels concerns crosses and mixed arms. In each cross all four arrow heads

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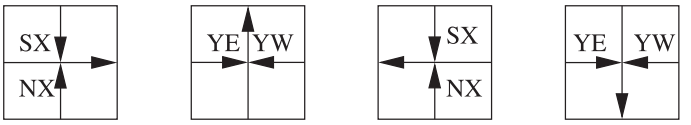


FIGURE 4.6. The possible orientation labels of the two side arrows of a bold arm. $X \in \{\text{E, W}\}$ and $Y \in \{\text{N, S}\}$.

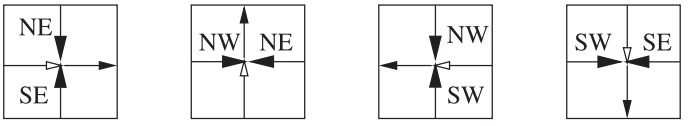


FIGURE 4.7. The possible orientation labels of the two side arrows of a mixed arm.

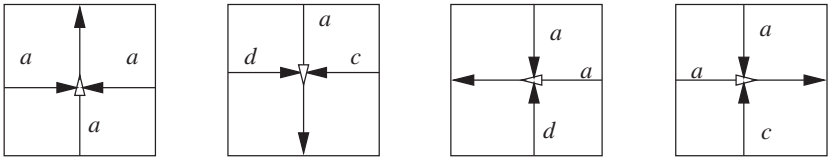


FIGURE 4.8. The labeling of mixed arms whose principal arrow has the label a.

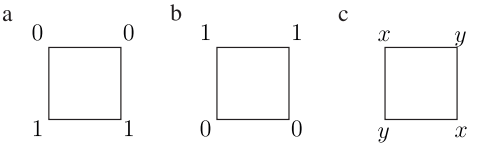


FIGURE 4.9. The possible corner parity labels on (a) horizontal arms, (b) vertical arms and (c) crosses ($x, y \in \{0, 1\}$).

must have the same Hilbert label. The Hilbert labels of the side arrows on the mixed arms are restricted according to the direction and Hilbert label of the principal arrow. The restrictions can be obtained from Figure 4.2 as follows. If the Hilbert label of the principal arrow is $x \in \{a, b, c, d\}$, and the principal arrow is facing right, then the upper side arrow has the label of the path in the upper right corner of the square of P^x_{n+1}, and the lower side arrow has the label of the path in the lower right corner of the square of P^x_{n+1}. For mixed arms facing the other directions, the restrictions are defined in a similar manner. For example, Figure 4.8 shows the allowed labels of side arrows for mixed arms whose primary arrow is labeled a.

Each tile has four *corner parity labels* in $\{0, 1\}$, each corresponding to a corner of the tile. In a tile which has a horizontal arm, the top corner labels are both 0, and the bottom corner labels are both 1. In a tile which has a vertical arm, the top corner labels are both 1 and the bottom corner labels are both 0. A tile with a cross has two possible labelings for the corner parities, with opposite corners having the same parity and adjacent corners having different parities. The possible corner parity labels are shown in Figure 4.9.

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There are two more labels for each tile: a *horizontal parity label* and a *vertical parity label*, both of which are in {0, 1}. The restrictions on these labels are as follows. For a blank cross, both these parity labels must be 0. For all other basic tiles (bold crosses and arms), at least one of these parity labels must be 1.

In addition each tile has a direction label, which is in {N, S, E, W}. These direction labels determine the direction function $d : S_H \to \{\text{N, S, E, W}\}$, which make S_H a directed set of tiles.

Let us describe the adjacency rules for these tiles, which define when a tiling is locally valid. For a configuration to be valid around a tile the following conditions must hold:

(1) for any arrow head facing outwards, the arrow head must meet an arrow tail;
(2) any arrow head facing inwards must have an arrow head which meets its tail;
(3) the orientation labels and Hilbert labels of meeting arrow heads and tails agree;
(4) corner parity labels of adjacent corners agree;
(5) both horizontal and vertical parity labels alternate;
(6) the direction label is valid, as explained below.

We now explain the rules which determine when the direction label is valid (rule 6). The direction label of a blank cross is valid if and only if one of the following conditions is fulfilled:

(1) the direction is N and either the NW neighbor is a bold cross with Hilbert label b or a vertical arm whose right side arrow has Hilbert label a or d;
(2) the direction is W and either the NW neighbor is a bold cross with Hilbert label d or a horizontal arm whose lower side arrow has Hilbert label c or b;
(3) the direction is S and either the SE neighbor is a bold cross with Hilbert label c or a vertical arm whose left side arrow has Hilbert label a or d;
(4) the direction is W and either the SE neighbor is a bold cross with Hilbert label a or a horizontal arm whose lower side arrow has Hilbert label c or b.

For a tile which is not a blank cross, the direction label is valid if it agrees with the direction label of a neighboring blank cross whose direction label points at this tile.

This completes the description of Kari's tiles and their adjacency rules.

It is possible to extend Kari's construction to a $\mathbb{Z}^d$ tiling with any $d > 2$. The details are lengthy but there are no conceptual difficulties. We comment that these extensions are related to 'multidimensional Hilbert paths'.

4.3. *Bounding the number of infinite valid paths.* To complete the proof of Theorem 3.4 for $d = 2$, it remains to show that $0 < I(S_H) < \infty$.

We now show that there exists an infinite valid path for S_H. By induction, we define sequences of valid S_H-configurations of squares of size $(2^{n-1} + 1) \times (2^{n-1} + 1)$, and denote these by $B_{XY}(n)$ with $X \in \{\text{N, S}\}$ and $Y \in \{\text{E, W}\}$. Thus $B_{XY}(0)$ is a blank cross with orientation label XY; and $B_{XY}(n + 1)$ is obtained by surrounding bold cross with orientation label XY by four configurations $B_{\text{SE}}(n)$, $B_{\text{SW}}(n)$, $B_{\text{NW}}(n)$, $B_{\text{NE}}(n)$, with arms labeled correctly as in Figure 4.10. The configuration $B_{XY}(2)$ with orientation labels on the crosses is illustrated in Figure 4.11. The reader can verify that it is possible to fill the parity labels and corner parity labels of these configurations so they remain valid. This was proved by Kari [**8**]. We now explain how to add Hilbert labels and directions to $B_{XY}(n)$

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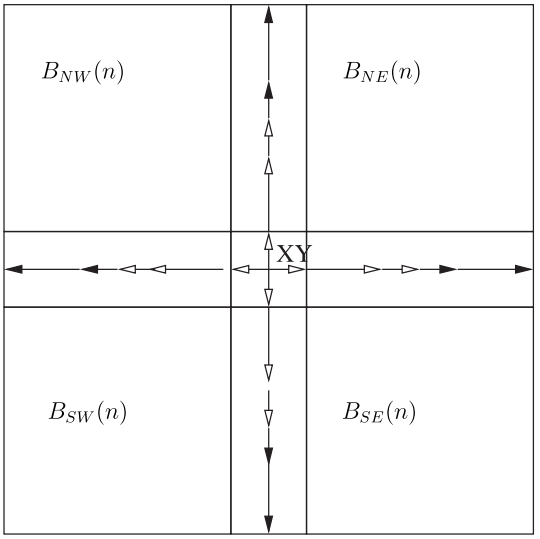


FIGURE 4.10. Inductive step in definition of $B_{XY}(n+1)$.

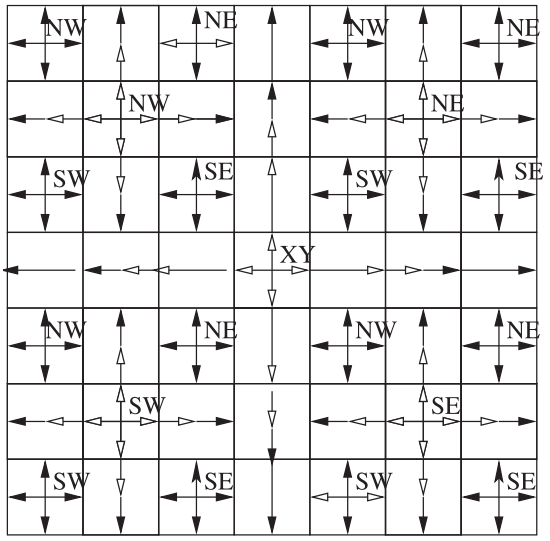


FIGURE 4.11. $B_{XY}(2)$ with orientation labels on the crosses.

so that the configuration remains valid. By induction on n, we can show that there exists a unique way to add these labels so that the central cross has Hilbert label $x \in \{a, b, c, d\}$. The Hilbert labels of each of the four surrounding squares $B_{XY}(n-1)$ are determined by the inductive definition of the Hilbert path (as in Figure 4.2). The labeling of the paths is determined by a Hilbert-path which traverses all blank crosses within this square. These configurations define arbitrarily long valid paths, and so by compactness there exists an infinite valid path for Kari's tiles (corresponding to a Hilbert path).

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Our next goal is to prove that $I(S_H) < \infty$. For this we quote the following technical lemma, about the structure of valid paths in the tiling system S_H.

LEMMA 4.2. (Kari [**8**, Lemma 5]) *For each $x \in S_H^{\mathbb{Z}^2}$, if $p_1, \ldots, p_N$ is a valid path in x, and $N \geq 2 \cdot 4^n$, then there are integers $1 \leq i < j \leq N$ with $j - i = 4^n$ so that the path $p_i, \ldots, p_j$ fills a $2^n \times 2^n$ square.*

Kari's proof of this lemma involves a delicate examination of the tiles S_H. We comment that it is easier to prove the corresponding statement about paths which are admissible configurations for the substitution system (S_H, ρ_H). However, applying Theorem 2.1 to deduce the result for the tiling system is non-trivial, since this lemma refers to configurations of which only some part (the path) is assumed to be valid. Using Lemma 4.2, we obtain the following result.

LEMMA 4.3. *Suppose $x \in S_H^{\mathbb{Z}^2}$ and the cells in $q_1, \ldots, q_N$ form a valid path in x. There exists a square $F \subset \mathbb{Z}^s$ centered at q_1 which contains all the cells of the path, such that the path fills up an ϵ-fraction of the cells in F. The constant ϵ is independent of x and of the path.*

Proof. First, some ad-hoc terminology is needed. If a sequence $q_1, \ldots, q_n$ of distinct points in $\mathbb{Z}^2$ has a subsequence $q_i, \ldots, q_{i+4^k}$ which fills a $2^k \times 2^k$ square, we say that $q_1, \ldots, q_n$ *sequentially contains a $2^k \times 2^k$-square*.

By applying Lemma 4.2, on every sub-interval length $2 \cdot 2^k$ for every $k \geq 0$, we deduce that the claim of this lemma follows from the following statement.

$\star$ Suppose $P = (p_1, \ldots, p_N)$ is a sequence of distinct points in $\mathbb{Z}^2$ with the property that, for any $k \geq 0$, any subsequence of P of length $2 \cdot 4^k$ sequentially contains a $2^k \times 2^k$ square. Then P intersects at most $C\sqrt{N}$ rows and at most $C\sqrt{N}$ columns in $\mathbb{Z}^2$, with $C \leq 34$ being a positive constant.

We now prove the statement $\star$. If $k \geq 0$ and $j \in [1 + 4^k, N - 4^k]$, then p_j is contained in a subsequence which fills a $2^k \times 2^k$ square. The contrary situation contradicts the fact that the sequence $(p_{j-4^k}, \ldots, p_{j+4^k-1})$ sequentially contains a $2^k \times 2^k$ square. Now denote by A_k the set of points in $\mathbb{Z}^2$ which are in the sequence P and are not part of a $2^k \times 2^k$ square subsequence. It follows that $|A_k| \leq 2 \cdot 4^k$, since $p_j \in A_k$ implies $j \leq 4^k$ or $j > N - 4^k$.

Denote by n_k the number of rows in $\mathbb{Z}^2$ which P intersects at least 2^k times and at most $2^{k+1} - 1$ times. Thus, since the points of the sequence P are distinct:

$$\sum_{k \geq 0} 2^k n_k \leq N. \tag{4.1}$$

Since any point in a row which P intersects at least 2^k times and at most $2^{k+1} - 1$ times is in A_{k+1}, then $2^k n_k \leq |A_{k+1}|$. We conclude that $2^k n_k \leq 2 \cdot 4^{k+1}$, or equivalently that

$$n_k \leq 8 \cdot 2^k. \tag{4.2}$$

Setting $l = \lceil \log_4 N \rceil$, by inequality (4.1) and the fact that the n_k are non-negative, we have

$$2^{l+i} n_{l+i} \leq N, \tag{4.3}$$

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for every $i \geq 0$. By summing up inequalities (4.2) up to $k = l$ and inequalities (4.3), we obtain

$$\sum_{k\geq 0} n_k \leq \sum_{k=0}^{l} 2^{k+3} + 2^l \leq 17 \cdot 2^l \leq 34 \cdot \sqrt{N}.$$

Now note that the quantity $\sum_{k\geq 0} n_k$ is the number of rows which P intersects. Thus, repeating argument with rows replaced by columns, the proof of $\star$ is complete.

To see that the lemma follows from $\star$, note that by Lemma 4.2 any valid $q_1, \ldots, q_N$ path satisfies the conditions of $\star$. Thus, the path intersects at most $C\sqrt{N}$ rows and $C\sqrt{N}$ columns, and so it is contained in a square of volume C^2N which intersects q_1. It is therefore contained in a square of volume at most $4C^2N$ centered in q_1, of which it fills an ϵ-fraction, with $\epsilon \geq 1/(4C^2)$. □

We now use Lemma 4.3 to show that the number of forward-infinite valid paths in any configuration on the Hilbert tiles is bounded.

LEMMA 4.4. *For each $x \in S_H^{\mathbb{Z}^2}$ there exist at most $M = \lceil 1/\epsilon \rceil$ disjoint forward-infinite valid paths in x, where ϵ is any constant satisfying Lemma 4.3.*

Proof. Fix some $x \in S_H^{\mathbb{Z}^2}$. Suppose there are more then $M = \lceil 1/\epsilon \rceil$ forward-infinite valid paths in x, and fix cells $c_1, \ldots, c_{M+1}$ with c_i a cell in the ith path. For any sufficiently large N, by Lemma 4.3 each of these paths fills up an ϵ-fraction of the cells in a $2^N \times 2^N$ square centered around c_i. As N tends to ∞, each of these paths also fills almost an ϵ-fraction of the same square of size $2^N \times 2^N$. As the paths are disjoint, we reach a contradiction. □

It follows from Lemma 4.4 that $I(S_H) \leq M$. On the other hand, we have seen that $I(S_H) > 0$. This concludes the proof of Theorem 3.4. We remark that it can actually be proved by a more detailed examination of this system that there can be at most four disjoint forward-infinite valid paths in any $x \in S_H^{\mathbb{Z}^2}$. Also, there exist $x \in S_H^{\mathbb{Z}^2}$ with exactly four disjoint infinite valid paths in x. Using these observations one can exactly compute the topological entropy of S_H.

5. *Measure-theoretic entropy of surjective CA*

This section contains a discussion of surjective CAs as measure-preserving dynamical systems, and the measure-theoretic entropy of these systems with respect to a 'natural' measure.

Denote the symmetric Bernoulli measure by μ: the state of each cell is distributed uniformly and independently of the other cells. The following simple proposition was already noted in the earliest dynamical systems study of CA for the case $\mathbb{G} = \mathbb{Z}$ (see Hedlund's fundamental paper [**7**]).

PROPOSITION 5.1. *If $\mathbb{G}$ is an amenable group, any surjective $\mathbb{G}$-CA is measure-preserving with respect to the symmetric Bernoulli measure μ.*

Proof. First, note that the action $\sigma_g : S^{\mathbb{G}} \to S^{\mathbb{G}}$ of $\mathbb{G}$ on $S^{\mathbb{G}}$ by translations preserves μ. The measure-theoretic entropy of this $\mathbb{G}$-action is equal to the topological entropy of

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this action, and μ is the unique σ-invariant measure with this property—μ is the unique probability measure of maximal entropy for σ. Now consider the set $\mathcal{S} = \{\nu \in \mathcal{P}(S^{\mathbb{G}}, \mathcal{B}, \sigma) : \mu = \nu \circ T^{-1}\}$ of σ-invariant probability measures. This set $\mathcal{S}$ is non-empty since $\mathbb{G}$ is amenable. For any $\nu \in \mathcal{S}$, $T : (S^{\mathbb{G}}, \mathcal{B}, \nu, \sigma) \to (S^{\mathbb{G}}, \mathcal{B}, \mu, \sigma)$ is a measure-theoretic factor map, and so $h_\nu(S^{\mathbb{G}}, \mathcal{B}, \sigma) \geq h_\mu(S^{\mathbb{G}}, \mathcal{B}, \sigma)$, but since μ is the unique measure of maximal entropy for σ, it follows that $\nu = \mu$. □

The closed support of μ is $S^{\mathbb{G}}$ for any countable $\mathbb{G}$, and so any continuous map which preserves μ must be surjective. It follows that, for an amenable group $\mathbb{G}$, a $\mathbb{G}$-CA is surjective if and only if it is μ-preserving.

It is interesting to note that the above characterization of surjective CA does not hold for general countable groups. Consider the free group on two generators, denoted by $\mathbb{F}_2$. We describe a surjective $\mathbb{F}_2$-CA which does not preserve the symmetric Bernoulli measure μ on $\{0, 1\}^{\mathbb{F}_2}$: denote two generators of $\mathbb{F}_2$ by a, b, and consider the CA $M : \{0, 1\}^{\mathbf{F}_2} \to \{0, 1\}^{\mathbf{F}_2}$ defined by the local rule determined by 'majority vote' of x_{wa}, x_{wb} and $x_{wa^{-1}}$.

PROPOSITION 5.2. *The cellular automaton $M : \{0, 1\}^{\mathbf{F}_2} \to \{0, 1\}^{\mathbf{F}_2}$ defined as above is surjective, but does not preserve μ.*

Proof. To see that M does not preserve μ, let

$$A = \{x \in \{0, 1\}^{\mathbf{F}_2} : x_{a^{-1}} \neq x_a\}.$$

Obviously, $\mu(A) = 1/2$. The pre-image of A under M is all the points $x \in \{0, 1\}^{\mathbf{F}_2}$ such that the state of the cell at the identity disagrees either with both the cells at a^{-2} and $a^{-1}b$ or with both the cells at a^2 and ab:

$$M^{-1}A = \{x_1 \neq x_{a^{-2}} = x_{a^{-1}b}\} \Delta \{x_1 \neq x_{a^2} = x_{ab}\}.$$

It follows that $\mu(A) = 1/2$ but $\mu(M^{-1}A) = 3/8$, so T does not preserve the measure μ.

It remains to prove that M is a surjective cellular automaton over $\mathbb{F}_2$; that is, for an arbitrary $y \in \{0, 1\}^{\mathbb{F}_2}$ find a point $x \in \{0, 1\}^{\mathbb{F}_2}$ with $Mx = y$. For an element $w \in \mathbb{F}_2$ denote by $|w|$ the minimal length of a word in $\{a, b, a^{-1}, b^{-1}\}^*$ which expresses w. Consider the directed graph G with vertex set $\mathbb{F}_2$ and edges

$$\{(w_1, w_2) \in \mathbb{F}_2 \times \mathbb{F}_2 : w_1^{-1} w_2 \in \{a, b, a^{-1}\}, |w_1| < |w_2|\}.$$

The graph G has no cycles, every $w \in F$ has at least two out-going edges and at most one in-going edge. A pre-image x can be found as follows. For every $w \in \mathbb{F}_2$ which has an in-going edge in G, let $p(w)$ denote the unique predecessor of w in G, and define $x_w = y_{p(w)}$. Define x_w in an arbitrary way for cells $w \in \mathbb{F}_2$ with no predecessor. Indeed $Mx = y$: for every $w \in \mathbb{F}_2$, at least two out of $x_{wa}, x_{wa^{-1}}, x_{wb}$ are equal to y_w. □

Since the symmetric Bernoulli measure is preserved by any surjective $\mathbb{Z}^d$-CA T, one can study the measure-theoretic entropy of T, which is bounded by the topological entropy.

PROPOSITION 5.3. *The $\mathbb{Z}^2$-CA associated with Kari's Hilbert-tiles which we denoted by T_H in §3 has measure-theoretic entropy zero with respect to the symmetric Bernoulli measure.*

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Proof. Recall that, for each $w \in S_H^{\mathbb{Z}^2}$, we denoted by X_w all the points $x \in X$ with directions determined by w. Since the sets X_w are all T_H-invariant, and form a partition of X, almost any ergodic component of μ with respect to T_H is contained in some X_w. Let F_1 and F_2 be squares centered around the origin of dimensions $2^N \times 2^N$ and $4 \cdot 2^N \times 4 \cdot 2^N$ respectively. Suppose there is a valid path starting inside F_1 and leaving F_2. By Lemma 4.2, such a path must fill a square of size $2^N \times 2^N$ contained inside F_2. But F_2 contains less than $16 \cdot 4^N$ squares of size $2^N \times 2^N$, and at least one of them must be valid. The μ-probability for a square of size $2^N \times 2^N$ to be valid is exponentially small in 4^N, and so the μ-probability of the event that a valid path starting in F_1 leaves F_2 tends to zero exponentially as $N \to \infty$. The Borel–Cantelli lemma implies that almost surely this does not happen for infinitely many N, and so with μ-probability 1 there is no infinite valid path. By Lemma 3.2, when w has no infinite valid path, μ_w is supported on a set with zero topological entropy. We conclude that μ-almost every ergodic component of μ has 0 measure-theoretic entropy. □

In a private communication, Ron Peled suggested that with an appropriate set of acyclic tiles, $T_{S,\Gamma}$ defined by equation (3.1) can be a (surjective) CA with finite non-zero measure-theoretic entropy with respect to μ. To see this, apply the $T_{S,\Gamma}$ construction with the acyclic tile set $S = \{\uparrow, \rightarrow\}$ with no adjacency restrictions. With probability 1 a configuration has one forward-infinite path, and any pair of forward paths eventually coincide. Using Lemma 3.3 we deduce that μ-almost any ergodic component of this CA has entropy $\log |\Gamma|$. On the other hand, this CA has infinite topological entropy.

6. *Concluding remarks and questions*

Our investigation of the existence of a multidimensional CA with finite non-zero entropy was motivated by the misleading intuition explained in the introduction. We conclude with some related questions and remarks.

Since the CA T_H described in §3 is not injective, a question which follows naturally is the following.

Question 6.1. Does there exist an automorphism of a $\mathbb{Z}^d$ full shift (an injective $\mathbb{Z}^d$-CA) with positive, finite topological entropy for $d > 1$?

The discussion of measure-theoretic entropy of surjective CA in §5 raises the following question.

Question 6.2. Does there exist a two-dimensional surjective cellular automaton with positive measure-theoretic entropy (with respect to the symmetric Bernoulli measure), and finite topological entropy?

One can study dynamical properties such as entropy of cellular automata over any countable group. We remark that, if $\mathbb{G}$ is a countable group which is not finitely generated, $X = \Sigma^G$ and $T = X \to X$ is a $\mathbb{G}$-cellular automaton, then $h(T) \in \{0, \infty\}$. Here is an explanation of this fact. Since T is given by some local rule, there exists some finitely generated subgroup $\mathbb{H} < \mathbb{G}$ such that each cell only interacts via T with other cells in the same $\mathbb{H}$-coset. Since $\mathbb{G}$ is not countably generated, $[\mathbb{G} : \mathbb{H}] = \infty$, and so (X, T)

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is conjugate to the infinite product $\prod_{a\in\mathbb{G}/\mathbb{H}}(S^{\mathbb{H}}, T)$, and so must have infinite or zero entropy. We ask the following question.

Question 6.3. Does there exist a finitely generated, countable group $\mathbb{G}$ such that any $\mathbb{G}$-CA has either zero or infinite topological entropy?

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References

[1] J. Alber and R. Niedermeier. On multidimensional curves with Hilbert property. *Theory Comput. Syst.* **33**(4) (2000), 295–312.
[2] S. Bandini, G. Mauri and R. Serra. Cellular automata: from a theoretical parallel computational model to its application to complex systems. *Parallel Comput.* **27**(5) (2001), 539–553 (Cellular Automata: From Modeling to Applications (Trieste, 1998)).
[3] F. Blanchard, P. Kurka and A. Maass. Topological and measure-theoretic properties of one-dimensional cellular automata. *Phys. D* **103**(1–4) (1997), 86–99 (Lattice Dynamics (Paris, 1995)).
[4] M. D'amico, G. Manzini and L. Margara. On computing the entropy of cellular automata. *Theoret. Comput. Sci.* **290**(3) (2003), 1629–1646.
[5] M. Denker, C. Grillenberger and K. Sigmund. *Ergodic Theory on Compact Spaces (Lecture Notes in Mathematics, 527)*. Springer, Berlin, 1976.
[6] C. Goodman-Strauss. Matching rules and substitution tilings. *Ann. of Math.* (2) **147**(1) (1998), 181–223.
[7] G. A. Hedlund. Endormorphisms and automorphisms of the shift dynamical system. *Math. Systems Theory* **3** (1969), 320–375.
[8] J. Kari. Reversibility and surjectivity problems of cellular automata. *J. Comput. System Sci.* **48**(1) (1994), 149–182.
[9] J. Kari. Theory of cellular automata: A survey. *Theoret. Comput. Sci.* **334**(1–3) (2005), 3–33.
[10] E. L. Lakshtanov and E. S. Langvagen. A criterion for the infinity of the topological entropy of multidimensional cellular automata. *Problemy Peredachi Informatsii* **40**(2) (2004), 70–72.
[11] G. Morris and T. Ward. Entropy bounds for endomorphisms commuting with K actions. *Israel J. Math.* **106** (1998), 1–11.
[12] S. Mozes. Tilings, substitution systems and dynamical systems generated by them. *J. Anal. Math.* **53** (1989), 139–186.
[13] R. M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Invent. Math.* **12** (1971), 177–209.
[14] M. A. Shereshevsky. Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms. *Indag. Math. (N.S.)* **4**(2) (1993), 203–210.
[15] J. von Neumann. *Theory of Self-Reproducing Automata*. University of Illinois Press, Urbana, IL, 1966.
[16] S. Wolfram. *A New Kind of Science*. Wolfram Media, Inc., Champaign, IL, 2002.

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