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RESEARCH ARTICLE

A Pride–Guba–Sapir exact sequence for the relation bimodule of an associative algebra

Benjamin Steinberg

Department of Mathematics, The City College of New York, Convent Avenue, New York, USA
Email: bsteinberg@ccny.cuny.edu

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Abstract

Given a presentation of a monoid M, combined work of Pride and of Guba and Sapir provides an exact sequence connecting the relation bimodule of the presentation (in the sense of Ivanov) with the first homology of the Squier complex of the presentation, which is naturally a $\mathbb{Z}M$-bimodule. This exact sequence was used by Kobayashi and Otto to prove the equivalence of Pride's finite homological type property with the homological finiteness condition bi-FP_3. Guba and Sapir used this exact sequence to describe the abelianization of a diagram group. We prove here a generalization of this exact sequence of bimodules for presentations of associative algebras. Our proof is more elementary than the original proof for the special case of monoids.

1. Introduction

Let K be a commutative ring. The relation bimodule arose out of the work of Bergman and Dicks on universal derivations [1], although the name seems to have been coined by Ivanov [5]. Let $\mathcal{P} = \langle X \mid R\rangle$ be a presentation of a monoid M. The Squier complex $D(\mathcal{P})$ [4, 8] is a certain 2-complex associated to the presentation with the property that homotopy classes of paths in the 2-complex correspond to isotopy classes of reduced monoid pictures (i.e., duals of van Kampen diagrams) for the presentation. The fundamental groups of the Squier complex are called diagram groups [4] and have been intensively studied in geometric group theory because they can be viewed as generalizations of Thompson's group F. Pride observed that $H_1(D(\mathcal{P}))$ has a natural $\mathbb{Z}M$-bimodule structure and denotes it $\pi_2^{(b)}(\mathcal{P})$. In the case of a group presentation, this bimodule corresponds in a natural way to π_2 of an Eilenberg–Mac Lane space for the group, whence the notation. Pride proved [8, Theorem 4.2] that there is an exact sequence $\pi_2^{(b)}(\mathcal{P}) \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Z}M \to \mathcal{M}^{(b)} \to 0$ where $\mathcal{M}^{(b)}$ is the relation bimodule of the presentation, and he asked whether the first map is injective. Guba and Sapir [4, Theorem 11.3] showed that $0 \to \pi_2^{(b)}(\mathcal{P}) \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Z}M \to \mathcal{M}^{(b)} \to 0$ is indeed exact, and used this to give a description of the abelianization of a diagram group.

The proof of Guba and Sapir is somewhat indirect. It suffices to show that the composition of the Hurewicz map on each diagram group with $H_1(D(\mathcal{P})) \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Z}M$ has kernel the commutator subgroup. Guba and Sapir first assume that R is a complete rewriting system and use their presentation for diagram groups of complete rewriting systems. Then they reduce the general case to this case using the Knuth–Bendix completion process and the behavior of diagram groups under Knuth–Bendix completion. Kobayashi and Otto [6] later gave a more homological proof of the exactness of this sequence, but they still used the Knuth–Bendix completion process.

Here I consider the more general situation of an associative K-algebra A given by a presentation $\mathcal{P} = \langle X \mid R\rangle$. I require A to be projective as a K-module, as is the case for a monoid ring KM. I define an A-bimodule $\pi_2(\mathcal{P})$ in analogy to the construction of $\pi_2^{(b)}(\mathcal{P})$. The main result gives an exact sequence

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$0 \to \pi_2(\mathcal{P}) \to A \otimes_K KR \otimes_K A \to \mathcal{M} \to 0$ where $\mathcal{M}$ is the relation bimodule of the presentation. The proof avoids using Knuth–Bendix completion or Gröbner bases. Instead, I use a long exact Tor-sequence and an explicit computation of the connecting map. Splicing this with the exact sequence $0 \to \mathcal{M} \to A \otimes_K KX \otimes_K A \to A \otimes_K A \to A \to 0$ of Bergman and Dicks [1], then yields the exact sequence $0 \to \pi_2(\mathcal{P}) \to A \otimes_K KR \otimes_K A \to A \otimes_K KX \otimes_K A \to A \otimes_K A \to A \to 0$.

In the final section, I recover the original exact sequence of Pride–Guba–Sapir by turning the monoid presentation into an algebra presentation and checking that the two notions of π_2 give the same bimodule. I also interpret $\pi_2^{(b)}(\mathcal{P})$ as the second homology of the two-sided Cayley complex of the monoid presentation $\mathcal{P}$ introduced in [3].

2. Relation bimodules

In what follows K is a commutative ring, and A is an associative K-algebra. Usually, A will be projective as a K-module. By an A-bimodule, we mean a left $A \otimes_K A^{op}$-module. We say that A is $bi\text{-FP}_n$ over K if A has a projective resolution as an A-bimodule that is finitely generated through degree n. We record here some elementary facts.

Proposition 2.1. *Let A be a K-algebra and I an ideal.*

1. *If B is a left A-module, $A/I \otimes_A B \cong B/IB$.*
2. *If B is a right A-module, $B \otimes_A A/I \cong B/BI$.*
3. *If B is an A-bimodule, then $A/I \otimes_A B \otimes_A A/I \cong B/(IB + BI)$.*
4. *If B is a flat right (respectively, left) A-module, then $B \otimes_A I \cong BI$ (respectively, $I \otimes_A B \cong IB$).*

Proof. The first two items are standard. The third follows from these as $(B/BI)/I(B/BI) = (B/BI)/((IB + BI)/BI) \cong B/(IB + BI)$. The final follows from tensoring B with the exact sequence $0 \to I \to A \to A/I \to 0$ and using the first/second items. □

The free monoid on a set X is denoted by X^*, and KX^* is then the free K-algebra generated by X. Following Ivanov [5], if A is X-generated and $J = \ker(KX^* \to A)$, then J/J^2 is an A-bimodule, termed the *relation bimodule*. Note that if $J = (R)$, then J/J^2 is generated by the $r + J^2$ with $r \in R$ as an A-bimodule. The relation bimodule was first studied by Bergman and Dicks [1], but not under that name.

Let $I = \ker(KX^* \otimes_K KX^* \xrightarrow{\mu} KX^*)$ be the kernel of the multiplication map. It is well known that I is a free KX^*-bimodule with basis the elements $x \otimes 1 - 1 \otimes x$ with $x \in X$. We give a quick sketch of the proof for completeness.

Proposition 2.2. *Let $I = \ker(KX^* \otimes_K KX^* \xrightarrow{\mu} KX^*)$. Then I is a free KX^*-bimodule with basis the elements $x \otimes 1 - 1 \otimes x$ with $x \in X$.*

Proof. First we show that I is generated by these elements. There is an isomorphism $K[X^* \times X^*] \to KX^* \otimes_K KX^*$ given by $(u, v) \mapsto u \otimes v$ for $u, v \in X^*$. Composing this with μ is the K-linear extension of the multiplication map $X^* \times X^* \to X^*$, which clearly has kernel with K-basis all differences $(u, v) - (1, uv)$ with $u, v \in X^*$ and $u \neq 1$. It follows that I has K-basis all elements of the form $u \otimes v - 1 \otimes uv$ with $u \neq 1$. But $u \otimes v - 1 \otimes uv = (u \otimes 1 - 1 \otimes u)v$, and so I is generated by all elements of the form $w \otimes 1 - 1 \otimes w$ with $1 \neq w \in X^*$ as a KX^*-bimodule. We show by induction on $|w|$ that $w \otimes 1 - 1 \otimes w$ is in the sub-bimodule generated by the $x \otimes 1 - 1 \otimes x$ with $x \in X$ with the case $|w| = 1$ trivial. If $w = xu$ with $x \in X$, then $w \otimes 1 - 1 \otimes w = xu \otimes 1 - 1 \otimes xu = x(u \otimes 1 - 1 \otimes u) + (x \otimes 1 - 1 \otimes x)u$ and the claim follows by induction.

Suppose now that B is a KX^*-bimodule and $f\colon X \to B$ is any map. Then

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$$A = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a \in KX^*, b \in B \right\}$$

is a K-algebra, and we can define a homomorphism $\psi : KX^* \to A$ such that

$$\psi(x) = \begin{bmatrix} x & f(x) \\ 0 & x \end{bmatrix}, \quad \psi(a) = \begin{bmatrix} a & d(a) \\ 0 & a \end{bmatrix}$$

for $x \in X$ and $a \in KX^*$. One easily verifies that d is a derivation, meaning that $d(aa') = ad(a') + d(a)a'$, and that $d(x) = f(x)$. Since $(a, a') \mapsto d(a)a'$ is clearly K-bilinear, we can define a K-linear map $KX^* \otimes_K KX^* \to B$ by $\alpha(a \otimes a') = d(a)a'$. We claim that $\alpha : I \to B$ is a KX^*-bimodule homomorphism with $\alpha(x \otimes 1 - 1 \otimes x) = f(x)$ for $x \in X$. This will complete the proof. Indeed, if $\sum u_i \otimes v_i \in I$, then $\sum u_i v_i = 0$. Trivially, $\alpha(\sum u_i \otimes v_i w) = \sum d(u_i)v_i w = d(\sum u_i \otimes v_i)w$. On the other hand, $\alpha(\sum w u_i \otimes v_i) = \sum d(wu_i)v_i = w \sum d(u_i)v_i + d(w) \sum u_i v_i = wd(\sum u_i \otimes v_i)$. Finally, $\alpha(x \otimes 1 - 1 \otimes x) = d(x) - d(1)x = f(x)$, as $d(1) = 0$. □

The following result is due to Bergman and Dicks [1]. We give a proof for completeness.

Theorem 2.3. *Let A be a K-algebra generated by X that is projective over K. Then there is an exact sequence*

$$0 \to J/J^2 \xrightarrow{d_2} A \otimes_K KX \otimes_K A \xrightarrow{d_1} A \otimes_A A \xrightarrow{\mu} A \to 0$$

where μ is the multiplication map, $d_1(1 \otimes x \otimes 1) = x \otimes 1 - 1 \otimes x$ and d_2 is induced by the derivation $KA^* \to A \otimes_K KX \otimes_K A$ given by $x_1 \cdots x_m \mapsto \sum_{i=1}^m (x_1 \cdots x_{i-1} + J) \otimes x_i \otimes (x_{i+1} \cdots x_m + J)$.

Proof. Since I is freely generated as a KX^*-bimodule by the elements of the form $x \otimes 1 - 1 \otimes x$ with $x \in X$, we have an exact sequence

$$0 \to KX^* \otimes_K KX \otimes_K KX^* \to KX^* \otimes_K KX^* \to KX^* \to 0 \tag{2.1}$$

where the first map takes $1 \otimes x \otimes 1$ to $x \otimes 1 - 1 \otimes x$ and the second map is the multiplication map. Since KX^* is free as a right KX^*-module, this sequence splits over $(KX^*)^{op}$, and so tensoring on the right with A yields the exact sequence $0 \to KX^* \otimes_K KX \otimes_K A \to KX^* \otimes_K A \to A \to 0$. Since A is projective over K, it follows that $KX^* \otimes_K KX \otimes_K A$ and $KX^* \otimes_K A$ are projective left KX^*-modules. Thus, tensoring on the left with $A = KX^*/J$ and recalling that $\mathrm{Tor}_1^{KX^*}(KX^*/J, KX^*/J) \cong J/J^2$ (cf. [2, Exercise 19, p. 126]), yields the exact sequence

$$0 \to J/J^2 \to A \otimes_K KX \otimes_K A \to A \otimes_K A \to A \to 0.$$

It is a straightforward exercise to check that the connecting map is given by d_2 above (or see [1]). □

In particular, if X is finite, then A is bi-FP$_2$ over K if and only if the relation bimodule is finitely generated. Next we shall need a result I learned of from Webb [9].

Proposition 2.4. *Let M be a left/right KX^*-module that is projective over K. Then* pd$M \leq 1$.

Proof. We just handle the case M is a left KX^*-module. Tensoring the split exact sequence of free right KX^*-modules (2.1) on the right with M yields an exact sequence $0 \to KX^* \otimes_K KX \otimes_K M \to KX^* \otimes_K M \to M \to 0$ of left KX^*-modules. Since M is a projective K-module, $KX^* \otimes_K KX \otimes_K M$ and $KX^* \otimes_K M$ are projective KX^*-modules. Therefore, pd$M \leq 1$. □

As a corollary, we can deduce that J is projective as both a left and right KX^*-module.

Corollary 2.5. *Let J be an ideal of KX^* and suppose that KX^*/J is projective over K. Then J is projective as both a left and right KX^*-module.*

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Proof. We handle the case of left KX^*-modules. We have an exact sequence $0 \to J \to KX^* \to KX^*/J \to 0$ with KX^* free. Also $\mathrm{pd}KX^*/J \leq 1$ by Proposition 2.4. It follows that $\mathrm{pd}J = 0$, that is, J is projective as a left KX^*-module. □

The next proposition will be used to show that J^m is projective as both a left and right module.

Proposition 2.6. *Let B, C be bimodules over a K-algebra A that are projective as both left and right A-modules. Then $B \otimes_A C$ is projective as both a left and right A-module.*

Proof. Suppose that $C \oplus C' = F$ with F a free left A-module. Then $(B \otimes_A C) \oplus (B \otimes_A C') \cong B \otimes_A F \cong \bigoplus B$. Thus $B \otimes_A C$ is projective as a left module (since B is). The other claim is dual. □

Corollary 2.7. *Let I be an ideal of K-algebra A that is projective as both a left and right A-module. If B is an A-bimodule that is projective as both a left and right A-module, then IB, BI are projective as both left and right A-modules. In particular, I^m is projective as both a left and right A-module for all $m \geq 1$.*

Proof. This follows from Proposition 2.1(4) and Proposition 2.6. □

3. The homotopy bimodule

A *presentation* of a K-algebra is a gadget $\mathcal{P} = \langle X \mid R\rangle$ where X is a set and $R \subseteq KX^*$. The presentation is finite if X and R are finite. A K-algebra A is *presented* by $\mathcal{P}$ if $A \cong KX^*/(R)$. As usual, we put $J = (R)$. We shall assume throughout this section that A is projective over K, and hence J is projective as both a left and right KX^*-module by Corollary 2.5. The relation bimodule J/J^2 is generated by the cosets $r + J$ with $r \in R$, whence A is bi-FP_2 if the presentation is finite. It follows there is an exact sequence $A \otimes_K KR \otimes_K A \to J/J^2 \to 0$, and we aim to compute the kernel. This was done in the case that A is an integral monoid ring and $\mathcal{P}$ comes from a monoid presentation in the work of Pride [8] and of Guba–Sapir [4], where it was shown that the kernel is the first homology of the Squier complex of the presentation. Here we construct the analog of the first homology of the Squier complex for any presentation of a K-algebra A that is projective over K. We then prove the analog of the Pride–Guba–Sapir exact sequence. In the next section, we shall derive anew the Pride–Guba–Sapir sequence in the monoid case from our more general result.

Since $J = (R)$, we have an exact sequence of KX^*-bimodules

$$0 \to H \to KX^* \otimes_K KR \otimes_K KX^* \xrightarrow{\alpha} J \to 0. \tag{3.1}$$

To keep R and its image in KX^* formally distinct, we write $\overline{r}$ for the basis element of KR corresponding to $r \in R \subseteq KX^*$. Then $\alpha(u \otimes \overline{r} \otimes v) = urv$. Note that since J is projective as both a left and right KX^*-module by Corollary 2.5, the sequence (3.1) splits when viewed as either a sequence of left or of right KX^*-modules. Moreover, since $KX^* \otimes_K KR \otimes_K KX^*$ is free as both a left and right KX^*-module, we deduce that H is projective as both a left and right KX^*-module.

If $r_i, r_j \in R$ and $u \in X^*$, then we put $[r_i, u, r_j] = 1 \otimes \overline{r}_i \otimes ur_j - r_iu \otimes \overline{r}_j \otimes 1 \in KX^* \otimes_K KR \otimes_K KX^*$. Note that $\alpha([r_i, u, r_j]) = r_iur_j - r_iur_j = 0$, and so $[r_i, u, r_j] \in H$. Let D be the sub-KX^*-bimodule of H generated by $\{[r_i, u, r_j] \mid r_i, r_j \in R, u \in X^*\}$. Note that D is spanned as a K-module by the elements $v[r_i, u, r_j]w$ with $v, w \in X^*$.

Proposition 3.1. *There is a containment $JH + HJ \subseteq D$, and hence H/D is an A-bimodule.*

Proof. We claim that if $r_i \in R$ and $h \in KX^* \otimes_K KR \otimes_K KX^*$, then $r_ih + D = 1 \otimes \overline{r}_i \otimes \alpha(h) + D$ and $hr_i + D = \alpha(h) \otimes \overline{r}_i \otimes 1 + D$. Since $H = \ker \alpha$, it will then follow that $JH + HJ \subseteq D$. We just prove the

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first equality, for which it suffices to handle the case $h = u \otimes \bar{r}_j \otimes v$ with $u, v \in X^*$ and $r_j \in R$. Then $r_i h + [r_i, u, r_j]v = r_i u \otimes \bar{r}_j \otimes v + 1 \otimes \bar{r}_i \otimes u r_j v - r_i u \otimes \bar{r}_j \otimes v = 1 \otimes \bar{r}_i \otimes \alpha(h)$, as was required. □

The A-bimodule H/D will be denoted $\pi_2(\mathcal{P})$ in analogy to the corresponding notation used by Pride in the case of integral monoid rings and monoid presentations.

The crucial idea is now the following. The exact sequence (3.1) splits over $(KX^*)^{op}$, and so tensoring on the right with $A = KX^*/J$ yields an exact sequence $0 \to H/HJ \to KX^* \otimes_K KR \otimes_K A \to J/J^2 \to 0$. Now tensoring on the left with A and observing that $KX^* \otimes_K KR \otimes_K A$ is a projective KX^*-module, because A is projective over K, yields an exact sequence $0 \to \mathrm{Tor}_1^{KX^*}(KX^*/J, J/J^2) \to H/(JH + HJ) \to A \otimes_K KR \otimes_K A \to J/J^2 \to 0$. Note that $0 \to J^2 \to J \to J/J^2 \to 0$ is a projective resolution of J/J^2 as a left KX^*-module by Corollary 2.7, and so $\mathrm{Tor}_1^{KX^*}(KX^*/J, J/J^2) = \ker(J^2/J^3 \to J/J^2) = J^2/J^3$. We will compute the connecting map $J^2/J^3 \to H/(JH + HJ)$ and see that its image is $D/(JH + HJ)$. This will then yield the exact sequence $0 \to \pi_2(\mathcal{P}) \to A \otimes_K KR \otimes_K A \to J/J^2 \to 0$, that we desire.

Theorem 3.2. *Let A be a K-algebra that is projective over K given by a presentation $\mathcal{P} = \langle X \mid R \rangle$. Let $J = (R)$. Then there is an exact sequence of A-bimodules $0 \to \pi_2(\mathcal{P}) \to A \otimes_K KR \otimes_K A \to J/J^2 \to 0$.*

Proof. Note that $F_R = KX^* \otimes_K KR \otimes_K KX^*$ is free as a left KX^*-module. We have already observed that H is projective as both a left and right KX^*-module. Therefore, HJ is also projective as both a left and right KX^*-module and $HJ \cong H \otimes_{KX^*} J$ by Corollary 2.7 and Proposition 2.1(4). Thus, $0 \to HJ \to H \to H/HJ \to 0$ is a projective resolution as both left and right KX^*-modules. Also, as already mentioned, $0 \to J^2 \to J \to J/J^2 \to 0$ is a projective resolution. Clearly, $0 \to F_R J \to F_R \to F_R/F_R J \to 0$ is exact. Moreover, $F_R/F_R J \cong KX^* \otimes_K KR \otimes_K A$ is projective as a left KX^*-module, since A is a projective K-module, and so this sequence splits over KX^*. Hence, $F_R J$ is also projective as a left KX^*-module (which also follows from Corollary 2.7). Note that since J is projective as a left KX^*-module (hence flat), tensoring (3.1) on the right with J and applying Proposition 2.1(4), we have that $0 \to HJ \to F_R J \to J^2 \to 0$ is exact. Finally, as we already observed, since (3.1) splits over $(KX^*)^{op}$, tensoring on the right with A yields the exact sequence $0 \to H/HJ \to F_R/F_R J \to J/J^2 \to 0$. Thus, we have the commutative diagram of left KX^*-modules displayed in Figure 1 with all rows exact, each column a projective resolution and with the middle column split.

Since each module in the first two rows is projective, these rows split. Hence, tensoring on the left with $A = KX^*/J$ yields the commutative diagram with exact rows and columns in Figure 2. Applying the snake lemma gives us an exact sequence $0 \to \ker a \to 0 \to \ker c \xrightarrow{\partial} \mathrm{coker}\, a \to \mathrm{coker}\, b \to \mathrm{coker}\, c \to 0$, where ∂ is the connecting map and the others are induced naturally. Notice that c is the zero map, and so $\ker c = J^2/J^3$, whereas $\mathrm{coker}\, a = H/(JH + HJ)$, $\mathrm{coker}\, b = F_R/(JF_R + F_R J)$ and $\mathrm{coker}\, c = J/J^2$. Note that J^2/J^3 is generated as a left A-module by elements of the form $r_i u r_j v + J^3$ with $u, v \in X^*$ and $r_i, r_j \in R$. We can take as a preimage in $F_R J/JF_R J$ the element $1 \otimes \bar{r}_i \otimes u r_j v + JF_R J$ (since $u r_j v \in J$), which maps to $1 \otimes \bar{r}_i \otimes u r_j v + JF_R = 1 \otimes \bar{r}_i \otimes u r_j v - r_i u \otimes \bar{r}_j \otimes v + JF_R = [r_i, u, r_j]v + JF_R$ (since $r_i u \in J$). Thus $\partial(r_i u r_j v + J^3) = [r_i, u, r_j]v + JH + HJ$. It follows that the image of ∂ is $(D + JH + HJ)/(JH + HJ) = D/(JH + HJ)$ (by Proposition 3.1), and hence $\mathrm{coker}\, \partial \cong H/D = \pi_2(\mathcal{P})$. Since $F_R/(JF_R + F_R J) \cong A \otimes_K KR \otimes_K A$, this completes the proof. □

Splicing together the exact sequences in Theorems 2.3 and 3.2 yields the following result.

Corollary 3.3. *Let A be a K-algebra that is projective over K with presentation $\mathcal{P} = \langle X \mid R \rangle$. Then there is an exact sequence of A-bimodules*

$$0 \to \pi_2(\mathcal{P}) \to A \otimes_K KR \otimes_K A \to A \otimes_K KX \otimes_K A \to A \otimes_K A \to A \to 0.$$

In particular, if $\mathcal{P}$ is a finite presentation, then A is of type bi-FP_3 over K if and only if $\pi_2(\mathcal{P})$ is finitely generated.

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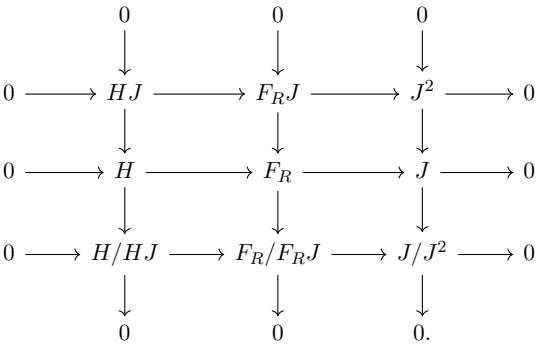


Figure 1. *A commutative diagram.*

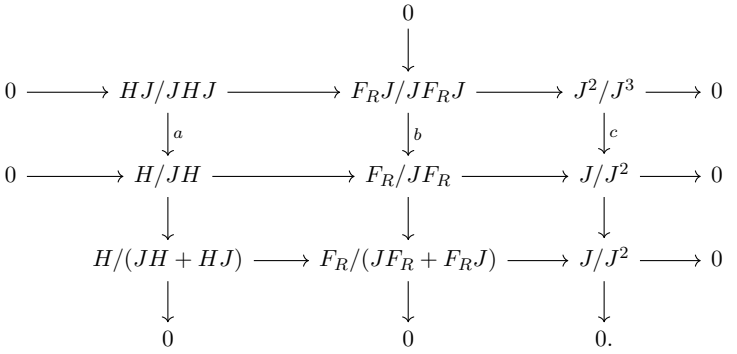


Figure 2. *A commutative diagram.*

4. Monoid presentations

We now consider the case of a monoid presentation and apply Theorem 3.2 to recover a result of Pride [8, Theorem 4.2] and Guba–Sapir [4, Theorem 11.3] (see also [6]). The proof of Theorem 3.2 is more elementary than the proofs found in [4] or [6] as we avoid Knuth–Bendix completions.

Let $\mathcal{P} = \langle X \mid R\rangle$ be a monoid presentation of a monoid M. We view $R \subseteq X^* \times X^*$. The *derivation graph* $\Gamma(\mathcal{P})$ has vertex set X^* and directed edge set $X^* \times R \times X^*$ where $(u, (\ell, r), v)\colon u\ell v \to urv$. It is evident that two vertices are in the same connected component of $\Gamma(\mathcal{P})$ if and only if they represent the same element of M. The *Squier complex* $D(\mathcal{P})$ has 1-skeleton $\Gamma(\mathcal{P})$ and has set of 2-cells $X^* \times R \times X^* \times R \times X^*$. The cell corresponding to $(u, (\ell, r), v, (\ell', r'), w)$ has boundary path

$$(u, (\ell, r), v\ell' w)(urv, (\ell', r'), w)[(u\ell v, (\ell', r'), w)(u, (\ell, r), vr' w)]^{-1}.$$

Roughly speaking, paths correspond to monoid pictures of a derivation from one word to another, and two paths are homotopic if and only if they can be obtained from each other by isotopy and removing or inserting dipoles. See [4] for details, as we will not need monoid pictures.

Notice that X^* has commuting left and right actions on $D(\mathcal{P})$ by regular cellular maps induced by multiplication.

The monoid algebra $\mathbb{Z}M$ has presentation as a $\mathbb{Z}$-algebra $\mathcal{P}' = \langle X \mid \widetilde{R}\rangle$ where $\widetilde{R} = \{r - \ell \mid (\ell, r) \in R\}$. Moreover, $\mathbb{Z}M$ is free over $\mathbb{Z}$ so the above theory applies. Put $J = (\widetilde{R})$ as usual. Then J/J^2 is the relation bimodule considered by Ivanov [5]. Notice that if we look at the cellular chain complex of $\Gamma(\mathcal{P})$, we get the exact sequence $0 \to H_1(\Gamma(\mathcal{P})) \to \mathbb{Z}[X^* \times R \times X^*] \xrightarrow{d_1} \mathbb{Z}X^*$, and the image of d_1 is J as $d_1((1, (\ell, r), 1)) = r - \ell$. Identifying $\mathbb{Z}[X^* \times R \times X^*]$ with $\mathbb{Z}X^* \otimes_{\mathbb{Z}} \mathbb{Z}\widetilde{R} \otimes_{\mathbb{Z}} \mathbb{Z}X^*$ via $(u, (\ell, r), v) \mapsto u \otimes \overline{r - \ell} \otimes v$ then identifies $H_1(\Gamma(\mathcal{P}))$ with H from (3.1).

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On the other hand, if $d_2\colon \mathbb{Z}[X^* \times R \times X^* \times R \times X^*] \to \mathbb{Z}[X^* \times R \times X^*]$ is the boundary map in the cellular chain complex of $D(\mathcal{P})$, then $d_2(u,(\ell,r),v,(\ell',r'),w) = (u,(\ell,r),v\ell' w) - (u,(\ell,r),vr'w) + (urv,(\ell',r'),w) - (u\ell v,(\ell',r'),w)$. Under our identification of $\mathbb{Z}[X^* \times R \times X^*]$ with $\mathbb{Z}X^* \otimes_{\mathbb{Z}} \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Z}X^*$ this maps to $-u \otimes \overline{r-\ell} \otimes v(r'-\ell')w + u(r-\ell)v \otimes \overline{r'-\ell'} \otimes w = -u[(r-\ell,v,r'-\ell')]w$. It follows that $H_1(D(\mathcal{P})) \cong H/D = \pi_2(\mathcal{P}')$ as $\mathbb{Z}X^*$-bimodules, and hence $H_1(D(\mathcal{P}))$ is a $\mathbb{Z}M$-bimodule (as was already observed by Pride). Pride puts $\pi_2^{(b)}(\mathcal{P}) = H_1(D(\mathcal{P}))$. Putting together these observations with Theorem 3.2 recovers the exact sequence of Pride [8] and Guba–Sapir [4].

Theorem 4.1 (Pride/Guba–Sapir). *Let $\mathcal{P} = \langle X \mid R\rangle$ be a monoid presentation. Then there is an exact sequence of $\mathbb{Z}M$-bimodules*

$$0 \to \pi_2^{(b)}(\mathcal{P}) \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Z}M \to J/J^2 \to 0$$

where J/J^2 is the relation bimodule of this presentation.

This theorem implies that the abelianization of a diagram group is free abelian [4, Theorem 11.2].

There is an augmentation homomorphism $\varepsilon\colon \mathbb{Z}M \to \mathbb{Z}$ with $\varepsilon(m) = 1$ for all $m \in M$. The left (respectively, right) relation module, studied by Ivanov [5], is $J/J^2 \otimes_{\mathbb{Z}M} \mathbb{Z}$ (respectively, $\mathbb{Z} \otimes_{\mathbb{Z}M} J/J^2$). Pride defined $\pi_2^{(\ell)}(\mathcal{P}) = \pi_2^{(b)}(\mathcal{P}) \otimes_{\mathbb{Z}M} \mathbb{Z}$ and $\pi_2^{(r)}(\mathcal{P}) = \mathbb{Z} \otimes_{\mathbb{Z}M} \pi_2^{(b)}(\mathcal{P})$. The following exact sequences are known [6, 8], but our proof is different.

Corollary 4.2. *Let $\mathcal{P} = \langle X \mid R\rangle$ be a presentation of a monoid M. Let $J = \ker(\mathbb{Z}X^* \to \mathbb{Z}M)$. Then there are exact sequences*

$$0 \to J/J^2 \otimes_{\mathbb{Z}M} \mathbb{Z} \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}X \to \mathbb{Z}M \to \mathbb{Z} \to 0$$
$$0 \to \pi_2^{(\ell)}(\mathcal{P}) \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}R \to J/J^2 \otimes_{\mathbb{Z}M} \mathbb{Z} \to 0$$

and dually for the right relation module and $\pi_2^{(r)}(\mathcal{P})$.

Proof. By Theorem 2.3, we have exact sequences of $\mathbb{Z}M$-bimodules $0 \to \Omega \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}M \to \mathbb{Z}M \to 0$ and $0 \to J/J^2 \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}M \to \Omega \to 0$. Since $\mathbb{Z}M$ and $\mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}M$ are free right $\mathbb{Z}M$-modules, the first sequence splits over $\mathbb{Z}M^{op}$ and Ω is a projective right $\mathbb{Z}M$-module. Therefore, the second sequence splits over $\mathbb{Z}M^{op}$ and J/J^2 is a projective right $\mathbb{Z}M$-module (since $\mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}X \otimes_{\mathbb{Z}M} \mathbb{Z}M$ is free as a right $\mathbb{Z}M$-module). Thus, tensoring on the right with $\mathbb{Z}$ yields exact sequences $0 \to \Omega \otimes_{\mathbb{Z}M} \mathbb{Z} \to \mathbb{Z}M \to \mathbb{Z} \to 0$ and $0 \to J/J^2 \otimes_{\mathbb{Z}M} \mathbb{Z} \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}X \to \Omega \otimes_{\mathbb{Z}M} \mathbb{Z} \to 0$, and these splice together to give the first exact sequence.

The exact sequence $0 \to \pi_2^{(b)}(\mathcal{P}) \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Z}M \to J/J^2 \to 0$ splits over $\mathbb{Z}M^{op}$ because we observed that J/J^2 is a projective right $\mathbb{Z}M$-module. Therefore, tensoring on the right with $\mathbb{Z}$ yields the second exact sequence. □

If we splice the two exact sequences in Corollary 4.2 together, we get the exact sequence $0 \to \pi_2^{(\ell)}(\mathcal{P}) \to \mathbb{Z}[M \times R] \to \mathbb{Z}[M \times X] \to \mathbb{Z}M \to \mathbb{Z} \to 0$. Recall that the (left) Cayley complex $C^{(\ell)}(\mathcal{P})$ of M with respect to the presentation $\mathcal{P}$ is the free M-CW complex with vertex set M, edge set $M \times X$ and set of 2-cells $M \times R$ [3]. Here $(m,x)\colon m \to mx$ and the boundary path of $(m,(\ell,r))$ is the path from m to $m\ell = mr$ labeled by r followed by the inverse of the path from m to $m\ell = mr$ labeled by ℓ. This complex is simply connected and the above exact sequence identifies $\pi_2^{(\ell)}(\mathcal{P})$ as a left $\mathbb{Z}M$-module with $H_2(C^{(\ell)}(\mathcal{P})) \cong \pi_2(C^{(\ell)}(\mathcal{P}))$.

We observe that Theorem 4.1 provides a similar result for the two-sided Cayley complex $C^{(b)}(\mathcal{P})$ [3]. The vertex set of $C^{(b)}(\mathcal{P})$ is $M \times M$, the edge set is $M \times X \times M$ and the set of 2-cells is $M \times R \times M$. The edge (m,x,m') goes from $(m,xm') \to (mx,m')$. If $(\ell,r) \in R$ and $(m,m') \in M \times M$, then $m\ell = mr$, $\ell m' = rm'$, and it is not difficult to see that there are paths labeled by ℓ and by r from $(m,\ell m') \to (m\ell,m')$. The boundary path of the cell $(m,(\ell,r),m')$ is the aforementioned path labeled by r followed by the inverse of the aforementioned path labeled by ℓ. There are natural commuting left and right cellular

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actions of M on $C^{(b)}$ induced by multiplication. The set of path components of $C^{(b)}(\mathcal{P})$ is in bijection with M via $(m, m') \mapsto mm'$ on vertices and each path component is simply connected. Moreover, one has $H_0(C^{(b)}(\mathcal{P})) \cong \mathbb{Z}M$ as a $\mathbb{Z}M$-bimodule.

The exact sequence in Theorem 4.1 can be spliced with that in Theorem 2.3 to yield an exact sequence $0 \to \pi_2^{(b)}(\mathcal{P}) \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Z}M \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}X \otimes_{\mathbb{Z}} ZM \to \mathbb{Z}M \otimes_{\mathbb{Z}} \mathbb{Z}M \to \mathbb{Z}M \to 0$. It is not difficult to check using Theorem 2.3 that this can be identified with the exact sequence $0 \to \pi_2^{(b)}(\mathcal{P}) \to \mathbb{Z}[M \times R \times M] \to \mathbb{Z}[M \times X \times M] \to \mathbb{Z}[M \times M] \to H_0(C^{(b)}(\mathcal{P})) \to 0$, where the maps come from the cellular chain complex of $C^{(b)}(\mathcal{P})$. Therefore, we can identify $\pi_2^{(b)}(\mathcal{P})$ with $H_2(C^{(b)}(\mathcal{P}))$ as a $\mathbb{Z}M$-bimodule. Since each path component of $C^{(b)}(\mathcal{P})$ is simply connected, as an abelian group this is the direct sum of the π_2 of the path components of $C^{(b)}(\mathcal{P})$, justifying the notation. It follows from the above exact sequence that if the presentation is finite, then M is bi-FP$_3$ if and only if $\pi_2^{(b)}(\mathcal{P})$ is finitely generated, as was observed by Kobayashi and Otto [7].

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