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A Markov partition for the Feigenbaum dynamics

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Abstract. We build a Markov partition of the Feigenbaum dynamical plane, and a system of 'external rays', which are invariant under rescaling. Applications to geometry of the Feigenbaum map are given.

1. *Introduction and main results*

A 'puzzle' is a Markov partition for the dynamics of a polynomial map $z \mapsto z^r + c$. It was introduced by J.-C. Yoccoz for non-renormalizable quadratic polynomials to study the problem of local connectivity of Julia sets and the Mandelbrot set [**9**].

We present here a different construction to build a countable Markov partition for the famous Feigenbaum map [**5, 6, 8**]. Recall that this is an infinitely renormalizable map. The key point is that our partition is invariant under the rescaling. To obtain this property, we first construct an 'external ray', or access (i.e. a simple arc disjoint from the Julia set of the map) to the critical point, which is invariant under the rescaling. Moreover, such a 'ray' appears to be unique; it is a fractal and a quasi-circle, see Theorem A. Taking preimages of this 'ray' we obtain the Markov partition; it is described in Theorem B (see also Theorem C(a)). Roughly speaking, it says that there exists a (in fact, unique) 'shape', such that its images under the action of univalent maps with uniformly bounded distortions (which are inverse branches and rescalings) tile the plane, and so that the boundary of any 'tile' touches a (rescaled) Julia set at a unique point, which is a (rescaled) preimage of the critical point. Then we show (Theorem C) that there is a system of 'external rays', which is invariant under the rescaling. For other applications, see Theorem C and Corollary.

The Feigenbaum map g is a unimodal interval map with a given criticality $r > 1$ at the critical point and such that g is a fixed point of the period-doubling renormalization operator [**8**]. In other words, it is a solution of the Cvitanovic-Feigenbaum equation

$$g(x) = \alpha g(g(x/\alpha)), \tag{1}$$

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where $\alpha < -1$ is the Feigenbaum universal constant (rescaling in the dynamical plane). The Feigenbaum map exists for every $r > 1$ [**7, 10**]. The uniqueness is proved for every r even integer [**14**]. Moreover, in this case the map is universal [**8, 11, 14**]. It means that the sequence of renormalizations of any smooth unimodal map (with the Feigenbaum combinatorics and fixed criticality r) converges to a map which depends on r only. This is the Feigenbaum map.

Let g be the Feigenbaum map with critical point of criticality r, where $r > 1$ is a fixed even integer. As usual, g is normalized so that it is an even unimodal map of $[-1, 1]$ into itself, with the critical point at 0 and the critical value $g(0) = 1$. It is proved [**5, 6**] that g extends to an analytic polynomial-like map [**4**]

$$g : \Omega_0 \to \Omega,$$

with the single critical point at 0, where $\Omega = \mathbb{C} \setminus \mathbb{R} \cup (\alpha, \alpha^2)$ is a slit complex plane, $\overline{\Omega_0} \subset \Omega$, and Ω_0 *is a bounded topological disc*; moreover, if one denotes

$$\Pi = \{z : \arg(z) \in (0, \pi/r)\},$$

then the boundary of Ω_0 contains exactly one *singular* point c in Π such that the limit values of g at c is infinity.

The dynamical system $g : \Omega_0 \to \Omega$ is the subject of the paper. (So, whenever we speak about iterates g^n or g^{-n}, $n \geq 0$, we mean iterates of the above map.)

There is a representation $g(z) = F(z^r)$ where F is analytic in an appropriate domain. In particular, Ω_0 is symmetric with respect to the rotation by the angle $2\pi/r$, as well as with respect to the complex conjugation

$$z \mapsto z^*.$$

Denote by J the Julia set of g: $J = \{z : g^n(z) \in \Omega_0, n = 0, 1, 2, \dots\}$. As is proved in [**11**], *the union*

$$J_\infty = \bigcup_{n=0}^{\infty} \alpha^n J$$

of an increasing sequence of compacts $\alpha^n J$, $n \geq 0$, is dense in the plane.

This is the reason for the uniqueness of the curves constructed in Theorem A and Theorem C(b) and (b′). Note that the dense set J_∞ is connected (but not closed).

By a curve, or arc, we usually mean an *open* one, i.e. without the end points. A curve with the end points is called *closed* (this will be clear from the context).

Denote by $\mathrm{HD}(E)$ the Hausdorff dimension of a subset E of the plane.

THEOREM A. (Invariant access at zero) *There exists a unique simple unbounded curve L which lies in Π and terminates at the critical point 0, and which has the following properties:*

(1) *L is invariant under the rescaling $z \mapsto |\alpha| \cdot z$;*

(2) *L is disjoint with the Julia set J of the polynomial-like map g; in fact,*

$$L \cap J_\infty = \emptyset.$$

Additionally, L satisfies the following properties:

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(3) $1 < \mathrm{HD}(L) < 2$ *and* L *has no tangent at any point. Moreover,* L *is a quasicircle (i.e. a quasiconformal image of the real line);*

(4) $c \in L$*; if one denotes by* L_n *the arc of* L *between the points* $c/|\alpha|^n$ *and* $c/|\alpha|^{n+1}$, $n = 0, 1, 2, \ldots$, *and by* L_∞ *the infinite arc of* L *between* c *and* ∞, *then, for* $n = 0, 1, \ldots,$*:*

(4a) $L_\infty \subset \Omega \setminus \Omega_0$*;*

(4b) *the arc* L_n *lies in the 'annulus'* $g^{-2^n}(\Omega) \setminus g^{-2^{n+1}}(\Omega)$ *and joins points* $c/|\alpha|^n \in \partial g^{-2^n}(\Omega)$ *and* $c/|\alpha|^{n+1} \in \partial g^{-2^{n+1}}(\Omega)$*;*

(4c) *the curve* $g^{2^n}(L_n)$ *is equal to either* $(L^{(n)})^*$ *or* $-(L^{(n)})^*$, *where* $L^{(n)} = L_\infty/|\alpha|^n$ *is the (infinite) arc of* L *between* $c/|\alpha|^n$ *and* ∞*;*

(5) *the intersection of the curve* $g^{2^n}(L \cap g^{-2^n}(\Omega))$ *with either* L^* *or* $-L^*$ *is equal to either* $(L^{(n)})^*$ *or* $-(L^{(n)})^*$*; moreover, for any* $k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_l}$ *with* $n_1 > n_2 > \cdots > n_l$, *the intersection of the curve* $g^k(L \cap g^{-k}(\Omega))$ *with one of the curves* $\pm L, \pm L^*$ *is equal to one of the infinite arcs among* $\pm L_\infty/|\alpha|^{n_l}, \pm L^*_\infty/|\alpha|^{n_l}$.

Considering all the preimages of $L \cup L^*$, we build a Markov partition (even tiling, see below) of g, which is invariant under the rescaling. Let us define it precisely. Consider two infinite open 'sectors' R_0, R_0^*, where R_0 is bounded by the curves L and $-L^*$. Let us rotate the curves L and L^* by the angles $2\pi j/r$, $0 \le j \le r-1$. Then we get $2r$ curves, which divide either 'sector' R_0, R_0^* into $r-1$ smaller 'sectors' $R_{0,1}, R_{0,2}, \ldots, R_{0,r-1}$ (in the counterclockwise direction) and $R^*_{0,1}, R^*_{0,2}, \ldots, R^*_{0,r-1}$ respectively. They are said to be the set $\mathcal{M}_0$ of $2(r-1)$ pieces of *depth* 0. Now, for every $n = 1, 2, \ldots$, we define the set $\mathcal{M}_n$ of pieces of *depth* n as the set of components of $g^{-n}(R)$, $R \in \mathcal{M}_0$. Observe that the set of 1-depth pieces consists of $2r(r-1)$ bounded Jordan domains. Hence, all pieces of all depths (≥ 1) are bounded Jordan domains. Closed pieces are said to be the closures of the pieces. For a piece R of depth $n \ge 1$, define its 'base points' $x_R, x_R^\infty \in \partial R$ as follows: if $g^n(R) \in \mathcal{M}_0$ then $g^n(x_R) = 0$, $g^n(x_R^\infty) = \infty$. For example, for the pieces of depth 1, their base points are rotated (by $2\pi j/r$, $j \in \mathbb{Z}$) and conjugated pairs of points to the pair x_0, c, where $x_0 > 0$ is such that $g(x_0) = 0$.

THEOREM B. (The Markov partition) *The set* $\mathcal{M} = \bigcup_{n \ge 1} \mathcal{M}_n$ *of all pieces satisfies the following properties.*

(a) *(Invariance) if* $R \in \mathcal{M}_n$, *then* $R/\alpha \in \mathcal{M}_{2n}$. *Also,* $\partial R \cap J_\infty = \{x_R\}$.

(b) *(Markov property) any two pieces* R, R' *are either disjoint or one contains the other one; moreover,* $\partial R \cap \partial R'$ *is either (i) empty, or (ii) a non-trivial arc, which contains one of the base points* $x_R^\infty, x_{R'}^\infty$, *or (iii)* $R \cap R' = \emptyset$ *and* $\partial R \cap \partial R' = \{x_R\} = \{x_{R'}\}$.

(c) *(Tiling) the union of all closed pieces together with the real line rotated by the angles* $2\pi j/r$, $j = 1, \ldots, r-1$, *covers a neighborhood of zero.*

(d) *(Bounded distortion) all pieces are roughly of the same shape; more precisely, for any piece* R *there exists a minimal* $n \ge 0$ *such that:*

(d1) $g^n(R)$ *is one of the pieces of the form* G/α^k, *for some piece* G *of depth one and for some* $k \ge 0$*;*

(d2) *the map* $g^n : R \to G/\alpha^k$ *extends to a univalent map onto* $V(G)/\alpha^k$, *where* $V(G)$ *is a neighborhood of* $\overline{G}$, *so that* $V(G)$ *is fixed for each of* $2r(r-1)$ *closed pieces of depth 1.*

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Moreover, there exist $C > 0, 0 < \lambda < 1$, *such that, for every* m *different pieces* $R_m \subset R_{m-1} \subset \cdots \subset R_1$, *one has* $\operatorname{diam}(R_m) \le C\lambda^m \operatorname{diam}(R_1)$.

Introduce the extended set of pieces $\mathcal{M}^{\text{ext}} = \bigcup_{n\in\mathbb{Z}}\{\alpha^n R : R \in \mathcal{M}\}$. The elements of $\mathcal{M}^{\text{ext}}$ are also called pieces. Theorem B(a), (c) shows that $\mathcal{M} \subset \mathcal{M}^{\text{ext}}$ and *the set of closures of all (pairwise disjoint) pieces of* $\mathcal{M}^{\text{ext}}$ *(united with the rotated real axis) forms a tiling of the plane.*

Let us arrange the set of all pieces $\mathcal{M}^{\text{ext}}$ by inclusion. Then there is a countable number of the pieces of the *first level* (so that they are not contained in bigger pieces), the *second level*, etc. Every piece of *level* n is contained in some pieces of levels up to n and contains pieces of higher levels.

Let us introduce the following sets.

$B = \bigcup_{n\ge 0} \alpha^n B_0$, where B_0 is the set of all z such that, for some $n \ge 0$, $g^n(z) = 0$.

$Y = \bigcup_{n\ge 0} \alpha^n Y_0$, where Y_0 is the set of all z such that, for some $n \ge 0$, $g^n(z)$ lies in the real axis. In particular, $B_0 \subset Y_0 \subset J$.

$\partial\mathcal{M}^{\text{ext}}$ is said to be the set of all z such that $z \in \partial R$ for some $R \in \mathcal{M}^{\text{ext}}$. In particular, $B \subset \partial\mathcal{M}^{\text{ext}}$.

Finally, $J^{\text{ess}} = \mathbb{C} \setminus (Y \cup \partial\mathcal{M}^{\text{ext}})$. Note that $J \setminus Y_0 \subset J^{\text{ess}}$.

We use Theorems A–B to prove the following.

THEOREM C. (Geometry of the Feigenbaum plane)

(a) *For every* $z \in J^{\text{ess}} \cup \partial\mathcal{M}^{\text{ext}} \setminus Y = \mathbb{C} \setminus Y$ *there is a sequence of closed nested pieces* $(\overline{R_n(z)})_{n\ge 0}$ *where* $R_n(z) \in \mathcal{M}^{\text{ext}}$ *is a piece of level* n *such that* $z \in \overline{R_n(z)}$; *moreover (see Theorem B(d)),* $\operatorname{diam} R_n(z) \le C\lambda^n \operatorname{diam} R_1(z)$, *with the universal* $C > 0, \lambda < 1$. *If* $z \in J \setminus Y_0$, *then all* $R_n(z) \in \mathcal{M}$.

(b) *Let* $x, y \in \overline{\Pi}$. *There exists a simple curve* $l(x, y)$, *such that:*

(b1) *it joins the point* x *and* y;

(b2) *it lies in* $\Pi \setminus J_\infty$.

Moreover, $l(x, y)$ *consists of arcs of the boundaries of pieces of* $\mathcal{M}^{\text{ext}}$, *and if* x, y *belong to a closed piece, then* $l(x, y)$ *lies in the same closed piece; finally, if* $x, y \in \Pi \setminus Y$, *then* $l(x, y)$ *is the unique simple curve that satisfies (b1) and (b2).*

(b′) *Given* $x \in \overline{\Pi}$, *define an 'external ray'* $l \subset \Pi$ *to the point* x *as the curve* $l(x, \infty)$ *(i.e. the simple curve that joins* x *and* ∞ *and is contained inside the curve* $l(x, c)$ *extended by the infinite arc* L_∞*). Then: (i) for any* $n \in \mathbb{Z}$, $l(|\alpha|^n x, \infty) = |\alpha|^n l(x, \infty)$, *and (ii) for any two 'external rays'* l_1, l_2, *the intersection* $l_1 \cap l_2$ *is equal to a simple curve joining a point* $y \in l_1 \cap l_2$ *and infinity, where* y *is such that, for some non-negative* n, k, *the point* $g^n(y/\alpha^k)$ *belongs to the set* $\{c \exp(i2\pi j/r), c^* \exp(i2\pi j/r), j = 0, \ldots, r-1\}$.

(c) *There exists* C *such that, for every point* $z \in \overline{\Pi}$, *there exists a rectifiable curve inside of* $J_\infty \cap \overline{\Pi}$ *with length less than* $C \cdot |z|$, *which connects* z *and* 0.

We complete the system of 'external rays' from Π to the whole plane in a natural way, as follows.

Complement to (b′). Applying to the 'external rays' in Π complex conjugation and rotations by the angles $2\pi j/r$ $(1 \le j \le r-1)$, we define the set of 'external rays' in

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the whole plane (in particular, to any point of the plane). Specifically, there are precisely $2r$ 'rays' to the critical point 0 (obtained as above from the 'ray' $L \subset \Pi$). Hence, one can define precisely $2r$ 'rays' to any point of the set B. Moreover, if $x \in \mathbb{R} \setminus B$, then, as is easy to see, there are precisely two 'rays' to x (at least two, by the complex conjugation, and at most two, by topological considerations). Therefore, one can define precisely two 'rays' to any point of $Y \setminus B$. To any other point of the plane, the number of 'rays' is 1 (by Theorem C(b)).

It is known [**6**], that g has a maximal analytic continuation to a domain W, which is simply connected. Moreover, $W = \bigcup_{n\geq 0} \alpha^n g^{-2^n}(\Omega)$ [**2, 6**], and W is dense in the plane [**11**] (because the dense set $J_\infty \subset W$). Note that W is invariant under the complex conjugation and rotation by $2\pi/r$. Then, Theorems A(4b) and C(b) give us the following.

COROLLARY 1.1.

(a) $L_\infty \subset \partial W$,

(b) ∂W *is a dendrite, that is:*

(b1) *for every two points* $x, y \in \partial W \cap \Pi$ *there exists a unique curve, namely* $l(x, y)$, *see Theorem C(b), that joins* x *and* y *inside* $\partial W \cap \Pi$;

(b2) *if* $x, y \to z \in \partial W$, *then the diameter of* $l(x, y)$ *tends to zero.*

Proof. Part (a) follows immediately from Theorem A(4b). To prove (b1), assume the contrary, i.e. assume that, for some $x \neq y$ in $\partial W \cap \Pi$, the curve $l(x, y)$ is not contained in ∂W. Since ∂W is connected (on the Riemann sphere), then $l(x, y) \cup \partial W$ separates the sphere. On the other hand, both sets ∂W and $\overline{l(x, y)}$ are disjoint with the dense and connected set J_∞, a contradiction. By the same reasoning, the curve $l(x, y)$ is the unique simple curve in the plane connecting x, y inside of ∂W.

For the proof of (b2), see §5. □

2. *Proof of Theorem A*

We construct the curve L as the union $\bigcup_{n\in\mathbb{Z}} |\alpha|^n I$, where I is a simple (closed, i.e. with the end points) curve joining c and $c/|\alpha|$. In turn, the curve I is going to be the limit set of a *finite* system of conformal maps (iterated function system).

2.1. *Some basic facts.* Denote by $\mathbf{H}^\pm$ the upper and lower half-planes. Let x_0 be the unique point $0 < x_0 < 1$, such that $g(x_0) = 0$. The following facts (1)–(3) (see [**5, 6**]) will be of crucial importance for the proof.

(1) Let u be the branch of g^{-1} such that $u(1) = 0$ and $u(1/\alpha) = 1$. Then u extends as a univalent (symmetric under the complex conjugacy) function to the slit plane $\mathbf{H}^+ \cup \mathbf{H}^- \cup (\alpha, 1)$, and $u(\mathbf{H}^-) \subset \Pi$. It satisfies the equation

$$u(z) = -\alpha u(u(z/\alpha)), \quad z \in \mathbf{H}^+ \cup \mathbf{H}^- \cup (\alpha, 1).$$

(2) Introduce the following antiholomorphic maps of $\mathbf{H}^+$ into itself:

$$u^* : z \mapsto u(z^*),$$
$$\chi : z \mapsto |\alpha| u^*(z).$$

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Then u^*, χ extend to homeomorphisms of $\overline{\mathbf{H}}^+$ into itself, and $\chi : \mathbf{H}^+ \to \mathbf{H}^+$ has a unique (globally attracting) fixed point, namely, the singular point $c \in \Pi$.

(3) Denote

$$\omega = u^*(\mathbf{H}^+). \tag{2}$$

Then ω is a Jordan domain. Its boundary consists of a real segment $[0, |\alpha|x_0] = u^*([\alpha, 1])$, a sequence of (closed) analytic arcs $\tau_n, n \geq 0$, and the point c. Here, τ_0 is the segment $\exp(i\pi/r)[0, |u(\alpha^2)|] = u^*([1, \alpha^2])$, and $\tau_{n+1} = \chi(\tau_n) = u^*((-1)^{n+1}[|\alpha|^{n+1}, |\alpha|^{n+3}])$, $n \geq 0$. In particular, $\{\tau_n\}$ converge to the point c exponentially fast. For every n, τ_n, τ_{n+2} meet at a joint end point at the angle π/r.

Observe (after [**5**]) that the map u^* conjugates $z \mapsto \alpha z^*$ and χ:

$$\chi \circ u^*(z) = u^*(\alpha z^*). \tag{3}$$

Hence,

$$\chi^2 \circ u^*(z) = u^*(\alpha^2 z). \tag{4}$$

In particular, we have the following.

Remark 1. If γ is any straight ray in $\mathbf{H}^+$ starting at zero, then $u^*(\gamma)$ is an (open) analytic arc in ω joining the points c and x_0 in $\partial\omega$, which has a finite length (because it is invariant under χ^2). Moreover, since the map u^* extends to a univalent map in a neighborhood of zero, $u^*(\gamma)$ extends to an analytic curve through the point x_0.

2.2. *Two iterated function systems.* Note that

$$u^*(\omega) = u(u(\mathbf{H}^+)) = u(u(\mathbf{H}^-/\alpha)) = u(\mathbf{H}^-)/(-\alpha) = \omega/|\alpha|.$$

Since $\omega \subset \Pi$, we have $\omega/|\alpha| = u^*(\omega) \subset u^*(\mathbf{H}^+) = \omega$ and $u^*(\omega) \subset u^*(\Pi)$.

Introduce a compact set

$$X = \overline{u^*(\Pi) \setminus u^*(\omega)} = \overline{u^*(\Pi) \setminus (\omega/|\alpha|)},$$

and the following *two* families $S_\infty = \{\psi_k\}_{k=1}^\infty$, $S = \{\varphi_1, \varphi_2, \varphi_3\}$ of conformal maps from $\mathbf{H}^+$ into itself defined as follows:

$$\psi_k(z) = u^*(|\alpha|^k z), \quad k = 1, 2, 3, 4, \ldots, \tag{5}$$

and

$$\varphi_1 = \psi_1, \quad \varphi_2 = \psi_2, \quad \varphi_3 = \chi^2. \tag{6}$$

The equation (3) implies

$$\psi_{2j}(z) = \chi^{2j}(u^*(z)) \ \ (j \geq 1), \quad \psi_{2j+1}(z) = \chi^{2j+1}(u^*(-z^*)) \ \ (j \geq 0). \tag{7}$$

Therefore,

$$\psi_{2j+2} = \varphi_3^j \circ \varphi_2, \quad \psi_{2j+1}(z) = \varphi_3^j \circ \varphi_1 \ \ (j \geq 0). \tag{8}$$

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Remark 2. The latter means that the infinite system S_∞ is *induced* by the finite system S. So, it is enough to study the system S (see also Remark 3). Although the system S_∞ arises naturally if one wants to construct the curve L, the system S is simpler to deal with. In the sequel, S_∞ is more convenient for the study of properties related to rescaling ((1) and (2), (4) and (5) of Theorem A) while S is used to prove the metric properties (3), Theorem A.

Denote by ρ the hyperbolic metric of $\mathbf{H}^+$. Note that maps u^* and χ as conformal maps of $\mathbf{H}^+$ into itself are contractions with respect to the metric ρ.

LEMMA 2.1. *The family S has the following properties.*

(1) $\mathrm{Int}(X)$ *is a Jordan domain. The boundary of X consists of: (i) a sequence of (closed) curves $\tau_n/|\alpha|$, $n \geq 2$, which converges to the points $c/|\alpha| \in \partial X$, so that $\bigcup_{n\geq 2} \tau_n/|\alpha| \cup \{c/|\alpha|\}$ is a (closed) curve, which we call the 'bottom' B_X of X; (ii) the curve $\tau_0 \setminus (\tau_0/|\alpha|)$ and a sequence of curves τ_{2j}, $j \geq 1$, which converges to the point $c \in \partial X$, so that $\tau_0 \setminus (\tau_0/|\alpha|) \cup \bigcup_{j\geq 1} \tau_{2j} \cup \{c\}$ is a curve, which we call the 'left side' L_X of X; (iii) the curve $u^*(\exp(i\pi/r)\mathbb{R}^+ \setminus \tau_0)$, which we call the 'right side' R_X of X. Finally, the compact X is contained in $\mathbf{H}^+$.*

(2) $|\alpha|X \cap X = \bigcup_{j\geq 1} \tau_{2j} \cup \{c\}$.

(3) *Each φ_i is either a holomorphic or an antiholomorphic injection of $\mathbf{H}^+$ into itself, which maps the compact X into itself; moreover, there exist a domain V and a number $\lambda < 1$, such that $X \subset V$, $\overline{V} \subset \mathbf{H}^+$, $\varphi_i(V) \subset V$, and*

$$\rho(\varphi_i(x), \varphi_i(y)) \leq \lambda \rho(x, y),$$

for $i = 1, 2, 3$ and all $x, y \in V$.

(4) $\varphi_i(\mathrm{Int}(X)) \cap \varphi_j(\mathrm{Int}(X)) = \emptyset$ *for* $1 \leq i < j \leq 3$.

(5) $\varphi_i(X) \cap \varphi_j(X) \neq \emptyset$ *if and only if* $|i - j| \leq 1$.

(6) *For any finite word $\bar{\varepsilon} = \{\varepsilon_1, \ldots, \varepsilon_n\}$ of n symbols $\varepsilon_i \in \{1, 2, 3\}$, denote $\varphi_{\bar{\varepsilon}} = \varphi_{\varepsilon_1} \circ \varphi_{\varepsilon_2} \circ \cdots \circ \varphi_{\varepsilon_n}$. Then $\varphi_{\bar{\varepsilon}}$ maps V into itself with uniformly bounded distortion: there is K such that $|\varphi'_{\bar{\varepsilon}}(x)| \leq K|\varphi'_{\bar{\varepsilon}}(y)|$ for every finite word $\bar{\varepsilon}$ and any $x, y \in V$.*

Proof. Parts (1) and (2) follow from the definition of X and from the description of ω.

Part (3). For every $k > 0$, $\psi_k(X) = u^*(|\alpha|^k X) \subset u^*(\overline{\Pi \setminus \omega}) = X$. In particular, $\varphi_i(X) \subset X$, $i = 1, 2$. As for $\varphi_3 = \chi^2$, one can use that $\psi_2 = \chi^2 \circ u^*$ and write $\chi^2(X) = \chi^2 \circ u^*(\overline{\Pi \setminus \omega}) = \psi_2(\overline{\Pi \setminus \omega}) = u^*(\alpha^2 \overline{\Pi \setminus \omega}) \subset u^*(\overline{\Pi \setminus \omega}) = X$.

Let us construct the domain V (the existence of λ then follows because $\overline{V} \subset \mathbf{H}^+$). We start with the following general claim.

CLAIM. *For a point $a \in \mathbb{C}$, a simply connected domain $U \ni a$, and a map f which is a holomorphic injection of $U^* = \{z^* : z \in U\}$ into U, define an antiholomorphic map $f^*(z) = f(z^*)$, $f^* : U \to U$; assume that $f^*(a) = a$ and $0 < |\rho| < 1$, where $\rho = f'(a^*)$. Then there is a holomorphic change of coordinates $K : U \to \mathbb{C}$, $K(a) = 0$, such that $K \circ f^* \circ K^{-1} : w \mapsto \rho w^*$. Moreover, if $Y \subset U$ is compact, such that $f^*(Y) \subset Y$, then there is an arbitrary small neighborhood W_Y of Y, such that $f^*(W_Y)$ is compactly contained in W_Y.*

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It is enough to prove the existence of K locally, around a. Notice that the second iterate $(f^*)^2$ of f^* is holomorphic around its fixed point a, and its multiplier $((f^*)^2)'(a)$ is equal to $|\rho|^2$. Hence, there is a holomorphic change K, $K(a) = 0$, $K'(a) = 1$, such that $K \circ (f^*)^2 \circ K^{-1} : w \mapsto |\rho|^2 w$. Then, the map $K \circ f^* \circ K^{-1}$ is antiholomorphic around a; hence, there is a univalent at $w = 0$ map F, such that $K \circ f^* \circ K^{-1} = F^*$. If $F(w) = \rho w + a_n w^n + \cdots$, then one sees by substitution that $a_n = 0$. Thus, $F(w) = \rho w$, which proves that $K \circ f^* \circ K^{-1}(w) = \rho w^*$. Now, for every $\epsilon > 0$, the neighborhood W_Y can be defined as $K^{-1}(W^\epsilon)$, where W^ϵ is the ϵ-neighborhood of $K(Y)$.

Let us apply the claim to the map φ_1 and the point $c/|\alpha|$. It is relevant, because $\varphi_1 = \psi_1 = |\alpha|^{-1} \circ \chi \circ |\alpha|$; hence, $c/|\alpha|$ is the (global) attracting fixed point of the map $\varphi_1 : \mathbf{H}^+ \to \mathbf{H}^+$. Also, $c/|\alpha|$ lies in the 'bottom' B_X of X. Note that $\varphi_1(B_X) \subset B_X$. Hence, there is a small enough neighborhood W_B of B_X, such that $\varphi_1(W_B)$ is compactly contained in W_B.

Now, note that the 'left side' L_X of X as well as its 'right side' R_X is invariant under the holomorphic map φ_3. Hence, one can choose a small enough neighborhood W_L of L_X, such that $\varphi_3(W_L)$ is compactly contained in W_L. Moreover, note the following:

(a) φ_1 maps the 'left side' L_X onto a curve, which is an arc in the 'right side' R_X extended by an arc in $\mathrm{Int}(X)$, and it maps R_X into $\mathrm{Int}(X)$ except for an end point in B_X;

(b) φ_2 maps L_X onto a curve, which is an arc in R_X extended by an arc in $\mathrm{Int}(X)$, and it maps $R_X \cup B_X$ into $\mathrm{Int}(X)$ except for an end point in B_X;

(c) φ_3 maps B_X into $\mathrm{Int}(X)$ except for end points in $L_X \cup R_X$.

It follows that there is a small neighborhood W_R of R_X, such that $\varphi_3(W_R)$ is compactly contained in W_R and $\varphi_1(W_R) \cup \varphi_2(W_R) \subset X \cup W_B$; moreover, W_L is chosen so small that $\varphi_1(W_L) \cup \varphi_2(W_L) \subset W_R$. Thus, one can set $V = X \cup W_R \cup W_L \cup W_B$. We have proved part (3). This implies also, by the Koebe distortion theorem, the bounded distortion property (6).

Let us now prove parts (4) and (5). For $i = 1$, $j = 2$, these follow directly from the definition of φ_k. For $i = 2$, $j = 3$, we have $\varphi_2(X) = \varphi_3 \circ u^*(X) \subset \varphi_3 \circ u^*(\omega) = \varphi_3(\omega/|\alpha|)$. Since $\mathrm{Int}(\omega/|\alpha|)$ is disjoint with $\mathrm{Int}(X)$, then $\varphi_2(\mathrm{Int}(X)) \subset \varphi_3(\mathrm{Int}(\omega/|\alpha|))$ is disjoint with $\varphi_3(\mathrm{Int}(X))$. On the other hand, $\varphi_2(X) \cap \varphi_3(X)$ contains $\varphi_2 \cap \varphi_3 \circ \varphi_1(X) = \psi_2(X) \cap \psi_3(X)$, which is a non-empty set, by the definition of ψ_k and by (2). Finally, for $i = 1$, $j = 3$, $\varphi_1(X) = u^*(|\alpha|X) \subset u^*(|\alpha|\omega)$ while $\varphi_3(X) = \varphi_3 \circ u^*(\overline{\Pi \setminus \omega})$, hence, by (4), $\varphi_3(X) = u^*(\alpha^2 \overline{\Pi \setminus \omega})$, and this set is disjoint with $u^*(|\alpha|\omega)$ because $|\alpha|\omega$ and $\alpha^2 \overline{\Pi \setminus \omega}$ are disjoint. □

2.3. *The limit set.* For any infinite word of symbols $\bar{\varepsilon} = \{\varepsilon_n\}_{n>0}$, $\varepsilon_n \in \{1, 2, 3\}$, denote $\bar{\varepsilon}|n = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ and define the limit set of S as

$$I = \bigcup_{\bar{\varepsilon}} \bigcap_{n=1}^{\infty} \varphi_{\bar{\varepsilon}|n}(X).$$

Since S is finite, I is a closed set. It is easy to see that

$$I = \bigcap_{n>0} X_n.$$

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Here, X_n is a decreasing sequence of compacts $X_n = \bigcup_{|\bar\varepsilon|=n} \varphi_{\bar\varepsilon}(X)$, where $|\bar\varepsilon| = n$ means that the number of symbols in the finite word $\bar\varepsilon$ is n. Since $\varphi_3(c) = c$ and $\varphi_1(c/|\alpha|) = c/|\alpha|$, so $c, c/|\alpha| \in I$.

Remark 3. Let us consider the closure $\overline{I}_\infty$ of the limit set I_∞ of the infinite system $S_\infty = \{\psi_k : X \to X\}_{k\in\mathbb{N}}$, where $I_\infty = \bigcup_{\bar w} \bigcap_{n=1}^\infty \psi_{\bar w|n}(X)$, and $\bar w$ are the infinite words of symbols $w_n \in \mathbb{N}$. Then it is easy to see that $I = \overline{I}_\infty$. Indeed, since S_∞ is induced by S, $\overline{I}_\infty \subset I$. To prove the opposite inclusion, notice that $\varphi_3^n(X) \to c$ $(n \to \infty)$, and consider any point $a \in I$, $a = \bigcap_{n=1}^\infty \varphi_{\varepsilon|n}(X)$, where ε is an infinite word of symbols $\varepsilon_n \in \{1, 2, 3\}$. If $\varepsilon_n = 3$ for all $n > n_0$, and $\varepsilon_{n_0} \in \{1, 2\}$, then $a = \varphi_{\varepsilon|n_0}(c)$ and $\varphi_{\varepsilon|n_0} = \psi_w$ for a finite word w (in the alphabet $\mathbb{N}$ of S_∞). If $\varepsilon_{n_j} \in \{1, 2\}$, for $n_j \to \infty$, then there is an infinite word $\bar w = \{w_n\}_{n=1}^\infty$, $w_n \in \mathbb{N}$, such that $\varphi_{\varepsilon|n_j} = \psi_{\bar w|j}$, and hence $a \in I_\infty$.

Our first aim is to show that I is a simple (closed, i.e. with the end points $c, c/|\alpha|$) curve. Since $X \setminus \bigcup_{1\le i\le 3} \varphi_i(X)$ contains an open non-empty disc, and because of contraction, the Lebesgue measure (area) of I is zero. In particular, I has no interior. Clearly, I is also locally connected. To be more precise, if $x \in \bigcup_{\bar\varepsilon} \bigcap_{n=1}^\infty \varphi_{\bar\varepsilon|n}(\mathrm{Int}(X))$, there is a unique word $\bar\tau$, such that $\varphi_{\bar\tau|n}(I), n \ge 1$, form a sequence of connected closed neighborhoods of x shrinking to x. Let now $x \in \varphi_\tau(\partial X) \cap I$, for a finite word τ, $|\tau| = n$. Assume first that $n = 1$. As follows from (a), (b), (c) (see the proof of Lemma 2.1), $X_n \cap \partial X \subset \partial\varphi_1^{n-1}(X) \cup \partial\varphi_3^{n-1}(X)$ for $n \ge 2$, and that $\varphi_1^n(X) \to c/|\alpha|$, $\varphi_3^n(X) \to c$ as $n \to \infty$. Therefore,

$$I \cap \partial X = \{c/|\alpha|, c\} \tag{9}$$

and, moreover, $I \cap \bigcup_{1\le i\le 3} \partial\varphi_i(X)$ consists of the following four points a_j, $1 \le j \le 4$, with corresponding sequences of closed connected neighborhoods:

$$a_4 = c = \bigcap_{n=1}^\infty \varphi_3^n(I), \quad a_1 = c/|\alpha| = \bigcap_{n=1}^\infty \varphi_1^n(I), \tag{10}$$

$$a_2 = \varphi_1(a_4) = \varphi_2(a_1) = \bigcap_{n=1}^\infty \varphi_1(\varphi_3^n(I)) \cup \varphi_2(\varphi_1^n(I)), \tag{11}$$

$$a_3 = \varphi_2(a_4) = \varphi_3(a_1) = \bigcap_{n=1}^\infty \varphi_2(\varphi_3^n(I)) \cup \varphi_3(\varphi_1^n(I)). \tag{12}$$

If now $n > 1$, then x is an image by $\varphi_{\tau|n-1}$ of one of the points a_j, $1 \le j \le 4$. Thus, I is locally connected at every point.

We have just proved that I is a dendrite. In particular, for every two points $x, y \in I$ there exists a unique simple arc $[x, y]$ connecting x, y inside I. Now we show that $I = [c/|\alpha|, c]$. Assume there is some $z \in I \setminus [c/|\alpha|, c]$. Then we get a (branch) point $d \in [c/|\alpha|, c]$, i.e. such that $[z, c] \cap [c/|\alpha|, c] = [d, c]$ and $[z, c/|\alpha|] \cap [c/|\alpha|, c] = [d, c/|\alpha|]$. If $d \in \bigcup_{\bar\varepsilon} \bigcap_{n=1}^\infty \varphi_{\bar\varepsilon|n}(\mathrm{Int}(X))$, then passing to a big scale by some $(\psi_{\bar\varepsilon})^{-1}$ we get that I has at least three different points with the boundary of X, a contradiction to (9). If $d \in \varphi_\tau(\partial X) \cap I$, for a finite word τ, $|\tau| = n$, then applying $\varphi_{\tau|n-1}^{-1}$ one can assume from the beginning that the branch point d is one of the points a_j, $1 \le j \le 4$, and also that, for some $1 \le i \le 3$, $[d, z] \cap \varphi_i(X)$ contains a non-trivial subarc $[d, z']$, and either

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$[c, d]$ or $[c/|\alpha|, d]$ intersects $\partial\varphi_i(X)$. Then, $\varphi_i^{-1}(d)$ is either c or $c/|\alpha|$. If, say, it is c, then $[c/|\alpha|, c] \cup \varphi_i^{-1}([d, z'])$ is a subset of I, and it intersets $\partial\varphi_3^n(X)$, n big, in at least three points, a contradiction to (9).

2.4. *The curve L. Proof of (1) and (2), (4) and (5) of Theorem A.* Define two curves: $L_0 = I \setminus \{c, c/|\alpha|\}$ and

$$L = \bigcup_{n\in\mathbb{Z}} |\alpha|^n I.$$

Then L is a curve which satisfies the property (1) of Theorem A. L is a simple curve because $c \in I \cap |\alpha|I \subset (\mathrm{Int}(X) \cup \{c, c/|\alpha|\}) \cap |\alpha|(\mathrm{Int}(X) \cup \{c, c/|\alpha|\}) = \{c\}$, by Lemma 2.1(2). By Remark 3 and from the definition of the maps ψ_k, we see that

$$u^*(L_\infty) = \bigcup_{k>0} u^*(|\alpha|^k(L_0 \cup \{c\})) = \bigcup_{k>0} \psi_k(L_0 \cup \{c\}) = L_0,$$

i.e.

$$g(L_0) = L_\infty^*.$$

Therefore, using the functional equation, properties (4a)–(4c) follow easily. In particular, (4c) implies that $L_n \cap J = \emptyset$ for all $n \geq 0$, thus proving also (2). Let us prove property (5) and the uniqueness.

The uniqueness: assume that l is a curve in Π satisfying (1) and (2) of Theorem A. Let $x \in l \cap \partial\Omega_0$. Since $\partial(\Pi \cap \Omega_0) \setminus J_\infty = \{c\}$, we must have $x = c$. Hence, also, $c/|\alpha|^n \in l$, $n \geq 0$. If, now, l is different from L, then (1) and (2) give us a contradiction, as in the proof of Corollary 1.1.

(5): for any $k > 0$, if x lies in the intersection of $\gamma = g^k(L \cap g^{-k}(\Omega))$ and, say, L, then the whole arc of L from x to ∞ also belongs to γ, because otherwise the dense set $\bigcup_{n\geq 0} \alpha^n J$ would be not connected—see the proof of Corollary 1.1. We now prove the other statements. First, it is easy to see that $g(L \cap g^{-1}(\Omega)) \cap L^* = L_\infty^*$ (because $g(L_0) = L_\infty^*$ and $g^2(L_1) = L_\infty^*/\alpha$). From this and from the functional equation, it follows that, for any $n \geq 0$, $g^{2^n}(L \cap g^{-2^n}(\Omega))$ intersects either L^* or $-L^*$ along either $(L^{(n)})^*$ or $-(L^{(n)})^*$. Let us apply this consequently for $n = n_1, n_2, \ldots, n_l$ when $k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_l}$ with $n_1 > n_2 > \cdots > n_l$. Then we get the statement. □

2.5. *Metric properties of L: proof of (3) of Theorem A.*

The curve L is nowhere differentiable. It is enough to show the following.

LEMMA 2.2. *The curve I has no tangent at any point.*

Proof. We follow [**3**]. Assume the contrary, i.e. I has a tangent at a point $a \in I$. Since S is finite, there is an infinite word $\epsilon = \epsilon_1\epsilon_2 \cdots \epsilon_n \cdots$, such that $a = \bigcap_{n=1}^\infty \phi_{\epsilon|n}(X)$. Then, the maps $f_n := \phi_{\epsilon|n} : V \to V$, $n \geq 1$, converge to the point a uniformly in V. Let $b = \lim_{j\to\infty} b_{n_j}$, where $b_n = f_n^{-1}(a)$. One can further assume (possibly passing to a subsequence) that all maps f_{n_j} are holomorphic (if they are all antiholomorphic, replace f_{n_j} by the complex conjugate to f_{n_j}). Define $F_j = (f_{n_j} - a)/f'_{n_j}(b_{n_j})$. Since $F_j(b_{n_j}) = 0$,

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$F_j'(b_{n_j}) = 1$ and F_j are univalent, one can assume (possibly passing to a subsequence) that F_j converge uniformly in V to a univalent map F, $F(b) = 0, F'(b) = 1$. Then, $f_{n_j}(z) - a = \lambda_j(F(z) + \delta_j(z))$, where $\lambda_j = f_{n_j}'(b_{n_j}) \to 0$ as $j \to \infty$, $\lambda_j \neq 0$ and $\delta_j \to 0$ uniformly in V. Consider any $z \in I$, $z \neq b$. Then, $F(z) \neq 0$. If one fixes a sequence $j_k \to \infty$, such that $\arg \lambda_{j_k} \to \theta$, then $\arg(f_{n_j}(z) - a) \to \theta + \arg F(z)$. By our assumption, we conclude that the function $\arg F(z)$ is a constant in $z \in I$. Therefore, I is an analytic arc. In particular, it is analytic in a neighborhood of the point $u^*(|\alpha|c) \in L_0 = I \setminus \{c, c/|\alpha|\}$. But, $u^*(L_\infty) = L_0$. Therefore, the curve $L = \bigcup_{n\in\mathbb{Z}} |\alpha|^n I$ is analytic in a neighborhood of the point c, and, hence, the curve L is analytic at any point. Consider the curve $u^*(L)$. It is also analytic and joins the points $x_0 \notin L$ and $c \in L$. Moreover, $L_0 \subset L \cap u^*(L)$. Thus, L_0 has two different analytic continuations, a contradiction. □

This proves that L has no tangent at any point.

Hausdorff dimension of L. Here we prove that $\mathrm{HD}(I) > 1$. The method of the proof is very similar to that for the Julia set of $z^2 + \varepsilon$ [**16**]. First, we show the existence of an h-conformal measure for the system S (Lemma 2.3), then we derive from here that $h = \mathrm{HD}(I)$ and that the h-Hausdorff measure of I is between 0 and ∞ (Corollary 2.1). Hence, if $h = 1$, then the curve I is rectifiable, and therefore, has tangents at many points, a contradiction to Lemma 2.2. Here, we only state Lemma 2.3 and Corollary 2.1. The proof is very close to the proof of corresponding statements in [**15**] and [**12**], and is postponed until the Appendix (one could also apply directly the theory developed in [**12**]).

LEMMA 2.3. *There exist* $0 < h \leq 2$ *and a probability measure* μ *with its support* $\mathrm{supp}(\mu) = I$, *which is h-conformal, i.e.* $\mu(\varphi_i(A)) = \int_A |\varphi_i'(x)|^h \, d\mu(x)$ *for* $i = 1, 2, 3$ *and any Borel set* $A \subset X$. *Moreover,* μ *has no atoms.*

COROLLARY 2.1. $h = \mathrm{HD}(I)$. *Moreover, the h-Hausdorff measure of I is finite and positive.*

As we noticed before, Lemma 2.3, Corollary 2.1, and Lemma 2.2 imply that $\mathrm{HD}(L) > 1$.

L is a quasi-circle. The inequality $\mathrm{HD}(I) < 2$ follows from Lemma 2.3 and [**12**, Theorem 4.5]. Hence, $\mathrm{HD}(L) = \mathrm{HD}(I) < 2$. But, we prove a stronger statement: L (or, equivalently, I) is a quasi-circle. According to [**1**], it is enough to prove that I satisfies the M-condition: there is an M such that for every three points ξ_1, ξ_2, ξ_3 of the curve I such that ξ_3 lies between ξ_1 and ξ_2, $|\xi_3 - \xi_1|/|\xi_2 - \xi_1| \leq M$. Let us agree that the point ξ_1 lies between $c/|\alpha|$ and ξ_3. The M-property follows from the following two observations.

(a) Because of the bounded distortion property Lemma 2.1(6), one can assume that the points ξ_1 and ξ_2 belong to different $\varphi_k(I), \varphi_n(I)$, $1 \leq k < n \leq 3$: $\xi_1 \in \varphi_k(I), \xi_2 \in \varphi_n(I)$.

(b) If $n - k = 2$, then there is $\sigma > 0$ such that $|\xi_1 - \xi_3| \geq \sigma$, for all ξ_1, ξ_3 as above, and the existence of some $M = M(\sigma)$ follows. Assume, now, that $n - k = 1$. For example, $\xi_1 \in \varphi_2(I)$, $\xi_2 \in \varphi_3(I)$. Note that $\varphi_2(I)$ and $\varphi_3(I)$ have in common a single boundary point $a = a_3 = \varphi_2(c) = \varphi_3(c/|\alpha|) = \bigcap_{n=1}^{\infty} \varphi_2(\varphi_3^n(X)) \cup \varphi_3(\varphi_1^n(X))$.

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By the triangle inequality, it is enough to prove the M-property for $\xi_3 = a$. So, assume $\xi_3 = a$. Note that $\xi_1 \in \varphi_2(\varphi_3^n(X))$, $\xi_2 \in \varphi_3(\varphi_1^m(X))$, for some $n, m \geq 0$, and the sets $\varphi_3^k(X)$, $\varphi_1^{2k}(X)$ converge (as $k \to \infty$) to the fixed points c, $c/|\alpha|$ of φ_3, φ_1 respectively with the same speed, because $\varphi_1^2 = |\alpha|^{-1} \circ \varphi_3 \circ |\alpha|$. Hence, $|\xi_1 - a|/|\xi_1 - \xi_2|$ is comparable to $|\tilde{\xi}_1 - a|/|\tilde{\xi}_1 - \tilde{\xi}_2|$, where $\tilde{\xi}_1 \in \varphi_2(\varphi_3^{n_0}(X) \setminus \varphi_3^{n_0+1}(X))$, $\tilde{\xi}_2 \in \varphi_3(\varphi_1^{m_0}(X) \setminus (\varphi_1^{m_0+2}(X)))$, with some $n_0, m_0 \geq 0$, and, additionally, either n_0 or m_0 is zero. Also, any two sets of the form $\varphi_2(\varphi_3^n(X) \setminus \varphi_3^{n+1}(X))$, $\varphi_3(\varphi_1^m(X) \setminus \varphi_1^{m+2}(X))$ are disjoint. All this implies that the distance between $\tilde{\xi}_1, \tilde{\xi}_2$ is bounded from below by the (positive) distance between the sets $\varphi_2(\varphi_3^{n_0}(X) \setminus \varphi_3^{n_0+1}(X))$, $\varphi_3(\varphi_1^{m_0}(X)) \setminus (\varphi_1^{m_0+2}(X))$. The M-property follows.

3. *Proof of Theorem B*

(a) Note that, for any $n \geq 0$, $\mathcal{M}_n$ is invariant with respect to the complex conjugation and with respect to the rotation by the rth root of unity. In particular, it is invariant under $z \mapsto -z$. Hence, Theorem A(1) implies that if $R \in \mathcal{M}_0$ then $R/\alpha \in \mathcal{M}_0$. If, now, $g^n(R) = R_0$ for some $R_0 \in \mathcal{M}_0$, then $g^{2n}(R/\alpha) = g^n(R)/\alpha = R_0/\alpha \in \mathcal{M}_0$. Let us show that $\partial R \cap J_\infty$ is the base point $\{x_R\}$ of R. First, $\partial R \cap J = \{x_R\}$, from the definition. If, now, $x \in \partial R \cap \alpha^n J$, for some n, then $x/\alpha^n \in \partial(R/\alpha^n) \cap J = \partial G \cap J = \{x_G\}$, for a piece G. Hence, $x = \alpha^n x_G = x_R$.

(b) This follows from Theorem A(5). In fact, by the same reasoning, a stronger statement holds. Namely, recall that R_0 is an infinite open 'sector' bounded by the curves L and $-L^*$. Introduce the set $\tilde{\mathcal{M}} = \{\tilde{R}\}$ of all components of $g^{-n}(R_0)$ and $g^{-n}(R_0^*)$ for all $n \geq 1$. Then, any two such components are either disjoint or one covers the other one. Repeating (a), we get also that if $\tilde{R} \in \tilde{\mathcal{M}}$, then $\pm\tilde{R}/\alpha \in \tilde{\mathcal{M}}$.

Note that any piece R is contained in one and only one element $\tilde{R} \in \tilde{\mathcal{M}}$ such that R and $\tilde{R}$ have the same boundary 'base points' (recall that if $g^n(R) \in \mathcal{M}_0$, then the 'base points' of R are x_R, x_R^∞ such that $g^n(x_R) = 0$, $g^n(x_R^\infty) = \infty$).

(c) and (d). It is enough to prove both statements, replacing $R \in \mathcal{M}$ by $\tilde{R} \in \tilde{\mathcal{M}}$.

Let us construct a 'machine' that produces new elements of $\tilde{\mathcal{M}}$. Consider a component $\tilde{R}_1$ of $g^{-1}(R_0^*)$, namely, $\tilde{R}_1 = u^*(R_0)$. Its boundary consists of the points c, x_0 and two arcs $\gamma^\pm$ joining the points c, x_0, such that $g(\gamma^+) = -L$ and $g(\gamma^-) = L^*$. Moreover, $\gamma^- \cap L = L_0$ (because $g(L_0) = L_\infty^* \subset L^*$), while γ^+ has no common arcs with L. Denote by γ_1^- the sub-arc of γ^- between the points $c/|\alpha|$ and x_0: $\gamma_1^- = \gamma^- \setminus \overline{L_0}$. In turn, it is divided into two arcs: β_1 and β_2 such that $g(\beta_1) = L_0^*$ and $g(\beta_2)$ is the arc of L^* between 0 and $c^*/|\alpha|$. Observe that

$$\gamma^+/|\alpha| \cap \gamma_1^- = \beta_1. \tag{13}$$

(Proof: $g^2(\gamma^+/|\alpha|) = g(\gamma^+)/\alpha = -L/\alpha = L$; hence, $g(\gamma^+/|\alpha|) = (\gamma^-)^*$ and $g(\gamma^+/|\alpha| \cap \gamma_1^-) = L_0^* = g(\beta_1)$.)

Denote by Δ_0 an open 'triangle' (with the 'vertices' $x_0, x_0/|\alpha|, c/|\alpha|$) bounded by the interval $(x_0/|\alpha|, x_0)$, the sub-arc $\beta_2 = \gamma_1^- \setminus \beta_1$ of γ_1^-, and the sub-arc $\gamma^+/|\alpha| \setminus \beta_1$ of $\gamma^+/|\alpha|$. Denote also by Δ a (bigger) open 'triangle' bounded by the interval $(0, x_0)$, the arc γ_1^-, and the arc $L \setminus \overline{L_0 \cup L_\infty}$ of L between 0 and $c/|\alpha|$. It is now easy to check that $g(\partial\Delta_0) = \partial\Delta^*$, i.e. $\Delta_0 = u^*(\Delta)$. Also, $R_2 := \tilde{R}_1/|\alpha| \subset \Delta$. We have now a dynamical system (the 'machine') consisting of two maps $A : z \mapsto z/|\alpha|$ and $B : z \mapsto u^*(z)$ of Δ

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into itself which produces new elements of $\tilde{\mathcal{M}}$: if $\tilde{R} \in \tilde{\mathcal{M}}$ is a subset of Δ, then $A(\tilde{R})$ and $B(\tilde{R})$ are again elements of $\tilde{\mathcal{M}}$. To study it, note that there is a neighborhood U of $\overline{\Delta}$, such that $B = u^*$ extends to a conformal map on U, $\overline{B(U)} \subset U$ and $\overline{A(U)} \subset U$ (for example, take a Poincaré neighborhood of an interval $(-\varepsilon, 1)$ in the slit plane $(\mathbb{C} \setminus \mathbb{R}) \cup (-\varepsilon, 1)$). Then, $A, B : \overline{\Delta} \to \overline{\Delta}$ are strict contractions in the hyperbolic metric of U. Also, $A(\Delta)$, $B(\Delta)$ and R_2 are pairwise disjoint, and $A(\overline{\Delta}) \cup B(\overline{\Delta}) \cup \overline{R}_2 = \overline{\Delta}$. It follows that if we apply to R_2 all possible finite compositions of A and B, then the union of the closures of all elements of $\tilde{\mathcal{M}}$ obtained like this, with the points of the segment $[0, x_0]$, is just the closure of Δ.

Let us finish the proof of (c). As we have just seen, the set $F = \cup \pm (\overline{\Delta \cup \Delta^*})$ is filled by the closed pieces (and the points of the segment $[-x_0, x_0]$). F is also a closed neighborhood of the point $g^2(0) = 1/\alpha \in (-x_0, 0)$. Pulling F back by a branch of g^{-2} we get a neighborhood of zero with the same property.

Let us prove (d). Let R be any piece. Since R is a preimage of a piece of depth 1, there indeed exists a minimal $n \geq 0$ such that $g^n(R) = G/\alpha^k$, for some piece G of depth 1 and for some $k \geq 0$. Then, $g^n(\tilde{R}) = \tilde{G}/\alpha^k$, for corresponding $\tilde{R} \supset R$ and $\tilde{G} \supset G$ (so that $g(\tilde{G})$ is either R_0 or R_0^*). To prove (d2), we find for each (among $2r$) component $\tilde{G}$ of $g^{-1}(R_0)$ or $g^{-1}(R_0^*)$ a fixed neighborhood $V(\tilde{G})$, so that the branch of g^{-n} from $\tilde{G}/\alpha^k$ onto $\tilde{R}$ extends to a map onto $V(\tilde{G})/\alpha^k$.

If $\tilde{G}$ does not touch the real axis, such a neighborhood obviously exists. The remaining possibility is that $\tilde{G}$ is one of $\pm\tilde{R}_1, \pm\tilde{R}_1^*$ (see the definition of $\tilde{R}_1$ above).

For example, let $\tilde{G} = \tilde{R}_1$. Let us fix a neighborhood V of the closure of $\tilde{R}_1$, so that V intersects the real axis along, say, the interval $(g^3(0), 1 + \epsilon)$, with $\epsilon > 0$ fixed and small enough. Assume that the branch of g^{-n} from $\tilde{R}_1/\alpha^k$ onto $\tilde{R}$ does not extend to a map on V/α^k, i.e. $g^{-n} = g^{-(n-m)} \circ g^{-m}$, where $g^{-m}(V/\alpha^k)$ covers 0 for the first time. Then, as the real Feigenbaum dynamics shows us, $m = 2^k$, and, moreover, the branch $g^{-m}(z) = g^{-2^k}(z) = (1/\alpha^k)g^{-1}(\alpha^k z)$; hence, the element $g^{n-m}(\tilde{R}) = g^{-m}(\tilde{R}_1/\alpha^k)$ has the form $\tilde{G}_1/\alpha^{k+1}$, for a component $\tilde{G}_1$ of either $g^{-1}(R_0)$ or $g^{-1}(R_0^*)$, a contradiction with the minimality of n.

Thus, (d1) and (d2) are proved.

To complete the proof, we show that if $R_m \subset R_{m-1} \subset \cdots \subset R_1$ are different pieces, then $\operatorname{diam}(R_m) \leq C\lambda^m \operatorname{diam}(R_1)$, for universal $C > 0, \lambda < 1$. First, observe that any piece G of depth 1 contains at least two (in fact, an infinite number of) disjoint pieces of level 2 (i.e. which are not contained in any other piece except G). Hence, by (d1) and (d2), there exists $\rho < 1$, so that $\operatorname{area}(R')/\operatorname{area}(R) \leq \rho$ for arbitrary different pieces $R' \subset R$. It follows that $\operatorname{area}(R_m) \leq \rho^m \operatorname{area}(R_1)$. Second, again by (d1) and (d2), and by compactness of an appropriate family of univalent maps, there is C, such that $C^{-1} \leq (\operatorname{diam}(R))^2/\operatorname{area}(R) \leq C$ for all pieces R. Therefore, $\operatorname{diam}(R_m) \leq C\rho^{m/2}\operatorname{diam}(R_1)$.

The proof of (c) and (d) yields the following.

Remark 4. We have seen that the 'machine' produces the covering of the triangle Δ by the pairwise disjoint pieces, and each of them touches the interval $(0, x_0)$. It follows that every piece R of level 1 touches at its 'base point' x_R one of the lines $\mathbb{R}\exp(i\pi j/r)$, $j = 0, 1, \ldots, r-1$.

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Remark 5. Since every point of $\overline{\Delta} \cap \mathbb{R}^+$ is a limit of real 'base points' $x_{\tilde{R}}$ of elements $\tilde{R}$ produced by the 'machine', for every point $x \in \overline{\Delta}$, which lies either in $\mathbb{R}^+$ or in the boundary of a piece of level 1, there exists a simple curve $l(x, c/|\alpha|)$ which joins the points x and $c/|\alpha|$, lies in the (open) 'triangle' Δ and is such that $l(x, c/|\alpha|)$ consists of closed sub-arcs of boundary curves of pieces of level 1.

Remark 6. The 'ray' $g^2(L \cap g^{-2}(\Omega))$, in a neighborhood of the landing point $g^2(0)$, consists of (closed, i.e. with their end points) arcs of boundaries of pieces of level 1 (follows from Theorem A(5)). This sequence of arcs accumulates only at the point $g^2(0)$, as follows from Theorem B(d) and the previous remark. Pulling this picture back, we see the same is true for the 'ray' L (with the obvious replacement of $g^2(0)$ by 0): L consists of a sequence of (closed) arcs of ∂R, where R are some pieces of level 1, which touch the straight ray $\mathbb{R}^+ \exp(i\pi/r)$. Recall also that at the same time L consists of a sequence of (closed) arcs of ∂R with ends at the points $c/|\alpha|^n, n \in \mathbb{Z}$, where R are some pieces of level 1, which touch the real ray $\mathbb{R}^+$.

4. *Some conclusions*

Let us summarize some useful facts that follow from the previous section.

4.1. *The structure of a piece.* Let $R \in \mathcal{M}^{\text{ext}}$ be any piece. By the definition, $g^m(R/\alpha^j) = G$, for some $m, j \geq 0$, where G is a piece of depth 0, i.e. one of the infinite 'sectors' $R_{0,1}, R_{0,2}, \ldots, R_{0,r-1}, R^*_{0,1}, R^*_{0,2}, \ldots, R^*_{0,r-1}$ (see the definition before Theorem B). The G contains one and only one straight ray of the form $\mathbb{R}^+ \exp(i\pi j/r)$, $j = \pm 1, \ldots, \pm(r-1)$. Denote it by γ_G. The following facts have been proved.

(1) The boundary of R, $\partial R = L(R) \cup \{x_R\}$, where $x_R \in B$ is a 'base point', i.e. $g^m(x_R/\alpha^j) = 0$, and $L(R)$ is a simple (open) curve which consists of arcs of preimages of the curve L. Hence, $L(R) \subset \mathbb{C} \setminus J_\infty$.

(2) Define the 'vein' of R, γ_R, so that $g^m(\gamma_R/\alpha^j) = \gamma_G$. Then γ_R is an analytic arc joining the 'base points' x_R, x_R^∞ of R. Moreover, $\gamma_R \subset J_\infty$, and the length of γ_R is finite (Remark 1 and Theorem B(d)).

(3) Let $n \geq 1$ be the level of the piece R. Let us look at the set $\mathcal{M}^{\text{ext}}(R)$ of all pieces of level $n+1$ which are contained in R. Then:

  (3a) they form a partition of R, i.e. they are pairwise disjoint, and the union of their closures $\overline{P}$ ($P \in \mathcal{M}^{\text{ext}}(R)$) united with the (closed) 'vein' $\gamma_R \cup \{x_R\} \cup \{x_R^\infty\}$ is the closure of R;

  (3b) each $P \in \mathcal{M}^{\text{ext}}(R)$ touches the 'vein' γ_R at the point x_P (Remark 4); the boundaries of P, R are either disjoint or have a common non-trivial arc;

  (3c) the latter is true for any two pieces $P, P' \in \mathcal{M}^{\text{ext}}(R)$, with an extra possibility that P, P' can lie on different sides of the 'vein' and have the same 'base point' $x_P = x_{P'}$;

  (3d) all $P \in \mathcal{M}^{\text{ext}}(R)$ as well as R have roughly the same shape (see Theorem B(d));

  (3e) the 'base points' x_R, x_R^∞ divide ∂R into two simple curves: $L^\pm(R)$ (in other words, $L^+(R) \cup L^-(R) \cup \{x_R^\infty\} = L(R)$). As follows from

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Remark 6, either curve $L^{\pm}(R)$ consists of a sequence $\{L_n^{\pm}(R)\}_{n=-\infty}^{+\infty}$ of (closed) sub-arcs of boundaries of pieces $R_n^{\pm}$ from $\mathcal{M}^{\text{ext}}(R)$ (so that $x_{R_n^+} = x_{R_n^-}$), which accumulate at the points x_R, x_R^{∞}: $\lim_{n\to-\infty} L_n^{\pm}(R) = \{x_R\}$, $\lim_{n\to+\infty} L_n^{\pm}(R) = \{x_R^{\infty}\}$.

For every large enough integer m, denote by $W_m(R)$ the domain bounded by the arcs $L_n^{\pm}$, $n \geq m$, the points x_R^{∞} and $x_{R_m^+} = x_{R_m^-}$, and arcs of $\partial R_m^{\pm}$ from $x_{R_m^{\pm}}$ to ∂R.

(3f) if y is an end point of an arc $L_n^{\pm}(R)$, then, for some non-negative n, j, the point $g^n(y/\alpha^j)$ belongs to the set $\{c\exp(i2\pi j/r), c^*\exp(i2\pi j/r), j = 0,\dots,r-1\}$. Indeed, it is enough to prove this when R is of depth 0. Note that y belongs to the boundaries of two different pieces. Then, iterating it further we apply Theorem A(5) and get the statement.

4.2. *Neighborhoods of points.* Fix a point $z \in \partial M^{\text{ext}} \setminus Y$. That is, for some non-negative n, j, the point $w = w(z) = g^n(z/\alpha^j)$ is either infinity or belongs to one of the curves $L\exp(i2\pi j/r)$, $L^*\exp(i2\pi j/r)$ $(0 \leq j \leq r-1)$. We want to describe a basis of neighborhoods at z as a union of pieces. Consider two cases:

- z is not a 'base point' x_R^{∞} of any piece R. Then, by Remark 4, the corresponding w is a boundary point of at least two and at most three closed pieces of level 1, such that their union forms a closed neighborhood of w. Returning to z and using (3e), we find a sequence of closed neighborhoods $W_n(z)$ of z shrinking to z, such that each $W_n(z)$ is the union of at least two and at most three closed pieces;
- z is a 'base point' x_R^{∞} of a piece R. It follows from the definition of x_R^{∞}, that then there are exactly $r-1$ pieces R_j, $j = 1,\dots,r-1$, so that $z = x_{R_j}^{\infty}$. Moreover, all R_j are of the same depth (and, in fact, of the same level). Similarly to the first case, we then get a sequence of closed neighborhoods $W_n(z)$ of z shrinking to z, such that each $W_n(z)$ is the union of at least 1 and at most two closed pieces *and* $r-1$ closures of domains $W_n(R_j)$, $j = 1,\dots,r-1$ defined in (3e) (see the previous subsection).

5. *Theorem C and Corollary 1.1*

5.1. *Proof of Theorem C.* If $z \in J^{\text{ess}}$, (a) follows directly from Theorem B(b) and (c). If $z \in \partial M^{\text{ext}} \setminus Y$, then (a) follows from §4.2.

To prove (c) we need the following.

FACT. *There is C' such that, for any piece $R \in \mathcal{M}$ and any point $w \in \overline{R}$, there exists a rectifiable curve $\sigma(w, x_R) \subset R \cap J_\infty$ (in fact, consisting of closed (arcs of) 'veins' of different pieces), which joins w and the 'base point' x_R and of length less than $C'\text{diam}R$.*

Indeed, let R be of level n; if $z \notin Y$, then, by (a), $\{w\} = \bigcap_{k\geq n} \overline{R_k(w)}$ where $R_k(w)$ is a piece of level k such that $w \in \overline{R_k(w)}$. Moreover, for any $k \geq n$, $R_{k+1}(w)$ is contained in $R_k(w)$ and touches the 'vein' of $R_k(w)$. With this representation, the curve $\sigma(w, x_R)$ arises naturally: we start at x_R and follow the 'vein' γ_R until the base point $x_{R_{n+1}(w)}$, then follow the 'vein' $\gamma_{R_{n+1}(w)}$ until the base point $x_{R_{n+2}(w)}$ etc. Now, the fact is a corollary of Theorem B(d). If $z \in Y$, then y belongs to a 'vein' of a piece R_m contained in $R_n(w)$. Then, we repeat the above construction, going along 'veins' from the piece R_m until the base point x_R.

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Then (c) follows from the fact and Theorem B(c).

It remains to prove (b) and (b′). The *existence* of the curve $l(x, y)$ $(x, y \in \overline{\Pi})$ follows from (i)–(vii).

(i) If $l(x, y)$ is constructed then one can set $l(|\alpha|^j x, |\alpha|^j y) = |\alpha|^j l(x, y)$, $j \in \mathbb{Z}$.

(ii) It is enough to construct $l(x, c)$, for any $x \in \overline{\Pi}$, because then $l(x, y)$ will be a composition of the curves $l(x, c), l(c, y)$ with cancelling out arcs passing in opposite directions.

(iii) If $x \in \mathbb{R}^+$ or $x \in \partial R$, where R is a piece of level 1, which lies in the closed 'sector' S bounded by the curves L and $\mathbb{R}^+$, then using (i) and Remark 5, we see that the curve $l(x, c)$ exists, belongs to S and goes inside some of the curves $L(R)$ where R runs over the pieces of level 1.

(iv) Let $x \in \overline{G}$, for a piece G of *depth* zero (since $x \in \overline{\Pi}$, it means that x belongs to the closed 'sector' bounded by L and $\exp(2\pi i/r)\mathbb{R}^+$). Moreover, let us assume that x lies either on the ray $\exp(2\pi i/r)\mathbb{R}^+$ or in ∂R, for a piece R of level 1. Consider the point $g(x)$. It belongs to the 'sector' S (defined in (iii)). Using (i) and (iii), we obtain the curve $l(g(x), c)$, and then define the curve $l(x, g^{-1}(c)) = g^{-1}(l(g(x), c))$, with a branch $g^{-1} : g(G) \to G$. Extending the curve $l(x, g^{-1}(c))$ along the boundary of G to infinity, we get a curve $l(x, \infty)$ which lies in $\overline{G}$ and consists of arcs of the curves $L(R)$ where R are pieces of level 1.
Cases (iii) and (iv) cover the points $x \in \overline{\Pi}$ belonging either to two rays $\partial\Pi$ or to the boundary of a piece of level 1.

(v) Now we can go into deeper levels. Let $x \in \overline{R}_n$, for some $R_n \in \mathcal{M}^{\text{ext}}$, a piece of level n, and such that x lies either on the 'vein' γ_{R_n} or in the boundary ∂R_{n+1} of a piece R_{n+1} of level $n+1$. Using (i) one can assume that $R_n \in \mathcal{M}$. If, now, $m \geq 0$ is so that $g^m(R_n) = G \in \mathcal{M}_0$, then we can apply (iv) to the point $g^m(x) \in \overline{G}$. Going back to $\overline{R}_n$ by a branch of g^{-m}, we get the curve $l(x, x^\infty_{R_n}) \subset \overline{R}_n$. It consists of arcs of curves $L(R)$, for a collection of pieces $R \subset R_n$ of level $n+1$.

(vi) Let $x \in Y \setminus \partial\Pi$. Then x lies in a 'vein' of a closed piece R_n of a maximal *level* n. Then use (v) to get $l(x, x^\infty_{R_n})$. Since $x^\infty_{R_n}$ lies in a closed piece R_{n-1} of level $n-1$, we get $l(x^\infty_{R_n}, x^\infty_{R_{n-1}})$, which composing with $l(x, x^\infty_{R_n})$ gives $l(x, x^\infty_{R_{n-1}})$ etc. until we get $l(x, x^\infty_R)$, where x^∞_R is a base point of a piece R of level 1. Then apply (iv). The above considerations also treat the case when $x \in \partial\mathcal{M}^{\text{ext}}$.

(vii) Finally, let $x \in J^{\text{ess}}$. Then $\{x\} = \bigcap_{n=1}^\infty R_n(x)$, and $l(x, c)$ can be defined as the composition of curves $l(x^\infty_{R_n(x)}, x^\infty_{R_{n+1}(x)})$ for all n.

Let us prove the uniqueness. If x, y are not in J_∞, then $l(x, y)$ is unique because otherwise (as in the proof of Corollary 1.1) J_∞ would not be connected. Let now $x \in |\alpha|^n J \setminus Y$. If there are two different simple curves joining x, y and satisfying (b1) and (b2) of Theorem C, then again because J_∞ is connected, for some $m \geq n$, the set $|\alpha|^m J \setminus \{x\}$ has at least two components. This is equivalent to saying that $x \in |\alpha|^m Y_0$, a contradiction.

(b′). (i) is clear from the construction. (ii): notice that, by the construction of the curves $l(x, y)$, any two such curves are either disjoint, or intersected along an arc, so that each end point belongs to two different pieces. It remains to apply the property (3f) of §4.1.

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5.2. *Proof of Corollary 1.1(b2).* We need to show that the curves $l(x, y) \to z$ whenever $x, y, z \in \partial W$ and $x, y \to z$.

If $z \in J^{\text{ess}}$, then z is the intersection of nested sequence of (open) pieces $R_n(z)$, so that $R_n(z)$ is a neighborhood of z. Then the statement follows from Theorem C(a).

If $z \in \partial M^{\text{ext}} \setminus Y$, then we use the sequence of closed neighborhoods $W_n(z)$ of z constructed in §4.2: $l(x, y) \subset W_n(z)$ whenever $x, y \in W_n(z)$. Since the $W_n(z)$ shrink to z, we get the result. □

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A. *Appendix. Proof of Lemma 2.3 and Corollary 2.1*

Recall that I is the limit set of the system of maps $S = \{\varphi_i : X \to X\}_{1 \le i \le 3}$.

Proof of Lemma 2.3. We need to find $0 < h \le 2$ and a non-atomic probability measure μ with its support $\text{supp}(\mu) = I$, which is h-conformal, i.e. $\mu(\varphi_i(A)) = \int_A |\varphi_i'(x)|^h \, d\mu(x)$ for $i = 1, 2, 3$ and any Borel set $A \subset X$. There are two ways of proving such statements. One method goes back to Bowen and Ruelle, and is developed e.g. in [**12**]; first, it shows the existence of a so-called semi-conformal measure, which is a fixed point of the corresponding adjoint Ruelle transfer operator, then the condition (5) of Lemma 2.1 can be used to prove that this semi-conformal measure is actually conformal (see the proof of [**12**, Lemma 3.10]). Here we follow the second, more direct, approach of [**15**], where the existence of conformal measure for rational maps is proved. The main difference from [**15**] is that the system of inverse maps φ_i^{-1} does not glue to a global map. On the other hand, the set of points where it does not happen, $\varphi_i(X) \cap \varphi_j(X), i \ne j$, is 'small' because of Lemma 2.1(5).

Fix a point $z \in \text{Int}(X) \setminus I$. For every $s > 0$, define the series

$$P(s) := \sum_{\varepsilon \in \Delta} |\varphi_\varepsilon'(z)|^s,$$

where Δ is the set of all finite words ε. Considering the area case ($s = 2$), we see that $P(s)$ is finite for $s \ge 2$. Define

$$h = \inf\{s : P(s) < \infty\}.$$

Then $h > 0$ because $\inf\{i, x \in X : |\varphi_i'(x)|\} > 0$. Pick a positive function $\tau(t), t > 0$, such that:

(i) $\tau(t) \to +\infty$ as $t \to 0$ in such a way that, for all $\delta > 0, 0 < \lambda < \infty$, there is $t_0(\delta, \lambda)$ so that $\tau(\lambda t)/\tau(t) \in (1 - \delta, 1 + \delta)$ for $0 < t \le t_0$;

(ii) $P_\tau(h) := \sum_{\varepsilon \in \Delta} \tau(|\varphi_\varepsilon'(z)|)|\varphi_\varepsilon'(z)|^h = \infty$.

(If $P(h) = \infty$, there is no need to introduce the function τ.)

Notice that (i) implies that $\tau(t) < t^{-a}$ for every $a > 0$ and all t small enough. Therefore, $P_\tau(s) := \sum_{\varepsilon \in \Delta} \tau(|\varphi_\varepsilon'(z)|)|\varphi_\varepsilon'(z)|^s < \infty$ for $s > h$.

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For every $s > h$, define a probability measure

$$\mu_s = \frac{1}{P_\tau(s)} \sum_{\varepsilon \in \Delta} \tau(|\varphi'_\varepsilon(z)|)|\varphi'_\varepsilon(z)|^s \delta_{\varphi_\varepsilon(z)}. \tag{14}$$

Let μ be any weak limit of μ_s as s decreases to h. Then, (ii) implies that μ is supported on I. Fix $i \in \{1, 2, 3\}$. First, consider any point $x_0 \in I \cap \mathrm{Int}(X)$ and consider a small enough neighborhood U of x_0. Then, $\varphi_j(x) \in \varphi_i(U)$, for some $x \in X$, implies that $j = i$ and $x \in U$. Because of (i), and since the length of words ε, such that $\varphi_\varepsilon(z) \in U$, tends to infinity as U shrinks to x_0, we derive that for any $s > h$,

$$\mu_s(\varphi_i(U))/|\varphi'_i(x_0)|^s \mu_s(U) \in [K^{-1}, K], \tag{15}$$

where the constant $K > 1$ tends to 1 uniformly in s as U shrinks to x_0.

Let us show that any $x \in I$ is not an atom of μ. Assume the contrary. If $x = \bigcap_{n=1}^\infty \varphi_{\varepsilon|n}(\mathrm{Int}(X))$ for an infinite word ε, then applying (15) finitely many times to the points $\varphi_{\varepsilon|n}^{-1}(x)$ we come to a contradiction to the fact that the measure μ is finite. As for the rest of the points x, the same argument shows that one can assume from the beginning that $x \in I \cap \bigcup_{1 \le j \le 3} \partial\varphi_j(X) = \{a_j, 1 \le j \le 4\}$. If x is either a_1 or a_4, then x is the attracting fixed point of either φ_1 or φ_3. If, say, $x = a_1$, then again, for small neighborhoods U of a_1, $\varphi_j(y) \in \varphi_1(U)$, for some $y \in X$, implies that $j = 1$ and $y \in U$. Then, we can apply the same argument as above to the map φ_1, and get a contradiction. Consider the remaining cases when x is either a_2 or a_3, say, $x = a_2$. Since $a_2 = \varphi_3(a_1) = \varphi_1(a_4)$, for a small neighborhood U of a_2, $\varphi_j(y) \in U$, for some $y \in X$, implies that j is either 3 or 1, and y lies in a neighborhood of a_1, a_4 respectively. It shows that either a_1 or a_4 is an atom, a contradiction.

Since μ has no atoms and $I \cap \bigcup_{1 \le i \le 3} \varphi_i(\partial X)$ is finite, it is easy to see that the measure μ^* defined by $\mu^*(A) = \mu(\varphi_i(A))$, for any Borel $A \subset X$, is absolutely continuous with respect to μ. By the same reasoning, x_0 as in (15) is a typical point of μ. Passing to the limit in (15) as $s \to h$ and U shrinks to x_0 we have $d\mu^*/d\mu = |\varphi'_i|^h$ μ-a.e. □

Proof of Corollary 2.1. Prove that $h = \mathrm{HD}(I)$ and the h-Hausdorff measure of I is finite and positive. This will follow in a usual way (see e.g. [**12**], Lemma 3.14) from the finiteness of the system S and the existence of the conformal measure. By a general fact from geometric measure theory, see e.g. [**13**], it is enough to show that there exists a constant $C > 0$, such that, for every $x \in I$ and every $0 < r < 1$,

$$C^{-1} \le \frac{\mu(B(x, r))}{r^h} \le C. \tag{16}$$

Notice that by the bounded distortion property there is C_0 such that, for every finite word τ, $|\tau| = n$, for every $y \in X$, we have $C_0^{-1} \le \mathrm{diam}(\varphi_\tau(X))/|\varphi'_\tau(y)| \le C_0$, $C_0^{-1} \le \mathrm{area}(\varphi_\tau(X))/\mathrm{diam}(\varphi_\tau(X))^2 \le C_0$, $C_0^{-1} \le \mathrm{diam}(\varphi_\tau(X))/\mathrm{diam}(\varphi_{\tau|n-1}(X)) \le C_0$ and $C_0^{-1} \le \mu(\varphi_\tau(X))/\mathrm{diam}(\varphi_\tau(X))^h \le C_0$. Let $x = \bigcap_{n=1}^\infty \varphi_{\varepsilon|n}(X)$. Let m be minimal such that $\varphi_{\varepsilon|m}(X) \subset B(x, r)$. From the minimality of m and because $x \in \varphi_{\varepsilon|m}(X)$, $\mathrm{diam}(\varphi_{\varepsilon|m}(X)) \ge r/C_0$; hence, $\mu(B(x, r)) \ge \mathrm{diam}(\varphi_{\varepsilon|m}(X))^h/C_0 \ge r^h/C_0^{h+1}$. To prove the opposite inequality, consider a collection Σ of sets of the form $\varphi_w(X)$, such that $B(x, r) \cap \varphi_w(X) \ne \emptyset$, $\varphi_w(X) \subset B(x, 2r)$, and the cardinalities $|w|$ are minimal

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(i.e. if $\varphi_w(X) \in \Sigma$, then $\varphi_{w||w|-1}(X)$ is not a subset of $B(x, 2r)$). Clearly, these sets form a cover of $B(x, r) \cap I$, and their interiors are pairwise disjoint. From the minimality of $|w|$, we conclude that $\operatorname{diam}(\varphi_w(X)) \geq r/C_0$. Hence, $4\pi r^2 = \operatorname{area} B(x, 2r) \geq \sum_{A\in\Sigma} \operatorname{area}(A) \geq C_0^{-1} \sum_{A\in\Sigma} \operatorname{diam}(A)^2 \geq Nr^2/C_0^3$, where N is the number of elements in Σ. Therefore, $N \leq 4\pi C_0^3$, and, finally, $\mu(B(x, r)) \leq \sum_{A\in\Sigma} \mu(A) \leq C_0 \sum_{A\in\Sigma} \operatorname{diam}(A)^h \leq NC_0(4r)^h \leq 4^{1+h}\pi C_0^4 r^h$. □

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