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Arcbodies

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Introduction

A Haken manifold is a compact, orientable, irreducible 3-manifold which contains a properly embedded 2-sided, incompressible surface of positive genus. These manifolds are important in connection with the work of Haken, Waldhausen and the more recent work of Thurston (8). Thus it is interesting to investigate criteria for testing incompressible surfaces on 3-manifolds.

The result of this paper is somewhat more general than the one of Gordon and Litherland (3). These authors show that, under certain circumstances, a closed incompressible surface in the complement of a link gives rise to one in any regular branched covering of the link. Here we introduce the concept of *arcbody* which generalizes the 'prime tangle' used by Lickorish (5) and Bleiler (1).

Our main result asserts that if there is a surface F dividing a link L into arcbodies, any regular branched covering over L has an incompressible surface of positive genus. The theorem in particular applies to the link 6_2^3 (borromean rings) which cannot be treated by Gordon and Litherland's method since, as we show in the paper, the complement of 6_2^3 does not have incompressible closed non-peripheral surfaces.

We shall work in the PL category.

1. *Preliminaries*

Through the paper we use the notation (A, l_A) to indicate a pair where l_A is a (possibly empty) 1-manifold properly embedded in an orientable, compact 3-manifold A with boundary.

Definition 1·1. The pair (A, l_A) will be called an *arcbody* if

(i) The inclusion $\partial A - l_A \to A - l_A$ is monic, i.e. any loop γ on $\partial A - l_A$ which is contractible in $A - l_A$ is contractible in $\partial A - l_A$.

(ii) No *component* $(C, C \cap l_A)$ of (A, l_A) is homeomorphic to $(D^2 \times I, \{0\} \times I)$ or $(D^2 \times S^1, \{0\} \times S^1)$, where C is a connected component of A.

(iii) Every sphere of ∂A must intersect l_A.

A *cutting disc* in (A, l_A), is a properly embedded disc D in A intersecting l_A transversely in exactly one interior point and if D, together with a disc E in ∂A, bound a ball $D^3 \subset A$ then $D^3 \cap l_A$ is not homeomorphic to $(D^2 \times I, \{0\} \times I)$. Then, the closure (B, l_B) of the complement of a relative regular neighbourhood of $(D, D \cap l_A)$ in (A, l_A) will be called *the result of cutting* (A, l_A) *along* D.

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LEMMA 1·1. *The result (B, l_B) of cutting (A, l_A) along a cutting disc D is an arcbody if (A, l_A) is an arcbody. Conversely, if (B, l_B) is an arcbody and every sphere of ∂A intersects l_A then (A, l_A) is an arcbody.*

Proof. Suppose (A, l_A) is an arcbody. To show that (B, l_B) satisfies property (i) of Definition 1·1, i.e. $\partial B - l_B \to B - l_B$ is a monic map, assume α is a loop in $\partial B - l_B$ bounding a disc Δ in $B - l_B$. Since $D \cap l_A$ is precisely one point we can suppose that α is contained in $(\partial B - l_B) \cap (\partial A - l_A)$. Then there exists a disc Δ' in $\partial A - l_A$ with $\partial \Delta' = \alpha$. If Δ' is not contained in $\partial B - l_B$, then $\partial D \subset \Delta' \subset \partial A - l_A$ and ∂D would not be essential in $\partial A - l_A$, which is not the case. Thus Δ' lies in $\partial B - l_B$.

The required conditions for D to be a cutting disc insure that (B, l_B) satisfies also properties (ii) and (iii) of Definition 1·1.

Now, suppose (B, l_B) is an arcbody. To prove that (A, l_A) is also an arcbody we will use the following result of González-Acuña (2) in which $N(f: X \to Y)$ means the set of loops γ in X such that $f(\gamma)$ is contractible in Y.

The Six lemma (González-Acuña). *Let W_1^n, W_2^n be submanifolds of a manifold W^n such that $W = W_1 \cup W_2$ and $W_1 \cap W_2$ is a submanifold of W_1 and W_2. If*

(1) *$\partial W_1 \hookrightarrow W_1$, $W_1 \cap W_2 \hookrightarrow W_2$, $\partial W_2 - W_1 \hookrightarrow W_2$ are monic maps and*

(2) $$N(\partial(W_1 \cap W_2) \to W_1 \cap W_2) = N(\partial(W_1 \cap W_2) \to \partial W_1 - W_2) = N(\partial(W_1 \cap W_2) \to \partial W_2 - W_1),$$

then $\partial W \hookrightarrow W$ is a monic map.

Then, to show that (A, l_A) verifies property (i) of Definition 1·1 use the Six lemma taking $W_1 = B - l_B$ and $W_2 = N - (N \cap l_A)$ where $(N, N \cap l_A)$ is the closure of a relative regular neighbourhood of $(D, D \cap l_A)$ in (A, l_A). Thus $W = W_1 \cup W_2 = A - l_A$. Property (ii) is easily verified. ∎

Let α be an essential loop in $\partial A - l_A$, for the pair (A, l_A). Attach a 2-handle H^2 along α and let c be the cocore of H^2; then we say $(A \cup H^2, l_A \cup c)$ *is obtained from (A, l_A) by attaching a 2-handle along α.*

LEMMA 1·2. *If (A, l_A) is an arcbody, so is $(A \cup H^2, l_A \cup c)$.*

Proof. To check that $(A \cup H^2, l_A \cup c)$ verifies property (i) of Definition 1·1 use the Six lemma taking $W_1 = A - l_A$, $W_2 = H^2 - c$, $W = A \cup H^2 - l_A \cup c$. The properties (ii) and (iii) are clear. ∎

Note. The converse of Lemma 1·2 is not true in general. The pair (A, l_A) of Fig. 1(*b*) is not an arcbody. But $(A \cup H^2, l_A \cup c)$ of Fig. 1(*a*) is an arcbody.

Assume (A, l_A) and (B, l_B) are connected arcbodies such that there is a homeomorphism $\psi: (X_1, X_1 \cap l_A) \to (X_2, X_2 \cap l_B)$ where X_1(resp. X_2) is a connected component of ∂A(resp. ∂B), then, we say $(C, l_C) = (B, l_B) \underset{\psi}{\cup} (A, l_A)$ *is the result of pasting (A, l_A) to (B, l_B).*

LEMMA 1·3. *The result (C, l_C) of pasting (A, l_A) to (B, l_B) is an arcbody if $\partial C \neq \varnothing$.*

Proof. To check property (i) of Definition 1·1, i.e. that $\partial C - l_C \hookrightarrow C - l_C$ is monic, use the Six lemma with $W_1 = B - l_B$, $W_2 = A - l_A$. Let us now check property (ii). Suppose $(C, l_C) \cong (D^2 \times I, o \times I)$ or $(D^2 \times S^1, o \times S^1)$ and $A \subset \operatorname{int} C$. The homomorphisms

$$\pi_1(\partial A - l_C) \to \pi_1(A - l_A), \quad \pi_1(\partial A - l_C) \to \pi_1(B - l_B)$$

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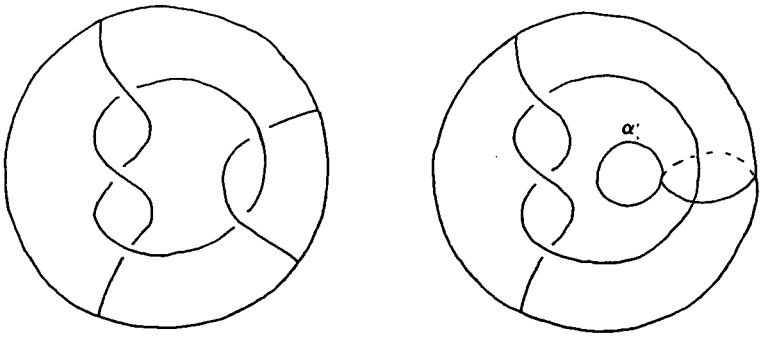


Fig. 1

are injective and, by Van Kampen's theorem, $\pi_1(\partial A - l_C) \to \pi_1(C - l_C)$ is also injective, with $\pi_1(C - l_C) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. Then $\pi_1(\partial A - l_C)$ is a free abelian subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ which is impossible if (A, l_A) is an arcbody. Thus property (ii) is true. Property (iii) is trivial. ∎

We define the *complexity* of an arcbody (A, l_A) with ∂A connected, to be the ordered pair $c(A, l_A) = (g, k)$ where g is the genus of ∂A and k is the number of points of $\partial A \cap l_A$. Complexities are ordered lexicographically as follows: $(g, k) < (g', k')$ if either $g < g'$ or $g = g'$ and $k < k'$.

Recall that (S^3, L) is a *prime link* if L is a link of disjoint simple closed curves in S^3 such that any S^2 meeting L transversally in two points bounds in S^3 exactly one ball intersecting L in a single unknotted spanning arc.

Recall also that a closed orientable 3-manifold M^3 is *sufficiently large* if it contains a closed incompressible surface of positive genus.

Let $\tilde{M}$ be a closed, oriented 3-manifold, and G a finite non-trivial group acting effectively on $\tilde{M}$ as a group of orientation-preserving diffeomorphisms. Suppose that the union of the exceptional orbits is a link $\tilde{L}$. (Equivalently, all the isotropy groups G_x, $x \in \tilde{M}$, are cyclic.) If the quotient manifold is S^3, and $L = p(\tilde{L})$, where $p: \tilde{M} \to S^3$ is the quotient map, then $\tilde{M}$ is a *regular branched covering over* L.

2. *Fracturable links*

Let (M^3, L) be a link in a closed orientable 3-manifold M^3. We say that (M^3, L) is *fracturable* if there is a compact closed surface F embedded in M^3 dividing (M^3, L) in two arcbodies. Then F gives a *fracture* of (M^3, L).

A fracture is *minimal* if the surface F is connected and there are no cutting discs in any of the two arcbodies of the fracture.

PROPOSITION 2·1. *A fracturable prime link (S^3, L) has a minimal fracture.*

Proof. Suppose F gives a fracture of (S^3, L) in two arcbodies $(A_1, L \cap A_1), (A_2, L \cap A_2)$. If F is not connected, let $C_1, \ldots, C_n$ be the connected component of A_1 and A_2, $n > 2$, then $(C_i, L \cap C_i)$, $i = 1, \ldots, n$, are also arcbodies. Let $(C_j, L \cap C_j)$ be one of them with ∂C_j connected. Then ∂C_j is strictly contained in ∂C_k for some $k \neq j$. Call $(C'_k, L \cap C'_k)$ the result of pasting $(C_j, L \cap C_j)$ to $(C_k, L \cap C_k)$, using the natural map. Then $(C'_k, L \cap C'_k)$ is an arcbody (Lemma 1·3) and the number of connected components of F has been reduced by one. Thus assume F is connected.

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Let D be a cutting disc in $(A_1, L \cap A_1)$ or $(A_2, L \cap A_2)$ (say $(A_1, L \cap A_1)$). Cut $(A_1, L \cap A_1)$ along D, and call $(B_1, L \cap B_1)$ the resulting arcbody (Lemma 1·1). Note that then $(A_2, L \cap A_2)$ becomes $(B_2, L \cap B_2)$ which is the result of attaching a 2-handle to $(A_2, L \cap A_2)$ along ∂D, so that $(B_2, L \cap B_2)$ is still an arcbody (Lemma 1·2). Two cases are possible according to whether ∂D is a non-separating or a separating curve in F. Assume ∂D is a non-separating curve in F. Then $\partial B_1 = \partial B_2$ is connected, ∂B_1 gives a fracture of the link (S^3, L), and the genus of ∂B, equals the genus of F minus one. The complexity of any of the arcbodies of the fracture given by ∂B_1 is strictly less than the complexity of any of the arcbodies of the fracture given by F.

If ∂D is a separating curve in F, then B_1 has two connected components Y_1, Y_2 and B_2 is connected with boundary $\partial Y_1 \cup \partial Y_2$. Compare the complexities of $(Y_1, L \cap Y_1)$ and $(Y_2, L \cap Y_2)$ with the complexity of $(A_1, L \cap A_1)$. We see that $c(Y_i, L \cap Y_i) \leqslant c(A_1, L \cap A_1)$, $i = 1, 2$. The case $c(Y_1, L \cap Y_1) = c(Y_2, L \cap Y_2) = c(A_1, L \cap A_1)$ contradicts the fact that L is prime. So, at least one of them, say $c(Y_1, L \cap Y_1)$, is strictly less than $c(A_1, L \cap A_1)$. Let $(Y, L \cap Y)$ be the arcbody $(B_2, L \cap B_2) \underset{\mathrm{id}.\partial Y_1}{\cup} (Y_2, L \cap Y_2)$ (Lemma 1·3). Then ∂Y_1 gives a fracture of (S^3, L) in two arcbodies of strictly smaller complexity than the complexity of the arcbodies of the fracture given by F.

Then, by eliminating cutting discs according to the preceding process, we arrive at a minimal fracture.

3. *The main result*

THEOREM 3·1. *Let (S^3, L) be a fracturable, prime link, then any regular covering $\tilde{M}$ of S^3 branched over L is a sufficiently large 3-manifold.*

Proof. Let F be a closed surface in S^3 which gives a minimal fracture of (S^3, L) in the two arcbodies $(M_1, L \cap M_1)$, $(M_2, L \cap M_2)$ (Proposition 2·1). And let $p: \tilde{M} \to S^3$ be a regular branched covering over L. Then $p^{-1}(F)$ is a possibly non-connected surface. Now, each component of $p^{-1}(F)$ has positive genus: this is clear if genus of F is positive; and if $F = S^2$ then $F \cap L$ has at least 4 points, and using the Riemann–Hurwitz formula (4) it follows that the components of $p^{-1}(F)$ have positive genus.

Suppose $p^{-1}(F)$ is compressible and let D be a compression disc transversal to $\tilde{L}$ in $\tilde{M}$, i.e. $D \cap p^{-1}(F) = \partial D$ with ∂D essential in $p^{-1}(F)$. Then $D \subset \tilde{M}_1$ or $D \subset \tilde{M}_2$. Suppose $D \subset \tilde{M}_1$. Using the equivariant loop theorem (6), we can assume D is a disc such that, for each covering transformation f, $fD = D$ or $fD \cap D = \varnothing$.

Two cases are possible:

(1) *$fD \cap D = \varnothing$, for every covering transformation f.* Then $D \cap \tilde{L} = \varnothing$ and $p(D)$ is a disc in M_1 such that $p(D) \cap L = \varnothing$. As $(M_1, L \cap M_1)$ is an arcbody, $\partial p(D)$ bounds a disc $D' \subset \partial M_1 - L \subset \partial M_1 = F$. Then, $p^{-1}(D')$ are discs in $p^{-1}(F)$, one of them has boundary ∂D, contradicting the essentiality of ∂D.

(2) *There are covering transformations f such that $fD = D$.* These elements form a cyclic group because the covering is regular, and $D \cap \tilde{L}$ consists of one interior point of D (compare ((3), p. 8)). Then $p(D)$ is a disc in M_1 intersecting L in one point, and (M_1, L_{M_1}) does not have cutting discs, then $\partial p(D)$ bounds a disc D' in F intersecting L in just one point. Thus $p^{-1}(D')$ are discs in $p^{-1}(F)$, one of them with boundary ∂D, which is impossible. ∎

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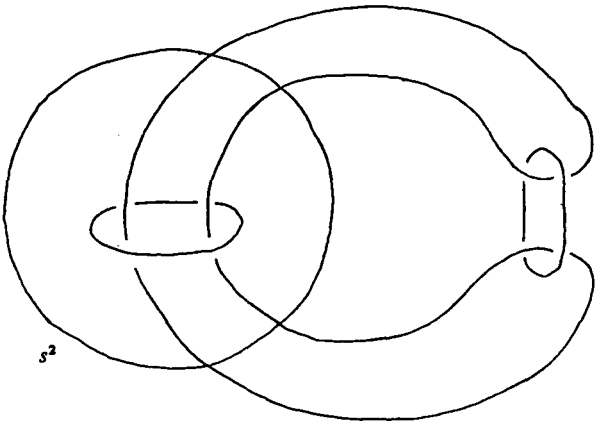


Fig. 2

PROPOSITION 3·1. *The following knots and links are fracturable.*

(i) *n-string composite* (1), *i.e. a prime knot K in S^2 such that there is a embedded S^2 intersecting the knot transversally which separates (B^3, K) into prime n-string tangles.*

(ii) *Fissionable links* (2), *i.e. a link L in a closed orientable 3-manifold M^3 such that there is a closed surface embedded in M^3 intersecting each component of L transversally, which separates the exterior of L, $E(L)$, in two manifolds with incompressible boundary.*

(iii) *A link L in an orientable closed 3-manifold M^3 such that $M - L$ is irreducible and contains a closed non-peripheral separating, incompressible surface. This is a particular case of a sufficiently large link in the sense of* (3).

Proof. (i) Recall that a n-string tangle (B, t) is a 3-ball with n properly imbedded disjoint spanning arcs t, and if it is prime (1), it is an arcbody. So part (i) is trivial.

(ii) Let F be the closed surface intersecting each component of L transversally, dividing M^3 in two parts A and B such that $A \cap E(L) = W_1$, $B \cap E(L) = W_2$, $\partial W_i \hookrightarrow W_i$ are monic, $\partial W_i \cap L = \varnothing$ and $\partial W_i \neq S^2$ $i = 1, 2$. Then $(W_i, \varnothing)$ are arcbodies, $i = 1, 2$. Note that $(A, L \cap A)$ is obtained from $(W_1, \varnothing)$ by attaching 2-handles along meridians of L, and thus $(A, L \cap A)$ is an arcbody. Analogously $(B, L \cap B)$ is also an arcbody. Then, F gives a fracture of (M^3, L). Note also that (ii) $\Rightarrow$ (iii) because ∂W_i is a closed non-peripheral incompressible surface in $M^3 - L$.

(iii) The closed non-peripheral incompressible surface gives a fracture.

4. *Example*: *the borromean rings*

Let N be the link 6_2^3 depicted in Fig. 2 and let $p: M \to S^3$ be the 2-fold covering of S^3 branched over N.

The surface S^2 of Fig. 2 gives a fracture of N, into two arcbodies. But the exterior $E(N)$ of N does not have any closed non-peripheral incompressible surface. For if F were such a surface there would exist an equivariant incompressible surface G in M obtained from $p^{-1}(F)$ by cutting along equivariant compressing discs which meet $p^{-1}(N)$ transversally in a single point, as in proof of (3). Then, F would be obtained from $p(G) \cap E(N)$ by pasting peripheral tubes to $p(G) \cap E(N)$ along their boundaries because the lifting of this operation to M^3 is the inverse of making surgery along equivariant compressing discs. The manifold M is $M_\psi(0/1)$ where ψ is equal to $\bar{\alpha}\beta\bar{\alpha}\beta\bar{\alpha}\beta$

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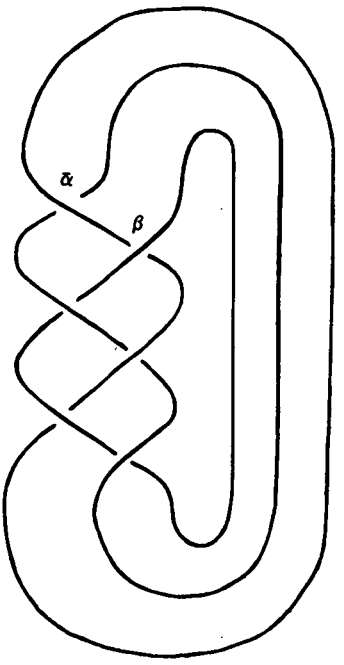


Fig. 3

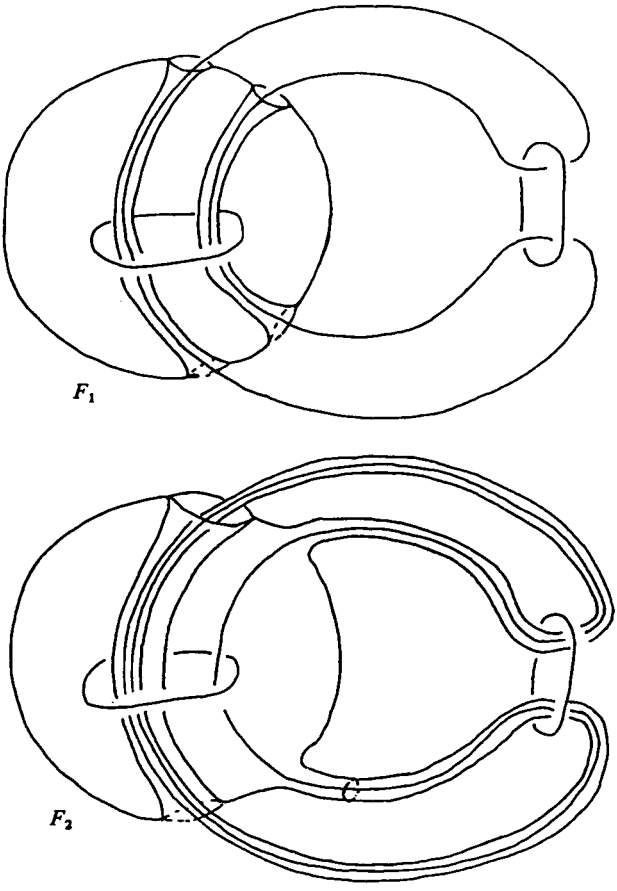


Fig. 4

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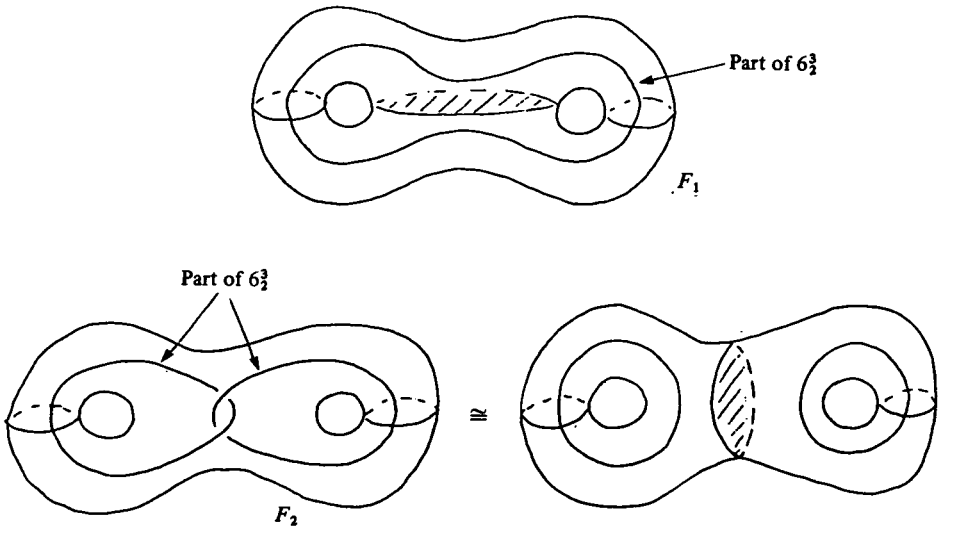


Fig. 5

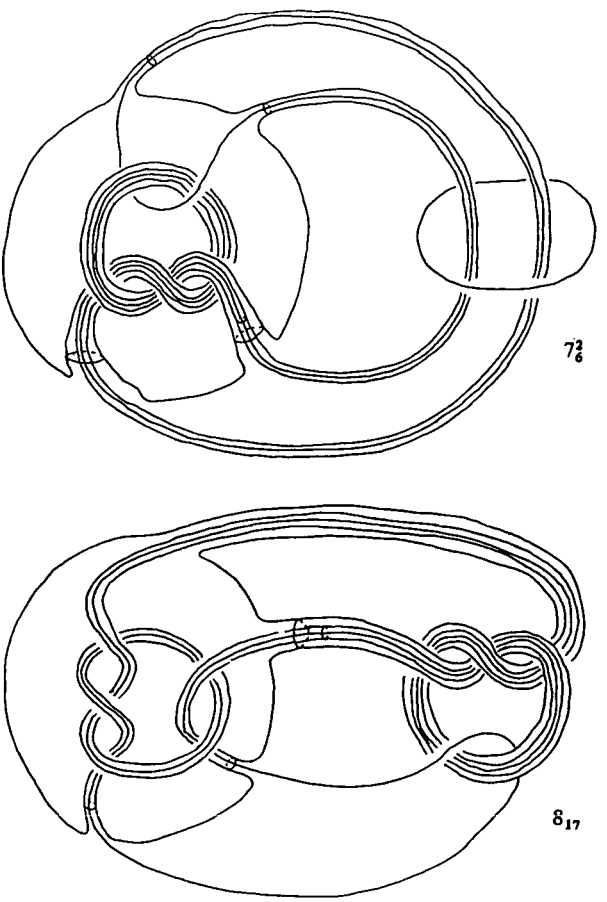


Fig. 6

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(Fig. 3) and by using the methods of Przytycki's thesis (7) we obtain that the only incompressible surfaces in M are three tori, one of them, by Theorem 3·1, projects onto S^2 of Fig. 2, and the other two onto the other two obvious S^2's, obtained by using the symmetry of 6_2^3.

Thus, F is one of the surfaces pictured in Fig. 4. Now F_1 and F_2 are compressible, because they bound the manifolds of Fig. 5 (which are genus 2 handlebodies minus curves). Thus F is compressible.

Notes. (1) $T^3 = S^1 \times S^1 \times S^1$ is a 4-fold covering of M, thus T^3 is an 8-fold regular covering of S^3 branched over N, and it has three incompressible tori which project onto the three S^2's fracturing N.

(2) It is easy to obtain more examples by pasting some arcbodies in different ways. In particular 7_6^2 and 8_{17} are fracturable, and the surfaces depicted in Fig. 6 are incompressible in the complement.

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